Finite Mathematics and Calculus with Applications

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Finite Mathematics and Calculus with Applications

NINTH EDITION

Margaret L. Lial
American River College

Raymond N. Greenwell
Hofstra University

Nathan P. Ritchey
Youngstown State University

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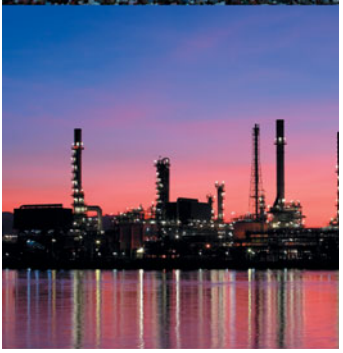

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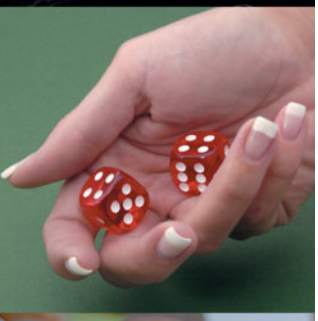

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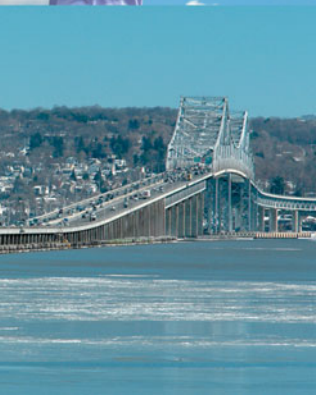

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Special Topics to Accompany Finite Mathematics and Calculus with Applications

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Digraphs and Networks
- Graphs and Digraphs
- Dominance Digraphs
- Communication Digraphs
- Networks

Review Exercises

Preface

Finite Mathematics and Calculus with Applications is a thorough, application-oriented text for students majoring in business, management, economics, or the life or social sciences. In addition to its clear exposition, this text consistently connects the mathematics to career and everyday-life situations. A prerequisite of two years of high school algebra is assumed. A renewed focus on quick and effective assessments, new applications and exercises, as well as other new learning tools make this 9th edition an even richer learning resource for students.

Our Approach

Our main goal is to present finite mathematics and applied calculus in a concise and meaningful way so that students can understand the full picture of the concepts they are learning and apply it to real-life situations. This is done through a variety of ways.

Focus on Applications Making this course meaningful to students is critical to their success. Applications of the mathematics are integrated throughout the text in the exposition, the examples, the exercise sets, and the supplementary resources. *Finite Mathematics and Calculus with Applications* presents students with a myriad of opportunities to relate what they're learning to career situations through the *Apply It* questions, the applied examples, and the *Extended Applications.* To get a sense of the breadth of applications presented, look at the Index of Applications in the back of the book or the extended list of sources of real-world data on www.pearsonhighered.com/mathstatsresources.

Pedagogy to Support Students Students need careful explanations of the mathematics along with examples presented in a clear and consistent manner. Additionally students and instructors should have a means to assess the basic prerequisite skills. This can now be done with the *Prerequisite Skills Diagnostic Test* located just before Chapter R. In addition, the students need a mechanism to check their understanding as they go and resources to help them remediate if necessary. *Finite Mathematics and Calculus with Applications* has this support built into the pedagogy of the text through fully developed and annotated examples, *Your Turn* exercises, *For Review* references, and supplementary material.

Beyond the Textbook Students today take advantage of a variety of resources and delivery methods for instruction. As such, we have developed a robust MyMathLab course for *Finite Mathematics and Calculus with Applications.* MyMathLab has a well-established and well-documented track record of helping students succeed in mathematics. The MyMathLab online course for *Finite Mathematics and Calculus with Applications* contains over 6700 exercises to challenge students and provides help when they need it. Students who learn best by seeing and hearing can view section- and example-level videos within MyMathLab or on the book-specific DVD-Rom. These and other resources are available to students as a unified and reliable tool for their success.

New to the Ninth Edition

Based on the authors' experience in the classroom along with feedback from many instructors across the country, the focus of this revision is to improve the clarity of the presentation and provide students with more opportunities to learn, practice, and apply what they've learned on their own. This is done in both the presentation of the content and in new features added to the text.

New and Revised Content

• **Chapter R** The flow of the material was improved by reordering some exercises and examples. Exercises were added to Section R.1 (on performing algebraic operations) and Section R.5 (on solving inequalities).

• **Chapter 1** Changes in the presentation were made throughout to increase clarity, including adding some examples and rewriting others. Terminology in Section 1.2 was adjusted to be more consistent with usage in economics.

• **Chapter 2** Section 2.1 was changed so that only systems of two equations are solved by the echelon method, while systems with three or more equations are solved using the Gauss-Jordan method in Section 2.2. The discussion of subtraction of matrices in Section 2.3 was simplified.

• **Chapter 3** The concept of bounded and unbounded regions was moved from Section 3.2 to Section 3.1, where such regions are first encountered. An *Extended Application* on sensitivity analysis was added to this chapter.

• **Chapter 4** Exercises 25 through 30 in Section 4.1 were modified to clarify the role of slack variables. Exercise 30 in Section 4.2 was modified to amplify how multiple solutions may occur. The method for handling ties in nonstandard problems in Section 4.4 was improved.

• **Chapter 5** In Section 5.1, examples and accompanying exercises were added covering how to solve for the interest rate and how to find the compounding time, both with a graphing calculator and with logarithms. The explanation of the rule of 70 and the rule of 72 was improved. Material on continuous compounding was also added to Section 5.1. In Section 5.3, an example and accompanying exercises were added on how a loan can be paid off early.

• **Chapter 6** Many exercises in this chapter were revised so that the information would be more relevant to students. For example, tax references include scholarships, tuition, paychecks, reporting tips, filing taxes, inheritances, and tuition deductions. Law references include car accidents, contracts, lawsuits, driver's licenses, and marriage, and warranty references cover iPhones and eBay. In Section 6.5, applications were revised to give more diversity in topics.

• **Chapter 7** Empirical probability was moved from Section 7.4 to 7.3 so that methods for determining probability are contained in the same section. In Section 7.4, probability distributions are emphasized more and a probability distribution example was added. The introduction to Bayes' Theorem was rewritten for brevity and clarity in Section 7.6.

• **Chapter 8** The notation for combinations was changed from $\binom{n}{r}$ to $C(n, r)$ to be more current and consistent with our notation throughout the book. Section 8.3 now includes an example illustrating probabilities using permutations and the multiplication principle.

• **Chapter 9** In Section 9.1, a new example was added illustrating a case in which the median is a truer representation of data than the mean.

• **Chapter 10** The material in Section 10.1 on the Dow Jones Average was updated. Material on even and odd functions was added. Material on identifying the degree of a polynomial has been rewritten as an example to better highlight the concept. The discussion of the Rule of 70 and the Rule of 72 was improved. A new *Extended Application* on Power Functions has been added.

• **Chapter 11** In Section 11.1, the introduction of limits was completely revised. The opening discussion and example were transformed into a series of examples that progress through different limit scenarios: a function defined at the limit, a function undefined at the limit (a hole in the graph), a function defined at the limit but with a different value than the limit (a piecewise function), and then finally, finding a limit when one does not exist. New figures were added to illustrate the different scenarios. In Section 11.2 the definition and example of continuity has been revised using a simple process to test for continuity. The opening discussion of Section 11.5, showing how to sketch the graph of the derivative given the graph of the original function, was rewritten as an example.

• **Chapter 12** The introduction to the chain rule was rewritten as an example in Section 12.3. Exercise topics were revised to cover subjects such as worldwide Internet users, online learning, and the Gateway Arch.

• **Chapter 13** In Section 13.1, the definition of increasing/decreasing functions has been moved to the beginning of the chapter, followed by the discussion of using derivatives to determine where the function increases and decreases. The determination of where a function is increasing or decreasing is divided into three examples: when the critical numbers are found by setting the derivative equal to zero, when the critical numbers are found by determining where the derivative is undefined, and when the function has no critical numbers.

• **Chapter 14** Changes in the presentation were made throughout to increase clarity and exercise sets were rearranged to improve progression and parity.

• **Chapter 15** The social sciences category of exercises was added to Section 15.1, including the topics of bachelor's degrees and the number of females earning degrees in dentistry. Color was added to the introduction and first example of substitution in Section 15.2 to enable students to follow the substitution more easily.

• **Chapter 16** In addition to exercises based on real data being updated, examples in this chapter were changed for pedagogical reasons.

• **Chapter 17** Graphs generated by Maple™ were added to Examples 2 and 4 in Section 17.3 to assist students in visualizing the concept of relative extrema. Material covering utility functions was added to Section 17.4. Many of the figures of three-dimensional surfaces were improved to make them clearer and more attractive.

• **Chapter 18** In Section 18.2, an example on how to calculate the probability within one standard deviation of the mean (which is required in many of the exercises) was added. The Social Sciences category was added to the exercise set, with exercises on calculating the median, expected value, and standard deviation. Topics include the time it takes to learn a task and the age of users of a social network.

Prerequisite Skills Diagnostic Test

The Prerequisite Skills Diagnostic Test gives students and instructors a means to assess the basic prerequisite skills needed to be successful in this course. In addition, the answers to the test include references to specific content in Chapter R as applicable so students can zero in on where they need improvement. Solutions to the questions in this test are in Appendix A.

More Applications and Exercises

This text is used in large part because of the enormous amounts of real data used in examples and exercises throughout the text. This 9th edition will not disappoint in this area. We have added or updated more than 20% of the applications and 32% of the examples throughout the text and added or updated more than 600 exercises.

Reference Tables for Exercises

The answers to odd-numbered exercises in the back of the textbook now contain a table referring students to a specific example in the section for help with most exercises. For the review exercises, the table refers to the section in the chapter where the topic of that exercise is first discussed.

Annotated Instructor's Edition

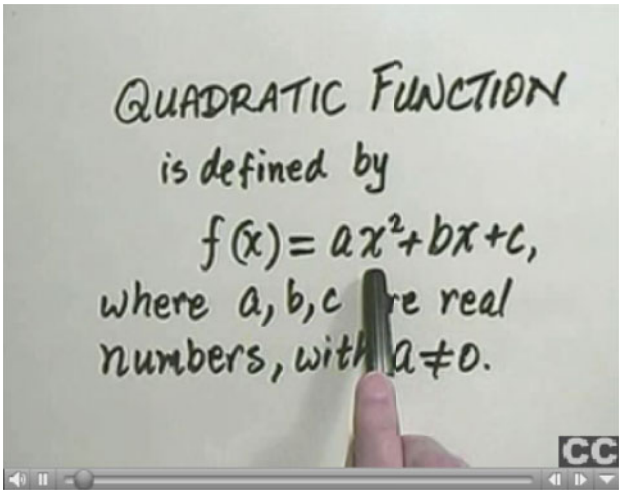
The annotated instructor's edition is filled with valuable teaching tips in the margins for those instructors who are new to teaching this course. In addition, answers to most exercises are provided directly on the exercise set page along with + symbol next to the most challenging exercises to make assigning and checking homework easier.

New to MyMathLab

Available now with *Finite Mathematics and Calculus with Applications* are the following resources within MyMathLab that will benefit students in this course.

- "Getting Ready for Finite Mathematics" and "Getting Ready for Applied Calculus" chapters cover basic prerequisite skills
- Personalized Homework allows you to create homework assignments based on the results of student assessments
- Videos with extensive section coverage
- Hundreds more assignable exercises than the previous edition of the text
- Application labels within exercise sets (e.g., "Bus/Econ") make it easy for you to find types of applications appropriate to your students
- Additional graphing calculator and Excel spreadsheet help

A detailed description of the overall capabilities of MyMathLab is provided on page xviii.



Source Lines

Sources for the exercises are now written in an abbreviated format within the actual exercise so that students immediately see that the problem comes from, or pulls data from, actual research or industry. The complete references are available at www.pearsonhighered.com/mathstatsresources as well as on page S-1.

Other New Features

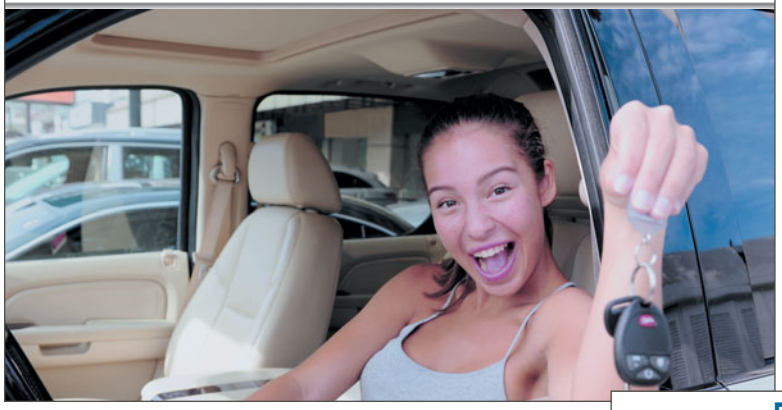
We have worked hard to meet the needs of today's students through this revision. In addition to the new content and resources listed above, there are many new features to this 9th edition including **new and enhanced examples, Your Turn** exercises, the inclusion of and instruction for **new technology**, and **new and updated Extended Applications**. You can view these new features in context in the following *Quick Walk-Through of Finite Mathematics and Calculus with Applications, 9e*.

5

Mathematics of Finance

5.1 Simple and Compound Interest
5.2 Future Value of an Annuity
5.3 Present Value of an Annuity; Amortization
Chapter 5 Review
Extended Application: Time, Money, and Polynomials

Buying a car usually requires both some savings for a down payment and a loan for the balance. An exercise in Section 2 calculates the regular deposits that would be needed to save up the full purchase price, and other exercises and examples in this chapter compute the payments required to amortize a loan.



Chapter Opener

Each chapter opens with a quick introduction that relates to an application presented in the chapter. In addition, a section-level table of contents is included.

Apply It

An **Apply It** question, typically at the start of a section, asks students to consider how to solve a real-life situation related to the math they are about to learn. The **Apply It** question is answered in an application within the section or the exercise set. ("Apply It" was labeled "Think About It" in the previous edition.)

NEW! Teaching Tips

Teaching Tips are provided in the margins of the Annotated Instructor's Edition for those who are new to teaching this course. In addition, answers to most exercises are provided directly on the exercise set page making it easier to assign and check homework.

For Review

For Review boxes are provided in the margin as appropriate, giving students just-in-time help with skills they should already know but may have forgotten. **For Review** comments sometimes include an explanation while others refer students back to earlier parts of the book for a more thorough review.

5.3 Present Value of an Annuity; Amortization

APPLY IT

What monthly payment will pay off a $17,000 car loan in 36 monthly payments at 6% annual interest?

The answer to this question is given in Example 2 in this section. We shall see that it involves finding the present value of an annuity.

Suppose that at the end of each year, for the next 10 years, $500 is deposited in a savings account paying 7% interest compounded annually. This is an example of an ordinary annuity. The **present value of an annuity** is the amount that would have to be deposited in one lump sum today (at the same compound interest rate) in order to produce exactly the same balance at the end of 10 years. We can find a formula for the present value of an annuity as follows.

Suppose deposits of R dollars are made at the end of each period for n periods at interest rate i per period. Then the amount in the account after n periods is the future value of this annuity:

$$S = R \cdot s_{\overline{n}|i} = R\left[\frac{(1+i)^n - 1}{i}\right].$$

Teaching Tip: This section extends the ideas of the previous section to show how to calculate payments on a loan and create an amortization table.

FOR REVIEW

Recall from Section R.6 that for any nonzero number a, $a^0 = 1$. Also, by the product rule for exponents, $a^x \cdot a^y = a^{x+y}$. In particular, if a is any nonzero number $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$.

On the other hand, if P dollars are deposited today at the same compound interest rate i, then at the end of n periods, the amount in the account is $P(1+i)^n$. If P is the present value of the annuity, this amount must be the same as the amount S in the formula above; that is,

$$P(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right].$$

To solve this equation for P, multiply both sides by $(1+i)^{-n}$.

$$P = R(1+i)^{-n}\left[\frac{(1+i)^n - 1}{i}\right]$$

Use the distributive property; also recall that $(1+i)^{-n}(1+i)^n = 1$.

$$P = R\left[\frac{(1+i)^{-n}(1+i)^n - (1+i)^{-n}}{i}\right] = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

The amount P is the *present value of the annuity*. The quantity in brackets is abbreviated as $a_{\overline{n}|i}$, so

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}.$$

(The symbol $a_{\overline{n}|i}$ is read "a-angle-n at i." Compare this quantity with $s_{\overline{n}|i}$ in the previous section.) The formula for the present value of an annuity is summarized on the next page.

Caution

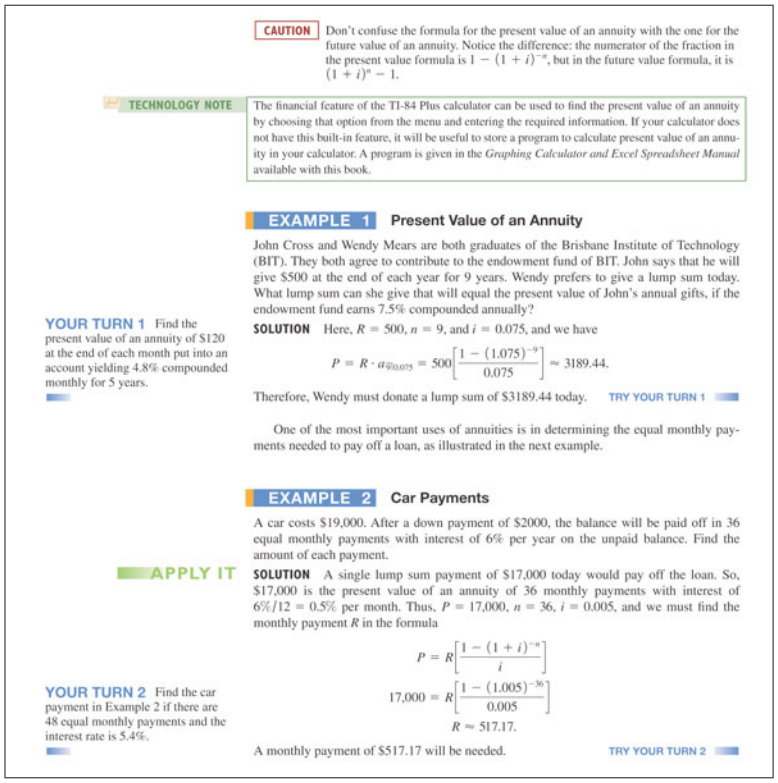
Caution boxes provide students with a quick "heads-up" to common student difficulties and errors.

NEW! "Your Turn" Exercises

The **Your Turn** exercises, following selected examples, provide students with an easy way to quickly stop and check their understanding of the skill or concept being presented. Answers are provided at the end of the section's exercises.

Apply It

The solution to the **Apply It** question often falls in the body of the text where it can be seen in context with the mathematics.



CAUTION Don't confuse the formula for the present value of an annuity with the one for the future value of an annuity. Notice the difference: the numerator of the fraction in the present value formula is $1 - (1 + i)^{-n}$, but in the future value formula, it is $(1 + i)^n - 1$.

TECHNOLOGY NOTE The financial feature of the TI-84 Plus calculator can be used to find the present value of an annuity by choosing that option from the menu and entering the required information. If your calculator does not have this built-in feature, it will be useful to store a program to calculate present value of an annuity in your calculator. A program is given in the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

EXAMPLE 1 Present Value of an Annuity

John Cross and Wendy Mears are both graduates of the Brisbane Institute of Technology (BIT). They both agree to contribute to the endowment fund of BIT. John says that he will give $500 at the end of each year for 9 years. Wendy prefers to give a lump sum today. What lump sum can she give that will equal the present value of John's annual gifts, if the endowment fund earns 7.5% compounded annually?

SOLUTION Here, $R = 500$, $n = 9$, and $i = 0.075$, and we have

$$P = R \cdot a_{\overline{9}|0.075} = 500\left[\frac{1 - (1.075)^{-9}}{0.075}\right] \approx 3189.44.$$

Therefore, Wendy must donate a lump sum of $3189.44 today. TRY YOUR TURN 1

YOUR TURN 1 Find the present value of an annuity of $120 at the end of each month put into an account yielding 4.8% compounded monthly for 5 years.

One of the most important uses of annuities is in determining the equal monthly payments needed to pay off a loan, as illustrated in the next example.

EXAMPLE 2 Car Payments

A car costs $19,000. After a down payment of $2000, the balance will be paid off in 36 equal monthly payments with interest of 6% per year on the unpaid balance. Find the amount of each payment.

APPLY IT

SOLUTION A single lump sum payment of $17,000 today would pay off the loan. So, $17,000 is the present value of an annuity of 36 monthly payments with interest of $6\%/12 = 0.5\%$ per month. Thus, $P = 17{,}000$, $n = 36$, $i = 0.005$, and we must find the monthly payment R in the formula

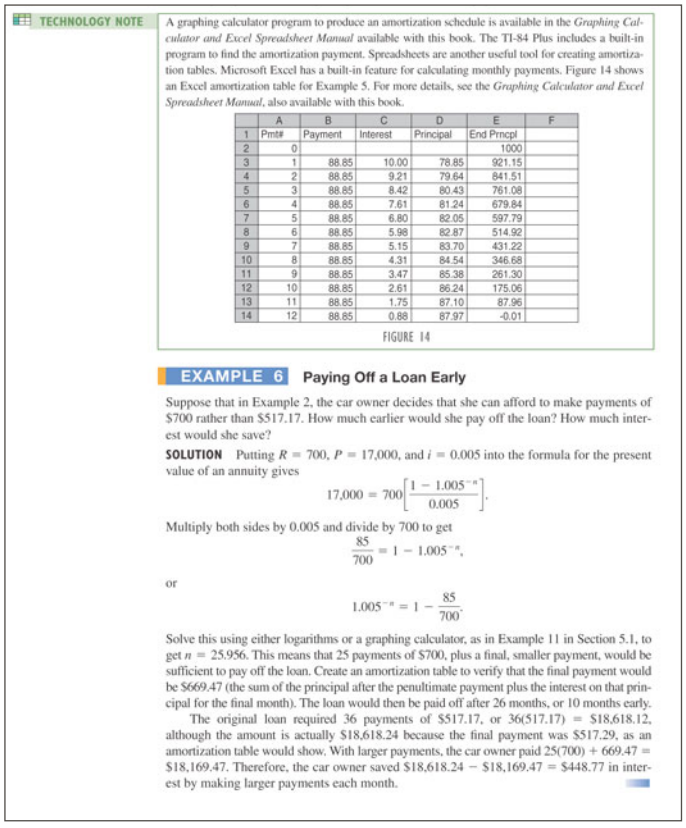
$$P = R\left[\frac{1 - (1 + i)^{-n}}{i}\right]$$

$$17{,}000 = R\left[\frac{1 - (1.005)^{-36}}{0.005}\right]$$

$$R \approx 517.17.$$

A monthly payment of $517.17 will be needed. TRY YOUR TURN 2

YOUR TURN 2 Find the car payment in Example 2 if there are 48 equal monthly payments and the interest rate is 5.4%.



TECHNOLOGY NOTE A graphing calculator program to produce an amortization schedule is available in the *Graphing Calculator and Excel Spreadsheet Manual* available with this book. The TI-84 Plus includes a built-in program to find the amortization payment. Spreadsheets are another useful tool for creating amortization tables. Microsoft Excel has a built-in feature for calculating monthly payments. Figure 14 shows an Excel amortization table for Example 5. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual*, also available with this book.

	A	B	C	D	E	F
1	Pmt#	Payment	Interest	Principal	End Prncpl	
2	0				1000	
3	1	88.85	10.00	78.85	921.15	
4	2	88.85	9.21	79.64	841.51	
5	3	88.85	8.42	80.43	761.08	
6	4	88.85	7.61	81.24	679.84	
7	5	88.85	6.80	82.05	597.79	
8	6	88.85	5.98	82.87	514.92	
9	7	88.85	5.15	83.70	431.22	
10	8	88.85	4.31	84.54	346.68	
11	9	88.85	3.47	85.38	261.30	
12	10	88.85	2.61	86.24	175.06	
13	11	88.85	1.75	87.10	87.96	
14	12	88.85	0.88	87.97	-0.01	

FIGURE 14

EXAMPLE 6 Paying Off a Loan Early

Suppose that in Example 2, the car owner decides that she can afford to make payments of $700 rather than $517.17. How much earlier would she pay off the loan? How much interest would she save?

SOLUTION Putting $R = 700$, $P = 17{,}000$, and $i = 0.005$ into the formula for the present value of an annuity gives

$$17{,}000 = 700\left[\frac{1 - 1.005^{-n}}{0.005}\right].$$

Multiply both sides by 0.005 and divide by 700 to get

$$\frac{85}{700} = 1 - 1.005^{-n},$$

or

$$1.005^{-n} = 1 - \frac{85}{700}.$$

Solve this using either logarithms or a graphing calculator, as in Example 11 in Section 5.1, to get $n = 25.956$. This means that 25 payments of $700, plus a final, smaller payment, would be sufficient to pay off the loan. Create an amortization table to verify that the final payment would be $669.47 (the sum of the principal after the penultimate payment plus the interest on that principal for the final month). The loan would then be paid off after 26 months, or 10 months early.

The original loan required 36 payments of $517.17, or $36(517.17) = \$18{,}618.12$, although the amount is actually $18,618.24 because the final payment was $517.29, as an amortization table would show. With larger payments, the car owner paid $25(700) + 669.47 = \$18{,}169.47$. Therefore, the car owner saved $18,618.24 − $18,169.47 = $448.77 in interest by making larger payments each month.

NEW! Recognizing New Technology

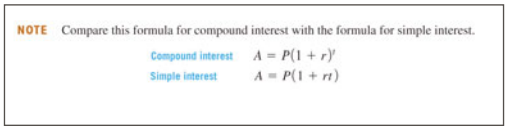
Material on graphing calculators or Microsoft Excel™ is now set off to make it easier for instructors to use this material or not. All of the figures depicting graphing calculator screens have been redrawn to create a more accurate depiction of the math. In addition, this edition references and provides students with a transition to the new MathPrint™ operating system of the TI-84 Plus through the technology notes, a new appendix, and the *Graphing Calculator and Excel Spreadsheet Manual.*

NEW! Enhanced Examples

Most learning from a textbook takes place within the examples of the text. The authors have taken advantage of this by adding more detailed annotations to the already well-developed examples to guide students through new concepts and skills.

Note

Note boxes highlight and emphasize important treatments and asides.



NOTE Compare this formula for compound interest with the formula for simple interest.

Compound interest $A = P(1 + r)^t$

Simple interest $A = P(1 + rt)$

Exercises

Skill-based problems are followed by **application exercises**, which are grouped by subject with subheads indicating the specific topic.

Connection exercises integrate topics presented in different sections or chapters and are indicated with ∞.

5.3 EXERCISES

1. Explain the difference between the present value of an annuity and the future value of an annuity. For a given annuity, which is larger? Why?
2. What does it mean to amortize a loan?

Find the present value of each ordinary annuity.

3. Payments of $890 each year for 16 years at 6% compounded annually
4. Payments of $1400 each year for 8 years at 6% compounded annually
5. Payments of $10,000 semiannually for 15 years at 5% compounded semiannually
6. Payments of $50,000 quarterly for 10 years at 4% compounded quarterly
7. Payments of $15,806 quarterly for 3 years at 6.8% compounded quarterly
8. Payments of $18,579 every 6 months for 8 years at 5.4% compounded semiannually

Find the lump sum deposited today that will yield the same total amount as payments of $10,000 at the end of each year for 15 years at each of the given interest rates.

9. 4% compounded annually
10. 6% compounded annually

Find (a) the payment necessary to amortize each loan; (b) the total payments and the total amount of interest paid based on the calculated monthly payments, and (c) the total payments and total amount of interest paid based upon an amortization table.

11. $2500; 6% compounded quarterly; 6 quarterly payments
12. $41,000; 8% compounded semiannually; 10 semiannual payments
13. $90,000; 6% compounded annually; 12 annual payments
14. $140,000; 8% compounded quarterly; 15 quarterly payments
15. $7400; 6.2% compounded semiannually; 18 semiannual payments
16. $5500; 10% compounded monthly; 24 monthly payments

Suppose that in the loans described in Exercises 13–16, the borrower paid off the loan after the time indicated below. Calculate the amount needed to pay off the loan, using either of the two methods described in Example 4.

ercise 13

Exercise 14

ercise 15

Exercise 16

table in Example 5 to answer the ques-
24.

ourth payment is interest?

leventh payment is used to reduce the debt?

23. How much interest is paid in the first 4 months of the loan?
24. How much interest is paid in the last 4 months of the loan?
25. What sum deposited today at 5% compounded annually for 8 years will provide the same amount as $1000 deposited at the end of each year for 8 years at 6% compounded annually?
26. What lump sum deposited today at 8% compounded quarterly for 10 years will yield the same final amount as deposits of $4000 at the end of each 6-month period for 10 years at 6% compounded semiannually?

Find the monthly house payments necessary to amortize each loan. Then calculate the total payments and the total amount of interest paid.

27. $199,000 at 7.01% for 25 years
28. $175,000 at 6.24% for 30 years
29. $253,000 at 6.45% for 30 years
30. $310,000 at 5.96% for 25 years

Suppose that in the loans described in Exercises 13–16, the borrower made a larger payment, as indicated below. Calculate (a) the time needed to pay off the loan, (b) the total amount of the payments, and (c) the amount of interest saved, compared with part c of Exercises 13–16.

31. $16,000 in Exercise 13
32. $18,000 in Exercise 14
33. $850 in Exercise 15
34. $400 in Exercise 16

APPLICATIONS

Business and Economics

35. **House Payments** Calculate the monthly payment and total amount of interest paid in Example 3 with a 15-year loan, and then compare with the results of Example 3.
36. **Installment Buying** Stereo Shack sells a stereo system for $600 down and monthly payments of $30 for the next 3 years. If the interest rate is 1.25% per month on the unpaid balance, find
    a. the cost of the stereo system.
    b. the total amount of interest paid.
37. **Car Payments** Hong Le buys a car costing $14,000. He agrees to make payments at the end of each monthly period for 4 years. He pays 7% interest, compounded monthly.
    a. What is the amount of each payment?
    b. Find the total amount of interest Le will pay.
38. **Credit Card Debt** Tom Shaffer charged $8430 on his credit card to relocate for his first job. When he realized that the interest rate for the unpaid balance was 27% compounded monthly, he decided not to charge any more on that account. He wants to have this account paid off by the end of 3 years,

48. **Loan Payments** When Nancy Hart opened her law office, she bought $14,000 worth of law books and $7200 worth of office furniture. She paid $1200 down and agreed to amortize the balance with semiannual payments for 5 years, at 8% compounded semiannually.
    a. Find the amount of each payment.
    b. Refer to the text and Figure 13. When her loan had been reduced below $5000, Nancy received a large tax refund and decided to pay off the loan. How many payments were left at this time?
49. **House Payments** Ian Desrosiers buys a house for $285,000. He pays $60,000 down and takes out a mortgage at 6.5% on the balance. Find his monthly payment and the total amount of interest he will pay if the length of the mortgage is
    a. 15 years;
    b. 20 years;
    c. 25 years.
    d. Refer to the text and Figure 13. When will half the 20-year loan in part b be paid off?
50. **House Payments** The Chavara family buys a house for $225,000. They pay $50,000 down and take out a 30-year mortgage on the balance. Find their monthly payment and the total amount of interest they will pay if the interest rate is
    a. 6%;
    b. 6.5%;
    c. 7%.
    d. Refer to the text and Figure 13. When will half the 7% loan in part c be paid off?
51. **Refinancing a Mortgage** Fifteen years ago, the Budai family bought a home and financed $150,000 with a 30-year mortgage at 8.2%.
    a. Find their monthly payment, the total amount of their payments, and the total amount of interest they will pay over the life of this loan.
    b. The Budais made payments for 15 years. Estimate the unpaid balance using the formula
    $$y = R\left[\frac{1-(1+i)^{-(n-x)}}{i}\right],$$
    and then calculate the total of their remaining payments.
    c. Suppose interest rates have dropped since the Budai family took out their original loan. One local bank now offers a 30-year mortgage at 6.5%. The bank fees for refinancing are $3400. If the Budais pay this fee up front and refinance the balance of their loan, find their monthly payment. Including the refinancing fee, what is the total amount of their payments? Discuss whether or not the family should refinance with this option.
    d. A different bank offers the same 6.5% rate but on a 15-year mortgage. Their fee for financing is $4500. If the Budais pay this fee up front and refinance the balance of their loan, find their monthly payment. Including the refinancing fee, what is the total amount of their payments? Discuss whether or not the family should refinance with this option.
52. **Inheritance** Deborah Harden has inherited $25,000 from her grandfather's estate. She deposits the money in an account offering 6% interest compounded annually. She wants to make equal annual withdrawals from the account so that the money (principal and interest) lasts exactly 8 years.
    a. Find the amount of each withdrawal.
    b. Find the amount of each withdrawal if the money must last 12 years.
53. **Charitable Trust** The trustees of a college have accepted a gift of $150,000. The donor has directed the trustees to deposit the money in an account paying 6% per year, compounded semiannually. The trustees may make equal withdrawals at the end of each 6-month period; the money must last 5 years.
    a. Find the amount of each withdrawal.
    b. Find the amount of each withdrawal if the money must last 6 years.

Amortization **Prepare an amortization schedule for each loan.**

54. A loan of $37,948 with interest at 6.5% compounded annually, to be paid with equal annual payments over 10 years.
55. A loan of $4836 at 7.25% interest compounded semi-annually, to be repaid in 5 years in equal semiannual payments.
56. **Perpetuity** A *perpetuity* is an annuity in which the payments go on forever. We can derive a formula for the present value of a perpetuity by taking the formula for the present value of an annuity and looking at what happens when n gets larger and larger. Explain why the present value of a perpetuity is given by
    $$P = \frac{R}{i}.$$
57. **Perpetuity** Using the result of Exercise 56, find the present value of perpetuities for each of the following.
    a. Payments of $1000 a year with 4% interest compounded annually
    b. Payments of $600 every 3 months with 6% interest compounded quarterly

YOUR TURN ANSWERS

1. $6389.86
2. $394.59
3. $1977.42, $135,935.60
4. $679.84

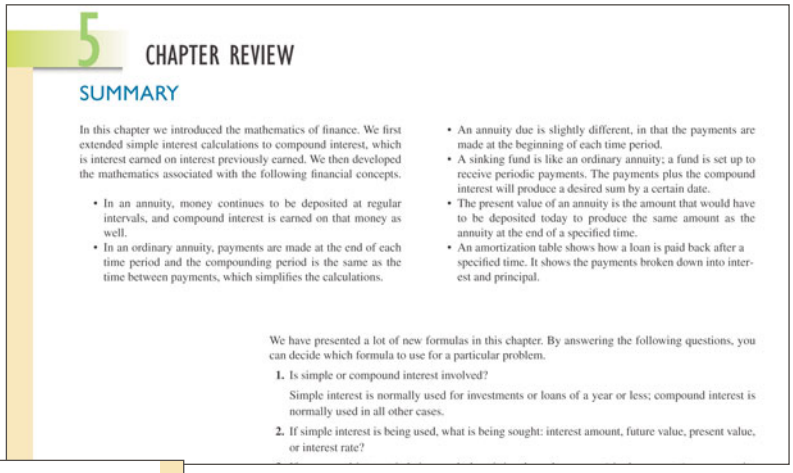
Technology exercises are labeled with for graphing calculator and for spreadsheet.

Writing exercises, labeled with , provide students with an opportunity to explain important mathematical ideas.

Exercises that are particularly **challenging** are denoted with + in the Annotated Instructor's Edition only.

End-of-Chapter Summary

End-of-Chapter Summary provides students with a quick summary of the key ideas of the chapter followed by a list of key definitions, terms, and examples.



5 CHAPTER REVIEW

SUMMARY

In this chapter we introduced the mathematics of finance. We first extended simple interest calculations to compound interest, which is interest earned on interest previously earned. We then developed the mathematics associated with the following financial concepts.

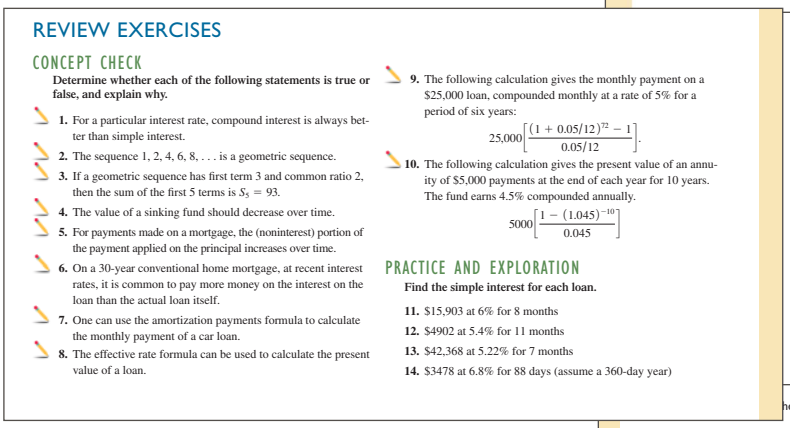
- In an annuity, money continues to be deposited at regular intervals, and compound interest is earned on that money as well.
- In an ordinary annuity, payments are made at the end of each time period and the compounding period is the same as the time between payments, which simplifies the calculations.
- An annuity due is slightly different, in that the payments are made at the beginning of each time period.
- A sinking fund is like an ordinary annuity; a fund is set up to receive periodic payments. The payments plus the compound interest will produce a desired sum by a certain date.
- The present value of an annuity is the amount that would have to be deposited today to produce the same amount as the annuity at the end of a specified time.
- An amortization table shows how a loan is paid back after a specified time. It shows the payments broken down into interest and principal.

We have presented a lot of new formulas in this chapter. By answering the following questions, you can decide which formula to use for a particular problem.

1. Is simple or compound interest involved?

   Simple interest is normally used for investments or loans of a year or less; compound interest is normally used in all other cases.

2. If simple interest is being used, what is being sought: interest amount, future value, present value, or interest rate?



REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. For a particular interest rate, compound interest is always better than simple interest.
2. The sequence 1, 2, 4, 6, 8, . . . is a geometric sequence.
3. If a geometric sequence has first term 3 and common ratio 2, then the sum of the first 5 terms is $S_5 = 93$.
4. The value of a sinking fund should decrease over time.
5. For payments made on a mortgage, the (noninterest) portion of the payment applied on the principal increases over time.
6. On a 30-year conventional home mortgage, at recent interest rates, it is common to pay more money on the interest on the loan than the actual loan itself.
7. One can use the amortization payments formula to calculate the monthly payment of a car loan.
8. The effective rate formula can be used to calculate the present value of a loan.
9. The following calculation gives the monthly payment on a $25,000 loan, compounded monthly at a rate of 5% for a period of six years:

$$25{,}000\left[\frac{(1+0.05/12)^{72}-1}{0.05/12}\right].$$

10. The following calculation gives the present value of an annuity of $5,000 payments at the end of each year for 10 years. The fund earns 4.5% compounded annually.

$$5000\left[\frac{1-(1.045)^{-10}}{0.045}\right]$$

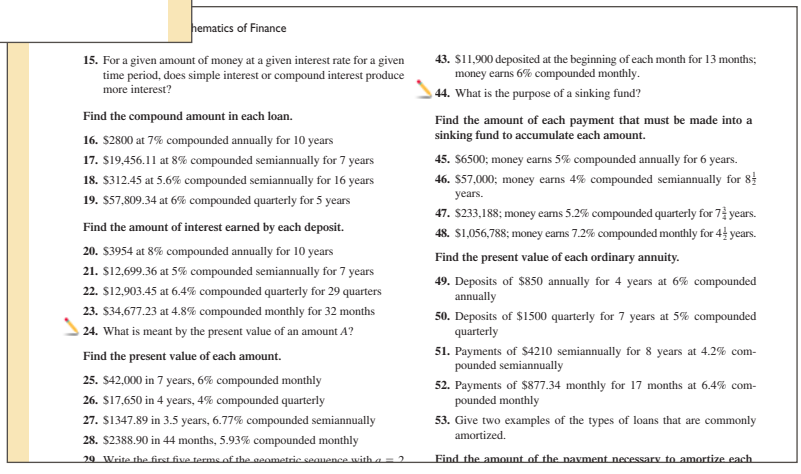
PRACTICE AND EXPLORATION

Find the simple interest for each loan.

11. $15,903 at 6% for 8 months
12. $4902 at 5.4% for 11 months
13. $42,368 at 5.22% for 7 months
14. $3478 at 6.8% for 88 days (assume a 360-day year)

Chapter Review Exercises

Chapter Review Exercises have been slightly reorganized so that the Concept Check exercises fall within the Chapter Review Exercises. This provides students with a more complete review of both the skills and the concepts they should have mastered in this chapter. These exercises in their entirety provide a comprehensive review for a chapter-level exam.



hematics of Finance

15. For a given amount of money at a given interest rate for a given time period, does simple interest or compound interest produce more interest?

Find the compound amount in each loan.

16. $2800 at 7% compounded annually for 10 years
17. $19,456.11 at 8% compounded semiannually for 7 years
18. $312.45 at 5.6% compounded semiannually for 16 years
19. $57,809.34 at 6% compounded quarterly for 5 years

Find the amount of interest earned by each deposit.

20. $3954 at 8% compounded annually for 10 years
21. $12,699.36 at 5% compounded semiannually for 7 years
22. $12,903.45 at 6.4% compounded quarterly for 29 quarters
23. $34,677.23 at 4.8% compounded monthly for 32 months
24. What is meant by the present value of an amount A?

Find the present value of each amount.

25. $42,000 in 7 years, 6% compounded monthly
26. $17,650 in 4 years, 4% compounded quarterly
27. $1347.89 in 3.5 years, 6.77% compounded semiannually
28. $2388.90 in 44 months, 5.93% compounded monthly
29. Write the first five terms of the geometric sequence with $a = 2$

43. $11,900 deposited at the beginning of each month for 13 months; money earns 6% compounded monthly.
44. What is the purpose of a sinking fund?

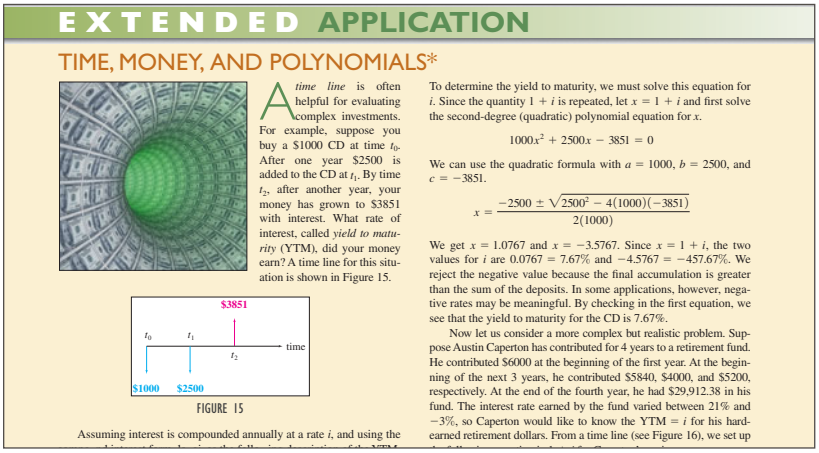
Find the amount of each payment that must be made into a sinking fund to accumulate each amount.

45. $6500; money earns 5% compounded annually for 6 years.
46. $57,000; money earns 4% compounded semiannually for $8\frac{1}{2}$ years.
47. $233,188; money earns 5.2% compounded quarterly for $7\frac{1}{4}$ years.
48. $1,056,788; money earns 7.2% compounded monthly for $4\frac{1}{2}$ years.

Find the present value of each ordinary annuity.

49. Deposits of $850 annually for 4 years at 6% compounded annually
50. Deposits of $1500 quarterly for 7 years at 5% compounded quarterly
51. Payments of $4210 semiannually for 8 years at 4.2% compounded semiannually
52. Payments of $877.34 monthly for 17 months at 6.4% compounded monthly
53. Give two examples of the types of loans that are commonly amortized.

Find the amount of the payment necessary to amortize each



EXTENDED APPLICATION

TIME, MONEY, AND POLYNOMIALS*

A *time line* is often helpful for evaluating complex investments. For example, suppose you buy a $1000 CD at time t_0. After one year $2500 is added to the CD at t_1. By time t_2, after another year, your money has grown to $3851 with interest. What rate of interest, called *yield to maturity* (YTM), did your money earn? A time line for this situation is shown in Figure 15.

FIGURE 15

Assuming interest is compounded annually at a rate i, and using the

To determine the yield to maturity, we must solve this equation for i. Since the quantity $1 + i$ is repeated, let $x = 1 + i$ and first solve the second-degree (quadratic) polynomial equation for x.

$$1000x^2 + 2500x - 3851 = 0$$

We can use the quadratic formula with $a = 1000$, $b = 2500$, and $c = -3851$.

$$x = \frac{-2500 \pm \sqrt{2500^2 - 4(1000)(-3851)}}{2(1000)}$$

We get $x = 1.0767$ and $x = -3.5767$. Since $x = 1 + i$, the two values for i are $0.0767 = 7.67\%$ and $-4.5767 = -457.67\%$. We reject the negative value because the final accumulation is greater than the sum of the deposits. In some applications, however, negative rates may be meaningful. By checking in the first equation, we see that the yield to maturity for the CD is 7.67%.

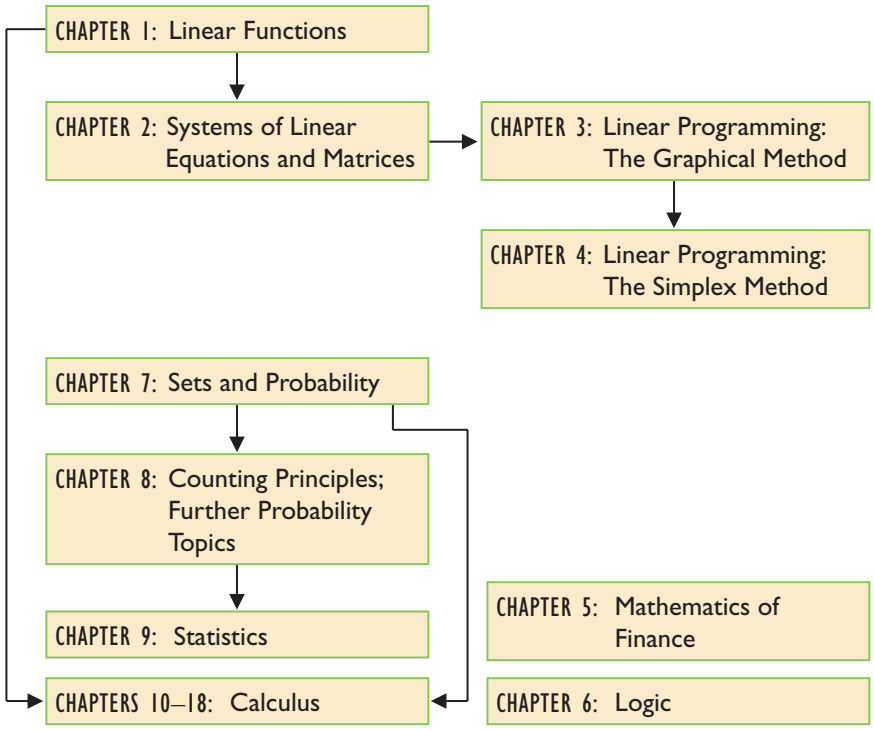
Now let us consider a more complex but realistic problem. Suppose Austin Caperton has contributed for 4 years to a retirement fund. He contributed $6000 at the beginning of the first year. At the beginning of the next 3 years, he contributed $5840, $4000, and $5200, respectively. At the end of the fourth year, he had $29,912.38 in his fund. The interest rate earned by the fund varied between 21% and -3%, so Caperton would like to know the YTM $= i$ for his hard-earned retirement dollars. From a time line (see Figure 16), we set up

Extended Applications

Extended Applications are provided now at the end of **every** chapter as in-depth applied exercises to help stimulate student interest. These activities can be completed individually or as a group project.

Flexible Syllabus

The flexibility of the text is indicated in the following chart of chapter prerequisites. As shown, the course could begin with either Chapter 1 or Chapter 7. Chapter 5 on the mathematics of finance and Chapter 6 on logic could be covered at any time, although Chapter 6 makes a nice introduction to ideas covered in Chapter 7.



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Margaret L. Lial
Raymond N. Greenwell
Nathan P. Ritchey

Dear Student,

Hello! The fact that you're reading this preface is good news. One of the keys to success in a math class is to read the book. Another is to answer all the questions correctly on your professor's tests. You've already started doing the first; doing the second may be more of a challenge, but by reading this book and working out the exercises, you'll be in a much stronger position to ace the tests. One last essential key to success is to go to class and actively participate.

You'll be happy to discover that we've provided the answers to the odd-numbered exercises in the back of the book. As you begin the exercises, you may be tempted to immediately look up the answer in the back of the book, and then figure out how to get that answer. It is an easy solution that has a consequence—you won't learn to do the exercises without that extra hint. Then, when you take a test, you will be forced to answer the questions without knowing what the answer is. Believe us, this is a lot harder! The learning comes from figuring out the exercises. Once you have an answer, look in the back and see if your answer agrees with ours. If it does, you're on the right path. If it doesn't, try to figure out what you did wrong. Once you've discovered your error, continue to work out more exercises to master the concept and skill.

Equations are a mathematician's way of expressing ideas in concise shorthand. The problem in reading mathematics is unpacking the shorthand. One useful technique is to read with paper and pencil in hand so you can work out calculations as you go along. When you are baffled, and you wonder, "How did they get that result?" try doing the calculation yourself and see what you get. You'll be amazed (or at least mildly satisfied) at how often that answers your question. Remember, math is not a spectator sport. You don't learn math by passively reading it or watching your professor. You learn mathematics by doing mathematics.

Finally, if there is anything you would like to see changed in the book, feel free to write to us at matrng@hofstra.edu or npritchey@ysu.edu. We're constantly trying to make this book even better. If you'd like to know more about us, we have Web sites that we invite you to visit: http://people.hofstra.edu/rgreenwell and http://people.ysu.edu/~npritchey.

Marge Lial
Ray Greenwell
Nate Ritchey

Prerequisite Skills Diagnostic Test

Below is a very brief test to help you recognize which, if any, prerequisite skills you may need to remediate in order to be successful in this course. After completing the test, check your answers in the back of the book. In addition to the answers, we have also provided the solutions to these problems in Appendix A. These solutions should help remind you how to solve the problems. For problems 5-26, the answers are followed by references to sections within Chapter R where you can find guidance on how to solve the problem and/or additional instruction. Addressing any weak prerequisite skills now will make a positive impact on your success as you progress through this course.

1. What percent of 50 is 10?

2. Simplify $\frac{13}{7} - \frac{2}{5}$.

3. Let x be the number of apples and y be the number of oranges. Write the following statement as an algebraic equation: "The total number of apples and oranges is 75."

4. Let s be the number of students and p be the number of professors. Write the following statement as an algebraic equation: "There are at least four times as many students as professors."

5. Solve for k: $7k + 8 = -4(3 - k)$.

6. Solve for x: $\frac{5}{8}x + \frac{1}{16}x = \frac{11}{16} + x$.

7. Write in interval notation: $-2 < x \leq 5$.

8. Using the variable x, write the following interval as an inequality: $(-\infty, -3]$.

9. Solve for y: $5(y - 2) + 1 \leq 7y + 8$.

10. Solve for p: $\frac{2}{3}(5p - 3) > \frac{3}{4}(2p + 1)$.

11. Carry out the operations and simplify: $(5y^2 - 6y - 4) - 2(3y^2 - 5y + 1)$.

12. Multiply out and simplify $(x^2 - 2x + 3)(x + 1)$.

13. Multiply out and simplify $(a - 2b)^2$.

14. Factor $3pq + 6p^2q + 9pq^2$.

15. Factor $3x^2 - x - 10$.

16. Perform the operation and simplify: $\dfrac{a^2 - 6a}{a^2 - 4} \cdot \dfrac{a - 2}{a}$.

17. Perform the operation and simplify: $\dfrac{x + 3}{x^2 - 1} + \dfrac{2}{x^2 + x}$.

18. Solve for x: $3x^2 + 4x = 1$.

19. Solve for z: $\dfrac{8z}{z + 3} \leq 2$.

20. Simplify $\dfrac{4^{-1}(x^2y^3)^2}{x^{-2}y^5}$.

21. Simplify $\dfrac{4^{1/4}(p^{2/3}q^{-1/3})^{-1}}{4^{-1/4}\,p^{4/3}q^{4/3}}$.

22. Simplify as a single term without negative exponents: $k^{-1} - m^{-1}$.

23. Factor $(x^2 + 1)^{-1/2}(x + 2) + 3(x^2 + 1)^{1/2}$.

24. Simplify $\sqrt[3]{64b^6}$.

25. Rationalize the denominator: $\dfrac{2}{4 - \sqrt{10}}$.

26. Simplify $\sqrt{y^2 - 10y + 25}$.

R Algebra Reference

R.1 Polynomials
R.2 Factoring
R.3 Rational Expressions
R.4 Equations
R.5 Inequalities
R.6 Exponents
R.7 Radicals

In this chapter, we will review the most important topics in algebra. Knowing algebra is a fundamental prerequisite to success in higher mathematics. This algebra reference is designed for self-study; study it all at once or refer to it when needed throughout the course. Since this is a review, answers to all exercises are given in the answer section at the back of the book.

R.1 Polynomials

An expression such as $9p^4$ is a **term**; the number 9 is the **coefficient**, p is the **variable**, and 4 is the **exponent**. The expression p^4 means $p \cdot p \cdot p \cdot p$, while p^2 means $p \cdot p$, and so on. Terms having the same variable and the same exponent, such as $9x^4$ and $-3x^4$, are **like terms**. Terms that do not have both the same variable and the same exponent, such as m^2 and m^4, are **unlike terms**.

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents, and no variables appear in denominators. Examples of polynomials include

$$5x^4 + 2x^3 + 6x, \qquad 8m^3 + 9m^2n - 6mn^2 + 3n^3, \qquad 10p, \qquad \text{and} \qquad -9.$$

Order of Operations

Algebra is a language, and you must be familiar with its rules to correctly interpret algebraic statements. The following order of operations have been agreed upon through centuries of usage.

- Expressions in **parentheses** are calculated first, working from the inside out. The numerator and denominator of a fraction are treated as expressions in parentheses.
- **Powers** are performed next, going from left to right.
- **Multiplication** and **division** are performed next, going from left to right.
- **Addition** and **subtraction** are performed last, going from left to right.

For example, in the expression $(6(x+1)^2 + 3x - 22)^2$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned}
(6(2+1)^2 + 3(2) - 22)^2 &= (6(3)^2 + 3(2) - 22)^2 && \text{Evaluate the expression in the innermost parentheses.}\\
&= (6(9) + 3(2) - 22)^2 && \text{Evaluate 3 raised to a power.}\\
&= (54 + 6 - 22)^2 && \text{Perform the multiplication.}\\
&= (38)^2 && \text{Perform the addition and subtraction from left to right.}\\
&= 1444 && \text{Evaluate the power.}
\end{aligned}$$

In the expression $\dfrac{x^2 + 3x + 6}{x + 6}$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned}
\frac{2^2 + 3(2) + 6}{2 + 6} &= \frac{16}{8} && \text{Evaluate the numerator and the denominator.}\\
&= 2 && \text{Simplify the fraction.}
\end{aligned}$$

Adding and Subtracting Polynomials

The following properties of real numbers are useful for performing operations on polynomials.

Properties of Real Numbers

For all real numbers a, b, and c:

1. $a + b = b + a$; $ab = ba$;	**Commutative properties**
2. $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$;	**Associative properties**
3. $a(b + c) = ab + ac$.	**Distributive property**

EXAMPLE 1 Properties of Real Numbers

(a) $2 + x = x + 2$ Commutative property of addition

(b) $x \cdot 3 = 3x$ Commutative property of multiplication

(c) $(7x)x = 7(x \cdot x) = 7x^2$ Associative property of multiplication

(d) $3(x + 4) = 3x + 12$ Distributive property

One use of the distributive property is to add or subtract polynomials. Only like terms may be added or subtracted. For example,

$$12y^4 + 6y^4 = (12 + 6)y^4 = 18y^4,$$

and

$$-2m^2 + 8m^2 = (-2 + 8)m^2 = 6m^2,$$

but the polynomial $8y^4 + 2y^5$ cannot be further simplified. To subtract polynomials, we use the facts that $-(a + b) = -a - b$ and $-(a - b) = -a + b$. In the next example, we show how to add and subtract polynomials.

EXAMPLE 2 Adding and Subtracting Polynomials

Add or subtract as indicated.

(a) $(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)$

SOLUTION Combine like terms.

$$\begin{aligned}(8x^3 - 4x^2 + 6x) &+ (3x^3 + 5x^2 - 9x + 8)\\ &= (8x^3 + 3x^3) + (-4x^2 + 5x^2) + (6x - 9x) + 8\\ &= 11x^3 + x^2 - 3x + 8\end{aligned}$$

(b) $2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)$

SOLUTION Multiply each polynomial by the coefficient in front of the polynomial, and then combine terms as before.

$$\begin{aligned}2(-4x^4 &+ 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)\\ &= -8x^4 + 12x^3 - 18x^2 - 24 - 9x^3 + 24x^2 - 33x + 21\\ &= -8x^4 + 3x^3 + 6x^2 - 33x - 3\end{aligned}$$

(c) $(2x^2 - 11x + 8) - (7x^2 - 6x + 2)$

SOLUTION Distributing the minus sign and combining like terms yields

$$\begin{aligned}(2x^2 &- 11x + 8) + (-7x^2 + 6x - 2)\\ &= -5x^2 - 5x + 6.\end{aligned}$$

TRY YOUR TURN 1

YOUR TURN 1 Perform the operation $3(x^2 - 4x - 5) - 4(3x^2 - 5x - 7)$.

Multiplying Polynomials

The distributive property is also used to multiply polynomials, along with the fact that $a^m \cdot a^n = a^{m+n}$. For example,

$$x \cdot x = x^1 \cdot x^1 = x^{1+1} = x^2 \quad \text{and} \quad x^2 \cdot x^5 = x^{2+5} = x^7.$$

EXAMPLE 3 Multiplying Polynomials

Multiply.

(a) $8x(6x - 4)$

SOLUTION Using the distributive property yields

$$\begin{aligned}8x(6x - 4) &= 8x(6x) - 8x(4)\\ &= 48x^2 - 32x.\end{aligned}$$

(b) $(3p - 2)(p^2 + 5p - 1)$

SOLUTION Using the distributive property yields

$$\begin{aligned}&(3p - 2)(p^2 + 5p - 1)\\ &= 3p(p^2 + 5p - 1) - 2(p^2 + 5p - 1)\\ &= 3p(p^2) + 3p(5p) + 3p(-1) - 2(p^2) - 2(5p) - 2(-1)\\ &= 3p^3 + 15p^2 - 3p - 2p^2 - 10p + 2\\ &= 3p^3 + 13p^2 - 13p + 2.\end{aligned}$$

(c) $(x + 2)(x + 3)(x - 4)$

SOLUTION Multiplying the first two polynomials and then multiplying their product by the third polynomial yields

$$\begin{aligned}&(x + 2)(x + 3)(x - 4)\\ &= [(x + 2)(x + 3)](x - 4)\\ &= (x^2 + 2x + 3x + 6)(x - 4)\\ &= (x^2 + 5x + 6)(x - 4)\\ &= x^3 + 5x^2 + 6x - 4x^2 - 20x - 24\\ &= x^3 + x^2 - 14x - 24.\end{aligned}$$

TRY YOUR TURN 2

YOUR TURN 2 Perform the operation $(3y + 2)(4y^2 - 2y - 5)$.

A **binomial** is a polynomial with exactly two terms, such as $2x + 1$ or $m + n$. When two binomials are multiplied, the FOIL method (First, Outer, Inner, Last) is used as a memory aid.

EXAMPLE 4 Multiplying Polynomials

Find $(2m - 5)(m + 4)$ using the FOIL method.

SOLUTION

$$\begin{aligned}(2m - 5)(m + 4) &= \underset{\text{F}}{(2m)(m)} + \underset{\text{O}}{(2m)(4)} + \underset{\text{I}}{(-5)(m)} + \underset{\text{L}}{(-5)(4)}\\ &= 2m^2 + 8m - 5m - 20\\ &= 2m^2 + 3m - 20\end{aligned}$$

EXAMPLE 5 Multiplying Polynomials

Find $(2k - 5m)^3$.

SOLUTION Write $(2k - 5m)^3$ as $(2k - 5m)(2k - 5m)(2k - 5m)$. Then multiply the first two factors using FOIL.

$$\begin{aligned}(2k - 5m)(2k - 5m) &= 4k^2 - 10km - 10km + 25m^2\\ &= 4k^2 - 20km + 25m^2\end{aligned}$$

Now multiply this last result by $(2k - 5m)$ using the distributive property, as in Example 3(b).

$$\begin{aligned}&(4k^2 - 20km + 25m^2)(2k - 5m)\\ &= 4k^2(2k - 5m) - 20km(2k - 5m) + 25m^2(2k - 5m)\\ &= 8k^3 - 20k^2m - 40k^2m + 100km^2 + 50km^2 - 125m^3\\ &= 8k^3 - 60k^2m + 150km^2 - 125m^3 \qquad \text{Combine like terms.}\end{aligned}$$

Notice in the first part of Example 5, when we multiplied $(2k - 5m)$ by itself, that the product of the square of a binomial is the square of the first term, $(2k)^2$, plus twice the product of the two terms, $(2)(2k)(-5m)$, plus the square of the last term, $(-5k)^2$.

CAUTION Avoid the common error of writing $(x + y)^2 = x^2 + y^2$. As the first step of Example 5 shows, the square of a binomial has three terms, so

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Furthermore, higher powers of a binomial also result in more than two terms. For example, verify by multiplication that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Remember, for any value of $n \neq 1$,

$$(x + y)^n \neq x^n + y^n.$$

R.1 EXERCISES

Perform the indicated operations.

1. $(2x^2 - 6x + 11) + (-3x^2 + 7x - 2)$
2. $(-4y^2 - 3y + 8) - (2y^2 - 6y - 2)$
3. $-6(2q^2 + 4q - 3) + 4(-q^2 + 7q - 3)$
4. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$
5. $(0.613x^2 - 4.215x + 0.892) - 0.47(2x^2 - 3x + 5)$
6. $0.5(5r^2 + 3.2r - 6) - (1.7r^2 - 2r - 1.5)$
7. $-9m(2m^2 + 3m - 1)$
8. $6x(-2x^3 + 5x + 6)$
9. $(3t - 2y)(3t + 5y)$
10. $(9k + q)(2k - q)$
11. $(2 - 3x)(2 + 3x)$
12. $(6m + 5)(6m - 5)$
13. $\left(\frac{2}{5}y + \frac{1}{8}z\right)\left(\frac{3}{5}y + \frac{1}{2}z\right)$
14. $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$
15. $(3p - 1)(9p^2 + 3p + 1)$
16. $(3p + 2)(5p^2 + p - 4)$
17. $(2m + 1)(4m^2 - 2m + 1)$
18. $(k + 2)(12k^3 - 3k^2 + k + 1)$
19. $(x + y + z)(3x - 2y - z)$
20. $(r + 2s - 3t)(2r - 2s + t)$
21. $(x + 1)(x + 2)(x + 3)$
22. $(x - 1)(x + 2)(x - 3)$
23. $(x + 2)^2$
24. $(2a - 4b)^2$
25. $(x - 2y)^3$
26. $(3x + y)^3$

YOUR TURN ANSWERS

1. $-9x^2 + 8x + 13$ 2. $12y^3 + 2y^2 - 19y - 10$

R.2 Factoring

Multiplication of polynomials relies on the distributive property. The reverse process, where a polynomial is written as a product of other polynomials, is called **factoring**. For example, one way to factor the number 18 is to write it as the product $9 \cdot 2$; both 9 and 2 are **factors** of 18. Usually, only integers are used as factors of integers. The number 18 can also be written with three integer factors as $2 \cdot 3 \cdot 3$.

The Greatest Common Factor

To factor the algebraic expression $15m + 45$, first note that both $15m$ and 45 are divisible by 15; $15m = 15 \cdot m$ and $45 = 15 \cdot 3$. By the distributive property,

$$15m + 45 = \mathbf{15} \cdot m + \mathbf{15} \cdot 3 = \mathbf{15}(m + 3).$$

Both 15 and $m + 3$ are factors of $15m + 45$. Since 15 divides into both terms of $15m + 45$ (and is the largest number that will do so), 15 is the **greatest common factor** for

the polynomial $15m + 45$. The process of writing $15m + 45$ as $15(m + 3)$ is often called **factoring out** the greatest common factor.

EXAMPLE 1 Factoring

Factor out the greatest common factor.

(a) $12p - 18q$

SOLUTION Both $12p$ and $18q$ are divisible by 6. Therefore,

$$12p - 18q = \mathbf{6} \cdot 2p - \mathbf{6} \cdot 3q = \mathbf{6}(2p - 3q).$$

(b) $8x^3 - 9x^2 + 15x$

SOLUTION Each of these terms is divisible by x.

$$8x^3 - 9x^2 + 15x = (8x^2) \cdot \boldsymbol{x} - (9x) \cdot \boldsymbol{x} + 15 \cdot \boldsymbol{x}$$
$$= \boldsymbol{x}(8x^2 - 9x + 15) \quad \text{or} \quad (8x^2 - 9x + 15)\boldsymbol{x}$$

YOUR TURN 1 Factor $4z^4 + 4z^3 + 18z^2$.

TRY YOUR TURN 1

One can always check factorization by finding the product of the factors and comparing it to the original expression.

CAUTION When factoring out the greatest common factor in an expression like $2x^2 + x$, be careful to remember the 1 in the second term.

$$2x^2 + x = 2x^2 + 1x = x(2x + 1), \qquad \text{not } x(2x).$$

Factoring Trinomials

A polynomial that has no greatest common factor (other than 1) may still be factorable. For example, the polynomial $x^2 + 5x + 6$ can be factored as $(x + 2)(x + 3)$. To see that this is correct, find the product $(x + 2)(x + 3)$; you should get $x^2 + 5x + 6$. A polynomial such as this with three terms is called a **trinomial**. To factor the trinomial $x^2 + 5x + 6$, where the coefficient of x^2 is 1, we use FOIL backwards.

EXAMPLE 2 Factoring a Trinomial

Factor $y^2 + 8y + 15$.

SOLUTION Since the coefficient of y^2 is 1, factor by finding two numbers whose *product* is 15 and whose *sum* is 8. Since the constant and the middle term are positive, the numbers must both be positive. Begin by listing all pairs of positive integers having a product of 15. As you do this, also form the sum of each pair of numbers.

Products	Sums
$15 \cdot 1 = 15$	$15 + 1 = 16$
$\mathbf{5 \cdot 3 = 15}$	$\mathbf{5 + 3 = 8}$

The numbers 5 and 3 have a product of 15 and a sum of 8. Thus, $y^2 + 8y + 15$ factors as

$$y^2 + 8y + 15 = (y + 5)(y + 3).$$

The answer also can be written as $(y + 3)(y + 5)$.

If the coefficient of the squared term is *not* 1, work as shown below.

EXAMPLE 3 Factoring a Trinomial

Factor $4x^2 + 8xy - 5y^2$.

SOLUTION The possible factors of $4x^2$ are $4x$ and x or $2x$ and $2x$; the possible factors of $-5y^2$ are $-5y$ and y or $5y$ and $-y$. Try various combinations of these factors until one works (if, indeed, any work). For example, try the product $(x + 5y)(4x - y)$.

$$(x + 5y)(4x - y) = 4x^2 - xy + 20xy - 5y^2$$
$$= 4x^2 + 19xy - 5y^2$$

This product is not correct, so try another combination.

$$(2x - y)(2x + 5y) = 4x^2 + 10xy - 2xy - 5y^2$$
$$= 4x^2 + 8xy - 5y^2$$

YOUR TURN 2 Factor $6a^2 + 5ab - 4b^2$.

Since this combination gives the correct polynomial,

$$4x^2 + 8xy - 5y^2 = (2x - y)(2x + 5y).$$ TRY YOUR TURN 2

Special Factorizations

Four special factorizations occur so often that they are listed here for future reference.

Special Factorizations

$x^2 - y^2 = (x + y)(x - y)$	**Difference of two squares**
$x^2 + 2xy + y^2 = (x + y)^2$	**Perfect square**
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	**Difference of two cubes**
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	**Sum of two cubes**

A polynomial that cannot be factored is called a **prime polynomial**.

EXAMPLE 4 Factoring Polynomials

Factor each polynomial, if possible.

(a) $64p^2 - 49q^2 = (8p)^2 - (7q)^2 = (8p + 7q)(8p - 7q)$ Difference of two squares

(b) $x^2 + 36$ is a prime polynomial.

(c) $x^2 + 12x + 36 = (x + 6)^2$ Perfect square

(d) $9y^2 - 24yz + 16z^2 = (3y - 4z)^2$ Perfect square

(e) $y^3 - 8 = y^3 - 2^3 = (y - 2)(y^2 + 2y + 4)$ Difference of two cubes

(f) $m^3 + 125 = m^3 + 5^3 = (m + 5)(m^2 - 5m + 25)$ Sum of two cubes

(g) $8k^3 - 27z^3 = (2k)^3 - (3z)^3 = (2k - 3z)(4k^2 + 6kz + 9z^2)$ Difference of two cubes

(h) $p^4 - 1 = (p^2 + 1)(p^2 - 1) = (p^2 + 1)(p + 1)(p - 1)$ Difference of two squares

CAUTION In factoring, always look for a common factor first. Since $36x^2 - 4y^2$ has a common factor of 4,

$$36x^2 - 4y^2 = 4(9x^2 - y^2) = 4(3x + y)(3x - y).$$

It would be incomplete to factor it as

$$36x^2 - 4y^2 = (6x + 2y)(6x - 2y),$$

since each factor can be factored still further. To *factor* means to factor completely, so that each polynomial factor is prime.

R.2 EXERCISES

Factor each polynomial. If a polynomial cannot be factored, write *prime*. Factor out the greatest common factor as necessary.

1. $7a^3 + 14a^2$

2. $3y^3 + 24y^2 + 9y$

3. $13p^4q^2 - 39p^3q + 26p^2q^2$

4. $60m^4 - 120m^3n + 50m^2n^2$

5. $m^2 - 5m - 14$

6. $x^2 + 4x - 5$

7. $z^2 + 9z + 20$
8. $b^2 - 8b + 7$
9. $a^2 - 6ab + 5b^2$
10. $s^2 + 2st - 35t^2$
11. $y^2 - 4yz - 21z^2$
12. $3x^2 + 4x - 7$
13. $3a^2 + 10a + 7$
14. $15y^2 + y - 2$
15. $21m^2 + 13mn + 2n^2$
16. $6a^2 - 48a - 120$
17. $3m^3 + 12m^2 + 9m$
18. $4a^2 + 10a + 6$
19. $24a^4 + 10a^3b - 4a^2b^2$
20. $24x^4 + 36x^3y - 60x^2y^2$
21. $x^2 - 64$
22. $9m^2 - 25$
23. $10x^2 - 160$
24. $9x^2 + 64$
25. $z^2 + 14zy + 49y^2$
26. $s^2 - 10st + 25t^2$
27. $9p^2 - 24p + 16$
28. $a^3 - 216$
29. $27r^3 - 64s^3$
30. $3m^3 + 375$
31. $x^4 - y^4$
32. $16a^4 - 81b^4$

YOUR TURN ANSWERS

1. $2z^2(2z^2 + 2z + 9)$ **2.** $(2a - b)(3a + 4b)$

R.3 Rational Expressions

Many algebraic fractions are **rational expressions**, which are quotients of polynomials with nonzero denominators. Examples include

$$\frac{8}{x-1}, \quad \frac{3x^2+4x}{5x-6}, \quad \text{and} \quad \frac{2y+1}{y^2}.$$

Next, we summarize properties for working with rational expressions.

Properties of Rational Expressions

For all mathematical expressions P, Q, R, and S, with $Q \neq 0$ and $S \neq 0$:

$\frac{P}{Q} = \frac{PS}{QS}$	**Fundamental property**
$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$	**Addition**
$\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$	**Subtraction**
$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$	**Multiplication**
$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} \quad (R \neq 0)$	**Division**

When writing a rational expression in lowest terms, we may need to use the fact that $\frac{a^m}{a^n} = a^{m-n}$. For example,

$$\frac{x^4}{3x} = \frac{1x^4}{3x} = \frac{1}{3} \cdot \frac{x^4}{x} = \frac{1}{3} \cdot x^{4-1} = \frac{1}{3}x^3.$$

EXAMPLE 1 Reducing Rational Expressions

Write each rational expression in lowest terms, that is, reduce the expression as much as possible.

(a) $\frac{8x+16}{4} = \frac{8(x+2)}{4} = \frac{4 \cdot 2(x+2)}{4} = 2(x+2)$

Factor both the numerator and denominator in order to identify any common factors, which have a quotient of 1. The answer could also be written as $2x + 4$.

YOUR TURN 1 Write in lowest terms $\dfrac{z^2 + 5z + 6}{2z^2 + 7z + 3}$.

(b) $\dfrac{k^2 + 7k + 12}{k^2 + 2k - 3} = \dfrac{(k + 4)(k + 3)}{(k - 1)(k + 3)} = \dfrac{k + 4}{k - 1}$

The answer cannot be further reduced. TRY YOUR TURN 1

CAUTION One of the most common errors in algebra involves incorrect use of the fundamental property of rational expressions. Only common *factors* may be divided or "canceled." It is essential to factor rational expressions before writing them in lowest terms. In Example 1(b), for instance, it is not correct to "cancel" k^2 (or cancel k, or divide 12 by -3) because the additions and subtraction must be performed first. Here they cannot be performed, so it is not possible to divide. After factoring, however, the fundamental property can be used to write the expression in lowest terms.

EXAMPLE 2 Combining Rational Expressions

Perform each operation.

(a) $\dfrac{3y + 9}{6} \cdot \dfrac{18}{5y + 15}$

SOLUTION Factor where possible, then multiply numerators and denominators and reduce to lowest terms.

$$\frac{3y + 9}{6} \cdot \frac{18}{5y + 15} = \frac{3(y + 3)}{6} \cdot \frac{18}{5(y + 3)}$$
$$= \frac{3 \cdot 18(y + 3)}{6 \cdot 5(y + 3)}$$
$$= \frac{3 \cdot \cancel{6} \cdot 3\cancel{(y + 3)}}{\cancel{6} \cdot 5\cancel{(y + 3)}} = \frac{3 \cdot 3}{5} = \frac{9}{5}$$

(b) $\dfrac{m^2 + 5m + 6}{m + 3} \cdot \dfrac{m}{m^2 + 3m + 2}$

SOLUTION Factor where possible.

$$\frac{(m + 2)(m + 3)}{m + 3} \cdot \frac{m}{(m + 2)(m + 1)}$$
$$= \frac{m\cancel{(m + 2)}\cancel{(m + 3)}}{\cancel{(m + 3)}\cancel{(m + 2)}(m + 1)} = \frac{m}{m + 1}$$

(c) $\dfrac{9p - 36}{12} \div \dfrac{5(p - 4)}{18}$

SOLUTION Use the division property of rational expressions.

$$\frac{9p - 36}{12} \cdot \frac{18}{5(p - 4)}$$ Invert and multiply.
$$= \frac{9\cancel{(p - 4)}}{\cancel{6} \cdot 2} \cdot \frac{\cancel{6} \cdot 3}{5\cancel{(p - 4)}} = \frac{27}{10}$$ Factor and reduce to lowest terms.

(d) $\dfrac{4}{5k} - \dfrac{11}{5k}$

SOLUTION As shown in the list of properties, to subtract two rational expressions that have the same denominators, subtract the numerators while keeping the same denominator.

$$\frac{4}{\cancel{5k}} - \frac{11}{\cancel{5k}} = \frac{4 - 11}{\cancel{5k}} = -\frac{7}{5k}$$

(e) $\dfrac{7}{p} + \dfrac{9}{2p} + \dfrac{1}{3p}$

SOLUTION These three fractions cannot be added until their denominators are the same. A **common denominator** into which p, $2p$, and $3p$ all divide is $6p$. Note that $12p$ is also a common denominator, but $6p$ is the **least common denominator**. Use the fundamental property to rewrite each rational expression with a denominator of $6p$.

$$\begin{aligned}\frac{7}{p} + \frac{9}{2p} + \frac{1}{3p} &= \frac{\mathbf{6}\cdot 7}{\mathbf{6}\cdot p} + \frac{\mathbf{3}\cdot 9}{\mathbf{3}\cdot 2p} + \frac{\mathbf{2}\cdot 1}{\mathbf{2}\cdot 3p} \\ &= \frac{42}{6p} + \frac{27}{6p} + \frac{2}{6p} \\ &= \frac{42 + 27 + 2}{6p} \\ &= \frac{71}{6p}\end{aligned}$$

(f) $\dfrac{x+1}{x^2+5x+6} - \dfrac{5x-1}{x^2-x-12}$

SOLUTION To find the least common denominator, we first factor each denominator. Then we change each fraction so they all have the same denominator, being careful to multiply only by quotients that equal 1.

$$\begin{aligned}&\frac{x+1}{x^2+5x+6} - \frac{5x-1}{x^2-x-12} \\ &= \frac{x+1}{(x+2)(x+3)} - \frac{5x-1}{(x+3)(x-4)} \\ &= \frac{x+1}{(x+2)(x+3)}\cdot\frac{\mathbf{(x-4)}}{\mathbf{(x-4)}} - \frac{5x-1}{(x+3)(x-4)}\cdot\frac{\mathbf{(x+2)}}{\mathbf{(x+2)}} \\ &= \frac{(x^2-3x-4)-(5x^2+9x-2)}{(x+2)(x+3)(x-4)} \\ &= \frac{-4x^2-12x-2}{(x+2)(x+3)(x-4)} \\ &= \frac{-2(2x^2+6x+1)}{(x+2)(x+3)(x-4)}\end{aligned}$$

Because the numerator cannot be factored further, we leave our answer in this form. We could also multiply out the denominator, but factored form is usually more useful.

TRY YOUR TURN 2

YOUR TURN 2 Perform each of the following operations.

(a) $\dfrac{z^2+5z+6}{2z^2-5z-3}\cdot\dfrac{2z^2-z-1}{z^2+2z-3}$.

(b) $\dfrac{a-3}{a^2+3a+2} + \dfrac{5a}{a^2-4}$.

R.3 EXERCISES

Write each rational expression in lowest terms.

1. $\dfrac{5v^2}{35v}$

2. $\dfrac{25p^3}{10p^2}$

3. $\dfrac{8k+16}{9k+18}$

4. $\dfrac{2(t-15)}{(t-15)(t+2)}$

5. $\dfrac{4x^3-8x^2}{4x^2}$

6. $\dfrac{36y^2+72y}{9y}$

7. $\dfrac{m^2-4m+4}{m^2+m-6}$

8. $\dfrac{r^2-r-6}{r^2+r-12}$

9. $\dfrac{3x^2+3x-6}{x^2-4}$

10. $\dfrac{z^2-5z+6}{z^2-4}$

11. $\dfrac{m^4-16}{4m^2-16}$

12. $\dfrac{6y^2+11y+4}{3y^2+7y+4}$

Perform the indicated operations.

13. $\dfrac{9k^2}{25}\cdot\dfrac{5}{3k}$

14. $\dfrac{15p^3}{9p^2}\div\dfrac{6p}{10p^2}$

15. $\dfrac{3a+3b}{4c}\cdot\dfrac{12}{5(a+b)}$

16. $\dfrac{a-3}{16}\div\dfrac{a-3}{32}$

17. $\dfrac{2k-16}{6} \div \dfrac{4k-32}{3}$

18. $\dfrac{9y-18}{6y+12} \cdot \dfrac{3y+6}{15y-30}$

19. $\dfrac{4a+12}{2a-10} \div \dfrac{a^2-9}{a^2-a-20}$

20. $\dfrac{6r-18}{9r^2+6r-24} \cdot \dfrac{12r-16}{4r-12}$

21. $\dfrac{k^2+4k-12}{k^2+10k+24} \cdot \dfrac{k^2+k-12}{k^2-9}$

22. $\dfrac{m^2+3m+2}{m^2+5m+4} \div \dfrac{m^2+5m+6}{m^2+10m+24}$

23. $\dfrac{2m^2-5m-12}{m^2-10m+24} \div \dfrac{4m^2-9}{m^2-9m+18}$

24. $\dfrac{4n^2+4n-3}{6n^2-n-15} \cdot \dfrac{8n^2+32n+30}{4n^2+16n+15}$

25. $\dfrac{a+1}{2} - \dfrac{a-1}{2}$

26. $\dfrac{3}{p} + \dfrac{1}{2}$

27. $\dfrac{6}{5y} - \dfrac{3}{2}$

28. $\dfrac{1}{6m} + \dfrac{2}{5m} + \dfrac{4}{m}$

29. $\dfrac{1}{m-1} + \dfrac{2}{m}$

30. $\dfrac{5}{2r+3} - \dfrac{2}{r}$

31. $\dfrac{8}{3(a-1)} + \dfrac{2}{a-1}$

32. $\dfrac{2}{5(k-2)} + \dfrac{3}{4(k-2)}$

33. $\dfrac{4}{x^2+4x+3} + \dfrac{3}{x^2-x-2}$

34. $\dfrac{y}{y^2+2y-3} - \dfrac{1}{y^2+4y+3}$

35. $\dfrac{3k}{2k^2+3k-2} - \dfrac{2k}{2k^2-7k+3}$

36. $\dfrac{4m}{3m^2+7m-6} - \dfrac{m}{3m^2-14m+8}$

37. $\dfrac{2}{a+2} + \dfrac{1}{a} + \dfrac{a-1}{a^2+2a}$

38. $\dfrac{5x+2}{x^2-1} + \dfrac{3}{x^2+x} - \dfrac{1}{x^2-x}$

YOUR TURN ANSWERS

1. $(z+2)/(2z+1)$

2a. $(z+2)/(z-3)$

2b. $6(a^2+1)/[(a-2)(a+2)(a+1)]$

R.4 Equations

Linear Equations

Equations that can be written in the form $ax + b = 0$, where a and b are real numbers, with $a \neq 0$, are **linear equations**. Examples of linear equations include $5y + 9 = 16$, $8x = 4$, and $-3p + 5 = -8$. Equations that are *not* linear include absolute value equations such as $|x| = 4$. The following properties are used to solve linear equations.

Properties of Equality

For all real numbers a, b, and c:

1. **If $a = b$, then $a + c = b + c$.** (The same number may be added to both sides of an equation.) **Addition property of equality**
2. **If $a = b$, then $ac = bc$.** (Both sides of an equation may be multiplied by the same number.) **Multiplication property of equality**

EXAMPLE 1 **Solving Linear Equations**

Solve the following equations.

(a) $x - 2 = 3$

SOLUTION The goal is to isolate the variable. Using the addition property of equality yields

$$x - 2 + 2 = 3 + 2, \quad \text{or} \quad x = 5.$$

(b) $\frac{x}{2} = 3$

SOLUTION Using the multiplication property of equality yields

$$2 \cdot \frac{x}{2} = 2 \cdot 3, \quad \text{or} \quad x = 6.$$

The following example shows how these properties are used to solve linear equations. The goal is to isolate the variable. The solutions should always be checked by substitution in the original equation.

EXAMPLE 2 Solving a Linear Equation

Solve $2x - 5 + 8 = 3x + 2(2 - 3x)$.

SOLUTION

$$\begin{aligned} 2x - 5 + 8 &= 3x + 4 - 6x && \text{Distributive property} \\ 2x + 3 &= -3x + 4 && \text{Combine like terms.} \\ 5x + 3 &= 4 && \text{Add } 3x \text{ to both sides.} \\ 5x &= 1 && \text{Add } -3 \text{ to both sides.} \\ x &= \frac{1}{5} && \text{Multiply both sides by } \tfrac{1}{5}. \end{aligned}$$

YOUR TURN 1 Solve $3x - 7 = 4(5x + 2) - 7x$.

Check by substituting in the original equation. The left side becomes $2(1/5) - 5 + 8$ and the right side becomes $3(1/5) + 2[2 - 3(1/5)]$. Verify that both of these expressions simplify to $17/5$. TRY YOUR TURN 1

Quadratic Equations

An equation with 2 as the highest exponent of the variable is a *quadratic equation*. A **quadratic equation** has the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. A quadratic equation written in the form $ax^2 + bx + c = 0$ is said to be in **standard form**.

The simplest way to solve a quadratic equation, but one that is not always applicable, is by factoring. This method depends on the **zero-factor property**.

Zero-Factor Property

If a and b are real numbers, with $ab = 0$, then either

$$a = 0 \text{ or } b = 0 \quad \text{(or both)}.$$

EXAMPLE 3 Solving a Quadratic Equation

Solve $6r^2 + 7r = 3$.

SOLUTION First write the equation in standard form.

$$6r^2 + 7r - 3 = 0$$

Now factor $6r^2 + 7r - 3$ to get

$$(3r - 1)(2r + 3) = 0.$$

By the zero-factor property, the product $(3r - 1)(2r + 3)$ can equal 0 if and only if

$$3r - 1 = 0 \quad \text{or} \quad 2r + 3 = 0.$$

YOUR TURN 2 Solve $2m^2 + 7m = 15$.

Solve each of these equations separately to find that the solutions are $1/3$ and $-3/2$. Check these solutions by substituting them in the original equation. TRY YOUR TURN 2

CAUTION Remember, the zero-factor property requires that the product of two (or more) factors be equal to *zero*, not some other quantity. It would be incorrect to use the zero-factor property with an equation in the form $(x + 3)(x - 1) = 4$, for example.

If a quadratic equation cannot be solved easily by factoring, use the *quadratic formula.* (The derivation of the quadratic formula is given in most algebra books.)

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4 Quadratic Formula

Solve $x^2 - 4x - 5 = 0$ by the quadratic formula.

SOLUTION The equation is already in standard form (it has 0 alone on one side of the equal sign), so the values of a, b, and c from the quadratic formula are easily identified. The coefficient of the squared term gives the value of a; here, $a = 1$. Also, $b = -4$ and $c = -5$. (Be careful to use the correct signs.) Substitute these values into the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} \qquad a = 1, b = -4, c = -5$$

$$x = \frac{4 \pm \sqrt{16 + 20}}{2} \qquad (-4)^2 = (-4)(-4) = 16$$

$$x = \frac{4 \pm 6}{2} \qquad \sqrt{16 + 20} = \sqrt{36} = 6$$

The $\pm$ sign represents the two solutions of the equation. To find both of the solutions, first use $+$ and then use $-$.

$$x = \frac{4 + 6}{2} = \frac{10}{2} = 5 \quad \text{or} \quad x = \frac{4 - 6}{2} = \frac{-2}{2} = -1$$

The two solutions are 5 and -1.

CAUTION Notice in the quadratic formula that the square root is added to or subtracted from the value of $-b$ *before* dividing by $2a$.

EXAMPLE 5 Quadratic Formula

Solve $x^2 + 1 = 4x$.

SOLUTION First, add $-4x$ on both sides of the equal sign in order to get the equation in standard form.

$$x^2 - 4x + 1 = 0$$

Now identify the letters a, b, and c. Here $a = 1$, $b = -4$, and $c = 1$. Substitute these numbers into the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{16 - 4}}{2}$$
$$= \frac{4 \pm \sqrt{12}}{2}$$

Simplify the solutions by writing $\sqrt{12}$ as $\sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$. Substituting $2\sqrt{3}$ for $\sqrt{12}$ gives

$$x = \frac{4 \pm 2\sqrt{3}}{2}$$
$$= \frac{2(2 \pm \sqrt{3})}{2} \quad \text{Factor } 4 \pm 2\sqrt{3}.$$
$$= 2 \pm \sqrt{3}. \quad \text{Reduce to lowest terms.}$$

The two solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

The exact values of the solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. The $\sqrt{\ }$ key on a calculator gives decimal approximations of these solutions (to the nearest thousandth):

$$2 + \sqrt{3} \approx 2 + 1.732 = 3.732^*$$
$$2 - \sqrt{3} \approx 2 - 1.732 = 0.268$$

YOUR TURN 3 Solve $z^2 + 6 = 8z$.

TRY YOUR TURN 3

NOTE Sometimes the quadratic formula will give a result with a negative number under the radical sign, such as $3 \pm \sqrt{-5}$. A solution of this type is a complex number. Since this text deals only with real numbers, such solutions cannot be used.

Equations with Fractions

When an equation includes fractions, first eliminate all denominators by multiplying both sides of the equation by a common denominator, a number that can be divided (with no remainder) by each denominator in the equation. When an equation involves fractions with variable denominators, it is *necessary* to check all solutions in the original equation to be sure that no solution will lead to a zero denominator.

EXAMPLE 6 Solving Rational Equations

Solve each equation.

(a) $\frac{r}{10} - \frac{2}{15} = \frac{3r}{20} - \frac{1}{5}$

SOLUTION The denominators are 10, 15, 20, and 5. Each of these numbers can be divided into 60, so 60 is a common denominator. Multiply both sides of the equation by 60 and use the distributive property. (If a common denominator cannot be found easily, all the denominators in the problem can be multiplied together to produce one.)

$$\frac{r}{10} - \frac{2}{15} = \frac{3r}{20} - \frac{1}{5}$$
$$60\left(\frac{r}{10} - \frac{2}{15}\right) = 60\left(\frac{3r}{20} - \frac{1}{5}\right) \quad \text{Multiply by the common denominator.}$$
$$60\left(\frac{r}{10}\right) - 60\left(\frac{2}{15}\right) = 60\left(\frac{3r}{20}\right) - 60\left(\frac{1}{5}\right) \quad \text{Distributive property}$$
$$6r - 8 = 9r - 12$$

*The symbol $\approx$ means "is approximately equal to."

Add $-9r$ and 8 to both sides.

$$6r - 8 + (-9r) + 8 = 9r - 12 + (-9r) + 8$$
$$-3r = -4$$
$$r = \frac{4}{3} \qquad \text{Multiply each side by } -\tfrac{1}{3}.$$

Check by substituting into the original equation.

(b) $\dfrac{3}{x^2} - 12 = 0$

SOLUTION Begin by multiplying both sides of the equation by x^2 to get $3 - 12x^2 = 0$. This equation could be solved by using the quadratic formula with $a = -12$, $b = 0$, and $c = 3$. Another method that works well for the type of quadratic equation in which $b = 0$ is shown below.

$$3 - 12x^2 = 0$$
$$3 = 12x^2 \qquad \text{Add } 12x^2.$$
$$\frac{1}{4} = x^2 \qquad \text{Multiply by } \tfrac{1}{12}.$$
$$\pm\frac{1}{2} = x \qquad \text{Take square roots.}$$

Verify that there are two solutions, $-1/2$ and $1/2$.

(c) $\dfrac{2}{k} - \dfrac{3k}{k+2} = \dfrac{k}{k^2+2k}$

SOLUTION Factor $k^2 + 2k$ as $k(k + 2)$. The least common denominator for all the fractions is $k(k + 2)$. Multiplying both sides by $k(k + 2)$ gives the following:

$$k(k+2)\cdot\left(\frac{2}{k} - \frac{3k}{k+2}\right) = k(k+2)\cdot\frac{k}{k^2+2k}$$
$$2(k+2) - 3k(k) = k$$
$$2k + 4 - 3k^2 = k \qquad \text{Distributive property}$$
$$-3k^2 + k + 4 = 0 \qquad \text{Add } -k\text{; rearrange terms.}$$
$$3k^2 - k - 4 = 0 \qquad \text{Multiply by } -1.$$
$$(3k-4)(k+1) = 0 \qquad \text{Factor.}$$
$$3k - 4 = 0 \quad \text{or} \quad k + 1 = 0$$
$$k = \frac{4}{3} \qquad k = -1$$

Verify that the solutions are $4/3$ and -1.

YOUR TURN 4 Solve $\dfrac{1}{x^2-4} + \dfrac{2}{x-2} = \dfrac{1}{x}$.

TRY YOUR TURN 4

CAUTION It is possible to get, as a solution of a rational equation, a number that makes one or more of the denominators in the original equation equal to zero. That number is not a solution, so it is *necessary* to check all potential solutions of rational equations. These introduced solutions are called **extraneous solutions**.

EXAMPLE 7 Solving a Rational Equation

Solve $\dfrac{2}{x-3} + \dfrac{1}{x} = \dfrac{6}{x(x-3)}$.

SOLUTION The common denominator is $x(x-3)$. Multiply both sides by $x(x-3)$ and solve the resulting equation.

$$x(x-3)\cdot\left(\frac{2}{x-3}+\frac{1}{x}\right)=x(x-3)\cdot\left[\frac{6}{x(x-3)}\right]$$
$$2x+x-3=6$$
$$3x=9$$
$$x=3$$

Checking this potential solution by substitution in the original equation shows that 3 makes two denominators 0. Thus, 3 cannot be a solution, so there is no solution for this equation.

R.4 EXERCISES

Solve each equation.

1. $2x+8=x-4$
2. $5x+2=8-3x$
3. $0.2m-0.5=0.1m+0.7$
4. $\frac{2}{3}k-k+\frac{3}{8}=\frac{1}{2}$
5. $3r+2-5(r+1)=6r+4$
6. $5(a+3)+4a-5=-(2a-4)$
7. $2[3m-2(3-m)-4]=6m-4$
8. $4[2p-(3-p)+5]=-7p-2$

Solve each equation by factoring or by using the quadratic formula. If the solutions involve square roots, give both the exact solutions and the approximate solutions to three decimal places.

9. $x^2+5x+6=0$
10. $x^2=3+2x$
11. $m^2=14m-49$
12. $2k^2-k=10$
13. $12x^2-5x=2$
14. $m(m-7)=-10$
15. $4x^2-36=0$
16. $z(2z+7)=4$
17. $12y^2-48y=0$
18. $3x^2-5x+1=0$
19. $2m^2-4m=3$
20. $p^2+p-1=0$
21. $k^2-10k=-20$
22. $5x^2-8x+2=0$
23. $2r^2-7r+5=0$
24. $2x^2-7x+30=0$
25. $3k^2+k=6$
26. $5m^2+5m=0$

Solve each equation.

27. $\frac{3x-2}{7}=\frac{x+2}{5}$
28. $\frac{x}{3}-7=6-\frac{3x}{4}$
29. $\frac{4}{x-3}-\frac{8}{2x+5}+\frac{3}{x-3}=0$
30. $\frac{5}{p-2}-\frac{7}{p+2}=\frac{12}{p^2-4}$
31. $\frac{2m}{m-2}-\frac{6}{m}=\frac{12}{m^2-2m}$
32. $\frac{2y}{y-1}=\frac{5}{y}+\frac{10-8y}{y^2-y}$
33. $\frac{1}{x-2}-\frac{3x}{x-1}=\frac{2x+1}{x^2-3x+2}$
34. $\frac{5}{a}+\frac{-7}{a+1}=\frac{a^2-2a+4}{a^2+a}$
35. $\frac{5}{b+5}-\frac{4}{b^2+2b}=\frac{6}{b^2+7b+10}$
36. $\frac{2}{x^2-2x-3}+\frac{5}{x^2-x-6}=\frac{1}{x^2+3x+2}$
37. $\frac{4}{2x^2+3x-9}+\frac{2}{2x^2-x-3}=\frac{3}{x^2+4x+3}$

YOUR TURN ANSWERS

1. $-3/2$
2. $3/2, -5$
3. $4\pm\sqrt{10}$
4. $-1, -4$

R.5 Inequalities

To write that one number is greater than or less than another number, we use the following symbols.

Inequality Symbols

$<$ means *is less than*
$>$ means *is greater than*
$\le$ means *is less than or equal to*
$\ge$ means *is greater than or equal to*

Linear Inequalities

An equation states that two expressions are equal; an **inequality** states that they are unequal. A **linear inequality** is an inequality that can be simplified to the form $ax < b$. (Properties introduced in this section are given only for $<$, but they are equally valid for $>$, $\le$, or $\ge$.) Linear inequalities are solved with the following properties.

Properties of Inequality

For all real numbers a, b, and c:

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and if $c > 0$, then $ac < bc$.
3. If $a < b$ and if $c < 0$, then $ac > bc$.

Pay careful attention to property 3; it says that if both sides of an inequality are multiplied by a negative number, the direction of the inequality symbol must be reversed.

EXAMPLE 1 Solving a Linear Inequality

Solve $4 - 3y \le 7 + 2y$.

SOLUTION Use the properties of inequality.

$$4 - 3y + (-4) \le 7 + 2y + (-4) \qquad \text{Add } -4 \text{ to both sides.}$$
$$-3y \le 3 + 2y$$

Remember that *adding* the same number to both sides never changes the direction of the inequality symbol.

$$-3y + (-2y) \le 3 + 2y + (-2y) \qquad \text{Add } -2y \text{ to both sides.}$$
$$-5y \le 3$$

Multiply both sides by $-1/5$. Since $-1/5$ is negative, change the direction of the inequality symbol.

$$-\frac{1}{5}(-5y) \ge -\frac{1}{5}(3)$$
$$y \ge -\frac{3}{5}$$

YOUR TURN 1 Solve $3z - 2 > 5z + 7$.

TRY YOUR TURN 1

CAUTION It is a common error to forget to reverse the direction of the inequality sign when multiplying or dividing by a negative number. For example, to solve $-4x \le 12$, we must multiply by $-1/4$ on both sides *and* reverse the inequality symbol to get $x \ge -3$.

The solution $y \ge -3/5$ in Example 1 represents an interval on the number line. **Interval notation** often is used for writing intervals. With interval notation, $y \ge -3/5$ is written as $[-3/5, \infty)$. This is an example of a **half-open interval**, since one endpoint, $-3/5$, is included. The **open interval** $(2, 5)$ corresponds to $2 < x < 5$, with neither endpoint included. The **closed interval** $[2, 5]$ includes both endpoints and corresponds to $2 \le x \le 5$.

The **graph** of an interval shows all points on a number line that correspond to the numbers in the interval. To graph the interval $[-3/5, \infty)$, for example, use a solid circle at $-3/5$, since $-3/5$ is part of the solution. To show that the solution includes all real numbers greater than or equal to $-3/5$, draw a heavy arrow pointing to the right (the positive direction). See Figure 1.

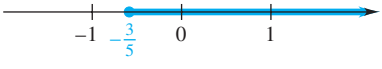


FIGURE 1

EXAMPLE 2 Graphing a Linear Inequality

Solve $-2 < 5 + 3m < 20$. Graph the solution.

SOLUTION The inequality $-2 < 5 + 3m < 20$ says that $5 + 3m$ is *between* -2 and 20. Solve this inequality with an extension of the properties given above. Work as follows, first adding -5 to each part.

$$-2 + (-5) < 5 + 3m + (-5) < 20 + (-5)$$
$$-7 < 3m < 15$$

Now multiply each part by $1/3$.

$$-\frac{7}{3} < m < 5$$

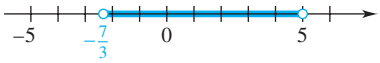


FIGURE 2

A graph of the solution is given in Figure 2; here open circles are used to show that $-7/3$ and 5 are *not* part of the graph.*

Quadratic Inequalities

A **quadratic inequality** has the form $ax^2 + bx + c > 0$ (or $<$, or $\le$, or $\ge$). The highest exponent is 2. The next few examples show how to solve quadratic inequalities.

EXAMPLE 3 Solving a Quadratic Inequality

Solve the quadratic inequality $x^2 - x < 12$.

SOLUTION Write the inequality with 0 on one side, as $x^2 - x - 12 < 0$. This inequality is solved with values of x that make $x^2 - x - 12$ negative (< 0). The quantity $x^2 - x - 12$ changes from positive to negative or from negative to positive at the points where it equals 0. For this reason, first solve the *equation* $x^2 - x - 12 = 0$.

$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$
$$x = 4 \quad \text{or} \quad x = -3$$

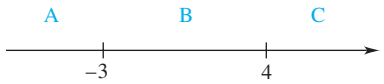


FIGURE 3

FIGURE 4

Locating -3 and 4 on a number line, as shown in Figure 3, determines three intervals A, B, and C. Decide which intervals include numbers that make $x^2 - x - 12$ negative by substituting any number from each interval in the polynomial. For example,

choose -4 from interval A: $(-4)^2 - (-4) - 12 = 8 > 0$;
choose 0 from interval B: $0^2 - 0 - 12 = -12 < 0$;
choose 5 from interval C: $5^2 - 5 - 12 = 8 > 0$.

Only numbers in interval B satisfy the given inequality, so the solution is $(-3, 4)$. A graph of this solution is shown in Figure 4.

YOUR TURN 2 Solve $3y^2 \le 16y + 12$.

TRY YOUR TURN 2

EXAMPLE 4 Solving a Polynomial Inequality

Solve the inequality $x^3 + 2x^2 - 3x \ge 0$.

SOLUTION This is not a quadratic inequality because of the x^3 term, but we solve it in a similar way by first factoring the polynomial.

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) \qquad \text{Factor out the common factor.}$$
$$= x(x - 1)(x + 3) \qquad \text{Factor the quadratic.}$$

*Some textbooks use brackets in place of solid circles for the graph of a closed interval, and parentheses in place of open circles for the graph of an open interval.

Now solve the corresponding equation.

$$x(x-1)(x+3)=0$$

$$x=0 \quad \text{or} \quad x-1=0 \quad \text{or} \quad x+3=0$$

$$x=1 \qquad\qquad x=-3$$

FIGURE 5

These three solutions determine four intervals on the number line: $(-\infty,-3)$, $(-3,0)$, $(0,1)$, and $(1,\infty)$. Substitute a number from each interval into the original inequality to determine that the solution consists of the numbers between -3 and 0 (including the endpoints) and all numbers that are greater than or equal to 1. See Figure 5. In interval notation, the solution is

$$[-3,0]\cup[1,\infty).*$$

Inequalities with Fractions

Inequalities with fractions are solved in a similar manner as quadratic inequalities.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{2x-3}{x}\geq 1$.

SOLUTION First solve the corresponding equation.

$$\frac{2x-3}{x}=1$$

$$2x-3=x$$

$$x=3$$

The solution, $x=3$, determines the intervals on the number line where the fraction may change from greater than 1 to less than 1. This change also may occur on either side of a number that makes the denominator equal 0. Here, the x-value that makes the denominator 0 is $x=0$. Test each of the three intervals determined by the numbers 0 and 3.

$$\text{For } (-\infty,0),\ \text{choose } -1\text{: } \frac{2(-1)-3}{-1}=5\geq 1.$$

$$\text{For } (0,3),\ \text{choose } 1\text{: } \frac{2(1)-3}{1}=-1\not\geq 1.$$

$$\text{For } (3,\infty),\ \text{choose } 4\text{: } \frac{2(4)-3}{4}=\frac{5}{4}\geq 1.$$

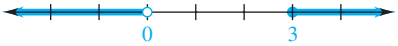


FIGURE 6

The symbol $\not\geq$ means "is *not* greater than or equal to." Testing the endpoints 0 and 3 shows that the solution is $(-\infty,0)\cup[3,\infty)$, as shown in Figure 6.

CAUTION A common error is to try to solve the inequality in Example 5 by multiplying both sides by x. The reason this is wrong is that we don't know in the beginning whether x is positive or negative. If x is negative, the $\geq$ would change to $\leq$ according to the third property of inequality listed at the beginning of this section.

*The symbol $\cup$ indicates the *union* of two sets, which includes all elements in either set.

EXAMPLE 6 Solving a Rational Inequality

Solve $\dfrac{(x-1)(x+1)}{x} \le 0$.

SOLUTION We first solve the corresponding equation.

$$\frac{(x-1)(x+1)}{x} = 0$$

$$(x-1)(x+1) = 0 \qquad \text{Multiply both sides by } x.$$

$$x = 1 \quad \text{or} \quad x = -1 \qquad \text{Use the zero-factor property.}$$

Setting the denominator equal to 0 gives $x = 0$, so the intervals of interest are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. Testing a number from each region in the original inequality and checking the endpoints, we find the solution is

$$(-\infty, -1] \cup (0, 1],$$

FIGURE 7

as shown in Figure 7.

CAUTION Remember to solve the equation formed by setting the *denominator* equal to zero. Any number that makes the denominator zero always creates two intervals on the number line. For instance, in Example 6, substituting $x = 0$ makes the denominator of the rational inequality equal to 0, so we know that there may be a sign change from one side of 0 to the other (as was indeed the case).

EXAMPLE 7 Solving a Rational Inequality

Solve $\dfrac{x^2 - 3x}{x^2 - 9} < 4$.

SOLUTION Solve the corresponding equation.

$$\frac{x^2 - 3x}{x^2 - 9} = 4$$

$$x^2 - 3x = 4x^2 - 36 \qquad \text{Multiply by } x^2 - 9.$$

$$0 = 3x^2 + 3x - 36 \qquad \text{Get 0 on one side.}$$

$$0 = x^2 + x - 12 \qquad \text{Multiply by } \tfrac{1}{3}.$$

$$0 = (x+4)(x-3) \qquad \text{Factor.}$$

$$x = -4 \quad \text{or} \quad x = 3$$

Now set the denominator equal to 0 and solve that equation.

$$x^2 - 9 = 0$$

$$(x-3)(x+3) = 0$$

$$x = 3 \quad \text{or} \quad x = -3$$

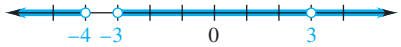


FIGURE 8

The intervals determined by the three (different) solutions are $(-\infty, -4)$, $(-4, -3)$, $(-3, 3)$, and $(3, \infty)$. Testing a number from each interval in the given inequality shows that the solution is

$$(-\infty, -4) \cup (-3, 3) \cup (3, \infty),$$

as shown in Figure 8. For this example, none of the endpoints are part of the solution because $x = 3$ and $x = -3$ make the denominator zero and $x = -4$ produces an equality.

YOUR TURN 3 Solve $\dfrac{k^2 - 35}{k} \ge 2$.

TRY YOUR TURN 3

R.5 EXERCISES

Write each expression in interval notation. Graph each interval.

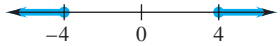
1. $x < 4$
2. $x \geq -3$
3. $1 \leq x < 2$
4. $-2 \leq x \leq 3$
5. $-9 > x$
6. $6 \leq x$

Using the variable x, write each interval as an inequality.

7. $[-7, -3]$
8. $[4, 10)$
9. $(-\infty, -1]$
10. $(3, \infty)$

11.

12.

13.


14.

Solve each inequality and graph the solution.

15. $6p + 7 \leq 19$
16. $6k - 4 < 3k - 1$
17. $m - (3m - 2) + 6 < 7m - 19$
18. $-2(3y - 8) \geq 5(4y - 2)$
19. $3p - 1 < 6p + 2(p - 1)$
20. $x + 5(x + 1) > 4(2 - x) + x$
21. $-11 < y - 7 < -1$
22. $8 \leq 3r + 1 \leq 13$
23. $-2 < \dfrac{1 - 3k}{4} \leq 4$
24. $-1 \leq \dfrac{5y + 2}{3} \leq 4$
25. $\dfrac{3}{5}(2p + 3) \geq \dfrac{1}{10}(5p + 1)$
26. $\dfrac{8}{3}(z - 4) \leq \dfrac{2}{9}(3z + 2)$

Solve each quadratic inequality. Graph each solution.

27. $(m - 3)(m + 5) < 0$
28. $(t + 6)(t - 1) \geq 0$
29. $y^2 - 3y + 2 < 0$
30. $2k^2 + 7k - 4 > 0$
31. $x^2 - 16 > 0$
32. $2k^2 - 7k - 15 \leq 0$
33. $x^2 - 4x \geq 5$
34. $10r^2 + r \leq 2$
35. $3x^2 + 2x > 1$
36. $3a^2 + a > 10$
37. $9 - x^2 \leq 0$
38. $p^2 - 16p > 0$
39. $x^3 - 4x \geq 0$
40. $x^3 + 7x^2 + 12x \leq 0$
41. $2x^3 - 14x^2 + 12x < 0$
42. $3x^3 - 9x^2 - 12x > 0$

Solve each inequality.

43. $\dfrac{m - 3}{m + 5} \leq 0$
44. $\dfrac{r + 1}{r - 1} > 0$
45. $\dfrac{k - 1}{k + 2} > 1$
46. $\dfrac{a - 5}{a + 2} < -1$
47. $\dfrac{2y + 3}{y - 5} \leq 1$
48. $\dfrac{a + 2}{3 + 2a} \leq 5$
49. $\dfrac{2k}{k - 3} \leq \dfrac{4}{k - 3}$
50. $\dfrac{5}{p + 1} > \dfrac{12}{p + 1}$
51. $\dfrac{2x}{x^2 - x - 6} \geq 0$
52. $\dfrac{8}{p^2 + 2p} > 1$
53. $\dfrac{z^2 + z}{z^2 - 1} \geq 3$
54. $\dfrac{a^2 + 2a}{a^2 - 4} \leq 2$

YOUR TURN ANSWERS

1. $z < -9/2$ **2.** $[-2/3, 6]$ **3.** $[-5, 0) \cup [7, \infty)$

R.6 Exponents

Integer Exponents

Recall that $a^2 = a \cdot a$, while $a^3 = a \cdot a \cdot a$, and so on. In this section, a more general meaning is given to the symbol a^n.

Definition of Exponent

If n is a natural number, then

$$a^n = a \cdot a \cdot a \cdot \cdots \cdot a,$$

where a appears as a factor n times.

In the expression a^n, the power n is the **exponent** and a is the **base**. This definition can be extended by defining a^n for zero and negative integer values of n.

Zero and Negative Exponents

If a is any nonzero real number, and if n is a positive integer, then

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

(The symbol 0^0 is meaningless.)

EXAMPLE 1 Exponents

(a) $6^0 = 1$

(b) $(-9)^0 = 1$

(c) $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(d) $9^{-1} = \dfrac{1}{9^1} = \dfrac{1}{9}$

(e) $\left(\dfrac{3}{4}\right)^{-1} = \dfrac{1}{(3/4)^1} = \dfrac{1}{3/4} = \dfrac{4}{3}$

The following properties follow from the definitions of exponents given above.

Properties of Exponents

For any integers m and n, and any real numbers a and b for which the following exist:

1. $a^m \cdot a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m \cdot b^m$
5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Note that $(-a)^n = a^n$ if n is an even integer, but $(-a)^n = -a^n$ if n is an odd integer.

EXAMPLE 2 Simplifying Exponential Expressions

Use the properties of exponents to simplify each expression. Leave answers with positive exponents. Assume that all variables represent positive real numbers.

(a) $7^4 \cdot 7^6 = 7^{4+6} = 7^{10}$ (or 282,475,249) — Property 1

(b) $\dfrac{9^{14}}{9^6} = 9^{14-6} = 9^8$ (or 43,046,721) — Property 2

(c) $\dfrac{r^9}{r^{17}} = r^{9-17} = r^{-8} = \dfrac{1}{r^8}$ — Property 2

(d) $(2m^3)^4 = 2^4 \cdot (m^3)^4 = 16m^{12}$ — Properties 3 and 4

(e) $(3x)^4 = 3^4 \cdot x^4 = 81x^4$ — Property 4

(f) $\left(\dfrac{x^2}{y^3}\right)^6 = \dfrac{(x^2)^6}{(y^3)^6} = \dfrac{x^{2\cdot 6}}{y^{3\cdot 6}} = \dfrac{x^{12}}{y^{18}}$ — Properties 3 and 5

(g) $\dfrac{a^{-3}b^5}{a^4b^{-7}} = \dfrac{b^{5-(-7)}}{a^{4-(-3)}} = \dfrac{b^{5+7}}{a^{4+3}} = \dfrac{b^{12}}{a^7}$ — Property 2

(h) $p^{-1} + q^{-1} = \frac{1}{p} + \frac{1}{q} = \frac{1}{p} \cdot \frac{q}{q} + \frac{1}{q} \cdot \frac{p}{p} = \frac{q}{pq} + \frac{p}{pq} = \frac{p+q}{pq}$

(i)
$$\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} - \frac{1}{y}}$$ Definition of a^{-n}

$$= \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y - x}{xy}}$$ Get common denominators and combine terms.

$$= \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y - x}$$ Invert and multiply.

$$= \frac{(y - x)(y + x)}{x^2y^2} \cdot \frac{xy}{y - x}$$ Factor.

$$= \frac{x + y}{xy}$$ Simplify.

YOUR TURN 1
Simplify
$\left(\frac{y^2z^{-4}}{y^{-3}z^4}\right)^{-2}$.

TRY YOUR TURN 1

CAUTION If Example 2(e) were written $3x^4$, the properties of exponents would not apply. When no parentheses are used, the exponent refers only to the factor closest to it. Also notice in Examples 2(c), 2(g), 2(h), and 2(i) that a negative exponent does *not* indicate a negative number.

Roots

For *even* values of n and nonnegative values of a, the expression $a^{1/n}$ is defined to be the **positive *n*th root** of a or the **principal *n*th root** of a. For example, $a^{1/2}$ denotes the positive second root, or **square root**, of a, while $a^{1/4}$ is the positive fourth root of a. When n is *odd*, there is only one nth root, which has the same sign as a. For example, $a^{1/3}$, the **cube root** of a, has the same sign as a. By definition, if $b = a^{1/n}$, then $b^n = a$. On a calculator, a number is raised to a power using a key labeled x^y, y^x, or $\wedge$. For example, to take the fourth root of 6 on a TI-84 Plus calculator, enter $6 \wedge (1/4)$ to get the result 1.56508458.

EXAMPLE 3 Calculations with Exponents

(a) $121^{1/2} = 11$, since 11 is positive and $11^2 = 121$.
(b) $625^{1/4} = 5$, since $5^4 = 625$.
(c) $256^{1/4} = 4$
(d) $64^{1/6} = 2$
(e) $27^{1/3} = 3$
(f) $(-32)^{1/5} = -2$
(g) $128^{1/7} = 2$
(h) $(-49)^{1/2}$ is not a real number.

Rational Exponents

In the following definition, the domain of an exponent is extended to include all rational numbers.

Definition of $a^{m/n}$

For all real numbers a for which the indicated roots exist, and for any rational number m/n,

$$a^{m/n} = (a^{1/n})^m.$$

EXAMPLE 4 Calculations with Exponents

(a) $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$

(b) $32^{2/5} = (32^{1/5})^2 = 2^2 = 4$

(c) $64^{4/3} = (64^{1/3})^4 = 4^4 = 256$

(d) $25^{3/2} = (25^{1/2})^3 = 5^3 = 125$

NOTE $27^{2/3}$ could also be evaluated as $(27^2)^{1/3}$, but this is more difficult to perform without a calculator because it involves squaring 27 and then taking the cube root of this large number. On the other hand, when we evaluate it as $(27^{1/3})^2$, we know that the cube root of 27 is 3 without using a calculator, and squaring 3 is easy.

All the properties for integer exponents given in this section also apply to any rational exponent on a nonnegative real-number base.

EXAMPLE 5 Simplifying Exponential Expressions

(a) $\dfrac{y^{1/3}y^{5/3}}{y^3} = \dfrac{y^{1/3+5/3}}{y^3} = \dfrac{y^2}{y^3} = y^{2-3} = y^{-1} = \dfrac{1}{y}$

(b) $m^{2/3}(m^{7/3} + 2m^{1/3}) = m^{2/3+7/3} + 2m^{2/3+1/3} = m^3 + 2m$

(c) $\left(\dfrac{m^7n^{-2}}{m^{-5}n^2}\right)^{1/4} = \left(\dfrac{m^{7-(-5)}}{n^{2-(-2)}}\right)^{1/4} = \left(\dfrac{m^{12}}{n^4}\right)^{1/4} = \dfrac{(m^{12})^{1/4}}{(n^4)^{1/4}} = \dfrac{m^{12/4}}{n^{4/4}} = \dfrac{m^3}{n}$

In calculus, it is often necessary to factor expressions involving fractional exponents.

EXAMPLE 6 Simplifying Exponential Expressions

Factor out the smallest power of the variable, assuming all variables represent positive real numbers.

(a) $4m^{1/2} + 3m^{3/2}$

SOLUTION The smallest exponent is $1/2$. Factoring out $\boldsymbol{m^{1/2}}$ yields

$$\begin{aligned} 4m^{1/2} + 3m^{3/2} &= \boldsymbol{m^{1/2}}(4m^{1/2-1/2} + 3m^{3/2-1/2}) \\ &= m^{1/2}(4 + 3m). \end{aligned}$$

Check this result by multiplying $m^{1/2}$ by $4 + 3m$.

(b) $9x^{-2} - 6x^{-3}$

SOLUTION The smallest exponent here is -3. Since 3 is a common numerical factor, factor out $\boldsymbol{3x^{-3}}$.

$$9x^{-2} - 6x^{-3} = \boldsymbol{3x^{-3}}(3x^{-2-(-3)} - 2x^{-3-(-3)}) = 3x^{-3}(3x - 2)$$

Check by multiplying. The factored form can be written without negative exponents as

$$\frac{\boldsymbol{3}(3x - 2)}{\boldsymbol{x^3}}.$$

(c) $(x^2 + 5)(3x - 1)^{-1/2}(2) + (3x - 1)^{1/2}(2x)$

SOLUTION There is a common factor of 2. Also, $(3x - 1)^{-1/2}$ and $(3x - 1)^{1/2}$ have a common factor. Always factor out the quantity to the *smallest* exponent. Here $-1/2 < 1/2$, so the common factor is $2(3x - 1)^{-1/2}$ and the factored form is

$$2(3x - 1)^{-1/2}[(x^2 + 5) + (3x - 1)x] = 2(3x - 1)^{-1/2}(4x^2 - x + 5).$$

YOUR TURN 2 Factor $5z^{1/3} + 4z^{-2/3}$.

TRY YOUR TURN 2

R.6 EXERCISES

Evaluate each expression. Write all answers without exponents.

1. 8^{-2}

2. 3^{-4}

3. 5^{0}

4. $\left(-\frac{3}{4}\right)^{0}$

5. $-(-3)^{-2}$

6. $-(-3^{-2})$

7. $\left(\frac{1}{6}\right)^{-2}$

8. $\left(\frac{4}{3}\right)^{-3}$

Simplify each expression. Assume that all variables represent positive real numbers. Write answers with only positive exponents.

9. $\frac{4^{-2}}{4}$

10. $\frac{8^{9}\cdot 8^{-7}}{8^{-3}}$

11. $\frac{10^{8}\cdot 10^{-10}}{10^{4}\cdot 10^{2}}$

12. $\left(\frac{7^{-12}\cdot 7^{3}}{7^{-8}}\right)^{-1}$

13. $\frac{x^{4}\cdot x^{3}}{x^{5}}$

14. $\frac{y^{10}\cdot y^{-4}}{y^{6}}$

15. $\frac{(4k^{-1})^{2}}{2k^{-5}}$

16. $\frac{(3z^{2})^{-1}}{z^{5}}$

17. $\frac{3^{-1}\cdot x\cdot y^{2}}{x^{-4}\cdot y^{5}}$

18. $\frac{5^{-2}m^{2}y^{-2}}{5^{2}m^{-1}y^{-2}}$

19. $\left(\frac{a^{-1}}{b^{2}}\right)^{-3}$

20. $\left(\frac{c^{3}}{7d^{-2}}\right)^{-2}$

Simplify each expression, writing the answer as a single term without negative exponents.

21. $a^{-1}+b^{-1}$

22. $b^{-2}-a$

23. $\frac{2n^{-1}-2m^{-1}}{m+n^{2}}$

24. $\left(\frac{m}{3}\right)^{-1}+\left(\frac{n}{2}\right)^{-2}$

25. $(x^{-1}-y^{-1})^{-1}$

26. $(x\cdot y^{-1}-y^{-2})^{-2}$

Write each number without exponents.

27. $121^{1/2}$

28. $27^{1/3}$

29. $32^{2/5}$

30. $-125^{2/3}$

31. $\left(\frac{36}{144}\right)^{1/2}$

32. $\left(\frac{64}{27}\right)^{1/3}$

33. $8^{-4/3}$

34. $625^{-1/4}$

35. $\left(\frac{27}{64}\right)^{-1/3}$

36. $\left(\frac{121}{100}\right)^{-3/2}$

Simplify each expression. Write all answers with only positive exponents. Assume that all variables represent positive real numbers.

37. $3^{2/3}\cdot 3^{4/3}$

38. $27^{2/3}\cdot 27^{-1/3}$

39. $\frac{4^{9/4}\cdot 4^{-7/4}}{4^{-10/4}}$

40. $\frac{3^{-5/2}\cdot 3^{3/2}}{3^{7/2}\cdot 3^{-9/2}}$

41. $\left(\frac{x^{6}y^{-3}}{x^{-2}y^{5}}\right)^{1/2}$

42. $\left(\frac{a^{-7}b^{-1}}{b^{-4}a^{2}}\right)^{1/3}$

43. $\frac{7^{-1/3}\cdot 7r^{-3}}{7^{2/3}\cdot (r^{-2})^{2}}$

44. $\frac{12^{3/4}\cdot 12^{5/4}\cdot y^{-2}}{12^{-1}\cdot (y^{-3})^{-2}}$

45. $\frac{3k^{2}\cdot (4k^{-3})^{-1}}{4^{1/2}\cdot k^{7/2}}$

46. $\frac{8p^{-3}\cdot (4p^{2})^{-2}}{p^{-5}}$

47. $\frac{a^{4/3}\cdot b^{1/2}}{a^{2/3}\cdot b^{-3/2}}$

48. $\frac{x^{3/2}\cdot y^{4/5}\cdot z^{-3/4}}{x^{5/3}\cdot y^{-6/5}\cdot z^{1/2}}$

49. $\frac{k^{-3/5}\cdot h^{-1/3}\cdot t^{2/5}}{k^{-1/5}\cdot h^{-2/3}\cdot t^{1/5}}$

50. $\frac{m^{7/3}\cdot n^{-2/5}\cdot p^{3/8}}{m^{-2/3}\cdot n^{3/5}\cdot p^{-5/8}}$

Factor each expression.

51. $3x^{3}(x^{2}+3x)^{2}-15x(x^{2}+3x)^{2}$

52. $6x(x^{3}+7)^{2}-6x^{2}(3x^{2}+5)(x^{3}+7)$

53. $10x^{3}(x^{2}-1)^{-1/2}-5x(x^{2}-1)^{1/2}$

54. $9(6x+2)^{1/2}+3(9x-1)(6x+2)^{-1/2}$

55. $x(2x+5)^{2}(x^{2}-4)^{-1/2}+2(x^{2}-4)^{1/2}(2x+5)$

56. $(4x^{2}+1)^{2}(2x-1)^{-1/2}+16x(4x^{2}+1)(2x-1)^{1/2}$

YOUR TURN ANSWERS

1. z^{16}/y^{10} **2.** $z^{-2/3}(5z+4)$

R.7 Radicals

We have defined $a^{1/n}$ as the positive or principal nth root of a for appropriate values of a and n. An alternative notation for $a^{1/n}$ uses radicals.

Radicals

If n is an even natural number and $a > 0$, or n is an odd natural number, then

$$a^{1/n} = \sqrt[n]{a}.$$

The symbol $\sqrt[n]{\ }$ is a **radical sign**, the number a is the **radicand**, and n is the **index** of the radical. The familiar symbol $\sqrt{a}$ is used instead of $\sqrt[2]{a}$.

EXAMPLE 1 Radical Calculations

(a) $\sqrt[4]{16} = 16^{1/4} = 2$

(b) $\sqrt[5]{-32} = -2$

(c) $\sqrt[3]{1000} = 10$

(d) $\sqrt[6]{\frac{64}{729}} = \frac{2}{3}$

With $a^{1/n}$ written as $\sqrt[n]{a}$, the expression $a^{m/n}$ also can be written using radicals.

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or} \qquad a^{m/n} = \sqrt[n]{a^m}$$

The following properties of radicals depend on the definitions and properties of exponents.

Properties of Radicals

For all real numbers a and b and natural numbers m and n such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers:

1. $(\sqrt[n]{a})^n = a$
2. $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$
3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
4. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} \quad (b \neq 0)$
5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Property 3 can be used to simplify certain radicals. For example, since $48 = 16 \cdot 3$,

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}.$$

To some extent, simplification is in the eye of the beholder, and $\sqrt{48}$ might be considered as simple as $4\sqrt{3}$. In this textbook, we will consider an expression to be simpler when we have removed as many factors as possible from under the radical.

EXAMPLE 2 Radical Calculations

(a) $\sqrt{1000} = \sqrt{100 \cdot 10} = \sqrt{100} \cdot \sqrt{10} = 10\sqrt{10}$

(b) $\sqrt{128} = \sqrt{64 \cdot 2} = 8\sqrt{2}$

(c) $\sqrt{2} \cdot \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = 6$

(d) $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$

(e) $\sqrt{288m^5} = \sqrt{144 \cdot m^4 \cdot 2m} = 12m^2\sqrt{2m}$

(f) $2\sqrt{18} - 5\sqrt{32} = 2\sqrt{9 \cdot 2} - 5\sqrt{16 \cdot 2}$

$$= 2\sqrt{9} \cdot \sqrt{2} - 5\sqrt{16} \cdot \sqrt{2}$$

$$= 2(3)\sqrt{2} - 5(4)\sqrt{2} = -14\sqrt{2}$$

(g) $\sqrt{x^5} \cdot \sqrt[3]{x^5} = x^{5/2} \cdot x^{5/3} = x^{5/2+5/3} = x^{25/6} = \sqrt[6]{x^{25}} = x^4\sqrt[6]{x}$ TRY YOUR TURN 1

YOUR TURN 1
Simplify $\sqrt{28x^9y^5}$.

When simplifying a square root, keep in mind that $\sqrt{x}$ is nonnegative by definition. Also, $\sqrt{x^2}$ is not x, but $|x|$, the **absolute value of x**, defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

For example, $\sqrt{(-5)^2} = |-5| = 5$. It is correct, however, to simplify $\sqrt{x^4} = x^2$. We need not write $|x^2|$ because x^2 is always nonnegative.

EXAMPLE 3 Simplifying by Factoring

Simplify $\sqrt{m^2 - 4m + 4}$.

SOLUTION Factor the polynomial as $m^2 - 4m + 4 = (m - 2)^2$. Then by property 2 of radicals and the definition of absolute value,

$$\sqrt{(m-2)^2} = |m-2| = \begin{cases} m-2 & \text{if } m-2 \geq 0 \\ -(m-2) = 2-m & \text{if } m-2 < 0. \end{cases}$$

CAUTION Avoid the common error of writing $\sqrt{a^2 + b^2}$ as $\sqrt{a^2} + \sqrt{b^2}$. We must add a^2 and b^2 *before* taking the square root. For example, $\sqrt{16 + 9} = \sqrt{25} = 5$, *not* $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$. This idea applies as well to higher roots. For example, in general,

$$\sqrt[3]{a^3 + b^3} \neq \sqrt[3]{a^3} + \sqrt[3]{b^3},$$

$$\sqrt[4]{a^4 + b^4} \neq \sqrt[4]{a^4} + \sqrt[4]{b^4}.$$

Also,

$$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}.$$

Rationalizing Denominators

The next example shows how to *rationalize* (remove all radicals from) the denominator in an expression containing radicals.

EXAMPLE 4 Rationalizing Denominators

Simplify each expression by rationalizing the denominator.

(a) $\dfrac{4}{\sqrt{3}}$

SOLUTION To rationalize the denominator, multiply by $\sqrt{3}/\sqrt{3}$ (or 1) so that the denominator of the product is a rational number.

$$\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \qquad \sqrt{3} \cdot \sqrt{3} = \sqrt{9} = 3$$

(b) $\dfrac{2}{\sqrt[3]{x}}$

SOLUTION Here, we need a perfect cube under the radical sign to rationalize the denominator. Multiplying by $\sqrt[3]{x^2}/\sqrt[3]{x^2}$ gives

$$\frac{2}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{2\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{2\sqrt[3]{x^2}}{x}.$$

(c) $\dfrac{1}{1-\sqrt{2}}$

SOLUTION The best approach here is to multiply both numerator and denominator by the number $1+\sqrt{2}$. The expressions $1+\sqrt{2}$ and $1-\sqrt{2}$ are conjugates,* and their product is $1^2-(\sqrt{2})^2=1-2=-1$. Thus,

$$\frac{1}{1-\sqrt{2}}=\frac{1(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})}=\frac{1+\sqrt{2}}{1-2}=-1-\sqrt{2}.$$

YOUR TURN 2 Rationalize the denominator in $\dfrac{5}{\sqrt{x}-\sqrt{y}}$.

TRY YOUR TURN 2

Sometimes it is advantageous to rationalize the *numerator* of a rational expression. The following example arises in calculus when evaluating a *limit*.

EXAMPLE 5 Rationalizing Numerators

Rationalize each numerator.

(a) $\dfrac{\sqrt{x}-3}{x-9}$.

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{x}+3$.

$$\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}=\frac{(\sqrt{x})^2-3^2}{(x-9)(\sqrt{x}+3)} \qquad (a-b)(a+b)=a^2-b^2$$

$$=\frac{x-9}{(x-9)(\sqrt{x}+3)}$$

$$=\frac{1}{\sqrt{x}+3}$$

(b) $\dfrac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}$

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{3}-\sqrt{x+3}$.

$$\frac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}\cdot\frac{\sqrt{3}-\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}=\frac{3-(x+3)}{3-2\sqrt{3}\sqrt{x+3}+(x+3)}$$

$$=\frac{-x}{6+x-2\sqrt{3(x+3)}}$$

*If a and b are real numbers, the *conjugate* of $a+b$ is $a-b$.

R.7 EXERCISES

Simplify each expression by removing as many factors as possible from under the radical. Assume that all variables represent positive real numbers.

1. $\sqrt[3]{125}$

2. $\sqrt[4]{1296}$

3. $\sqrt[5]{-3125}$

4. $\sqrt{50}$

5. $\sqrt{2000}$

6. $\sqrt{32y^5}$

7. $\sqrt{27}\cdot\sqrt{3}$

8. $\sqrt{2}\cdot\sqrt{32}$

9. $7\sqrt{2}-8\sqrt{18}+4\sqrt{72}$

10. $4\sqrt{3}-5\sqrt{12}+3\sqrt{75}$

11. $4\sqrt{7}-\sqrt{28}+\sqrt{343}$

12. $3\sqrt{28} - 4\sqrt{63} + \sqrt{112}$

13. $\sqrt[3]{2} - \sqrt[3]{16} + 2\sqrt[3]{54}$

14. $2\sqrt[3]{5} - 4\sqrt[3]{40} + 3\sqrt[3]{135}$

15. $\sqrt{2x^3y^2z^4}$

16. $\sqrt{160r^7s^9t^{12}}$

17. $\sqrt[3]{128x^3y^8z^9}$

18. $\sqrt[4]{x^8y^7z^{11}}$

19. $\sqrt{a^3b^5} - 2\sqrt{a^7b^3} + \sqrt{a^3b^9}$

20. $\sqrt{p^7q^3} - \sqrt{p^5q^9} + \sqrt{p^9q}$

21. $\sqrt{a} \cdot \sqrt[3]{a}$

22. $\sqrt{b^3} \cdot \sqrt[4]{b^3}$

Simplify each root, if possible.

23. $\sqrt{16 - 8x + x^2}$

24. $\sqrt{9y^2 + 30y + 25}$

25. $\sqrt{4 - 25z^2}$

26. $\sqrt{9k^2 + h^2}$

Rationalize each denominator. Assume that all radicands represent positive real numbers.

27. $\dfrac{5}{\sqrt{7}}$

28. $\dfrac{5}{\sqrt{10}}$

29. $\dfrac{-3}{\sqrt{12}}$

30. $\dfrac{4}{\sqrt{8}}$

31. $\dfrac{3}{1 - \sqrt{2}}$

32. $\dfrac{5}{2 - \sqrt{6}}$

33. $\dfrac{6}{2 + \sqrt{2}}$

34. $\dfrac{\sqrt{5}}{\sqrt{5} + \sqrt{2}}$

35. $\dfrac{1}{\sqrt{r} - \sqrt{3}}$

36. $\dfrac{5}{\sqrt{m} - \sqrt{5}}$

37. $\dfrac{y - 5}{\sqrt{y} - \sqrt{5}}$

38. $\dfrac{\sqrt{z} - 1}{\sqrt{z} - \sqrt{5}}$

39. $\dfrac{\sqrt{x} + \sqrt{x + 1}}{\sqrt{x} - \sqrt{x + 1}}$

40. $\dfrac{\sqrt{p} + \sqrt{p^2 - 1}}{\sqrt{p} - \sqrt{p^2 - 1}}$

Rationalize each numerator. Assume that all radicands represent positive real numbers.

41. $\dfrac{1 + \sqrt{2}}{2}$

42. $\dfrac{3 - \sqrt{3}}{6}$

43. $\dfrac{\sqrt{x} + \sqrt{x + 1}}{\sqrt{x} - \sqrt{x + 1}}$

44. $\dfrac{\sqrt{p} - \sqrt{p - 2}}{\sqrt{p}}$

YOUR TURN ANSWERS

1. $2x^4y^2\sqrt{7xy}$

2. $5(\sqrt{x} + \sqrt{y})/(x - y)$

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Linear Functions

1.1 Slopes and Equations of Lines

1.2 Linear Functions and Applications

1.3 The Least Squares Line

Chapter 1 Review

Extended Application: Using Extrapolation to Predict Life Expectancy

Over short time intervals, many changes in the economy are well modeled by linear functions. In an exercise in the first section of this chapter, we will examine a linear model that predicts the number of cellular telephone users in the United States. Such predictions are important tools for cellular telephone company executives and planners.

Before using mathematics to solve a real-world problem, we must usually set up a **mathematical model,** a mathematical description of the situation. In this chapter we look at some mathematics of *linear* models, which are used for data whose graphs can be approximated by straight lines. Linear models have an immense number of applications, because even when the underlying phenomenon is not linear, a linear model often provides an approximation that is sufficiently accurate and much simpler to use.

1.1 Slopes and Equations of Lines

APPLY IT **How fast has tuition at public colleges been increasing in recent years, and how well can we predict tuition in the future?**

In Example 14 of this section, we will answer these questions using the equation of a line.

There are many everyday situations in which two quantities are related. For example, if a bank account pays 6% simple interest per year, then the interest I that a deposit of P dollars would earn in one year is given by

$$I = 0.06 \cdot P, \qquad \text{or} \qquad I = 0.06P.$$

The formula $I = 0.06P$ describes the relationship between interest and the amount of money deposited.

Using this formula, we see, for example, that if $P = \$100$, then $I = \$6$, and if $P = \$200$, then $I = \$12$. These corresponding pairs of numbers can be written as **ordered pairs**, $(100, 6)$ and $(200, 12)$, whose order is important. The first number denotes the value of P and the second number the value of I.

Ordered pairs are graphed with the perpendicular number lines of a **Cartesian coordinate system**, shown in Figure 1.* The horizontal number line, or ***x*-axis**, represents the first components of the ordered pairs, while the vertical or ***y*-axis** represents the second components. The point where the number lines cross is the zero point on both lines; this point is called the **origin**.

Each point on the xy-plane corresponds to an ordered pair of numbers, where the x-value is written first. From now on, we will refer to the point corresponding to the ordered pair (x, y) as "the point (x, y)."

Locate the point $(-2, 4)$ on the coordinate system by starting at the origin and counting 2 units to the left on the horizontal axis and 4 units upward, parallel to the vertical axis. This point is shown in Figure 1, along with several other sample points. The number -2 is the ***x*-coordinate** and the number 4 is the ***y*-coordinate** of the point $(-2, 4)$.

The x-axis and y-axis divide the plane into four parts, or **quadrants**. For example, quadrant I includes all those points whose x- and y-coordinates are both positive. The quadrants are numbered as shown in Figure 1. The points on the axes themselves belong to no quadrant. The set of points corresponding to the ordered pairs of an equation is the **graph** of the equation.

The x- and y-values of the points where the graph of an equation crosses the axes are called the ***x*-intercept** and ***y*-intercept**, respectively.** See Figure 2.

*The name "Cartesian" honors René Descartes (1596–1650), one of the greatest mathematicians of the seventeenth century. According to legend, Descartes was lying in bed when he noticed an insect crawling on the ceiling and realized that if he could determine the distance from the bug to each of two perpendicular walls, he could describe its position at any given moment. The same idea can be used to locate a point in a plane.

**Some people prefer to define the intercepts as ordered pairs, rather than as numbers.

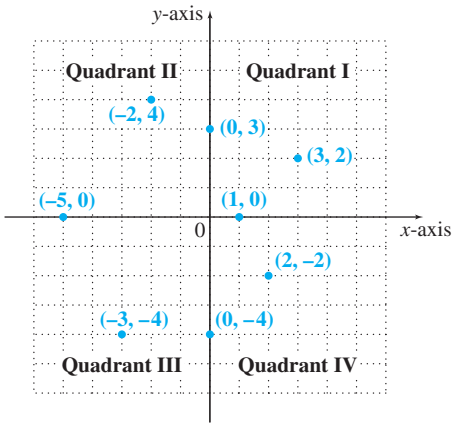


FIGURE 1

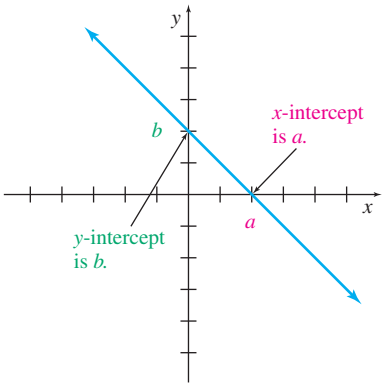


FIGURE 2

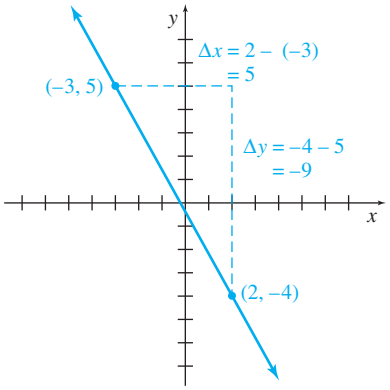


FIGURE 3

Slope of a Line

An important characteristic of a straight line is its *slope*, a number that represents the "steepness" of the line. To see how slope is defined, look at the line in Figure 3. The line goes through the points $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (2, -4)$. The difference in the two x-values,

$$x_2 - x_1 = 2 - (-3) = 5$$

in this example, is called the **change in x**. The symbol Δx (read "delta x") is used to represent the change in x. In the same way, Δy represents the **change in y**. In our example,

$$\begin{aligned} \Delta y &= y_2 - y_1 \\ &= -4 - 5 \\ &= -9. \end{aligned}$$

These symbols, Δx and Δy, are used in the following definition of slope.

Slope of a Line

The **slope** of a line is defined as the vertical change (the "rise") over the horizontal change (the "run") as one travels along the line. In symbols, taking two different points (x_1, y_1) and (x_2, y_2) on the line, the slope is

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1},$$

where $x_1 \neq x_2$.

By this definition, the slope of the line in Figure 3 is

$$m = \frac{\Delta y}{\Delta x} = \frac{-4 - 5}{2 - (-3)} = -\frac{9}{5}.$$

The slope of a line tells how fast y changes for each unit of change in x.

NOTE Using similar triangles, it can be shown that the slope of a line is independent of the choice of points on the line. That is, the same slope will be obtained for *any* choice of two different points on the line.

EXAMPLE 1 Slope

Find the slope of the line through each pair of points.

(a) $(7, 6)$ and $(-4, 5)$

SOLUTION Let $(x_1, y_1) = (7, 6)$ and $(x_2, y_2) = (-4, 5)$. Use the definition of slope.

$$m = \frac{\Delta y}{\Delta x} = \frac{5 - 6}{-4 - 7} = \frac{-1}{-11} = \frac{1}{11}$$

(b) $(5, -3)$ and $(-2, -3)$

SOLUTION Let $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (-2, -3)$. Then

$$m = \frac{-3 - (-3)}{-2 - 5} = \frac{0}{-7} = 0.$$

Lines with zero slope are horizontal (parallel to the x-axis).

(c) $(2, -4)$ and $(2, 3)$

SOLUTION Let $(x_1, y_1) = (2, -4)$ and $(x_2, y_2) = (2, 3)$. Then

$$m = \frac{3 - (-4)}{2 - 2} = \frac{7}{0},$$

which is undefined. This happens when the line is vertical (parallel to the y-axis).

TRY YOUR TURN 1

YOUR TURN 1 Find the slope of the line through (1, 5) and (4, 6).

CAUTION The phrase "no slope" should be avoided; specify instead whether the slope is zero or undefined.

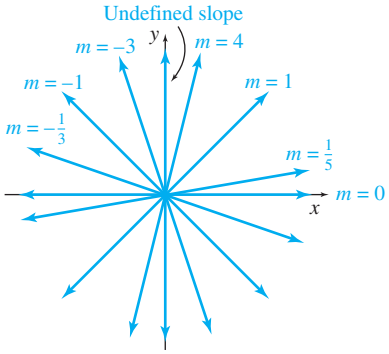


FIGURE 4

In finding the slope of the line in Example 1(a), we could have let $(x_1, y_1) = (-4, 5)$ and $(x_2, y_2) = (7, 6)$. In that case,

$$m = \frac{6 - 5}{7 - (-4)} = \frac{1}{11},$$

the same answer as before. The order in which coordinates are subtracted does not matter, as long as it is done consistently.

Figure 4 shows examples of lines with different slopes. Lines with positive slopes go up from left to right, while lines with negative slopes go down from left to right.

It might help you to compare slope with the percent grade of a hill. If a sign says a hill has a 10% grade uphill, this means the slope is 0.10, or $1/10$, so the hill rises 1 foot for every 10 feet horizontally. A 15% grade downhill means the slope is -0.15.

Equations of a Line

An equation in two first-degree variables, such as $4x + 7y = 20$, has a line as its graph, so it is called a **linear equation**. In the rest of this section, we consider various forms of the equation of a line.

Suppose a line has a slope m and y-intercept b. This means that it goes through the point $(0, b)$. If (x, y) is any other point on the line, then the definition of slope tells us that

$$m = \frac{y - b}{x - 0}.$$

We can simplify this equation by multiplying both sides by x and adding b to both sides. The result is

$$y = mx + b,$$

which we call the *slope-intercept* form of a line. This is the most common form for writing the equation of a line.

FOR REVIEW

For review on solving a linear equation, see Section R.4.

Slope-Intercept Form

If a line has slope m and y-intercept b, then the equation of the line in **slope-intercept form** is

$$y = mx + b.$$

When $b = 0$, we say that y is **proportional** to x.

EXAMPLE 2 Equation of a Line

Find an equation in slope-intercept form for each line.

(a) Through $(0, -3)$ with slope $3/4$

SOLUTION We recognize $(0, -3)$ as the y-intercept because it's the point with 0 as its x-coordinate, so $b = -3$. The slope is $3/4$, so $m = 3/4$. Substituting these values into $y = mx + b$ gives

$$y = \frac{3}{4}x - 3.$$

(b) With x-intercept 7 and y-intercept 2

SOLUTION Notice that $b = 2$. To find m, use the definition of slope after writing the x-intercept as $(7, 0)$ (because the y-coordinate is 0 where the line crosses the x-axis) and the y-intercept as $(0, 2)$.

$$m = \frac{0 - 2}{7 - 0} = -\frac{2}{7}$$

Substituting these values into $y = mx + b$, we have

$$y = -\frac{2}{7}x + 2.$$

YOUR TURN 2 Find the equation of the line with x-intercept -4 and y-intercept 6.

TRY YOUR TURN 2

EXAMPLE 3 Finding the Slope

Find the slope of the line whose equation is $3x - 4y = 12$.

SOLUTION To find the slope, solve the equation for y.

$$\begin{aligned} 3x - 4y &= 12 \\ -4y &= -3x + 12 && \text{Subtract } 3x \text{ from both sides.} \\ y &= \frac{3}{4}x - 3 && \text{Divide both sides by } -4. \end{aligned}$$

The coefficient of x is $3/4$, which is the slope of the line. Notice that this is the same line as in Example 2 (a).

YOUR TURN 3 Find the slope of the line whose equation is $8x + 3y = 5$.

TRY YOUR TURN 3

The slope-intercept form of the equation of a line involves the slope and the y-intercept. Sometimes, however, the slope of a line is known, together with one point (perhaps *not* the y-intercept) that the line goes through. The *point-slope form* of the equation of a line is used to find the equation in this case. Let (x_1, y_1) be any fixed point on the line, and let (x, y) represent any other point on the line. If m is the slope of the line, then by the definition of slope,

$$\frac{y - y_1}{x - x_1} = m,$$

or

$$y - y_1 = m(x - x_1). \qquad \text{Multiply both sides by } x - x_1.$$

Point-Slope Form

If a line has slope m and passes through the point (x_1, y_1), then an equation of the line is given by

$$y - y_1 = m(x - x_1),$$

the **point-slope form** of the equation of a line.

EXAMPLE 4 Point-Slope Form

Find an equation of the line that passes through the point $(3, -7)$ and has slope $m = 5/4$.

SOLUTION Use the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-7) &= \frac{5}{4}(x - 3) && y_1 = -7, m = \tfrac{5}{4}, x_1 = 3 \\ y + 7 &= \frac{5}{4}(x - 3) \\ 4y + 28 &= 5(x - 3) && \text{Multiply both sides by 4.} \\ 4y + 28 &= 5x - 15 && \text{Distribute.} \\ 4y &= 5x - 43 && \text{Combine constants.} \\ y &= \frac{5}{4}x - \frac{43}{4} && \text{Divide both sides by 4.} \end{aligned}$$

FOR REVIEW

See Section R.4 for details on eliminating denominators in an equation.

The equation of the same line can be given in many forms. To avoid confusion, the linear equations used in the rest of this section will be written in slope-intercept form, $y = mx + b$, which is often the most useful form.

The point-slope form also can be useful to find an equation of a line if we know two different points that the line goes through, as in the next example.

EXAMPLE 5 Using Point-Slope Form to Find an Equation

Find an equation of the line through $(5, 4)$ and $(-10, -2)$.

SOLUTION Begin by using the definition of slope to find the slope of the line that passes through the given points.

$$\text{Slope} = m = \frac{-2 - 4}{-10 - 5} = \frac{-6}{-15} = \frac{2}{5}$$

Either $(5, 4)$ or $(-10, -2)$ can be used in the point-slope form with $m = 2/5$. If $(x_1, y_1) = (5, 4)$, then

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= \frac{2}{5}(x - 5) && y_1 = 4, m = \tfrac{2}{5}, x_1 = 5 \\ 5y - 20 &= 2(x - 5) && \text{Multiply both sides by 5.} \\ 5y - 20 &= 2x - 10 && \text{Distributive property} \\ 5y &= 2x + 10 && \text{Add 20 to both sides.} \\ y &= \frac{2}{5}x + 2 && \text{Divide by 5 to put in slope-intercept form.} \end{aligned}$$

Check that the same result is found if $(x_1, y_1) = (-10, -2)$.

YOUR TURN 4 Find the equation of the line through (2, 9) and (5, 3). Put your answer in slope-intercept form.

TRY YOUR TURN 4

EXAMPLE 6 Horizontal Line

Find an equation of the line through $(8, -4)$ and $(-2, -4)$.

SOLUTION Find the slope.

$$m = \frac{-4 - (-4)}{-2 - 8} = \frac{0}{-10} = 0$$

Choose, say, $(8, -4)$ as (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 0(x - 8) && y_1 = -4, m = 0, x_1 = 8 \\ y + 4 &= 0 && 0(x - 8) = 0 \\ y &= -4 \end{aligned}$$

Plotting the given ordered pairs and drawing a line through the points show that the equation $y = -4$ represents a horizontal line. See Figure 5(a). Every horizontal line has a slope of zero and an equation of the form $y = k$, where k is the y-value of all ordered pairs on the line.

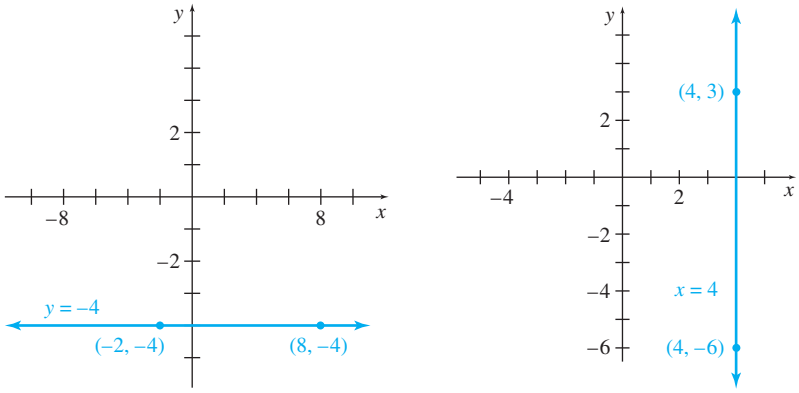


FIGURE 5

EXAMPLE 7 Vertical Line

Find an equation of the line through $(4, 3)$ and $(4, -6)$.

SOLUTION The slope of the line is

$$m = \frac{-6 - 3}{4 - 4} = \frac{-9}{0},$$

which is undefined. Since both ordered pairs have x-coordinate 4, the equation is $x = 4$. Because the slope is undefined, the equation of this line cannot be written in the slope-intercept form.

Again, plotting the given ordered pairs and drawing a line through them show that the graph of $x = 4$ is a vertical line. See Figure 5(b).

Slope of Horizontal and Vertical Lines

The slope of a horizontal line is 0.

The slope of a vertical line is undefined.

The different forms of linear equations discussed in this section are summarized below. The slope-intercept and point-slope forms are equivalent ways to express the equation of a nonvertical line. The slope-intercept form is simpler for a final answer, but you may find the point-slope form easier to use when you know the slope of a line and a point through which the line passes. The slope-intercept form is often considered the standard form. Any line that is not vertical has a unique slope-intercept form but can have many point-slope forms for its equation.

Equations of Lines

Equation	Description
$y = mx + b$	**Slope-intercept form:** slope m, y-intercept b
$y - y_1 = m(x - x_1)$	**Point-slope form:** slope m, line passes through (x_1, y_1)
$x = k$	**Vertical line:** x-intercept k, no y-intercept (except when $k = 0$), undefined slope
$y = k$	**Horizontal line:** y-intercept k, no x-intercept (except when $k = 0$), slope 0

Parallel and Perpendicular Lines

One application of slope involves deciding whether two lines are parallel, which means that they never intersect. Since two parallel lines are equally "steep," they should have the same slope. Also, two lines with the same "steepness" are parallel.

Parallel Lines

Two lines are **parallel** if and only if they have the same slope, or if they are both vertical.

EXAMPLE 8 Parallel Line

Find the equation of the line that passes through the point $(3, 5)$ and is parallel to the line $2x + 5y = 4$.

SOLUTION The slope of $2x + 5y = 4$ can be found by writing the equation in slope-intercept form.

$$2x + 5y = 4$$

$$y = -\frac{2}{5}x + \frac{4}{5}$$ Subtract $2x$ from both sides and divide both sides by 5.

This result shows that the slope is $-2/5$. Since the lines are parallel, $-2/5$ is also the slope of the line whose equation we want. This line passes through $(3, 5)$. Substituting $m = -2/5$, $x_1 = 3$, and $y_1 = 5$ into the point-slope form gives

$$y - y_1 = m(x - x_1)$$

$$y - 5 = -\frac{2}{5}(x - 3) = -\frac{2}{5}x + \frac{6}{5}$$

$$y = -\frac{2}{5}x + \frac{6}{5} + 5$$

$$y = -\frac{2}{5}x + \frac{31}{5}.$$ Multiply 5 by $5/5$ to get a common denominator.

YOUR TURN 5 Find the equation of the line that passes through the point (4, 5) and is parallel to the line $3x - 6y = 7$. Put your answer in slope-intercept form.

TRY YOUR TURN 5

As already mentioned, two nonvertical lines are parallel if and only if they have the same slope. Two lines having slopes with a product of -1 are perpendicular. A proof of this fact, which depends on similar triangles from geometry, is given as Exercise 43 in this section.

Perpendicular Lines

Two lines are **perpendicular** if and only if the product of their slopes is -1, or if one is vertical and the other horizontal.

EXAMPLE 9 Perpendicular Line

Find the equation of the line L passing through the point $(3, 7)$ and perpendicular to the line having the equation $5x - y = 4$.

SOLUTION To find the slope, write $5x - y = 4$ in slope-intercept form:

$$y = 5x - 4.$$

The slope is 5. Since the lines are perpendicular, if line L has slope m, then

$$5m = -1$$
$$m = -\frac{1}{5}.$$

Now substitute $m = -1/5$, $x_1 = 3$, and $y_1 = 7$ into the point-slope form.

$$y - 7 = -\frac{1}{5}(x - 3)$$
$$y - 7 = -\frac{1}{5}x + \frac{3}{5}$$
$$y = -\frac{1}{5}x + \frac{3}{5} + 7 \cdot \frac{5}{5}$$

Add 7 to both sides and get a common denominator.

$$y = -\frac{1}{5}x + \frac{38}{5}$$

TRY YOUR TURN 6

YOUR TURN 6 Find the equation of the line passing through the point $(3, 2)$ and perpendicular to the line having the equation $2x + 3y = 4$.

The next example uses the equation of a line to analyze real-world data. In this example, we are looking at how one variable changes over time. To simplify the arithmetic, we will *rescale* the variable representing time, although computers and calculators have made rescaling less important than in the past. Here it allows us to work with smaller numbers, and, as you will see, find the y-intercept of the line more easily. We will use rescaling on many examples throughout this book. When we do, it is important to be consistent.

EXAMPLE 10 Prevalence of Cigarette Smoking

In recent years, the percentage of the U.S. population age 18 and older who smoke has decreased at a roughly constant rate, from 24.1% in 1998 to 20.6% in 2008. ***Source: Centers for Disease Control and Prevention.***

(a) Find the equation describing this linear relationship.

SOLUTION Let t represent time in years, with $t = 0$ representing 1990. With this rescaling, the year 1998 corresponds to $t = 8$ and the year 2008 corresponds to $t = 2008 - 1990 = 18$. Let y represent the percentage of the population who smoke. The two ordered pairs representing the given information are then $(8, 24.1)$ and $(18, 20.6)$. The slope of the line through these points is

$$m = \frac{20.6 - 24.1}{18 - 8} = \frac{-3.5}{10} = -0.35.$$

This means that, on average, the percentage of the adult population who smoke is decreasing by about 0.35% per year.

Using $m = -0.35$ in the point-slope form, and choosing $(t_1, y_1) = (8, 24.1)$, gives the required equation.

$$\begin{aligned} y - 24.1 &= -0.35\,(t - 8) \\ y - 24.1 &= -0.35\,t + 2.8 \\ y &= -0.35\,t + 26.9 \end{aligned}$$

We could have used the other point (18, 20.6) and found the same answer. Instead, we'll use this to check our answer by observing that $-0.35(18) + 26.9 = 20.6$, which agrees with the y-value at $t = 18$.

(b) One objective of Healthy People 2010 (a campaign of the U.S. Department of Health and Human Services) was to reduce the percentage of U.S. adults who smoke to 12% or less by the year 2010. If this decline in smoking continued at the same rate, did they meet this objective?

SOLUTION Using the same rescaling, $t = 20$ corresponds to the year 2010. Substituting this value into the above equation gives

$$y = -0.35(20) + 26.9 = 19.9.$$

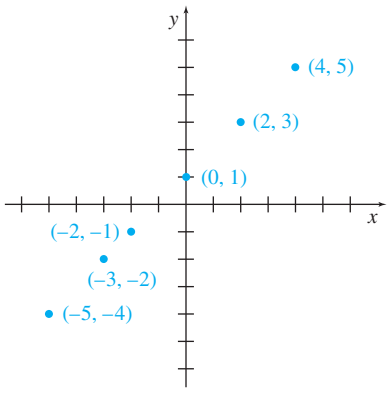
Continuing at this rate, an estimated 19.9% of the adult population still smoked in 2010, and the objective of Healthy People 2010 was not met.

Notice that if the formula from part (b) of Example 10 is valid for all nonnegative t, then eventually y becomes 0:

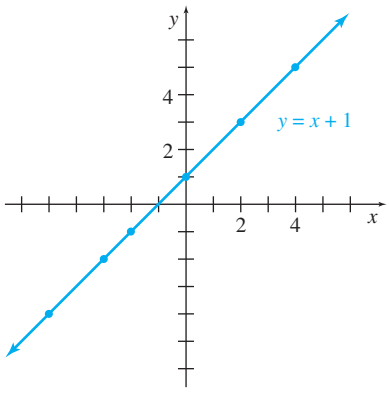
$$\begin{aligned} -0.35\,t + 26.9 &= 0 \\ -0.35\,t &= -26.9 && \text{Subtract 26.9 from both sides.} \\ t = \frac{-26.9}{-0.35} &= 76.857 \approx 77^*, && \text{Divide both sides by } -0.35. \end{aligned}$$

which indicates that 77 years from 1990 (in the year 2067), 0% of the U.S. adult population will smoke. Of course, it is still possible that in 2067 there will be adults who smoke; the trend of recent years may not continue. Most equations are valid for some specific set of numbers. It is highly speculative to extrapolate beyond those values.

On the other hand, people in business and government often need to make some prediction about what will happen in the future, so a tentative conclusion based on past trends may be better than no conclusion at all. There are also circumstances, particularly in the physical sciences, in which theoretical reasons imply that the trend will continue.



(a)



(b)

FIGURE 6

Graph of a Line

We can graph the linear equation defined by $y = x + 1$ by finding several ordered pairs that satisfy the equation. For example, if $x = 2$, then $y = 2 + 1 = 3$, giving the ordered pair $(2, 3)$. Also, $(0, 1)$, $(4, 5)$, $(-2, -1)$, $(-5, -4)$, $(-3, -2)$, among many others, satisfy the equation.

To graph $y = x + 1$, we begin by locating the ordered pairs obtained above, as shown in Figure 6(a). All the points of this graph appear to lie on a straight line, as in Figure 6(b). This straight line is the graph of $y = x + 1$.

It can be shown that every equation of the form $ax + by = c$ has a straight line as its graph, assuming a and b are not both 0. Although just two points are needed to determine a line, it is a good idea to plot a third point as a check. It is often convenient to use the x- and y-intercepts as the two points, as in the following example.

*The symbol $\approx$ means "is approximately equal to."

EXAMPLE 11 Graph of a Line

Graph $3x + 2y = 12$.

SOLUTION To find the y-intercept, let $x = 0$.

$$3(0) + 2y = 12$$
$$2y = 12 \qquad \text{Divide both sides by 2.}$$
$$y = 6$$

Similarly, find the x-intercept by letting $y = 0$, which gives $x = 4$. Verify that when $x = 2$, the result is $y = 3$. These three points are plotted in Figure 7(a). A line is drawn through them in Figure 7(b).

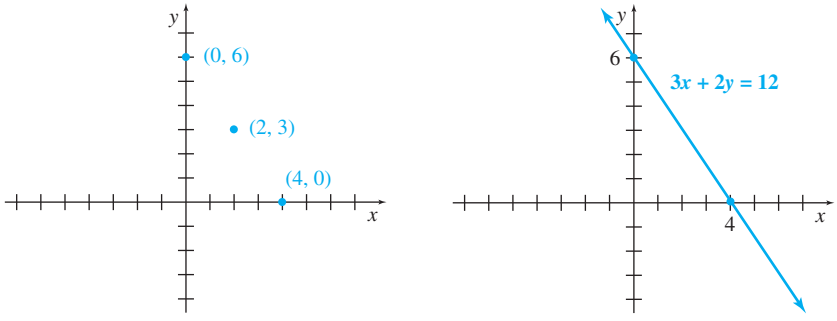


FIGURE 7

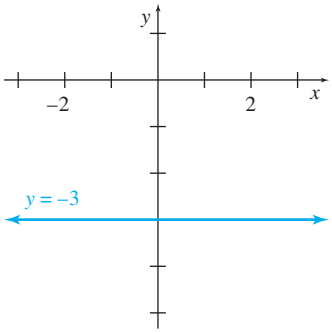


FIGURE 8

Not every line has two distinct intercepts; the graph in the next example does not cross the x-axis, and so it has no x-intercept.

EXAMPLE 12 Graph of a Horizontal Line

Graph $y = -3$.

SOLUTION The equation $y = -3$, or equivalently, $y = 0x - 3$, always gives the same y-value, -3, for any value of x. Therefore, no value of x will make $y = 0$, so the graph has no x-intercept. As we saw in Example 6, the graph of such an equation is a horizontal line parallel to the x-axis. In this case the y-intercept is -3, as shown in Figure 8.

The graph in Example 12 has only one intercept. Another type of linear equation with coinciding intercepts is graphed in Example 13.

EXAMPLE 13 Graph of a Line Through the Origin

Graph $y = -3x$.

SOLUTION Begin by looking for the x-intercept. If $y = 0$, then

$$y = -3x$$
$$0 = -3x \qquad \text{Let } y = 0.$$
$$0 = x. \qquad \text{Divide both sides by } -3.$$

We have the ordered pair $(0, 0)$. Starting with $x = 0$ gives exactly the same ordered pair, $(0, 0)$. Two points are needed to determine a straight line, and the intercepts have led to only one point. To get a second point, we choose some other value of x (or y). For example, if $x = 2$, then

$$y = -3x = -3(2) = -6, \qquad \text{Let } x = 2.$$

giving the ordered pair $(2, -6)$. These two ordered pairs, $(0, 0)$ and $(2, -6)$, were used to get the graph shown in Figure 9.

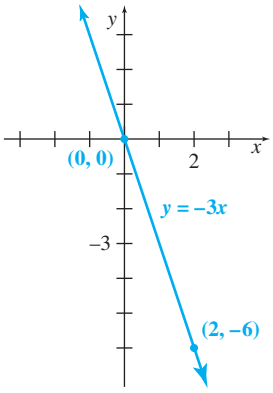


FIGURE 9

Linear equations allow us to set up simple mathematical models for real-life situations. In almost every case, linear (or any other reasonably simple) equations provide only approximations to real-world situations. Nevertheless, these are often remarkably useful approximations.

EXAMPLE 14 Tuition

APPLY IT

The table on the left lists the average annual cost (in dollars) of tuition and fees at public four-year colleges for selected years. ***Source: The College Board.***

Cost of Public College	
Year	**Tuition and Fees**
2000	3508
2001	3766
2002	4098
2003	4645
2004	5126
2005	5492
2006	5804
2007	6191
2008	6591
2009	7020

(a) Plot the cost of public colleges by letting $t = 0$ correspond to 2000. Are the data *exactly* linear? Could the data be *approximated* by a linear equation?

SOLUTION The data is plotted in Figure 10(a) in a figure known as a **scatterplot**. Although it is not exactly linear, it is approximately linear and could be approximated by a linear equation.

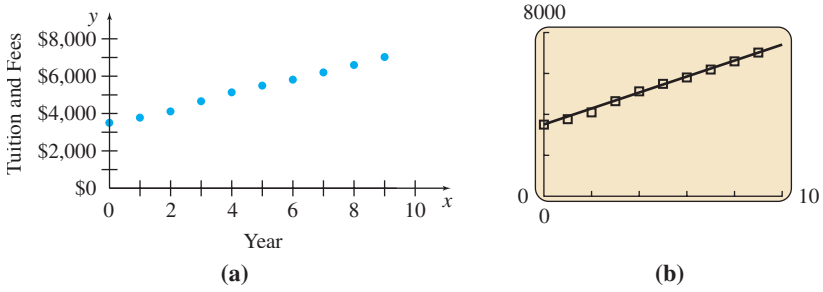


FIGURE 10

(b) Use the points (0, 3508) and (9, 7020) to determine an equation that models the data.

SOLUTION We first find the slope of the line as follows:

$$m = \frac{7020 - 3508}{9 - 0} = \frac{3512}{9} \approx 390.2.$$

We have rounded to four digits, noting that we cannot expect more accuracy in our answer than in our data, which is accurate to four digits. Using the slope-intercept form of the line, $y = mt + b$, with $m = 390.2$ and $b = 3508$, gives

$$y = 390.2t + 3508.$$

TECHNOLOGY NOTE

A graphing calculator plot of this line and the data points are shown in Figure 10(b). Notice that the points closely fit the line. More details on how to construct this graphing calculator plot are given at the end of this example.

(c) Discuss the accuracy of using this equation to estimate the cost of public colleges in the year 2030.

SOLUTION The year 2030 corresponds to the year $t = 30$, for which the equation predicts a cost of

$$y = 390.2(30) + 3508 = 15{,}214, \quad \text{or} \quad \$15{,}214.$$

The year 2030 is many years in the future, however. Many factors could affect the tuition, and the actual figure for 2030 could turn out to be very different from our prediction.

TECHNOLOGY NOTE

You can plot data with a TI-84 Plus graphing calculator using the following steps.

1. Store the data in lists.
2. Define the stat plot.
3. Turn off $Y =$ functions (unless you also want to graph a function).
4. Turn on the plot you want to display.
5. Define the viewing window.
6. Display the graph.

Consult the calculator's instruction booklet or the *Graphing Calculator and Excel Spreadsheet Manual*, available with this book, for specific instructions. See the calculator-generated graph in Figure 10(b), which includes the points and line from Example 14. Notice how the line closely approximates the data.

1.1 EXERCISES

Find the slope of each line.

1. Through $(4, 5)$ and $(-1, 2)$
2. Through $(5, -4)$ and $(1, 3)$
3. Through $(8, 4)$ and $(8, -7)$
4. Through $(1, 5)$ and $(-2, 5)$
5. $y = x$
6. $y = 3x - 2$
7. $5x - 9y = 11$
8. $4x + 7y = 1$
9. $x = 5$
10. The x-axis
11. $y = 8$
12. $y = -6$
13. A line parallel to $6x - 3y = 12$
14. A line perpendicular to $8x = 2y - 5$

In Exercises 15–24, find an equation in slope-intercept form for each line.

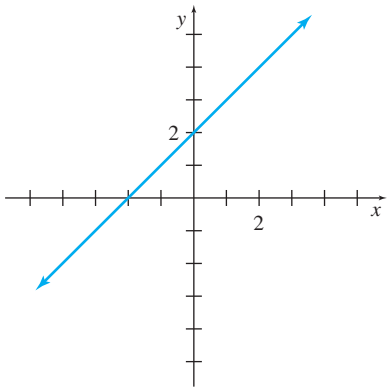
15. Through $(1, 3)$, $m = -2$
16. Through $(2, 4)$, $m = -1$
17. Through $(-5, -7)$, $m = 0$
18. Through $(-8, 1)$, with undefined slope
19. Through $(4, 2)$ and $(1, 3)$
20. Through $(8, -1)$ and $(4, 3)$
21. Through $(2/3, 1/2)$ and $(1/4, -2)$
22. Through $(-2, 3/4)$ and $(2/3, 5/2)$
23. Through $(-8, 4)$ and $(-8, 6)$
24. Through $(-1, 3)$ and $(0, 3)$

In Exercises 25–34, find an equation for each line in the form $ax + by = c$, where a, b, and c are integers with no factor common to all three and $a \geq 0$.

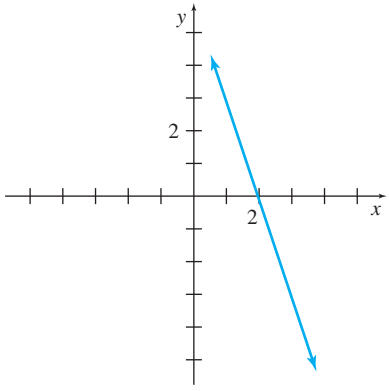
25. x-intercept -6, y-intercept -3
26. x-intercept -2, y-intercept 4
27. Vertical, through $(-6, 5)$
28. Horizontal, through $(8, 7)$
29. Through $(-4, 6)$, parallel to $3x + 2y = 13$
30. Through $(2, -5)$, parallel to $2x - y = -4$
31. Through $(3, -4)$, perpendicular to $x + y = 4$
32. Through $(-2, 6)$, perpendicular to $2x - 3y = 5$
33. The line with y-intercept 4 and perpendicular to $x + 5y = 7$
34. The line with x-intercept $-2/3$ and perpendicular to $2x - y = 4$
35. Do the points $(4, 3)$, $(2, 0)$, and $(-18, -12)$ lie on the same line? Explain why or why not. (*Hint:* Find the slopes between the points.)
36. Find k so that the line through $(4, -1)$ and $(k, 2)$ is
    a. parallel to $2x + 3y = 6$,
    b. perpendicular to $5x - 2y = -1$.
37. Use slopes to show that the quadrilateral with vertices at $(1, 3)$, $(-5/2, 2)$, $(-7/2, 4)$, and $(2, 1)$ is a parallelogram.
38. Use slopes to show that the square with vertices at $(-2, 5)$, $(4, 5)$, $(4, -1)$, and $(-2, -1)$ has diagonals that are perpendicular.

For the lines in Exercises 39 and 40, which of the following is closest to the slope of the line? (a) 1 (b) 2 (c) 3 (d) 21 (e) 22 (f) −3

39.

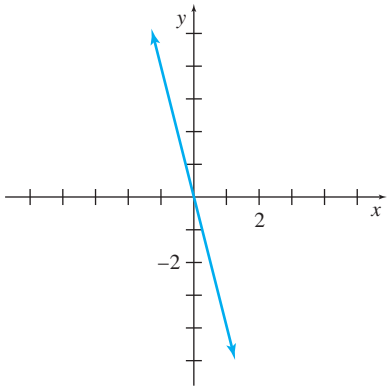


40.

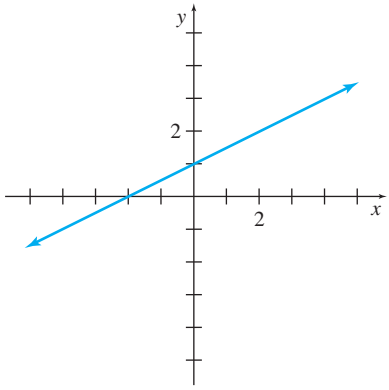


In Exercises 41 and 42, estimate the slope of the lines.

41.



42.



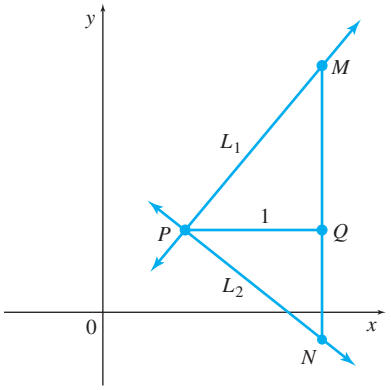
43. To show that two perpendicular lines, neither of which is vertical, have slopes with a product of −1, go through the following steps. Let line L_1 have equation $y = m_1x + b_1$, and let L_2 have equation $y = m_2x + b_2$, with $m_1 > 0$ and $m_2 < 0$. Assume that L_1 and L_2 are perpendicular, and use right triangle MPN shown in the figure. Prove each of the following statements.

a. MQ has length m_1.

b. QN has length $-m_2$.

c. Triangles MPQ and PNQ are similar.

d. $m_1/1 = 1/(-m_2)$ and $m_1m_2 = -1$



44. Consider the equation $\frac{x}{a} + \frac{y}{b} = 1$.

a. Show that this equation represents a line by writing it in the form $y = mx + b$.

b. Find the x- and y-intercepts of this line.

c. Explain in your own words why the equation in this exercise is known as the intercept form of a line.

Graph each equation.

45. $y = x - 1$

46. $y = 4x + 5$

47. $y = -4x + 9$

48. $y = -6x + 12$

49. $2x - 3y = 12$

50. $3x - y = -9$

51. $3y - 7x = -21$

52. $5y + 6x = 11$

53. $y = -2$

54. $x = 4$

55. $x + 5 = 0$

56. $y + 8 = 0$

57. $y = 2x$

58. $y = -5x$

59. $x + 4y = 0$

60. $3x - 5y = 0$

APPLICATIONS

Business and Economics

61. Sales The sales of a small company were $27,000 in its second year of operation and $63,000 in its fifth year. Let y represent sales in the xth year of operation. Assume that the data can be approximated by a straight line.

a. Find the slope of the sales line, and give an equation for the line in the form $y = mx + b$.

b. Use your answer from part a to find out how many years must pass before the sales surpass $100,000.

62. Use of Cellular Telephones The following table shows the subscribership of cellular telephones in the United States (in millions) for even-numbered years between 2000 and 2008. ***Source: Time Almanac 2010.***

Year	2000	2002	2004	2006	2008
Subscribers (in millions)	109.48	140.77	182.14	233.04	270.33

a. Plot the data by letting $t = 0$ correspond to 2000. Discuss how well the data fit a straight line.

b. Determine a linear equation that approximates the number of subscribers using the points (0, 109.48) and (8, 270.33).

c. Repeat part b using the points (2, 140.77) and (8, 270.33).

d. Discuss why your answers to parts b and c are similar but not identical.

e. Using your equations from parts b and c, approximate the number of cellular phone subscribers in the year 2007. Compare your result with the actual value of 255.40 million.

63. Consumer Price Index The Consumer Price Index (CPI) is a measure of the change in the cost of goods over time. The index was 100 for the three-year period centered on 1983. For simplicity, we will assume that the CPI was exactly 100 in 1983. Then the CPI of 215.3 in 2008 indicates that an item that cost \$1.00 in 1983 would cost \$2.15 in 2008. The CPI has been increasing approximately linearly over the last few decades. ***Source: Time Almanac 2010.***

a. Use this information to determine an equation for the CPI in terms of t, which represents the years since 1980.

b. Based on the answer to part a, what was the predicted value of the CPI in 2000? Compare this estimate with the actual CPI of 172.2.

c. Describe the rate at which the annual CPI is changing.

Life Sciences

64. HIV Infection The time interval between a person's initial infection with HIV and that person's eventual development of AIDS symptoms is an important issue. The method of infection with HIV affects the time interval before AIDS develops. One study of HIV patients who were infected by intravenous drug use found that 17% of the patients had AIDS after 4 years, and 33% had developed the disease after 7 years. The relationship between the time interval and the percentage of patients with AIDS can be modeled accurately with a linear equation. ***Source: Epidemiologic Review.***

a. Write a linear equation $y = mt + b$ that models this data, using the ordered pairs $(4, 0.17)$ and $(7, 0.33)$.

b. Use your equation from part a to predict the number of years before half of these patients will have AIDS.

65. Exercise Heart Rate To achieve the maximum benefit for the heart when exercising, your heart rate (in beats per minute) should be in the target heart rate zone. The lower limit of this zone is found by taking 70% of the difference between 220 and your age. The upper limit is found by using 85%. ***Source: Physical Fitness.***

a. Find formulas for the upper and lower limits (u and l) as linear equations involving the age x.

b. What is the target heart rate zone for a 20-year-old?

c. What is the target heart rate zone for a 40-year-old?

d. Two women in an aerobics class stop to take their pulse and are surprised to find that they have the same pulse. One woman is 36 years older than the other and is working at the upper limit of her target heart rate zone. The younger woman is working at the lower limit of her target heart rate zone. What are the ages of the two women, and what is their pulse?

e. Run for 10 minutes, take your pulse, and see if it is in your target heart rate zone. (After all, this is listed as an exercise!)

66. Ponies Trotting A 1991 study found that the peak vertical force on a trotting Shetland pony increased linearly with the pony's speed, and that when the force reached a critical level, the pony switched from a trot to a gallop. For one pony, the critical force was 1.16 times its body weight. It experienced a force of 0.75 times its body weight at a speed of 2 meters per second and a force of 0.93 times its body weight at 3 meters per second. At what speed did the pony switch from a trot to a gallop? ***Source: Science.***

67. Life Expectancy Some scientists believe there is a limit to how long humans can live. One supporting argument is that during the last century, life expectancy from age 65 has increased more slowly than life expectancy from birth, so eventually these two will be equal, at which point, according to these scientists, life expectancy should increase no further. In 1900, life expectancy at birth was 46 yr, and life expectancy at age 65 was 76 yr. In 2004, these figures had risen to 77.8 and 83.7, respectively. In both cases, the increase in life expectancy has been linear. Using these assumptions and the data given, find the maximum life expectancy for humans. ***Source: Science.***

Social Sciences

68. Child Mortality Rate The mortality rate for children under 5 years of age around the world has been declining in a roughly linear fashion in recent years. The rate per 1000 live births was 90 in 1990 and 65 in 2008. ***Source: World Health Organization.***

a. Determine a linear equation that approximates the mortality rate in terms of time t, where t represents the number of years since 1900.

b. If this trend continues, in what year will the mortality rate first drop to 50 or below per 1000 live births?

69. Health Insurance The percentage of adults in the United States without health insurance increased at a roughly linear rate from 1999, when it was 17.2%, to 2008, when it was 20.3%. ***Source: The New York Times.***

a. Determine a linear equation that approximates the percentage of adults in the United States without health insurance in terms of time t, where t represents the number of years since 1990.

b. If this trend were to continue, in what year would the percentage of adults without health insurance be at least 25%?

70. Marriage The following table lists the U.S. median age at first marriage for men and women. The age at which both groups marry for the first time seems to be increasing at a roughly linear rate in recent decades. Let t correspond to the number of years since 1980. ***Source: U.S. Census Bureau.***

Age at First Marriage						
Year	1980	1985	1990	1995	2000	2005
Men	24.7	25.5	26.1	26.9	26.8	27.1
Women	22.0	23.3	23.9	24.5	25.1	25.3

a. Find a linear equation that approximates the data for men, using the data for the years 1980 and 2005.

b. Repeat part a using the data for women.

c. Which group seems to have the faster increase in median age at first marriage?

d. In what year will the men's median age at first marriage reach 30?

e. When the men's median age at first marriage is 30, what will the median age be for women?

71. Immigration In 1950, there were 249,187 immigrants admitted to the United States. In 2008, the number was 1,107,126. ***Source: 2008 Yearbook of Immigration Statistics.***

a. Assuming that the change in immigration is linear, write an equation expressing the number of immigrants, y, in terms of t, the number of years after 1900.

b. Use your result in part a to predict the number of immigrants admitted to the United States in 2015.

c. Considering the value of the y-intercept in your answer to part a, discuss the validity of using this equation to model the number of immigrants throughout the entire 20th century.

Physical Sciences

72. Global Warming In 1990, the Intergovernmental Panel on Climate Change predicted that the average temperature on Earth would rise 0.3°C per decade in the absence of international controls on greenhouse emissions. Let t measure the time in years since 1970, when the average global temperature was 15°C. ***Source: Science News.***

a. Find a linear equation giving the average global temperature in degrees Celsius in terms of t, the number of years since 1970.

b. Scientists have estimated that the sea level will rise by 65 cm if the average global temperature rises to 19°C. According to your answer to part a, when would this occur?

73. Galactic Distance The table lists the distances (in megaparsecs where 1 megaparsec $\approx 3.1 \times 10^{19}$ km) and velocities (in kilometers per second) of four galaxies moving rapidly away from Earth. ***Source: Astronomical Methods and Calculations, and Fundamental Astronomy.***

Galaxy	Distance	Velocity
Virga	15	1600
Ursa Minor	200	15,000
Corona Borealis	290	24,000
Bootes	520	40,000

a. Plot the data points letting x represent distance and y represent velocity. Do the points lie in an approximately linear pattern?

b. Write a linear equation $y = mx$ to model this data, using the ordered pair $(520, 40{,}000)$.

c. The galaxy Hydra has a velocity of 60,000 km per sec. Use your equation to approximate how far away it is from Earth.

d. The value of m in the equation is called the *Hubble constant*. The Hubble constant can be used to estimate the age of the universe A (in years) using the formula

$$A = \frac{9.5 \times 10^{11}}{m}.$$

Approximate A using your value of m.

General Interest

74. News/Talk Radio From 2001 to 2007, the number of stations carrying news/talk radio increased at a roughly linear rate, from 1139 in 2001 to 1370 in 2007. ***Source: State of the Media.***

a. Find a linear equation expressing the number of stations carrying news/talk radio, y, in terms of t, the years since 2000.

b. Use your answer from part a to predict the number of stations carrying news/talk radio in 2008. Compare with the actual number of 2046. Discuss how the linear trend from 2001 to 2007 might have changed in 2008.

75. Tuition The table lists the annual cost (in dollars) of tuition and fees at private four-year colleges for selected years. (See Example 14.) ***Source: The College Board.***

Year	Tuition and Fees
2000	16,072
2002	18,060
2004	20,045
2006	22,308
2008	25,177
2009	26,273

a. Sketch a graph of the data. Do the data appear to lie roughly along a straight line?

b. Let $t = 0$ correspond to the year 2000. Use the points (0, 16,072) and (9, 26,273) to determine a linear equation that models the data. What does the slope of the graph of the equation indicate?

c. Discuss the accuracy of using this equation to estimate the cost of private college in 2025.

YOUR TURN ANSWERS

1. $1/3$ **2.** $y = (3/2)x + 6$
3. $-8/3$ **4.** $y = -2x + 13$
5. $y = (1/2)x + 3$ **6.** $y = (3/2)x - 5/2$

1.2 Linear Functions and Applications

APPLY IT **How many units must be sold for a firm to break even?**

In Example 6 in this section, this question will be answered using a linear function.

As we saw in the previous section, many situations involve two variables related by a linear equation. For such a relationship, when we express the variable y in terms of x, we say that y is a **linear function** of x. This means that for any allowed value of x (the **independent variable**), we can use the equation to find the corresponding value of y (the **dependent variable**). Examples of equations defining linear functions include $y = 2x + 3$, $y = -5$, and $2x - 3y = 7$, which can be written as $y = (2/3)x - (7/3)$. Equations in the form $x = k$, where k is a constant, do not define linear functions. All other linear equations define linear functions.

$f(x)$ Notation

Letters such as f, g, or h are often used to name functions. For example, f might be used to name the function defined by

$$y = 5 - 3x.$$

To show that this function is named f, it is common to replace y with $f(x)$ (read "f of x") to get

$$f(x) = 5 - 3x.$$

By choosing 2 as a value of x, $f(x)$ becomes $5 - 3 \cdot 2 = 5 - 6 = -1$, written

$$f(2) = -1.$$

The corresponding ordered pair is $(2, -1)$. In a similar manner,

$$f(-4) = 5 - 3(-4) = 17, \qquad f(0) = 5, \qquad f(-6) = 23,$$

and so on.

EXAMPLE 1 Function Notation

Let $g(x) = -4x + 5$. Find $g(3)$, $g(0)$, $g(-2)$, and $g(b)$.

SOLUTION To find $g(3)$, substitute 3 for x.

$$g(3) = -4(3) + 5 = -12 + 5 = -7$$

Similarly,

$$g(0) = -4(0) + 5 = 0 + 5 = 5,$$
$$g(-2) = -4(-2) + 5 = 8 + 5 = 13,$$

and

$$g(b) = -4b + 5.$$

YOUR TURN 1 Calculate $g(-5)$.

TRY YOUR TURN 1

We summarize the discussion below.

Linear Function

A relationship f defined by

$$y = f(x) = mx + b,$$

for real numbers m and b, is a **linear function**.

Supply and Demand

Linear functions are often good choices for supply and demand curves. Typically, as the price of an item increases, consumers are less likely to buy an increasingly expensive item, and so the demand for the item decreases. On the other

hand, as the price of an item increases, producers are more likely to see a profit in selling the item, and so the supply of the item increases. The increase in the quantity supplied and decrease in the quantity demanded can eventually result in a surplus, which causes the price to fall. These countervailing trends tend to move the price, as well as the quantity supplied and demanded toward an equilibrium value.

For example, during the late 1980s and early 1990s, the consumer demand for cranberries (and all of their healthy benefits) soared. The quantity demanded surpassed the quantity supplied, causing a shortage, and cranberry prices rose dramatically. As prices increased, growers wanted to increase their profits, so they planted more acres of cranberries. Unfortunately, cranberries take 3 to 5 years from planting until they can first be harvested. As growers waited and prices increased, consumer demand decreased. When the cranberries were finally harvested, the supply overwhelmed the demand and a huge surplus occurred, causing the price of cranberries to drop in the late 1990s. ***Source: Agricultural Marketing Resource Center***. Other factors were involved in this situation, but the relationship between price, supply, and demand was nonetheless typical.

Although economists consider price to be the independent variable, they have the unfortunate habit of plotting price, usually denoted by p, on the vertical axis, while everyone else graphs the independent variable on the horizontal axis. This custom was started by the English economist Alfred Marshall (1842–1924). In order to abide by this custom, we will write p, the price, as a function of q, the quantity produced, and plot p on the vertical axis. But remember, it is really *price* that determines how much consumers demand and producers supply, not the other way around.

Supply and demand functions are not necessarily linear, the simplest kind of function. Yet most functions are approximately linear if a small enough piece of the graph is taken, allowing applied mathematicians to often use linear functions for simplicity. That approach will be taken in this chapter.

EXAMPLE 2 Supply and Demand

Suppose that Greg Tobin, manager of a giant supermarket chain, has studied the supply and demand for watermelons. He has noticed that the demand increases as the price decreases. He has determined that the quantity (in thousands) demanded weekly, q, and the price (in dollars) per watermelon, p, are related by the linear function

$$p = D(q) = 9 - 0.75q. \qquad \text{Demand function}$$

(a) Find the quantity demanded at a price of \$5.25 per watermelon and at a price of \$3.75 per watermelon.

SOLUTION To find the quantity demanded at a price of \$5.25 per watermelon, replace p in the demand function with 5.25 and solve for q.

$$\begin{aligned} 5.25 &= 9 - 0.75q \\ -3.75 &= -0.75q && \text{Subtract 9 from both sides.} \\ 5 &= q && \text{Divide both sides by } -0.75. \end{aligned}$$

Thus, at a price of \$5.25, the quantity demanded is 5000 watermelons.

Similarly, replace p with 3.75 to find the demand when the price is \$3.75. Verify that this leads to $q = 7$. When the price is lowered from \$5.25 to \$3.75 per watermelon, the quantity demanded increases from 5000 to 7000 watermelons.

(b) Greg also noticed that the quantity of watermelons supplied decreased as the price decreased. Price p and supply q are related by the linear function

$$p = S(q) = 0.75q. \qquad \text{Supply function}$$

Find the quantity supplied at a price of \$5.25 per watermelon and at a price of \$3.00 per watermelon.

SOLUTION Substitute 5.25 for p in the supply function, $p = 0.75q$, to find that $q = 7$, so the quantity supplied is 7000 watermelons. Similarly, replacing p with 3 in the supply equation gives a quantity supplied of 4000 watermelons. If the price decreases from \$5.25 to \$3.00 per watermelon, the quantity supplied also decreases, from 7000 to 4000 watermelons.

(c) Graph both functions on the same axes.

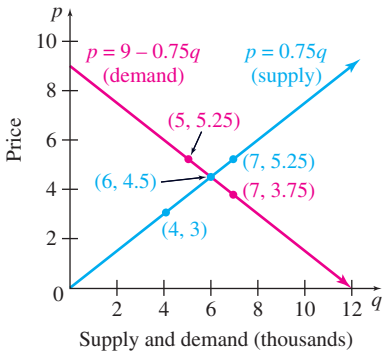
SOLUTION The results of part (a) are written as the ordered pairs (5, 5.25) and (7, 3.75). The line through those points is the graph of the demand function, $p = 9 - 0.75q$, shown in red in Figure 11(a). We used the ordered pairs (7, 5.25) and (4, 3) from the work in part (b) to graph the supply function, $p = 0.75q$, shown in blue in Figure 11(a).

TRY YOUR TURN 2

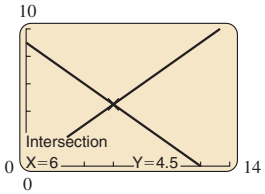
YOUR TURN 2 Find the quantity of watermelon demanded and supplied at a price of \$3.30 per watermelon.

TECHNOLOGY NOTE

A calculator-generated graph of the lines representing the supply and demand functions in Example 2 is shown in Figure 11(b). To get this graph, the equation of each line, using x and y instead of q and p, was entered, along with an appropriate window. A special menu choice gives the coordinates of the intersection point, as shown at the bottom of the graph.



(a)



(b)

FIGURE 11

NOTE Not all supply and demand problems will have the same scale on both axes. It helps to consider the intercepts of both the supply graph and the demand graph to decide what scale to use. For example, in Figure 11, the y-intercept of the demand function is 9, so the scale should allow values from 0 to at least 9 on the vertical axis. The x-intercept of the demand function is 12, so the values on the x-axis must go from 0 to 12.

As shown in the graphs of Figure 11, both the supply graph and the demand graph pass through the point (6, 4.5). If the price of a watermelon is more than \$4.50, the quantity supplied will exceed the quantity demanded and there will be a **surplus** of watermelons. At a price less than \$4.50, the quantity demanded will exceed the quantity supplied and there will be a **shortage** of watermelons. Only at a price of \$4.50 will quantity demanded and supplied be equal. For this reason, \$4.50 is called the *equilibrium price*. When the price is \$4.50, quantity demanded and supplied both equal 6000 watermelons, the *equilibrium quantity*. In general, the **equilibrium price** of the commodity is the price found at the point where the supply and demand graphs for that commodity intersect. The **equilibrium quantity** is the quantity demanded and supplied at that same point. Figure 12 illustrates a general supply and demand situation.

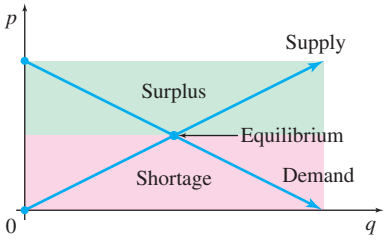


FIGURE 12

EXAMPLE 3 Equilibrium Quantity

Use algebra to find the equilibrium quantity and price for the watermelons in Example 2.

SOLUTION The equilibrium quantity is found when the prices from both supply and demand are equal. Set the two expressions for p equal to each other and solve.

$$\begin{aligned} 9 - 0.75q &= 0.75q \\ 9 &= 1.5q \qquad \text{Add } 0.75q \text{ to both sides.} \\ 6 &= q \end{aligned}$$

The equilibrium quantity is 6000 watermelons, the same answer found earlier.

The equilibrium price can be found by plugging the value of $q = 6$ into either the demand or the supply function. Using the demand function,

$$p = D(6) = 9 - 0.75(6) = 4.5.$$

The equilibrium price is \$4.50, as we found earlier. Check your work by also plugging $q = 6$ into the supply function.

TRY YOUR TURN 3

YOUR TURN 3 Repeat Example 3 using the demand equation $D(q) = 10 - 0.85q$ and the supply equation $S(q) = 0.4q$.

TECHNOLOGY NOTE You may prefer to find the equilibrium quantity by solving the equation with your calculator. Or, if your calculator has a TABLE feature, you can use it to find the value of q that makes the two expressions equal.

Another important issue is how, in practice, the equations of the supply and demand functions can be found. Data need to be collected, and if they lie perfectly along a line, then the equation can easily be found with any two points. What usually happens, however, is that the data are scattered, and there is no line that goes through all the points. In this case we must find a line that approximates the linear trend of the data as closely as possible (assuming the points lie approximately along a line) as in Example 14 in the previous section. This is usually done by the *method of least squares*, also referred to as *linear regression.* We will discuss this method in Section 1.3.

Cost Analysis

The cost of manufacturing an item commonly consists of two parts. The first is a **fixed cost** for designing the product, setting up a factory, training workers, and so on. Within broad limits, the fixed cost is constant for a particular product and does not change as more items are made. The second part is a *cost per item* for labor, materials, packing, shipping, and so on. The total value of this second cost *does* depend on the number of items made.

EXAMPLE 4 Cost Analysis

A small company decides to produce video games. The owners find that the fixed cost for creating the game is \$5000, after which they must spend \$12 to produce each individual copy of the game. Find a formula $C(x)$ for the cost as a linear function of x, the number of games produced.

SOLUTION Notice that $C(0) = 5000$, since \$5000 must be spent even if no games are produced. Also, $C(1) = 5000 + 12 = 5012$, and $C(2) = 5000 + 2 \cdot 12 = 5024$. In general,

$$C(x) = 5000 + 12x,$$

because every time x increases by 1, the cost should increase by \$12. The number 12 is also the slope of the graph of the cost function; the slope gives us the cost to produce one additional item.

In economics, **marginal cost** is the rate of change of cost $C(x)$ at a level of production x and is equal to the slope of the cost function at x. It approximates the cost of producing one additional item. In fact, some books define the marginal cost to be the cost of producing one additional item. With *linear functions*, these two definitions are equivalent, and the marginal cost, which is equal to the slope of the cost function, is *constant.* For instance, in the video game example, the marginal cost of each game is \$12. For other types of functions, these two definitions are only approximately equal. Marginal cost is important to management in making decisions in areas such as cost control, pricing, and production planning.

The work in Example 4 can be generalized. Suppose the total cost to make x items is given by the linear cost function $C(x) = mx + b$. The fixed cost is found by letting $x = 0$:

$$C(0) = m \cdot 0 + b = b;$$

thus, the fixed cost is b dollars. The additional cost of each additional item, the marginal cost, is m, the slope of the line $C(x) = mx + b$.

Linear Cost Function

In a cost function of the form $C(x) = mx + b$, the m represents the marginal cost and b the fixed cost. Conversely, if the fixed cost of producing an item is b and the marginal cost is m, then the **linear cost function** $C(x)$ for producing x items is $C(x) = mx + b$.

EXAMPLE 5 Cost Function

The marginal cost to make x batches of a prescription medication is \$10 per batch, while the cost to produce 100 batches is \$1500. Find the cost function $C(x)$, given that it is linear.

SOLUTION Since the cost function is linear, it can be expressed in the form $C(x) = mx + b$. The marginal cost is \$10 per batch, which gives the value for m. Using $x = 100$ and $C(x) = 1500$ in the point-slope form of the line gives

$$C(x) - 1500 = 10(x - 100)$$
$$C(x) - 1500 = 10x - 1000$$
$$C(x) = 10x + 500. \qquad \text{Add 1500 to both sides.}$$

The cost function is given by $C(x) = 10x + 500$, where the fixed cost is \$500.

TRY YOUR TURN 4

YOUR TURN 4 Repeat Example 5, using a marginal cost of \$15 per batch and a cost of \$1930 to produce 80 batches.

Break-Even Analysis

The **revenue** $R(x)$ from selling x units of an item is the product of the price per unit p and the number of units sold (demand) x, so that

$$R(x) = px.$$

The corresponding **profit** $P(x)$ is the difference between revenue $R(x)$ and cost $C(x)$. That is,

$$P(x) = R(x) - C(x).$$

A company can make a profit only if the revenue received from its customers exceeds the cost of producing and selling its goods and services. The number of units at which revenue just equals cost is the **break-even quantity**; the corresponding ordered pair gives the **break-even point**.

EXAMPLE 6 Break-Even Analysis

APPLY IT

A firm producing poultry feed finds that the total cost $C(x)$ in dollars of producing and selling x units is given by

$$C(x) = 20x + 100.$$

Management plans to charge \$24 per unit for the feed.

(a) How many units must be sold for the firm to break even?

SOLUTION The firm will break even (no profit and no loss) as long as revenue just equals cost, or $R(x) = C(x)$. From the given information, since $R(x) = px$ and $p = \$24$,

$$R(x) = 24x.$$

Substituting for $R(x)$ and $C(x)$ in the equation $R(x) = C(x)$ gives

$$24x = 20x + 100,$$

from which $x = 25$. The firm breaks even by selling 25 units, which is the break-even quantity. The graphs of $C(x) = 20x + 100$ and $R(x) = 24x$ are shown in Figure 13.

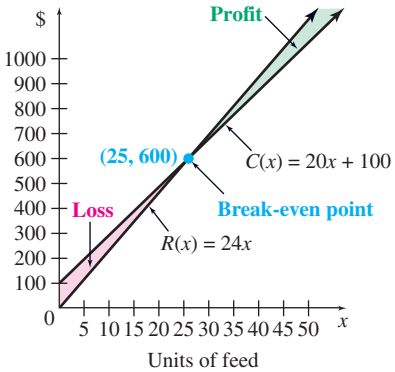


FIGURE 13

The break-even point (where $x = 25$) is shown on the graph. If the company sells more than 25 units (if $x > 25$), it makes a profit. If it sells fewer than 25 units, it loses money.

(b) What is the profit if 100 units of feed are sold?

SOLUTION Use the formula for profit $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 24x - (20x + 100) \\ &= 4x - 100 \end{aligned}$$

Then $P(100) = 4(100) - 100 = 300$. The firm will make a profit of \$300 from the sale of 100 units of feed.

(c) How many units must be sold to produce a profit of \$900?

SOLUTION Let $P(x) = 900$ in the equation $P(x) = 4x - 100$ and solve for x.

$$\begin{aligned} 900 &= 4x - 100 \\ 1000 &= 4x \\ x &= 250 \end{aligned}$$

Sales of 250 units will produce \$900 profit. TRY YOUR TURN 5

YOUR TURN 5 Repeat Example 6(c), using a cost function $C(x) = 35x + 250$, a charge of \$58 per unit, and a profit of \$8030.

Temperature

One of the most common linear relationships found in everyday situations deals with temperature. Recall that water freezes at 32° Fahrenheit and 0° Celsius, while it boils at 212° Fahrenheit and 100° Celsius.* The ordered pairs $(0, 32)$ and $(100, 212)$ are graphed in Figure 14 on axes showing Fahrenheit (F) as a function of Celsius (C). The line joining them is the graph of the function.

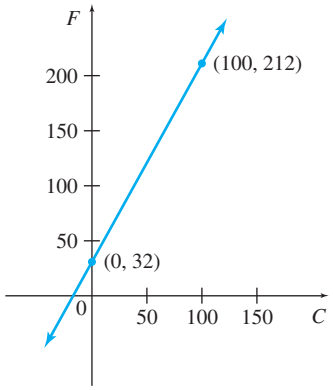


FIGURE 14

EXAMPLE 7 Temperature

Derive an equation relating F and C.

SOLUTION To derive the required linear equation, first find the slope using the given ordered pairs, $(0, 32)$ and $(100, 212)$.

$$m = \frac{212 - 32}{100 - 0} = \frac{9}{5}$$

*Gabriel Fahrenheit (1686–1736), a German physicist, invented his scale with 0° representing the temperature of an equal mixture of ice and ammonium chloride (a type of salt), and 96° as the temperature of the human body. (It is often said, erroneously, that Fahrenheit set 100° as the temperature of the human body. Fahrenheit's own words are quoted in *A History of the Thermometer and Its Use in Meteorology* by W. E. Knowles, Middleton: The Johns Hopkins Press, 1966, p. 75.) The Swedish astronomer Anders Celsius (1701–1744) set 0° and 100° as the freezing and boiling points of water.

The F-intercept of the graph is 32, so by the slope-intercept form, the equation of the line is

$$F = \frac{9}{5}C + 32.$$

With simple algebra this equation can be rewritten to give C in terms of F:

$$C = \frac{5}{9}(F - 32).$$

1.2 EXERCISES

For Exercises 1–10, let $f(x) = 7 - 5x$ and $g(x) = 2x - 3$. Find the following.

1. $f(2)$
2. $f(4)$
3. $f(-3)$
4. $f(-1)$
5. $g(1.5)$
6. $g(2.5)$
7. $g(-1/2)$
8. $g(-3/4)$
9. $f(t)$
10. $g(k^2)$

In Exercises 11–14, decide whether the statement is true or false.

11. To find the x-intercept of the graph of a linear function, we solve $y = f(x) = 0$, and to find the y-intercept, we evaluate $f(0)$.
12. The graph of $f(x) = -5$ is a vertical line.
13. The slope of the graph of a linear function cannot be undefined.
14. The graph of $f(x) = ax$ is a straight line that passes through the origin.
15. Describe what fixed costs and marginal costs mean to a company.
16. In a few sentences, explain why the price of a commodity not already at its equilibrium price should move in that direction.
17. Explain why a linear function may not be adequate for describing the supply and demand functions.
18. In your own words, describe the break-even quantity, how to find it, and what it indicates.

Write a linear cost function for each situation. Identify all variables used.

19. A Lake Tahoe resort charges a snowboard rental fee of \$10 plus \$2.25 per hour.
20. An Internet site for downloading music charges a \$10 registration fee plus 99 cents per downloaded song.
21. A parking garage charges 2 dollars plus 75 cents per half-hour.
22. For a one-day rental, a car rental firm charges \$44 plus 28 cents per mile.

Assume that each situation can be expressed as a linear cost function. Find the cost function in each case.

23. Fixed cost: \$100; 50 items cost \$1600 to produce.
24. Fixed cost: \$35; 8 items cost \$395 to produce.
25. Marginal cost: \$75; 50 items cost \$4300 to produce.
26. Marginal cost: \$120; 700 items cost \$96,500 to produce.

APPLICATIONS

Business and Economics

27. **Supply and Demand** Suppose that the demand and price for a certain model of a youth wristwatch are related by

$$p = D(q) = 16 - 1.25q,$$

where p is the price (in dollars) and q is the quantity demanded (in hundreds). Find the price at each level of demand.

a. 0 watches **b.** 400 watches **c.** 800 watches

Find the quantity demanded for the watch at each price.

d. \$8 **e.** \$10 **f.** \$12

g. Graph $p = 16 - 1.25q$.

Suppose the price and supply of the watch are related by

$$p = S(q) = 0.75q,$$

where p is the price (in dollars) and q is the quantity supplied (in hundreds) of watches. Find the quantity supplied at each price.

h. \$0 **i.** \$10 **j.** \$20

k. Graph $p = 0.75q$ on the same axis used for part g.

l. Find the equilibrium quantity and the equilibrium price.

28. **Supply and Demand** Suppose that the demand and price for strawberries are related by

$$p = D(q) = 5 - 0.25q,$$

where p is the price (in dollars) and q is the quantity demanded (in hundreds of quarts). Find the price at each level of demand.

a. 0 quarts **b.** 400 quarts **c.** 840 quarts

Find the quantity demanded for the strawberries at each price.

d. \$4.50 **e.** \$3.25 **f.** \$2.40

g. Graph $p = 5 - 0.25q$.

Suppose the price and supply of strawberries are related by

$$p = S(q) = 0.25q,$$

where p is the price (in dollars) and q is the quantity supplied (in hundreds of quarts) of strawberries. Find the quantity supplied at each price.

h. \$0 **i.** \$2 **j.** \$4.50

k. Graph $p = 0.75q$ on the same axis used for part g.

l. Find the equilibrium quantity and the equilibrium price.

29. Supply and Demand Let the supply and demand functions for butter pecan ice cream be given by

$$p = S(q) = \frac{2}{5}q \quad \text{and} \quad p = D(q) = 100 - \frac{2}{5}q,$$

where p is the price in dollars and q is the number of 10-gallon tubs.

a. Graph these on the same axes.

b. Find the equilibrium quantity and the equilibrium price. (*Hint:* The way to divide by a fraction is to multiply by its reciprocal.)

30. Supply and Demand Let the supply and demand functions for sugar be given by

$$p = S(q) = 1.4q - 0.6 \quad \text{and}$$
$$p = D(q) = -2q + 3.2,$$

where p is the price per pound and q is the quantity in thousands of pounds.

a. Graph these on the same axes.

b. Find the equilibrium quantity and the equilibrium price.

31. Supply and Demand Suppose that the supply function for honey is $p = S(q) = 0.3q + 2.7$, where p is the price in dollars for an 8-oz container and q is the quantity in barrels. Suppose also that the equilibrium price is \$4.50 and the demand is 2 barrels when the price is \$6.10. Find an equation for the demand function, assuming it is linear.

32. Supply and Demand Suppose that the supply function for walnuts is $p = S(q) = 0.25q + 3.6$, where p is the price in dollars per pound and q is the quantity in bushels. Suppose also that the equilibrium price is \$5.85, and the demand is 4 bushels when the price is \$7.60. Find an equation for the demand function, assuming it is linear.

33. T-Shirt Cost Joanne Wendelken sells silk-screened T-shirts at community festivals and crafts fairs. Her marginal cost to produce one T-shirt is \$3.50. Her total cost to produce 60 T-shirts is \$300, and she sells them for \$9 each.

a. Find the linear cost function for Joanne's T-shirt production.

b. How many T-shirts must she produce and sell in order to break even?

c. How many T-shirts must she produce and sell to make a profit of \$500?

34. Publishing Costs Alfred Juarez owns a small publishing house specializing in Latin American poetry. His fixed cost to produce a typical poetry volume is \$525, and his total cost to produce 1000 copies of the book is \$2675. His books sell for \$4.95 each.

a. Find the linear cost function for Alfred's book production.

b. How many poetry books must he produce and sell in order to break even?

c. How many books must he produce and sell to make a profit of \$1000?

35. Marginal Cost of Coffee The manager of a restaurant found that the cost to produce 100 cups of coffee is \$11.02, while the cost to produce 400 cups is \$40.12. Assume the cost $C(x)$ is a linear function of x, the number of cups produced.

a. Find a formula for $C(x)$.

b. What is the fixed cost?

c. Find the total cost of producing 1000 cups.

d. Find the total cost of producing 1001 cups.

e. Find the marginal cost of the 1001st cup.

f. What is the marginal cost of *any* cup and what does this mean to the manager?

36. Marginal Cost of a New Plant In deciding whether to set up a new manufacturing plant, company analysts have decided that a linear function is a reasonable estimation for the total cost $C(x)$ in dollars to produce x items. They estimate the cost to produce 10,000 items as \$547,500, and the cost to produce 50,000 items as \$737,500.

a. Find a formula for $C(x)$.

b. Find the fixed cost.

c. Find the total cost to produce 100,000 items.

d. Find the marginal cost of the items to be produced in this plant and what does this mean to the manager?

37. Break-Even Analysis Producing x units of tacos costs $C(x) = 5x + 20$; revenue is $R(x) = 15x$, where $C(x)$ and $R(x)$ are in dollars.

a. What is the break-even quantity?

b. What is the profit from 100 units?

c. How many units will produce a profit of \$500?

38. Break-Even Analysis To produce x units of a religious medal costs $C(x) = 12x + 39$. The revenue is $R(x) = 25x$. Both $C(x)$ and $R(x)$ are in dollars.

a. Find the break-even quantity.

b. Find the profit from 250 units.

c. Find the number of units that must be produced for a profit of \$130.

Break-Even Analysis **You are the manager of a firm. You are considering the manufacture of a new product, so you ask the accounting department for cost estimates and the sales department for sales estimates. After you receive the data, you must decide whether to go ahead with production of the new product. Analyze the data in Exercises 39–42 (find a break-even**

quantity) and then decide what you would do in each case. Also write the profit function.

39. $C(x) = 85x + 900$; $R(x) = 105x$; no more than 38 units can be sold.

40. $C(x) = 105x + 6000$; $R(x) = 250x$; no more than 400 units can be sold.

41. $C(x) = 70x + 500$; $R(x) = 60x$ (*Hint*: What does a negative break-even quantity mean?)

42. $C(x) = 1000x + 5000$; $R(x) = 900x$

43. Break-Even Analysis Suppose that the fixed cost for a product is \$400 and the break-even quantity is 80. Find the marginal profit (the slope of the linear profit function).

44. Break-Even Analysis Suppose that the fixed cost for a product is \$650 and the break-even quantity is 25. Find the marginal profit (the slope of the linear profit function).

Physical Sciences

45. Temperature Use the formula for conversion between Fahrenheit and Celsius derived in Example 7 to convert each temperature.

a. 58°F to Celsius

b. −20°F to Celsius

c. 50°C to Fahrenheit

46. Body Temperature You may have heard that the average temperature of the human body is 98.6°. Recent experiments show that the actual figure is closer to 98.2°. The figure of 98.6 comes from experiments done by Carl Wunderlich in 1868. But Wunderlich measured the temperatures in degrees Celsius and rounded the average to the nearest degree, giving 37°C as the average temperature. ***Source: Science News.***

a. What is the Fahrenheit equivalent of 37°C?

b. Given that Wunderlich rounded to the nearest degree Celsius, his experiments tell us that the actual average human body temperature is somewhere between 36.5°C and 37.5°C. Find what this range corresponds to in degrees Fahrenheit.

47. Temperature Find the temperature at which the Celsius and Fahrenheit temperatures are numerically equal.

General Interest

48. Education Cost The 2009–2010 budget for the California State University system projected a fixed cost of \$486,000 at each of five off-campus centers, plus a marginal cost of \$1140 per student. ***Source: California State University.***

a. Find a formula for the cost at each center, $C(x)$, as a linear function of x, the number of students.

b. The budget projected 500 students at each center. Calculate the total cost at each center.

c. Suppose, due to budget cuts, that each center is limited to \$1 million. What is the maximum number of students that each center can then support?

YOUR TURN ANSWERS

1. 25 **2.** 7600 and 4400

3. 8000 watermelons and \$3.20 per watermelon

4. $C(x) = 15x + 730$ **5.** 360

1.3 The Least Squares Line

APPLY IT

How has the accidental death rate in the United States changed over time?

In Example 1 in this section, we show how to answer such questions using the method of least squares.

We use past data to find trends and to make tentative predictions about the future. The only assumption we make is that the data are related linearly—that is, if we plot pairs of data, the resulting points will lie close to some line. This method cannot give exact answers. The best we can expect is that, if we are careful, we will get a reasonable approximation.

The table lists the number of accidental deaths per 100,000 people in the United States through the past century. ***Source: National Center for Health Statistics.*** If you were a manager at an insurance company, these data could be very important. You might need to make some predictions about how much you will pay out next year in accidental death benefits, and even a very tentative prediction based on past trends is better than no prediction at all.

Accidental Death Rate

Year	Death Rate
1910	84.4
1920	71.2
1930	80.5
1940	73.4
1950	60.3
1960	52.1
1970	56.2
1980	46.5
1990	36.9
2000	34.0

The first step is to draw a scatterplot, as we have done in Figure 15 on the next page. Notice that the points lie approximately along a line, which means that a linear function may give a good approximation of the data. If we select two points and find the line that passes through them, as we did in Section 1.1, we will get a different line for each pair of points, and in some cases the lines will be very different. We want to draw one line that is simultaneously close to all the points on the graph, but many such lines are possible, depending upon how we define the phrase "simultaneously close to all the points." How do we decide on the best possible line? Before going on, you might want to try drawing the line you think is best on Figure 15.

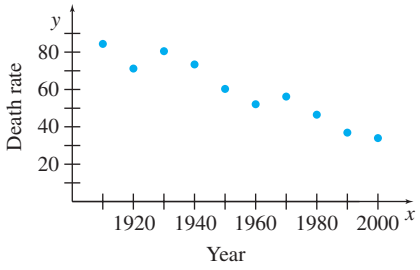


FIGURE 15

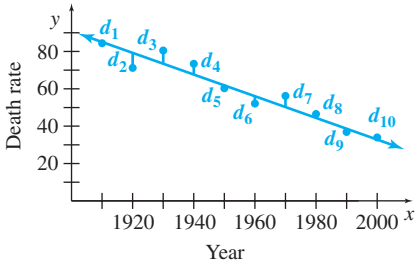


FIGURE 16

The line used most often in applications is that in which the sum of the squares of the vertical distances from the data points to the line is as small as possible. Such a line is called the **least squares line**. The least squares line for the data in Figure 15 is drawn in Figure 16. How does the line compare with the one you drew on Figure 15? It may not be exactly the same, but should appear similar.

In Figure 16, the vertical distances from the points to the line are indicated by d_1, d_2, and so on, up through d_{10} (read "d-sub-one, d-sub-two, d-sub-three," and so on). For n points, corresponding to the n pairs of data, the least squares line is found by minimizing the sum $(d_1)^2 + (d_2)^2 + (d_3)^2 + \cdots + (d_n)^2$.

We often use **summation notation** to write the sum of a list of numbers. The Greek letter sigma, Σ, is used to indicate "the sum of." For example, we write the sum $x_1 + x_2 + \cdots + x_n$, where n is the number of data points, as

$$x_1 + x_2 + \cdots + x_n = \Sigma x.$$

Similarly, Σxy means $x_1y_1 + x_2y_2 + \cdots + x_ny_n$, and so on.

CAUTION Note that Σx^2 means $x_1^2 + x_2^2 + \cdots + x_n^2$, which is *not* the same as squaring Σx. When we square Σx, we write it as $(\Sigma x)^2$.

For the least squares line, the sum of the distances we are to minimize, $d_1^2 + d_2^2 + \cdots + d_n^2$, is written as

$$d_1^2 + d_2^2 + \cdots + d_n^2 = \Sigma d^2.$$

To calculate the distances, we let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be the actual data points and we let the least squares line be $Y = mx + b$. We use Y in the equation instead of y to distinguish the predicted values (Y) from the y-value of the given data points. The predicted value of Y at x_1 is $Y_1 = mx_1 + b$, and the distance, d_1, between the actual y-value y_1 and the predicted value Y_1 is

$$d_1 = |Y_1 - y_1| = |mx_1 + b - y_1|.$$

Likewise,

$$d_2 = |Y_2 - y_2| = |mx_2 + b - y_2|,$$

and

$$d_n = |Y_n - y_n| = |mx_n + b - y_n|.$$

The sum to be minimized becomes

$$\begin{aligned}\Sigma d^2 &= (mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + \cdots + (mx_n + b - y_n)^2 \\ &= \Sigma(mx + b - y)^2,\end{aligned}$$

where $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are known and m and b are to be found.

The method of minimizing this sum requires advanced techniques and is not given here. To obtain the equation for the least squares line, a system of equations must be solved, producing the following formulas for determining the slope m and y-intercept b.*

Least Squares Line

The **least squares line** $Y = mx + b$ that gives the best fit to the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ has slope m and y-intercept b given by

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} \quad \text{and} \quad b = \frac{\Sigma y - m(\Sigma x)}{n}.$$

EXAMPLE 1 Least Squares Line

APPLY IT Calculate the least squares line for the accidental death rate data.

SOLUTION

Method 1 Calculating by Hand

To find the least squares line for the given data, we first find the required sums. To reduce the size of the numbers, we rescale the year data. Let x represent the years since 1900, so that, for example, $x = 10$ corresponds to the year 1910. Let y represent the death rate. We then calculate the values in the xy, x^2, and y^2 columns and find their totals. (The column headed y^2 will be used later.) Note that the number of data points is $n = 10$.

Least Squares Calculations

x	y	xy	x^2	y^2
10	84.4	844	100	7123.36
20	71.2	1424	400	5069.44
30	80.5	2415	900	6480.25
40	73.4	2936	1600	5387.56
50	60.3	3015	2500	3636.09
60	52.1	3126	3600	2714.41
70	56.2	3934	4900	3158.44
80	46.5	3720	6400	2162.25
90	36.9	3321	8100	1361.61
100	34.0	3400	10,000	1156.00
$\Sigma x = 550$	$\Sigma y = 595.5$	$\Sigma xy = 28{,}135$	$\Sigma x^2 = 38{,}500$	$\Sigma y^2 = 38{,}249.41$

Putting the column totals into the formula for the slope m, we get

$$\begin{aligned} m &= \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} && \text{Formula for } m \\ &= \frac{10(28{,}135) - (550)(595.5)}{10(38{,}500) - (550)^2} && \text{Substitute from the table.} \\ &= \frac{281{,}350 - 327{,}525}{385{,}000 - 302{,}500} && \text{Multiply.} \\ &= \frac{-46{,}175}{82{,}500} && \text{Subtract.} \\ &= -0.5596970 \approx -0.560. \end{aligned}$$

*See Exercise 9 at the end of this section.

The significance of m is that the death rate per 100,000 people is tending to drop (because of the negative) at a rate of 0.560 per year.

Now substitute the value of m and the column totals in the formula for b:

$$b = \frac{\Sigma y - m(\Sigma x)}{n} \qquad \text{Formula for } b$$

$$= \frac{595.5 - (-0.559697)(550)}{10} \qquad \text{Substitute.}$$

$$= \frac{595.5 - (-307.83335)}{10} \qquad \text{Multiply.}$$

$$= \frac{903.33335}{10} = 90.333335 \approx 90.3$$

Substitute m and b into the least squares line, $Y = mx + b$; the least squares line that best fits the 10 data points has equation

$$Y = -0.560x + 90.3.$$

This gives a mathematical description of the relationship between the year and the number of accidental deaths per 100,000 people. The equation can be used to predict y from a given value of x, as we will show in Example 2. As we mentioned before, however, caution must be exercised when using the least squares equation to predict data points that are far from the range of points on which the equation was modeled.

CAUTION In computing m and b, we rounded the final answer to three digits because the original data were known only to three digits. It is important, however, *not* to round any of the intermediate results (such as Σx^2) because round-off error may have a detrimental effect on the accuracy of the answer. Similarly, it is important not to use a rounded-off value of m when computing b.

Method 2 Graphing Calculator

The calculations for finding the least squares line are often tedious, even with the aid of a calculator. Fortunately, many calculators can calculate the least squares line with just a few keystrokes. For purposes of illustration, we will show how the least squares line in the previous example is found with a TI-84 Plus graphing calculator.

We begin by entering the data into the calculator. We will be using the first two lists, called L_1 and L_2. Choosing the `STAT` menu, then choosing the fourth entry `ClrList`, we enter L_1, L_2, to indicate the lists to be cleared. Now we press `STAT` again and choose the first entry `EDIT`, which brings up the blank lists. As before, we will only use the last two digits of the year, putting the numbers in L_1. We put the death rate in L_2, giving the two screens shown in Figure 17.

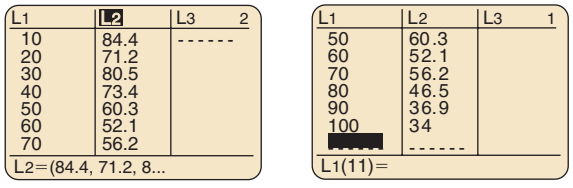


FIGURE 17

Quit the editor, press `STAT` again, and choose `CALC` instead of `EDIT`. Then choose item `4 LinReg(ax+b)` to get the values of a (the slope) and b (the y-intercept) for the least squares line, as shown in Figure 18. With a and b rounded to three decimal places, the least squares line is $Y = -0.560x + 90.3$. A graph of the data points and the line is shown in Figure 19.

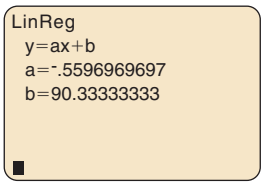


FIGURE 18

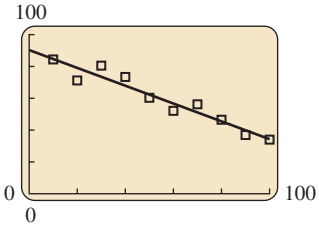


FIGURE 19

For more details on finding the least squares line with a graphing calculator, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

Method 3 Spreadsheet

Many computer spreadsheet programs can also find the least squares line. Figure 20 shows the scatterplot and least squares line for the accidental death rate data using an Excel spreadsheet. The scatterplot was found using the Marked Scatter chart from the Gallery and the line was found using the Add Trendline command under the Chart menu. For details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

YOUR TURN 1 Repeat Example 1, deleting the last pair of data (100, 34.0) and changing the second to last pair to (90, 40.2).

TRY YOUR TURN 1

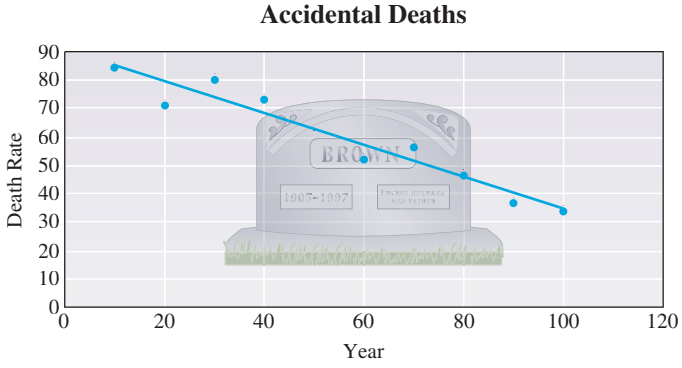


FIGURE 20

EXAMPLE 2 Least Squares Line

What do we predict the accidental death rate to be in 2012?

SOLUTION Use the least squares line equation given above with $x = 112$.

$$\begin{aligned} Y &= -0.560x + 90.3 \\ &= -0.56(112) + 90.3 \\ &= 27.6 \end{aligned}$$

The accidental death rate in 2012 is predicted to be about 27.6 per 100,000 population. In this case, we will have to wait until the 2012 data become available to see how accurate our prediction is. We have observed, however, that the accidental death rate began to go up after 2000 and was 40.6 per 100,000 population in 2006. This illustrates the danger of extrapolating beyond the data.

EXAMPLE 3 Least Squares Line

In what year is the death rate predicted to drop below 26 per 100,000 population?

SOLUTION Let $Y = 26$ in the equation above and solve for x.

$$\begin{aligned} 26 &= -0.560x + 90.3 \\ -64.3 &= -0.560x && \text{Subtract 90.3 from both sides.} \\ x &= 115 && \text{Divide both sides by } -0.560. \end{aligned}$$

This corresponds to the year 2015 (115 years after 1900), when our equation predicts the death rate to be $-0.560(115) + 90.3 = 25.9$ per 100,000 population.

Correlation

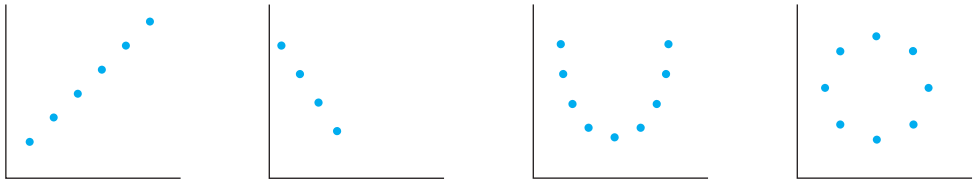
Although the least squares line can always be found, it may not be a good model. For example, if the data points are widely scattered, no straight line will model the data accurately. One measure of how well the original data fits a straight line is the **correlation coefficient,** denoted by r, which can be calculated by the following formula.

Correlation Coefficient

$$r = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2} \cdot \sqrt{n(\Sigma y^2) - (\Sigma y)^2}}$$

Although the expression for r looks daunting, remember that each of the summations, Σx, Σy, Σxy, and so on, are just the totals from a table like the one we prepared for the data on accidental deaths. Also, with a calculator, the arithmetic is no problem! Furthermore, statistics software and many calculators can calculate the correlation coefficient for you.

The correlation coefficient measures the strength of the linear relationship between two variables. It was developed by statistics pioneer Karl Pearson (1857–1936). The correlation coefficient r is between 1 and -1 or is equal to 1 or -1. Values of exactly 1 or -1 indicate that the data points lie *exactly* on the least squares line. If $r = 1$, the least squares line has a positive slope; $r = -1$ gives a negative slope. If $r = 0$, there is no linear correlation between the data points (but some *nonlinear* function might provide an excellent fit for the data). A correlation coefficient of zero may also indicate that the data fit a horizontal line. To investigate what is happening, it is always helpful to sketch a scatterplot of the data. Some scatterplots that correspond to these values of r are shown in Figure 21.



r close to 1 | r close to –1 | r close to 0 | r close to 0

FIGURE 21

A value of r close to 1 or -1 indicates the presence of a linear relationship. The exact value of r necessary to conclude that there is a linear relationship depends upon n, the number of data points, as well as how confident we want to be of our conclusion. For details, consult a text on statistics.*

*For example, see *Introductory Statistics*, 8th edition, by Neil A. Weiss, Boston, Mass.: Pearson, 2008.

EXAMPLE 4 Correlation Coefficient

Find r for the data on accidental death rates in Example 1.

SOLUTION

Method 1 Calculating by Hand

From the table in Example 1,

$\Sigma x = 550$, $\Sigma y = 595.5$, $\Sigma xy = 28{,}135$, $\Sigma x^2 = 38{,}500$,

$\Sigma y^2 = 38{,}249.41$, and $n = 10$.

Substituting these values into the formula for r gives

$$r = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2} \cdot \sqrt{n(\Sigma y^2) - (\Sigma y)^2}} \qquad \text{Formula for } r$$

$$= \frac{10(28{,}135) - (550)(595.5)}{\sqrt{10(38{,}500) - (550)^2} \cdot \sqrt{10(38{,}249.41) - (595.5)^2}} \qquad \text{Substitute.}$$

$$= \frac{281{,}350 - 327{,}525}{\sqrt{385{,}000 - 302{,}500} \cdot \sqrt{382{,}494.1 - 354{,}620.25}} \qquad \text{Multiply.}$$

$$= \frac{-46{,}175}{\sqrt{82{,}500} \cdot \sqrt{27{,}873.85}} \qquad \text{Subtract.}$$

$$= \frac{-46{,}175}{47{,}954.06787} \qquad \text{Take square roots and multiply.}$$

$$= -0.9629005849 \approx -0.963.$$

This is a high correlation, which agrees with our observation that the data fit a line quite well.

Method 2 Graphing Calculator

Most calculators that give the least squares line will also give the correlation coefficient. To do this on the TI-84 Plus, press the second function `CATALOG` and go down the list to the entry `DiagnosticOn`. Press `ENTER` at that point, then press `STAT`, `CALC`, and choose item 4 to get the display in Figure 22. The result is the same as we got by hand. The command `DiagnosticOn` need only be entered once, and the correlation coefficient will always appear in the future.

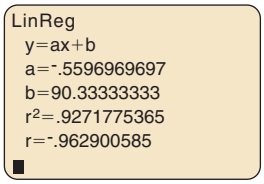


FIGURE 22

Method 3 Spreadsheet

Many computer spreadsheet programs have a built-in command to find the correlation coefficient. For example, in Excel, use the command "= `CORREL(A1:A10,B1:B10)`" to find the correlation of the 10 data points stored in columns A and B. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this text.

YOUR TURN 2 Repeat Example 4, deleting the last pair of data (100, 34.0) and changing the second to last pair to (90, 40.2).

TRY YOUR TURN 2

The square of the correlation coefficient gives the fraction of the variation in y that is explained by the linear relationship between x and y. Consider Example 4, where $r^2 = (-0.963)^2 = 0.927$. This means that 92.7% of the variation in y is explained by the linear relationship found earlier in Example 1. The remaining 7.3% comes from the scattering of the points about the line.

EXAMPLE 5 Average Expenditure per Pupil Versus Test Scores

Many states and school districts debate whether or not increasing the amount of money spent per student will guarantee academic success. The following scatterplot shows the average eighth grade reading score on the National Assessment of Education Progress (NAEP) for the 50 states and the District of Columbia in 2007 plotted against the average expenditure per pupil in 2007. Explore how the correlation coefficient is affected by the inclusion of the District of Columbia, which spent $14,324 per pupil and had a NAEP score of 241. ***Source: U.S. Census Bureau and National Center for Education Statistics.***

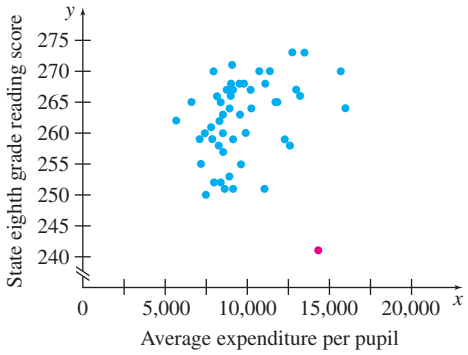


FIGURE 23

SOLUTION A spreadsheet was used to create a plot of the points shown in Figure 23. Washington D.C. corresponds to the red point in the lower right, which is noticeably separate from all the other points. Using the original data, the correlation coefficient when Washington D.C. is included is 0.1981, indicating that there is not a strong linear correlation. Excluding Washington D.C. raises the correlation coefficient to 0.3745, which is a somewhat stronger indication of a linear correlation. This illustrates that one extreme data point that is separate from the others, known as an **outlier**, can have a strong effect on the correlation coefficient.

Even if the correlation between average expenditure per pupil and reading score in Example 5 was high, this would not prove that spending more per pupil causes high reading scores. To prove this would require further research. It is a common statistical fallacy to assume that correlation implies causation. Perhaps the correlation is due to a third underlying variable. In Example 5, perhaps states with wealthier families spend more per pupil, and the students read better because wealthier families have greater access to reading material. Determining the truth requires careful research methods that are beyond the scope of this textbook.

1.3 EXERCISES

1. Suppose a positive linear correlation is found between two quantities. Does this mean that one of the quantities increasing causes the other to increase? If not, what does it mean?

2. Given a set of points, the least squares line formed by letting x be the independent variable will not necessarily be the same as the least squares line formed by letting y be the independent variable. Give an example to show why this is true.

3. For the following table of data,

x	1	2	3	4	5	6	7	8	9	10
y	0	0.5	1	2	2.5	3	3	4	4.5	5

a. draw a scatterplot.

b. calculate the correlation coefficient.

c. calculate the least squares line and graph it on the scatterplot.

d. predict the y-value when x is 11.

The following problem is reprinted from the November 1989 *Actuarial Examination on Applied Statistical Methods. Source: Society of Actuaries.*

4. You are given

X	6.8	7.0	7.1	7.2	7.4
Y	0.8	1.2	0.9	0.9	1.5

Determine r^2, the coefficient of determination for the regression of Y on X. Choose one of the following. (*Note:* The coefficient of determination is defined as the square of the correlation coefficient.)

a. 0.3 **b.** 0.4 **c.** 0.5 **d.** 0.6 **e.** 0.7

5. Consider the following table of data.

x	1	1	2	2	9
y	1	2	1	2	9

a. Calculate the least squares line and the correlation coefficient.

b. Repeat part a, but this time delete the last point.

c. Draw a graph of the data, and use it to explain the dramatic difference between the answers to parts a and b.

6. Consider the following table of data.

x	1	2	3	4	9
y	1	2	3	4	−20

a. Calculate the least squares line and the correlation coefficient.

b. Repeat part a, but this time delete the last point.

c. Draw a graph of the data, and use it to explain the dramatic difference between the answers to parts a and b.

7. Consider the following table of data.

x	1	2	3	4
y	1	1	1	1.1

a. Calculate the correlation coefficient.

b. Sketch a graph of the data.

c. Based on how closely the data fits a straight line, is your answer to part a surprising? Discuss the extent to which the correlation coefficient describes how well the data fit a horizontal line.

8. Consider the following table of data.

x	0	1	2	3	4
y	4	1	0	1	4

a. Calculate the least squares line and the correlation coefficient.

b. Sketch a graph of the data.

c. Comparing your answers to parts a and b, does a correlation coefficient of 0 mean that there is no relationship between the x and y values? Would some curve other than a line fit the data better? Explain.

9. The formulas for the least squares line were found by solving the system of equations

$$nb + (\Sigma x)m = \Sigma y$$
$$(\Sigma x)b + (\Sigma x^2)m = \Sigma xy.$$

Solve the above system for b and m to show that

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2} \quad \text{and}$$

$$b = \frac{\Sigma y - m(\Sigma x)}{n}.$$

APPLICATIONS

Business and Economics

10. Consumer Durable Goods The total value of consumer durable goods has grown at an approximately linear rate in recent years. The annual data for the years 2002 through 2008 can be summarized as follows, where x represents the years since 2000 and y the total value of consumer durable goods in trillions of dollars. ***Source: Bureau of Economic Analysis.***

$n = 7$ $\Sigma x = 35$ $\Sigma x^2 = 203$

$\Sigma y = 28.4269$ $\Sigma y^2 = 116.3396$ $\Sigma xy = 147.1399$

a. Find an equation for the least squares line.

b. Use your result from part **a** to predict the total value of consumer durable goods in the year 2015.

c. If this growth continues linearly, in what year will the total value of consumer durable goods first reach at least 6 trillion dollars?

d. Find and interpret the correlation coefficient.

11. Decrease in Banks The number of banks in the United States has been dropping steadily since 1984, and the trend in recent years has been roughly linear. The annual data for the years 1999 through 2008 can be summarized as follows, where x represents the years since 1990 and y the number of banks, in thousands, in the United States. ***Source: FDIC.***

$n = 10$ $\Sigma x = 235$ $\Sigma x^2 = 5605$

$\Sigma y = 77.564$ $\Sigma y^2 = 603.60324$ $\Sigma xy = 1810.095$

a. Find an equation for the least squares line.

b. Use your result from part a to predict the number of U.S. banks in the year 2020.

c. If this trend continues linearly, in what year will the number of U.S. banks drop below 6000?

d. Find and interpret the correlation coefficient.

12. Digital Cable Subscribers The number of subscribers to digital cable television has been growing steadily, as shown by the

following table. ***Source: National Cable and Telecommunications Association.***

Year	2000	2002	2004	2006	2008
Customers (in millions)	8.5	19.3	25.4	32.6	40.4

a. Find an equation for the least squares line, letting x equal the number of years since 2000.

b. Based on your answer to part a, at approximately what rate is the number of subscribers to digital cable television growing per year?

c. Use your result from part a to predict the number of digital cable subscribers in the year 2012.

d. If this trend continues linearly, in what year will the number of digital cable subscribers first exceed 70 million?

e. Find and interpret the correlation coefficient.

13. Consumer Credit The total amount of consumer credit has been increasing steadily in recent years. The following table gives the total U.S. outstanding consumer credit. ***Source: Federal Reserve.***

Year	2004	2005	2006	2007	2008
Consumer credit (in billions of dollars)	2219.5	2319.8	2415.0	2551.9	2592.1

a. Find an equation for the least squares line, letting x equal the number of years since 2000.

b. Based on your answer to part a, at approximately what rate is the consumer credit growing per year?

c. Use your result from part a to predict the amount of consumer credit in the year 2015.

d. If this trend continues linearly, in what year will the total debt first exceed $4000 billion?

e. Find and interpret the correlation coefficient.

14. New Car Sales New car sales have increased at a roughly linear rate. Sales, in millions of vehicles, from 2000 to 2007, are given in the table below. ***Source: National Automobile Dealers Association.*** Let x represent the number of years since 2000.

Year	Sales
2000	17.3
2001	17.1
2002	16.8
2003	16.6
2004	16.9
2005	16.9
2006	16.5
2007	16.1

a. Find the equation of the least squares line and the correlation coefficient.

b. Find the equation of the least squares line using only the data for every other year starting with 2000, 2002, and so on. Find the correlation coefficient.

c. Compare your results for parts a and b. What do you find? Why do you think this happens?

15. Air Fares In 2006, for passengers who made early reservations, American Airlines offered lower prices on one-way fares from New York to various cities. Fourteen of the cities are listed in the following table, with the distances from New York to the cities included. ***Source: American Airlines.***

a. Plot the data. Do the data points lie in a linear pattern?

b. Find the correlation coefficient. Combining this with your answer to part a, does the cost of a ticket tend to go up with the distance flown?

c. Find the equation of the least squares line, and use it to find the approximate marginal cost per mile to fly.

d. For similar data in a January 2000 *New York Times* ad, the equation of the least squares line was $Y = 113 + 0.0243x$. ***Source: The New York Times.*** Use this information and your answer to part b to compare the cost of flying American Airlines for these two time periods.

e. Identify the outlier in the scatterplot. Discuss the reason why there would be a difference in price to this city.

City	Distance (x) (miles)	Price (y) (dollars)
Boston	206	95
Chicago	802	138
Denver	1771	228
Kansas City	1198	209
Little Rock	1238	269
Los Angeles	2786	309
Minneapolis	1207	202
Nashville	892	217
Phoenix	2411	109
Portland	2885	434
Reno	2705	399
St. Louis	948	206
San Diego	2762	239
Seattle	2815	329

Life Sciences

16. Bird Eggs The average length and width of various bird eggs are given in the following table. ***Source: National Council of Teachers of Mathematics.***

Bird Name	Width (cm)	Length (cm)
Canada goose	5.8	8.6
Robin	1.5	1.9
Turtledove	2.3	3.1
Hummingbird	1.0	1.0
Raven	3.3	5.0

a. Plot the points, putting the length on the y-axis and the width on the x-axis. Do the data appear to be linear?

b. Find the least squares line, and plot it on the same graph as the data.

c. Suppose there are birds with eggs even smaller than those of hummingbirds. Would the equation found in part b continue to make sense for all positive widths, no matter how small? Explain.

d. Find the correlation coefficient.

17. Size of Hunting Parties In the 1960s, the famous researcher Jane Goodall observed that chimpanzees hunt and eat meat as part of their regular diet. Sometimes chimpanzees hunt alone, while other times they form hunting parties. The following table summarizes research on chimpanzee hunting parties, giving the size of the hunting party and the percentage of successful hunts. *Source: American Scientist and Mathematics Teacher.*

Number of Chimps in Hunting Party	Percentage of Successful Hunts
1	20
2	30
3	28
4	42
5	40
6	58
7	45
8	62
9	65
10	63
12	75
13	75
14	78
15	75
16	82

a. Plot the data. Do the data points lie in a linear pattern?

b. Find the correlation coefficient. Combining this with your answer to part a, does the percentage of successful hunts tend to increase with the size of the hunting party?

c. Find the equation of the least squares line, and graph it on your scatterplot.

18. Crickets Chirping Biologists have observed a linear relationship between the temperature and the frequency with which a cricket chirps. The following data were measured for the striped ground cricket. *Source: The Song of Insects.*

Temperature °F (x)	Chirps Per Second (y)
88.6	20.0
71.6	16.0
93.3	19.8
84.3	18.4
80.6	17.1
75.2	15.5
69.7	14.7
82.0	17.1
69.4	15.4
83.3	16.2
79.6	15.0
82.6	17.2
80.6	16.0
83.5	17.0
76.3	14.4

a. Find the equation for the least squares line for the data.

b. Use the results of part a to determine how many chirps per second you would expect to hear from the striped ground cricket if the temperature were 73°F.

c. Use the results of part a to determine what the temperature is when the striped ground crickets are chirping at a rate of 18 times per sec.

d. Find the correlation coefficient.

Social Sciences

19. Pupil-Teacher Ratios The following table gives the national average pupil-teacher ratio in public schools over selected years. *Source: National Center for Education Statistics.*

Year	Ratio
1990	17.4
1994	17.7
1998	16.9
2002	16.2
2006	15.8

a. Find the equation for the least squares line. Let x correspond to the number of years since 1990 and let y correspond to the average number of pupils per 1 teacher.

b. Use your answer from part a to predict the pupil-teacher ratio in 2020. Does this seem realistic?

c. Calculate and interpret the correlation coefficient.

20. Poverty Levels The following table lists how poverty level income cutoffs (in dollars) for a family of four have changed over time. ***Source: U.S. Census Bureau.***

Year	Income
1980	8414
1985	10,989
1990	13,359
1995	15,569
2000	17,604
2005	19,961
2008	22,207

Let x represent the year, with $x = 0$ corresponding to 1980 and y represent the income in thousands of dollars.

a. Plot the data. Do the data appear to lie along a straight line?

b. Calculate the correlation coefficient. Does your result agree with your answer to part a?

c. Find the equation of the least squares line.

d. Use your answer from part c to predict the poverty level in the year 2018.

21. Ideal Partner Height In an introductory statistics course at Cornell University, 147 undergraduates were asked their own height and the ideal height for their ideal spouse or partner. For this exercise, we are including the data for only a representative sample of 10 of the students, as given in the following table. All heights are in inches. ***Source: Chance.***

Height	Ideal Partner's Height
59	66
62	71
66	72
68	73
71	75
67	63
70	63
71	67
73	66
75	66

a. Find the regression line and correlation coefficient for this data. What strange phenomenon do you observe?

b. The first five data pairs are for female students and the second five for male students. Find the regression line and correlation coefficient for each set of data.

c. Plot all the data on one graph, using different types of points to distinguish the data for the males and for the females. Using this plot and the results from part b, explain the strange phenomenon that you observed in part a.

22. SAT Scores At Hofstra University, all students take the math SAT before entrance, and most students take a mathematics placement test before registration. Recently, one professor collected the following data for 19 students in his Finite Mathematics class:

Math SAT	Placement Test	Math SAT	Placement Test	Math SAT	Placement Test
540	20	580	8	440	10
510	16	680	15	520	11
490	10	560	8	620	11
560	8	560	13	680	8
470	12	500	14	550	8
600	11	470	10	620	7
540	10				

a. Find an equation for the least squares line. Let x be the math SAT and y be the placement test score.

b. Use your answer from part a to predict the mathematics placement test score for a student with a math SAT score of 420.

c. Use your answer from part a to predict the mathematics placement test score for a student with a math SAT score of 620.

d. Calculate the correlation coefficient.

e. Based on your answer to part d, what can you conclude about the relationship between a student's math SAT and mathematics placement test score?

Physical Sciences

23. Length of a Pendulum Grandfather clocks use pendulums to keep accurate time. The relationship between the length of a pendulum L and the time T for one complete oscillation can be determined from the data in the table.* ***Source: Gary Rockswold.***

L (ft)	T (sec)
1.0	1.11
1.5	1.36
2.0	1.57
2.5	1.76
3.0	1.92
3.5	2.08
4.0	2.22

a. Plot the data from the table with L as the horizontal axis and T as the vertical axis.

b. Find the least squares line equation and graph it simultaneously, if possible, with the data points. Does it seem to fit the data?

c. Find the correlation coefficient and interpret it. Does it confirm your answer to part b?

24. Air Conditioning While shopping for an air conditioner, Adam Bryer consulted the following table, which gives a machine's BTUs and the square footage (ft^2) that it would cool.

*The actual relationship is $L = 0.81T^2$, which is not a linear relationship. This illustration that even if the relationship is not linear, a line can give a good approximation.

ft^2 (x)	BTUs (y)
150	5000
175	5500
215	6000
250	6500
280	7000
310	7500
350	8000
370	8500
420	9000
450	9500

a. Find the equation for the least squares line for the data.

b. To check the fit of the data to the line, use the results from part a to find the BTUs required to cool a room of 150 ft^2, 280 ft^2, and 420 ft^2. How well does the actual data agree with the predicted values?

c. Suppose Adam's room measures 230 ft^2. Use the results from part a to decide how many BTUs it requires. If air conditioners are available only with the BTU choices in the table, which would Adam choose?

d. Why do you think the table gives ft^2 instead of ft^3, which would give the volume of the room?

General Interest

25. Football The following data give the expected points for a football team with first down and 10 yards to go from various points on the field. ***Source: Operations Research.*** (*Note:* $\sum x = 500$, $\sum x^2 = 33{,}250$, $\sum y = 20.668$, $\sum y^2 = 91.927042$, $\sum xy = 399.16$.)

Yards from Goal (x)	Expected Points (y)
5	6.041
15	4.572
25	3.681
35	3.167
45	2.392
55	1.538
65	0.923
75	0.236
85	−0.637
95	−1.245

a. Calculate the correlation coefficient. Does there appear to be a linear correlation?

b. Find the equation of the least squares line.

c. Use your answer from part a to predict the expected points when a team is at the 50-yd line.

26. Athletic Records The table shows the men's and women's outdoor world records (in seconds) in the 800-m run. ***Source: Nature, Track and Field Athletics, Statistics in Sports, and The World Almanac and Book of Facts.***

Year	Men's Record	Women's Record
1905	113.4	—
1915	111.9	—
1925	111.9	144
1935	109.7	135.6
1945	106.6	132
1955	105.7	125
1965	104.3	118
1975	103.7	117.48
1985	101.73	113.28
1995	101.11	113.28
2005	101.11	113.28

Let x be the year, with $x = 0$ corresponding to 1900.

a. Find the equation for the least squares line for the men's record (y) in terms of the year (x).

b. Find the equation for the least squares line for the women's record.

c. Suppose the men's and women's records continue to improve as predicted by the equations found in parts a and b. In what year will the women's record catch up with the men's record? Do you believe that will happen? Why or why not?

d. Calculate the correlation coefficient for both the men's and the women's record. What do these numbers tell you?

e. Draw a plot of the data, and discuss to what extent a linear function describes the trend in the data.

27. Running If you think a marathon is a long race, consider the Hardrock 100, a 100.5 mile running race held in southwestern Colorado. The chart at right lists the times that the 2008 winner, Kyle Skaggs, arrived at various mileage points along the way. ***Source: www.run100s.com.***

a. What was Skagg's average speed?

b. Graph the data, plotting time on the x-axis and distance on the y-axis. You will need to convert the time from hours and minutes into hours. Do the data appear to lie approximately on a straight line?

c. Find the equation for the least squares line, fitting distance as a linear function of time.

d. Calculate the correlation coefficient. Does it indicate a good fit of the least squares line to the data?

e. Based on your answer to part d, what is a good value for Skagg's average speed? Compare this with your answer to part a. Which answer do you think is better? Explain your reasoning.

Time (hr:min)	Miles
0	0
2:19	11.5
3:43	18.9
5:36	27.8
7:05	32.8
7:30	36.0
8:30	43.9
10:36	51.5
11:56	58.4
15:14	71.8
17:49	80.9
18:58	85.2
20:50	91.3
23:23	100.5

YOUR TURN ANSWERS

1. $Y = -0.535x + 89.5$ **2.** -0.949

CHAPTER REVIEW

SUMMARY

In this chapter we studied linear functions, whose graphs are straight lines. We developed the slope-intercept and point-slope formulas, which can be used to find the equation of a line, given a point and the slope or given two points. We saw that lines have many applications in virtually every discipline. Lines are used through the rest of this book, so fluency in their use is important. We concluded the chapter by introducing the method of least squares, which is used to find an equation of the line that best fits a given set of data.

Slope of a Line

The slope of a line is defined as the vertical change (the "rise") over the horizontal change (the "run") as one travels along the line. In symbols, taking two different points (x_1, y_1) and (x_2, y_2) on the line, the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

where $x_1 \neq x_2$.

Equations of Lines

Equation	Description
$y = mx + b$	Slope intercept form: slope m and y-intercept b.
$y - y_1 = m(x - x_1)$	Point-slope form: slope m and line passes through (x_1, y_1).
$x = k$	Vertical line: x-intercept k, no y-intercept (except when $k = 0$), undefined slope.
$y = k$	Horizontal line: y-intercept k, no x-intercept (except when $k = 0$), slope 0.

Parallel Lines

Two lines are parallel if and only if they have the same slope, or if they are both vertical.

Perpendicular Lines

Two lines are perpendicular if and only if the product of their slopes is -1, or if one is vertical and the other horizontal.

Linear Function A relationship f defined by

$$y = f(x) = mx + b,$$

for real numbers m and b, is a linear function.

Linear Cost Function In a cost function of the form $C(x) = mx + b$, the m represents the marginal cost and b represents the fixed cost.

Least Squares Line The least squares line $Y = mx + b$ that gives the best fit to the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ has slope m and y-intercept b given by the equations

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2}$$

$$b = \frac{\Sigma y - m(\Sigma x)}{n}$$

Correlation Coefficient

$$r = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2}\ \sqrt{n(\Sigma y^2) - (\Sigma y)^2}}$$

KEY TERMS

To understand the concepts presented in this chapter, you should know the meaning and use of the following terms. For easy reference, the section in the chapter where a word (or expression) was first used is provided.

mathematical model

1.1
ordered pair
Cartesian coordinate system
axes
origin
coordinates
quadrants
graph
intercepts
slope
linear equation
slope-intercept form
proportional
point-slope form
parallel
perpendicular
scatterplot

1.2
linear function
independent variable
dependent variable
surplus
shortage
equilibrium price
equilibrium quantity
fixed cost
marginal cost
linear cost function
revenue
profit
break-even quantity
break-even point

1.3
least squares line
summation notation
correlation coefficient
outlier

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each statement is true or false, and explain why.

1. A given line can have more than one slope.

2. The equation $y = 3x + 4$ represents the equation of a line with slope 4.

3. The line $y = -2x + 5$ intersects the point $(3, -1)$.

4. The line that intersects the points $(2, 3)$ and $(2, 5)$ is a horizontal line.

5. The line that intersects the points $(4, 6)$ and $(5, 6)$ is a horizontal line.

6. The x-intercept of the line $y = 8x + 9$ is 9.

7. The function $f(x) = \pi x + 4$ represents a linear function.

8. The function $f(x) = 2x^2 + 3$ represents a linear function.

9. The lines $y = 3x + 17$ and $y = -3x + 8$ are perpendicular.

10. The lines $4x + 3y = 8$ and $4x + y = 5$ are parallel.

11. A correlation coefficient of zero indicates a perfect fit with the data.

12. It is not possible to get a correlation coefficient of -1.5 for a set of data.

PRACTICE AND EXPLORATIONS

13. What is marginal cost? Fixed cost?

14. What six quantities are needed to compute a correlation coefficient?

Find the slope for each line that has a slope.

15. Through $(-3, 7)$ and $(2, 12)$

16. Through $(4, -1)$ and $(3, -3)$

17. Through the origin and $(11, -2)$

18. Through the origin and $(0, 7)$

19. $4x + 3y = 6$ **20.** $4x - y = 7$

21. $y + 4 = 9$ **22.** $3y - 1 = 14$

23. $y = 5x + 4$ **24.** $x = 5y$

Find an equation in the form $y = mx + b$ for each line.

25. Through $(5, -1)$; slope $= 2/3$

26. Through $(8, 0)$; slope $= -1/4$

27. Through $(-6, 3)$ and $(2, -5)$

28. Through $(2, -3)$ and $(-3, 4)$

29. Through $(2, -10)$, perpendicular to a line with undefined slope

30. Through $(-2, 5)$; slope $= 0$

Find an equation for each line in the form $ax + by = c$, where a, b, and c are integers with no factor common to all three and $a \geq 0$.

31. Through $(3, -4)$, parallel to $4x - 2y = 9$

32. Through $(0, 5)$, perpendicular to $8x + 5y = 3$

33. Through $(-1, 4)$; undefined slope

34. Through $(7, -6)$, parallel to a line with undefined slope

35. Through $(3, -5)$, parallel to $y = 4$

36. Through $(-3, 5)$, perpendicular to $y = -2$

Graph each linear equation defined as follows.

37. $y = 4x + 3$ **38.** $y = 6 - 2x$

39. $3x - 5y = 15$ **40.** $4x + 6y = 12$

41. $x - 3 = 0$ **42.** $y = 1$

43. $y = 2x$ **44.** $x + 3y = 0$

APPLICATIONS

Business and Economics

45. Profit To manufacture x thousand computer chips requires fixed expenditures of \$352 plus \$42 per thousand chips. Receipts from the sale of x thousand chips amount to \$130 per thousand.

a. Write an expression for expenditures.

b. Write an expression for receipts.

c. For profit to be made, receipts must be greater than expenditures. How many chips must be sold to produce a profit?

46. Supply and Demand The supply and demand for crabmeat in a local fish store are related by the equations

$$\text{Supply: } p = S(q) = 6q + 3$$

and

$$\text{Demand: } p = D(q) = 19 - 2q,$$

where p represents the price in dollars per pound and q represents the quantity of crabmeat in pounds per day. Find the quantity supplied and demanded at each of the following prices.

a. \$10 **b.** \$15 **c.** \$18

d. Graph both the supply and the demand functions on the same axes.

e. Find the equilibrium price.

f. Find the equilibrium quantity.

47. Supply For a new diet pill, 60 pills will be supplied at a price of \$40, while 100 pills will be supplied at a price of \$60. Write a linear supply function for this product.

48. Demand The demand for the diet pills in Exercise 47 is 50 pills at a price of \$47.50 and 80 pills at a price of \$32.50. Determine a linear demand function for these pills.

49. Supply and Demand Find the equilibrium price and quantity for the diet pills in Exercises 47 and 48.

Cost **In Exercises 50–53, find a linear cost function.**

50. Eight units cost \$300; fixed cost is \$60.

51. Fixed cost is \$2000; 36 units cost \$8480.

52. Twelve units cost \$445; 50 units cost \$1585.

53. Thirty units cost \$1500; 120 units cost \$5640.

54. Break-Even Analysis The cost of producing x cartons of CDs is $C(x)$ dollars, where $C(x) = 200x + 1000$. The CDs sell for \$400 per carton.

a. Find the break-even quantity.

b. What revenue will the company receive if it sells just that number of cartons?

55. Break-Even Analysis The cost function for flavored coffee at an upscale coffeehouse is given in dollars by $C(x) = 3x + 160$, where x is in pounds. The coffee sells for \$7 per pound.

a. Find the break-even quantity.

b. What will the revenue be at that point?

56. U.S. Imports from China The United States is China's largest export market. Imports from China have grown from about 102 billion dollars in 2001 to 338 billion dollars in 2008. This growth has been approximately linear. Use the given data pairs to write a linear equation that describes this growth in imports over the years. Let $t = 1$ represent 2001 and $t = 8$ represent 2008. ***Source: TradeStats Express™.***

57. U.S. Exports to China U.S. exports to China have grown (although at a slower rate than imports) since 2001. In 2001, about 19.1 billion dollars of goods were exported to China. By 2008, this amount had grown to 69.7 billion dollars. Write a linear equation describing the number of exports each year, with $t = 1$ representing 2001 and $t = 8$ representing 2008. ***Source: TradeStats Express™.***

58. Median Income The U.S. Census Bureau reported that the median income for all U.S. households in 2008 was \$50,303. In 1988, the median income (in 2008 dollars) was \$47,614. The median income is approximately linear and is a function of time. Find a formula for the median income, I, as a function of the year t, where t is the number of years since 1900. ***Source: U.S Census Bureau.***

59. New Car Cost The average new car cost (in dollars) for selected years from 1980 to 2005 is given in the table. ***Source: Chicago Tribune and National Automobile Dealers Association.***

Year	1980	1985	1990	1995	2000	2005
Cost	7500	12,000	16,000	20,450	24,900	28,400

a. Find a linear equation for the average new car cost in terms of x, the number of years since 1980, using the data for 1980 and 2005.

b. Repeat part a, using the data for 1995 and 2005.

c. Find the equation of the least squares line using all the data.

d. Use a graphing calculator to plot the data and the three lines from parts a-c.

e. Discuss which of the three lines found in parts a–c best describes the data, as well as to what extent a linear model accurately describes the data.

f. Calculate the correlation coefficient.

Life Sciences

60. World Health In general, people tend to live longer in countries that have a greater supply of food. Listed below is the 2003–2005 daily calorie supply and 2005 life expectancy at birth for 10 randomly selected countries. ***Source: Food and Agriculture Organization.***

Country	Calories (x)	Life Expectancy (y)
Belize	2818	75.4
Cambodia	2155	59.4
France	3602	80.4
India	2358	62.7
Mexico	3265	75.5
New Zealand	3235	79.8
Peru	2450	72.5
Sweden	3120	80.5
Tanzania	2010	53.7
United States	3826	78.7

a. Find the correlation coefficient. Do the data seem to fit a straight line?

b. Draw a scatterplot of the data. Combining this with your results from part a, do the data seem to fit a straight line?

c. Find the equation of the least squares line.

d. Use your answer from part c to predict the life expectancy in the United Kingdom, which has a daily calorie supply of 3426. Compare your answer with the actual value of 79.0 years.

e. Briefly explain why countries with a higher daily calorie supply might tend to have a longer life expectancy. Is this trend likely to continue to higher calorie levels? Do you think that an American who eats 5000 calories a day is likely to live longer than one who eats 3600 calories? Why or why not?

f. (For the ambitious!) Find the correlation coefficient and least squares line using the data for a larger sample of countries, as found in an almanac or other reference. Is the result in general agreement with the previous results?

61. Blood Sugar and Cholesterol Levels The following data show the connection between blood sugar levels and cholesterol levels for eight different patients.

Patient	Blood Sugar Level (x)	Cholesterol Level (y)
1	130	170
2	138	160
3	142	173
4	159	181
5	165	201
6	200	192
7	210	240
8	250	290

For the data given in the preceding table, $\Sigma x = 1394$, $\Sigma y = 1607$, $\Sigma xy = 291{,}990$, $\Sigma x^2 = 255{,}214$, and $\Sigma y^2 = 336{,}155$.

a. Find the equation of the least squares line.

b. Predict the cholesterol level for a person whose blood sugar level is 190.

c. Find the correlation coefficient.

Social Sciences

62. Beef Consumption The per capita consumption of beef in the United States decreased from 115.7 lb in 1974 to 92.9 lb in 2007. Assume a linear function describes the decrease. Write a linear equation defining the function. Let t represent the number of years since 1950 and y represent the number of pounds of red meat consumed. ***Source: U.S. Department of Agriculture.***

63. Marital Status More people are staying single longer in the United States. In 1995, the number of never-married adults, age 15 and over, was 55.0 million. By 2009, it was 72.1 million. Assume the data increase linearly, and write an equation that defines a linear function for this data. Let t represent the number of years since 1990. ***Source: U.S. Census Bureau.***

64. Poverty The following table gives the number of families under the poverty level in the U.S. in recent years. ***Source: U.S. Census Bureau.***

Year	Families Below Poverty Level (in thousands)
2000	6400
2001	6813
2002	7229
2003	7607
2004	7623
2005	7657
2006	7668
2007	7835
2008	8147

a. Find a linear equation for the number of families below poverty level (in thousands) in terms of x, the number of years since 2000, using the data for 2000 and 2008.

b. Repeat part a, using the data for 2004 and 2008.

c. Find the equation of the least squares line using all the data. Then plot the data and the three lines from parts a–c on a graphing calculator.

d. Discuss which of the three lines found in parts a–c best describes the data, as well as to what extent a linear model accurately describes the data.

e. Calculate the correlation coefficient.

65. Governors' Salaries In general, the larger a state's population, the more the governor earns. Listed in the table below are the estimated 2008 populations (in millions) and the salary of the governor (in thousands of dollars) for eight randomly selected states. ***Source: U.S. Census Bureau and Alaska Department of Administration.***

State	AZ	DE	MD	MA	NY	PA	TN	WY
Population (x)	6.50	0.88	5.54	6.45	19.30	12.39	5.92	0.53
Governor's Salary (y)	95	133	150	141	179	170	160	105

a. Find the correlation coefficient. Do the data seem to fit a straight line?

b. Draw a scatterplot of the data. Compare this with your answer from part a.

c. Find the equation for the least squares line.

d. Based on your answer to part c, how much does a governor's salary increase, on average, for each additional million in population?

e. Use your answer from part c to predict the governor's salary in your state. Based on your answers from parts a and b, would this prediction be very accurate? Compare with the actual salary, as listed in an almanac or other reference.

f. (For the ambitious!) Find the correlation coefficient and least squares line using the data for all 50 states, as found in an almanac or other reference. Is the result in general agreement with the previous results?

66. Movies A mathematician exploring the relationship between ratings of movies, their year of release, and their length discovered a paradox. Rather than list the data set of 100 movies in the original research, we have created a sample of size 10 that captures the properties of the original dataset. In the following table, the rating is a score from 1 to 10, and the length is in minutes. ***Source: Journal of Statistics Education.***

Year	Rating	Length
2001	10	120
2003	5	85
2004	3	100
2004	6	105
2005	4	110
2005	8	115
2006	6	135
2007	2	105
2007	5	125
2008	6	130

a. Find the correlation coefficient between the years since 2000 and the length.

b. Find the correlation coefficient between the length and the rating.

c. Given that you found a positive correlation between the year and the length in part a, and a positive correlation between the length and the rating in part b, what would you expect about the correlation between the year and the rating? Calculate this correlation. Are you surprised?

d. Discuss the paradoxical result in part c. Write out in words what each correlation tells you. Try to explain what is happening. You may want to look at a scatterplot between the year and the rating, and consider which points on the scatterplot represent movies of length no more than 110 minutes, and which represent movies of length 115 minutes or more.

EXTENDED APPLICATION

USING EXTRAPOLATION TO PREDICT LIFE EXPECTANCY

One reason for developing a mathematical model is to make predictions. If your model is a least squares line, you can predict the y-value corresponding to some new x by substituting this x into an equation of the form $Y = mx + b$. (We use a capital Y to remind us that we're getting a predicted value rather than an actual data value.) Data analysts distinguish between two very different kinds of prediction, *interpolation*, and *extrapolation*. An interpolation uses a new x inside the x range of your original data. For example, if you have inflation data at 5-year intervals from 1950 to 2000, estimating the rate of inflation in 1957 is an interpolation problem. But if you use the same data to estimate what the inflation rate was in 1920, or what it will be in 2020, you are extrapolating.

In general, interpolation is much safer than extrapolation, because data that are approximately linear over a short interval may be nonlinear over a larger interval. One way to detect nonlinearity is to look at *residuals,* which are the differences between the actual data values and the values predicted by the line of best fit. Here is a simple example:

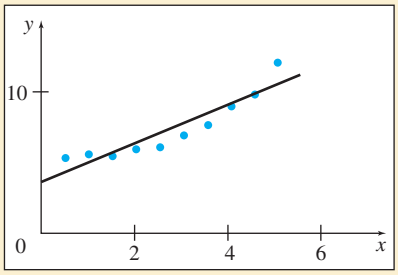


FIGURE 24

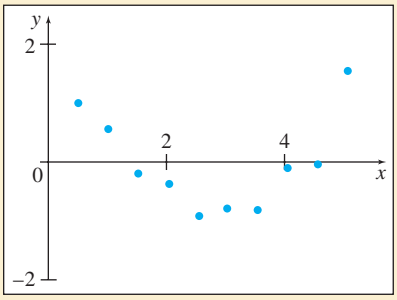


FIGURE 25

The regression equation for the linear fit in Figure 24 is $Y = 3.431 + 1.334x$. Since the r-value for this regression line is 0.93, our linear model fits the data very well. But we might notice that the predictions are a bit low at the ends and high in the middle. We can get a better look at this pattern by plotting the residuals. To find them, we put each value of the independent variable into the regression equation, calculate the predicted value Y, and subtract it from the actual y-value. The residual plot is shown in Figure 25, with the vertical axis rescaled to exaggerate the pattern. The residuals indicate that our data have a nonlinear, U-shaped component that is not captured by the linear fit. Extrapolating from this data set is probably not a good idea; our linear prediction for the value of y when x is 10 may be much too low.

EXERCISES

The following table gives the life expectancy at birth of females born in the United States in various years from 1970 to 2005. ***Source: National Center for Health Statistics.***

Year of Birth	Life Expectancy (years)
1970	74.7
1975	76.6
1980	77.4
1985	78.2
1990	78.8
1995	78.9
2000	79.3
2005	79.9

1. Find an equation for the least squares line for these data, using year of birth as the independent variable.
2. Use your regression equation to guess a value for the life expectancy of females born in 1900.
3. Compare your answer with the actual life expectancy for females born in 1900, which was 48.3 years. Are you surprised?
4. Find the life expectancy predicted by your regression equation for each year in the table, and subtract it from the actual value in the second column. This gives you a table of residuals. Plot your residuals as points on a graph.
5. Now look at the residuals as a fresh data set, and see if you can sketch the graph of a smooth function that fits the residuals well. How easy do you think it will be to predict the life expectancy at birth of females born in 2015?
6. What will happen if you try linear regression on the *residuals*? If you're not sure, use your calculator or software to find the regression equation for the residuals. Why does this result make sense?
7. Since most of the females born in 1995 are still alive, how did the Public Health Service come up with a life expectancy of 78.9 years for these women?
8. Go to the website WolframAlpha.com and enter: "linear fit {1970,74.7}, {1975,76.6}, etc.," putting in all the data from the table. Discuss how the solution compares with the solutions provided by a graphing calculator and by Microsoft Excel.

DIRECTIONS FOR GROUP PROJECT

Assume that you and your group (3–5 students) are preparing a report for a local health agency that is interested in using linear regression to predict life expectancy. Using the questions above as a guide, write a report that addresses the spirit of each question and any issues related to that question. The report should be mathematically sound, grammatically correct, and professionally crafted. Provide recommendations as to whether the health agency should proceed with the linear equation or whether it should seek other means of making such predictions.

2 Systems of Linear Equations and Matrices

2.1 Solution of Linear Systems by the Echelon Method

2.2 Solution of Linear Systems by the Gauss-Jordan Method

2.3 Addition and Subtraction of Matrices

2.4 Multiplication of Matrices

2.5 Matrix Inverses

2.6 Input-Output Models

Chapter 2 Review

Extended Application: Contagion

The synchronized movements of band members marching on a field can be modeled using matrix arithmetic. An exercise in Section 5 in this chapter shows how multiplication by a matrix inverse transforms the original positions of the marchers into their new coordinates as they change direction.

Many mathematical models require finding the solutions of two or more equations. The solutions must satisfy *all* of the equations in the model. A set of equations related in this way is called a **system of equations**. In this chapter we will discuss systems of equations, introduce the idea of a *matrix*, and then show how matrices are used to solve systems of equations.

2.1 Solution of Linear Systems by the Echelon Method

APPLY IT **How much of each ingredient should be used in an animal feed to meet dietary requirements?**

APPLY IT Suppose that an animal feed is made from two ingredients: corn and soybeans. One serving of each ingredient provides the number of grams of protein and fiber shown in the table. For example, the entries in the first column, 3 and 11, indicate that one serving of corn provides 3 g of protein and 11 g of fiber.

Nutritional Content of Ingredients	Corn	Soybeans
Protein	3	10
Fiber	11	4

Now suppose we want to know how many servings of corn and soybeans should be used to make a feed that contains 115 g of protein and 95 g of fiber. Let x represent the number of servings of corn used and y the number of servings of soybeans. Each serving of corn provides 3 g of protein, so the amount of protein provided by x servings of corn is $3x$. Similarly, the amount of protein provided by y servings of soybeans is $10y$. Since the total amount of protein is to be 115 g,

$$3x + 10y = 115.$$

The feed must supply 95 g of fiber, so

$$11x + 4y = 95.$$

Solving this problem means finding values of x and y that satisfy this system of equations. Verify that $x = 5$ and $y = 10$ is a solution of the system, since these numbers satisfy both equations. In fact, this is the only solution of this system. Many practical problems lead to such a system of *first-degree equations*.

A **first-degree equation in n unknowns** is any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = k,$$

where $a_1, a_2, \ldots, a_n$ and k are real numbers and $x_1, x_2, \ldots, x_n$ represent variables.* Each of the two equations from the animal feed problem is a first-degree equation. For example, the first equation

$$3x + 10y = 115$$

*a_1 is read "a-sub-one." The notation $a_1, a_2, \ldots, a_n$ represents n real-number coefficients (some of which may be equal), and the notation $x_1, x_2, \ldots, x_n$ represents n different variables, or unknowns.

is a first-degree equation with $n = 2$ where

$$a_1 = 3, \qquad a_2 = 10, \qquad k = 115,$$

and the variables are x and y. We use x_1, x_2, etc., rather than x, y, etc., in the general case because we might have any number of variables. When n is no more than 4, we usually use x, y, z, and w to represent x_1, x_2, x_3, and x_4.

A *solution* of the first-degree equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = k$$

is a sequence of numbers $s_1, s_2, \ldots, s_n$ such that

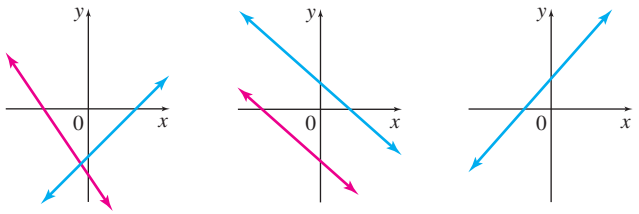
$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = k.$$

A solution of an equation is usually written in parentheses as $(s_1, s_2, \ldots, s_n)$. For example, $(1, 6, 2)$ is a solution of the equation $3x_1 + 2x_2 - 4x_3 = 7$, since $3(1) + 2(6) - 4(2) = 7$. This is an extension of the idea of an ordered pair, which was introduced in Chapter 1. A solution of a first-degree equation in two unknowns is an ordered pair, and the graph of the equation is a straight line. For this reason, all first-degree equations are also called linear equations.

Because the graph of a linear equation in two unknowns is a straight line, there are three possibilities for the solutions of a system of two linear equations in two unknowns.

Types of Solutions for Two Equations in Two Unknowns

1. The two graphs are lines intersecting at a single point. The system has a **unique solution**, and it is given by the coordinates of this point. See Figure 1(a).
2. The graphs are distinct parallel lines. When this is the case, the system is **inconsistent**; that is, there is no solution common to both equations. See Figure 1(b).
3. The graphs are the same line. In this case, the equations are said to be **dependent**, since any solution of one equation is also a solution of the other. There are infinitely many solutions. See Figure 1(c).



(a) **(b)** **(c)**

FIGURE 1

In larger systems, with more variables or more equations (or both, as is usually the case), there also may be exactly one solution, no solution, or infinitely many solutions. With more than two variables, the geometrical interpretation becomes more complicated, so we will only consider the geometry of two variables in two unknowns. We will, however, look at the algebra of two equations in three unknowns. In the next section, we will show how to handle any number of equations in any number of unknowns.

In Chapter 1, we used the graphing method and the substitution method to find the solution to a pair of equations describing the supply and demand for a commodity. The graphing method does not necessarily give an exact answer, and the substitution method becomes awkward with larger systems. In this section we will demonstrate a method to determine solutions using the addition property of equality to eliminate variables. This method forms the basis of the more general method presented in the next section.

Transformations

To solve a linear system of equations, we use properties of algebra to change, or transform, the system into a simpler *equivalent* system. An **equivalent system** is one that has the same solution(s) as the given system. Algebraic properties are the basis of the following transformations.

Transformations of a System

The following transformations can be applied to a system of equations to get an equivalent system:

1. exchanging any two equations;
2. multiplying both sides of an equation by any nonzero real number;
3. replacing any equation by a nonzero multiple of that equation plus a nonzero multiple of any other equation.

Use of these transformations leads to an equivalent system because each transformation can be reversed or "undone," allowing a return to the original system.

The Echelon Method

A systematic approach for solving systems of equations using the three transformations is called the **echelon method**. The goal of the echelon method is to use the transformations to rewrite the equations of the system until the system has a triangular form.

For a system of two equations in two variables, for example, the system should be transformed into the form

$$\begin{aligned} x + ay &= b \\ y &= c, \end{aligned}$$

where a, b, and c are constants. Then the value for y from the second equation can be substituted into the first equation to find x. This is called **back substitution**. In a similar manner, a system of three equations in three variables should be transformed into the form

$$\begin{aligned} x + ay + bz &= c \\ y + dz &= e, \\ z &= f. \end{aligned}$$

EXAMPLE 1 Solving a System of Equations with a Unique Solution

Solve the system of equations from the animal feed example that began this section:

$$3x + 10y = 115 \quad (1)$$

$$11x + 4y = 95. \quad (2)$$

SOLUTION We first use transformation 3 to eliminate the x-term from equation (2). We multiply equation (1) by 11 and add the results to -3 times equation (2).

$$\begin{aligned} \mathbf{11}(3x + 10y) &= \mathbf{11} \cdot 115 \\ \mathbf{-3}(11x + 4y) &= \mathbf{-3} \cdot 95 \end{aligned} \quad \rightarrow \quad \begin{aligned} 33x + 110y &= 1265 \\ -33x - 12y &= -285 \\ \hline 98y &= 980 \end{aligned}$$

FOR REVIEW

For review of how to combine like terms using properties of real numbers, see Section R.1.

We will indicate this process by the notation $11R_1 + (-3)R_2 \rightarrow R_2$. (R stands for the row.) The new system is

$$3x + 10y = 115 \quad (1)$$

$$\mathbf{11R_1 + (-3)R_2 \rightarrow R_2} \qquad 98y = 980 \quad (3)$$

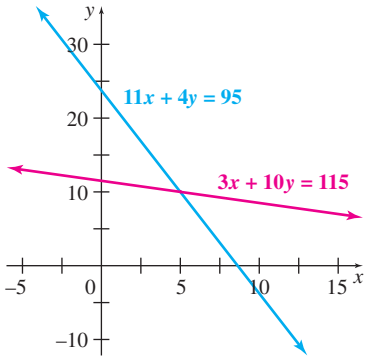


FIGURE 2

YOUR TURN 1

Solve the system

$2x + 3y = 12$

$3x - 4y = 1.$

FOR REVIEW

For review on graphing lines, see Section 1.1.

Now we use transformation 2 to make the coefficient of the first term in each row equal to 1. Here, we must multiply equation (1) by 1/3 and equation (3) by 1/98 to accomplish this.

We get the system

$$\tfrac{1}{3}\mathbf{R}_1 \to \mathbf{R}_1 \qquad x + \frac{10}{3}y = \frac{115}{3}$$

$$\tfrac{1}{98}\mathbf{R}_2 \to \mathbf{R}_2 \qquad y = 10.$$

Back-substitution gives

$$x + \frac{10}{3}(10) = \frac{115}{3} \qquad \text{Substitute } y = 10.$$

$$x + \frac{100}{3} = \frac{115}{3}$$

$$x = \frac{115}{3} - \frac{100}{3}$$

$$= \frac{15}{3} = 5.$$

The solution of the system is (5, 10). The graphs of the two equations in Figure 2 suggest that (5, 10) satisfies both equations in the system. **TRY YOUR TURN 1**

The echelon method, as illustrated in Example 1, is a specific case of a procedure known as the *elimination method*. The elimination method allows for any variable to be eliminated. For example, the variable y in Equations (1) and (2) could be eliminated by calculating $2\mathbf{R}_1 - 5\mathbf{R}_2$. In the echelon method, the goal is to get the system into the specific form described just before Example 1.

EXAMPLE 2 Solving a System of Equations with No Solution

Solve the system

$$2x - 3y = 6 \qquad (1)$$

$$-4x + 6y = 8. \qquad (2)$$

SOLUTION Eliminate x in equation (2) to get the system

$$2x - 3y = 6 \qquad (1)$$

$$2\mathbf{R}_1 + \mathbf{R}_2 \to \mathbf{R}_2 \qquad 0 = 20. \qquad (3)$$

In equation (3), both variables have been eliminated, leaving a *false statement*. This means it is impossible to have values of x and y that satisfy both equations, because this leads to a contradiction. To see why this occurs, see Figure 3, where we have shown that the graph of the system consists of two parallel lines. A solution would be a point lying on both lines, which is impossible. We conclude that the system is inconsistent and has no solution.

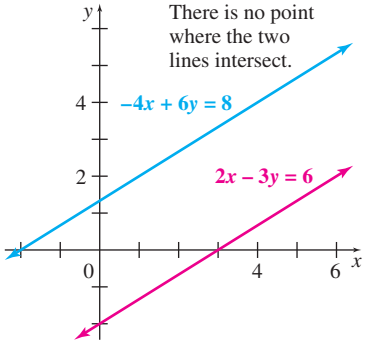


FIGURE 3

EXAMPLE 3 Solving a System of Equations with an Infinite Number of Solutions

Solve the system

$$3x - y = 4 \quad (1)$$
$$-6x + 2y = -8. \quad (2)$$

SOLUTION We use transformation 3 to eliminate x in equation (2), getting the system

$$3x - y = 4 \quad (1)$$
$$2\mathbf{R}_1 + \mathbf{R}_2 \rightarrow \mathbf{R}_2 \qquad 0 = 0. \quad (3)$$

The system becomes

$$\tfrac{1}{3}\mathbf{R}_1 \rightarrow \mathbf{R}_1 \qquad x - \frac{1}{3}y = \frac{4}{3} \quad (4)$$
$$0 = 0. \quad (3)$$

In equation (3), both variables have been eliminated, leaving a *true statement*. If we graph the original equations of the system on the same axes, as shown in Figure 4, we see that the graphs are the same line, and any point on the line will satisfy the system. This indicates that one equation was simply a multiple of the other. This system is dependent and has an infinite number of solutions.

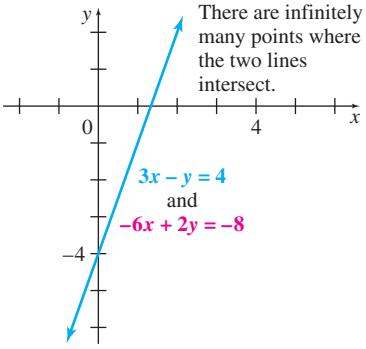


FIGURE 4

We will express the solutions in terms of y, where y can be any real number. The variable y in this case is called a **parameter**. (We could also let x be the parameter. In this text, we will follow the common practice of letting the rightmost variable be the parameter.) Solving equation (4) for x gives $x = (1/3)y + 4/3 = (y + 4)/3$, and all ordered pairs of the form

$$\left(\frac{y + 4}{3}, y\right)$$

are solutions. For example, if we let $y = 5$, then $x = (5 + 4)/3 = 3$ and one solution is $(3, 5)$. Similarly, letting $y = -10$ and $y = 3$ gives the solutions $(-2, -10)$ and $(7/3, 3)$.

Note that the original two equations are solved not only by the particular solutions like $(3, 5)$, $(-2, -10)$, and $(7/3, 3)$ but also by the general solution $(x, y) = ((y + 4)/3, y)$. For example, substituting this general solution into the first equation gives

$$3\left(\frac{y + 4}{3}\right) - y = y + 4 - y = 4,$$

which verifies that this general solution is indeed a solution. ■

In some applications, x and y must be nonnegative integers. For instance, in Example 3, if x and y represent the number of male and female workers in a factory, it makes no sense to have $x = 7/3$ or $x = -2$. To make both x and y nonnegative, we solve the inequalities

$$\frac{y + 4}{3} \geq 0 \qquad \text{and} \qquad y \geq 0,$$

yielding

$$y \geq -4 \qquad \text{and} \qquad y \geq 0.$$

To make these last two inequalities true, we require $y \geq 0$, from which $y \geq -4$ automatically follows. Furthermore, to ensure $(y + 4)/3$ is an integer, it is necessary that y be 2 more than a whole-number multiple of 3. Therefore, the possible values of y are 2, 5, 8, 11, and so on, and the corresponding values of x are 2, 3, 4, 5, and so on.

The echelon method can be generalized to systems with more equations and unknowns. Because systems with three or more equations are more complicated, however,

we will postpone those to the next section, where we will show a procedure for solving them based on the echelon system. Meanwhile, we summarize the echelon method here. To solve a linear system in n variables, perform the following steps using the three transformations given earlier.

Echelon Method of Solving a Linear System

1. If possible, arrange the equations so that there is an x_1-term in the first equation, an x_2-term in the second equation, and so on.
2. Eliminate the x_1-term in all equations after the first equation.
3. Eliminate the x_2-term in all equations after the second equation.
4. Eliminate the x_3-term in all equations after the third equation.
5. Continue in this way until the last equation has the form $ax_n = k$, for constants a and k, if possible.
6. Multiply each equation by the reciprocal of the coefficient of its first term.
7. Use back-substitution to find the value of each variable.

Applications

The mathematical techniques in this text will be useful to you only if you are able to apply them to practical problems. To do this, always begin by reading the problem carefully. Next, identify what must be found. Let each unknown quantity be represented by a variable. (It is a good idea to *write down* exactly what each variable represents.) Now reread the problem, looking for all necessary data. Write those down, too. Finally, look for one or more sentences that lead to equations or inequalities. The next example illustrates these steps.

EXAMPLE 4 Flight Time

A flight leaves New York at 8 P.M. and arrives in Paris at 9 A.M. (Paris time). This 13-hour difference includes the flight time plus the change in time zones. The return flight leaves Paris at 1 P.M. and arrives in New York at 3 P.M. (New York time). This 2-hour difference includes the flight time *minus* time zones, plus an extra hour due to the fact that flying westward is against the wind. Find the actual flight time eastward and the difference in time zones.

SOLUTION Let x be the flight time eastward and y be the difference in time zones. For the trip east, the flight time plus the change in time zones is 13 hours, so

$$x + y = 13.$$

For the trip west, the flight time (which is $x + 1$ hours due to the wind) minus the time zone is 2 hours, so

$$(x + 1) - y = 2.$$

Subtract 1 from both sides of this equation, and then solve the system

$$\begin{aligned} x + y &= 13 \\ x - y &= 1 \end{aligned}$$

using the echelon method.

$$\begin{aligned} & & x + y &= 13 \\ \mathbf{R_1 + (-1)R_2 \to R_2} & & 2y &= 12 \end{aligned}$$

Dividing the last equation by 2 gives $y = 6$. Substituting this into the first equation gives $x + 6 = 13$, so $x = 7$. Therefore, the flight time eastward is 7 hours, and the difference in time zones is 6 hours. TRY YOUR TURN 2

YOUR TURN 2 Solve Example 4 for the case in which there is a 16-hour time difference on the trip from New York, a 2-hour difference on the return trip, and no wind.

EXAMPLE 5 Integral Solutions

A restaurant owner orders a replacement set of knives, forks, and spoons. The box arrives containing 40 utensils and weighing 141.3 oz (ignoring the weight of the box). A knife, fork, and spoon weigh 3.9 oz, 3.6 oz, and 3.0 oz, respectively.

(a) How many solutions are there for the number of knives, forks, and spoons in the box?

SOLUTION Let

$$x = \text{the number of knives;}$$
$$y = \text{the number of forks;}$$
$$z = \text{the number of spoons.}$$

A chart is useful for organizing the information in a problem of this type.

Number and Weight of Utensils				
	Knives	**Forks**	**Spoons**	**Total**
Number	x	y	z	40
Weight	3.9	3.6	3.0	141.3

Because the box contains 40 utensils,

$$x + y + z = 40.$$

The x knives weigh $3.9x$ ounces, the y forks weigh $3.6y$ ounces, and the z spoons weigh $3.0z$ ounces. Since the total weight is 141.3 oz, we have the system

$$\begin{aligned} x + \quad y + \quad z &= 40 \\ 3.9x + 3.6y + 3.0z &= 141.3. \end{aligned}$$

Solve using the echelon method.

$$\begin{aligned} & x + \quad y + \quad z = 40 \\ \mathbf{3.9R_1 + (-1)R_2 \rightarrow R_2} \qquad & 0.3y + 0.9z = 14.7 \end{aligned}$$

We do not have a third equation to solve for z, as we did in Example 4. We could thus let z be any real number, and then use this value of z in the second equation to find a corresponding value of y. These values of y and z could be put into the first equation to find a corresponding value of x. This system, then, has an infinite number of solutions. Letting z be the parameter, solve the second equation for y to get

$$y = \frac{14.7 - 0.9z}{0.3} = 49 - 3z.$$

Substituting this into the first equation, we get

$$x + (49 - 3z) + z = 40.$$

Solving this for x gives

$$x = 2z - 9.$$

Thus, the solutions are $(2z - 9, 49 - 3z, z)$, where z is any real number.

Now that we have solved for x and y in terms of z, let us investigate what values z can take on. This application demands that the solutions be nonnegative integers. The number of forks cannot be negative, so set

$$49 - 3z \geq 0.$$

Solving for z gives

$$z \leq \frac{49}{3} \approx 16.33.$$

Also, the number of knives cannot be negative, so set

$$2z - 9 \geq 0.$$

Solving for z gives

$$z \geq \frac{9}{2} = 4.5.$$

Therefore, the permissible values of z are 5, 6, 7, . . . , 16, for a total of 12 solutions.

(b) Find the solution with the smallest number of spoons.

SOLUTION The smallest value of z is $z = 5$, from which we find $x = 2(5) - 9 = 1$ and $y = 49 - 3(5) = 34$. This solution has 1 knife, 34 forks, and 5 spoons.

TRY YOUR TURN 3

YOUR TURN 3 Find the solution with the largest number of spoons.

2.1 EXERCISES

Use the echelon method to solve each system of two equations in two unknowns. Check your answers.

1. $x + y = 5$
$2x - 2y = 2$

2. $4x + y = 9$
$3x - y = 5$

3. $3x - 2y = -3$
$5x - y = 2$

4. $2x + 7y = -8$
$-2x + 3y = -12$

5. $3x + 2y = -6$
$5x - 2y = -10$

6. $-3x + y = 4$
$2x - 2y = -4$

7. $6x - 2y = -4$
$3x + 4y = 8$

8. $4m + 3n = -1$
$2m + 5n = 3$

9. $5p + 11q = -7$
$3p - 8q = 25$

10. $12s - 5t = 9$
$3s - 8t = -18$

11. $6x + 7y = -2$
$7x - 6y = 26$

12. $3a - 8b = 14$
$a - 2b = 2$

13. $3x + 2y = 5$
$6x + 4y = 8$

14. $9x - 5y = 1$
$-18x + 10y = 1$

15. $3x - 2y = -4$
$-6x + 4y = 8$

16. $3x + 5y + 2 = 0$
$9x + 15y + 6 = 0$

17. $x - \frac{3y}{2} = \frac{5}{2}$
$\frac{4x}{3} + \frac{2y}{3} = 6$

18. $\frac{x}{5} + 3y = 31$
$2x - \frac{y}{5} = 8$

19. $\frac{x}{2} + y = \frac{3}{2}$
$\frac{x}{3} + y = \frac{1}{3}$

20. $\frac{x}{9} + \frac{y}{6} = \frac{1}{3}$
$2x + \frac{8y}{5} = \frac{2}{5}$

21. An inconsistent system has _____ solutions.

22. The solution of a system with two dependent equations in two variables is _____.

For each of the following systems of equations in echelon form, tell how many solutions there are in nonnegative integers.

23. $x + 2y + 3z = 90$
$3y + 4z = 36$

24. $x - 7y + 4z = 75$
$2y + 7z = 60$

25. $3x + 2y + 4z = 80$
$y - 3z = 10$

26. $4x + 2y + 3z = 72$
$2y - 3z = 12$

27. Describe what a parameter is and why it is used in systems of equations with an infinite number of solutions.

28. In your own words, describe the echelon method as used to solve a system of two equations in three variables.

Solve each system of equations. Let z be the parameter.

29. $2x + 3y - z = 1$
$3x + 5y + z = 3$

30. $3x + y - z = 0$
$2x - y + 3z = -7$

31. $x + 2y + 3z = 11$
$2x - y + z = 2$

32. $-x + y - z = -7$
$2x + 3y + z = 7$

33. In Exercise 9 in Section 1.3, you were asked to solve the system of least squares line equations

$$nb + (\Sigma x)m = \Sigma y$$
$$(\Sigma x)b + (\Sigma x^2)m = \Sigma xy$$

by the method of substitution. Now solve the system by the echelon method to get

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2}$$

$$b = \frac{\Sigma y - m(\Sigma x)}{n}.$$

34. The examples in this section did not use the first transformation. How might this transformation be used in the echelon method?

APPLICATIONS

Business and Economics

35. Groceries If 20 lb of rice and 10 lb of potatoes cost $16.20, and 30 lb of rice and 12 lb of potatoes cost $23.04, how much will 10 lb of rice and 50 lb of potatoes cost?

36. Sales An apparel shop sells skirts for $45 and blouses for $35. Its entire stock is worth $51,750. But sales are slow and only half the skirts and two-thirds of the blouses are sold, for a total of $30,600. How many skirts and blouses are left in the store?

37. Sales A theater charges $8 for main floor seats and $5 for balcony seats. If all seats are sold, the ticket income is $4200. At one show, 25% of the main floor seats and 40% of the balcony seats were sold and ticket income was $1200. How many seats are on the main floor and how many are in the balcony?

38. Stock Lorri Morgan has $16,000 invested in Disney and Exxon stock. The Disney stock currently sells for $30 a share and the Exxon stock for $70 a share. Her stockbroker points out that if Disney stock goes up 50% and Exxon stock goes up by $35 a share, her stock will be worth $25,500. Is this possible? If so, tell how many shares of each stock she owns. If not, explain why not.

39. Production A company produces two models of bicycles, model 201 and model 301. Model 201 requires 2 hours of assembly time and model 301 requires 3 hours of assembly time. The parts for model 201 cost $18 per bike and the parts for model 301 cost $27 per bike. If the company has a total of 34 hours of assembly time and $335 available per day for these two models, how many of each should be made in a day to use up all available time and money? If it is not possible, explain why not.

40. Banking A bank teller has a total of 70 bills in five-, ten-, and twenty-dollar denominations. The total value of the money is $960.

a. Find the total number of solutions.

b. Find the solution with the smallest number of five-dollar bills.

c. Find the solution with the largest number of five-dollar bills.

41. Production Felsted Furniture makes dining room furniture. A buffet requires 30 hours for construction and 10 hours for finishing. A chair requires 10 hours for construction and 10 hours for finishing. A table requires 10 hours for construction and 30 hours for finishing. The construction department has 350 hours of labor and the finishing department has 150 hours of labor available each week. How many pieces of each type of furniture should be produced each week if the factory is to run at full capacity?

42. Rug Cleaning Machines Kelly Karpet Kleaners sells rug cleaning machines. The EZ model weighs 10 lb and comes in a 10-cubic-ft box. The compact model weighs 20 lb and comes in an 8-cubic-ft box. The commercial model weighs 60 lb and comes in a 28-cubic-ft box. Each of their delivery vans has 248 cubic ft of space and can hold a maximum of 440 lb. In order for a van to be fully loaded, how many of each model should it carry?

43. Production Turley Tailor, Inc. makes long-sleeve, short-sleeve, and sleeveless blouses. A long-sleeve blouse requires 1.5 hours of cutting and 1.2 hours of sewing. A short-sleeve blouse requires 1 hour of cutting and 0.9 hour of sewing. A sleeveless blouse requires 0.5 hour of cutting and 0.6 hour of sewing. There are 380 hours of labor available in the cutting department each day and 330 hours in the sewing department. If the plant is to run at full capacity, how many of each type of blouse should be made each day?

44. Broadway Economics When Neil Simon opens a new play, he has to decide whether to open the show on Broadway or Off Broadway. For example, in his play *London Suite*, he decided to open it Off Broadway. From information provided by Emanuel Azenberg, his producer, the following equations were developed:

$$43{,}500x - y = 1{,}295{,}000$$
$$27{,}000x - y = 440{,}000,$$

where x represents the number of weeks that the show has run and y represents the profit or loss from the show (first equation is for Broadway and second equation is for Off Broadway). ***Source: The Mathematics Teacher.***

a. Solve this system of equations to determine when the profit/loss from the show will be equal for each venue. What is the profit at that point?

b. Discuss which venue is favorable for the show.

Life Sciences

45. Birds The date of the first sighting of robins has been occurring earlier each spring over the past 25 years at the Rocky Mountain Biological Laboratory. Scientists from this laboratory have developed two linear equations that estimate the date of the first sighting of robins:

$$y = 759 - 0.338x$$
$$y = 1637 - 0.779x,$$

where x is the year and y is the estimated number of days into the year when a robin can be expected. ***Source: Proceedings of the National Academy of Science.***

a. Compare the date of first sighting in 2000 for each of these equations. (*Hint:* 2000 was a leap year.)

b. Solve this system of equations to find the year in which the two estimates agree.

Physical Sciences

46. Stopping Distance The stopping distance of a car traveling 25 mph is 61.7 ft, and for a car traveling 35 mph it is 106 ft. The stopping distance in feet can be described by the equation $y = ax^2 + bx$, where x is the speed in mph. ***Source: National Traffic Safety Institute.***

a. Find the values of a and b.

b. Use your answers from part a to find the stopping distance for a car traveling 55 mph.

General Interest

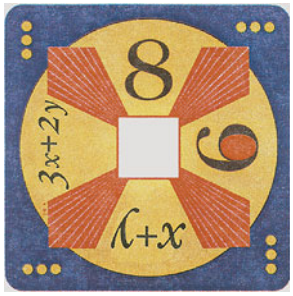
47. Basketball Wilt Chamberlain holds the record for the highest number of points scored in a single NBA basketball game.

Chamberlain scored 100 points for Philadelphia against the New York Knicks on March 2, 1962. This is an amazing feat, considering he scored all of his points without the help of three-point shots. Chamberlain made a total of 64 baskets, consisting of field goals (worth two points) and foul shots (worth one point). Find the number of field goals and the number of foul shots that Chamberlain made. ***Source: ESPN.***

48. The 24® Game The object of the 24® Game, created by Robert Sun, is to combine four numbers, using addition, subtraction, multiplication, and/or division, to get the number 24. For example, the numbers 2, 5, 5, 4 can be combined as $2(5 + 5) + 4 = 24$. For the algebra edition of the game and the game card shown to the right, the object is to find single-digit positive integer values x and y so the four numbers $x + y$, $3x + 2y$, 8, and 9 can be combined to make 24. ***Source: Suntex.***

a. Using the game card, write a system of equations that, when solved, can be used to make 24 from the game card. What is the solution to this system, and how can it be used to make 24 on the game card?

b. Repeat part a and develop a second system of equations.



YOUR TURN ANSWERS

1. (3, 2)
2. Flight time is 9 hours, difference in time zones is 7 hours.
3. 23 knives, 1 fork, and 16 spoons

2.2 Solution of Linear Systems by the Gauss-Jordan Method

APPLY IT

How can an auto manufacturer with more than one factory and several dealers decide how many cars to send to each dealer from each factory?

Questions like this are called transportation problems; they frequently lead to a system of equations that must be satisfied. In Exercise 52 in this section we use a further refinement of the echelon method to answer this question.

When we use the echelon method, since the variables are in the same order in each equation, we really need to keep track of just the coefficients and the constants. For example, consider the following system of three equations in three unknowns.

$$\begin{aligned} 2x + \ y - \ z &= 2 \\ x + 3y + 2z &= 1 \\ x + \ y + \ z &= 2 \end{aligned}$$

This system can be written in an abbreviated form as

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Rows

Columns

Such a rectangular array of numbers enclosed by brackets is called a **matrix** (plural: **matrices**).* Each number in the array is an **element** or **entry**. To separate the constants in

*The word matrix, Latin for "womb," was coined by James Joseph Sylvester (1814–1897) and made popular by his friend Arthur Cayley (1821–1895). Both mathematicians were English, although Sylvester spent much of his life in the United States.

the last column of the matrix from the coefficients of the variables, we use a vertical line, producing the following **augmented matrix**.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

The rows of the augmented matrix can be transformed in the same way as the equations of the system, since the matrix is just a shortened form of the system. The following **row operations** on the augmented matrix correspond to the transformations of systems of equations given earlier.

Row Operations

For any augmented matrix of a system of equations, the following operations produce the augmented matrix of an equivalent system:

1. interchanging any two rows;
2. multiplying the elements of a row by any nonzero real number;
3. adding a nonzero multiple of the elements of one row to the corresponding elements of a nonzero multiple of some other row.

In steps 2 and 3, we are replacing a row with a new, modified row, which the old row helped to form, just as we replaced an equation with a new, modified equation in the previous section.

Row operations, like the transformations of systems of equations, are reversible. If they are used to change matrix A to matrix B, then it is possible to use row operations to transform B back into A. In addition to their use in solving equations, row operations are very important in the simplex method to be described in Chapter 4.

In the examples in this section, we will use the same notation as in Section 1 to show the row operation used. For example, the notation R_1 indicates row 1 of the previous matrix, and $-3R_1 + R_2$ means that row 1 is multiplied by -3 and added to row 2.

By the first row operation, interchanging two rows, the matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ -2 & -6 & -10 & -12 \\ 2 & 1 & -2 & -5 \end{array}\right] \quad \text{becomes} \quad \left[\begin{array}{ccc|c} -2 & -6 & -10 & -12 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & -2 & -5 \end{array}\right] \qquad \text{Interchange } R_1 \text{ and } R_2$$

by interchanging the first two rows. Row 3 is left unchanged.

The second row operation, multiplying a row by a number, allows us to change

$$\left[\begin{array}{ccc|c} -2 & -6 & -10 & -12 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & -2 & -5 \end{array}\right] \quad \text{to} \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & -2 & -5 \end{array}\right] \qquad (-1/2)R_1 \to R_1$$

by multiplying the elements of row 1 of the original matrix by $-1/2$. Note that rows 2 and 3 are left unchanged.

Using the third row operation, adding a multiple of one row to another, we change

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & -2 & -5 \end{array}\right] \quad \text{to} \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -12 & -17 \end{array}\right] \qquad -2R_1 + R_3 \to R_3$$

by first multiplying each element in row 1 of the original matrix by -2 and then adding the results to the corresponding elements in the third row of that matrix. Work as follows.

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \\ (-2)1+2 & (-2)3+1 & (-2)5-2 & (-2)6-5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -12 & -17 \end{array}\right]$$

Notice that rows 1 and 2 are left unchanged, *even though the elements of row 1 were used to transform row 3.*

The Gauss-Jordan Method

The **Gauss-Jordan method** is an extension of the echelon method of solving systems.* Before the Gauss-Jordan method can be used, the system must be in proper form: the terms with variables should be on the left and the constants on the right in each equation, with the variables in the same order in each equation.

The system is then written as an augmented matrix. Using row operations, the goal is to transform the matrix so that it has zeros above and below a diagonal of 1's on the left of the vertical bar. Once this is accomplished, the final solution can be read directly from the last matrix. The following example illustrates the use of the Gauss-Jordan method to solve a system of equations.

EXAMPLE 1 Gauss-Jordan Method

Solve the system

$$3x - 4y = 1 \quad (1)$$
$$5x + 2y = 19. \quad (2)$$

SOLUTION

Method I
1's on Diagonal

The system is already in the proper form to use the Gauss-Jordan method. Our goal is to transform this matrix, if possible, into the form

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right],$$

where m and n are real numbers. To begin, we change the 3 in the first row to 1 using the second row operation. (Notice that the same notation is used to indicate each transformation, as in the previous section.)

$$\left[\begin{array}{cc|c} 3 & -4 & 1 \\ 5 & 2 & 19 \end{array}\right] \quad \text{Augmented matrix}$$

$$\tfrac{1}{3}R_1 \to R_1 \qquad \left[\begin{array}{cc|c} 1 & -\frac{4}{3} & \frac{1}{3} \\ 5 & 2 & 19 \end{array}\right]$$

Using the third row operation, we change the 5 in row 2 to 0.

$$-5R_1 + R_2 \to R_2 \qquad \left[\begin{array}{cc|c} 1 & -\frac{4}{3} & \frac{1}{3} \\ 0 & \frac{26}{3} & \frac{52}{3} \end{array}\right]$$

We now change $26/3$ in row 2 to 1 to complete the diagonal of 1's.

$$\tfrac{3}{26}R_2 \to R_2 \qquad \left[\begin{array}{cc|c} 1 & -\frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 2 \end{array}\right]$$

*The great German mathematician Carl Friedrich Gauss (1777–1855), sometimes referred to as the "Prince of Mathematicians," originally developed his elimination method for use in finding least squares coefficients. (See Section 1.3.) The German geodesist Wilhelm Jordan (1842–1899) improved his method and used it in surveying problems. Gauss's method had been known to the Chinese at least 1800 years earlier and was described in the *Jiuahang Suanshu (Nine Chapters on the Mathematical Art).*

The final transformation is to change the $-4/3$ in row 1 to 0.

$$\tfrac{4}{3}\mathbf{R}_2 + \mathbf{R}_1 \to \mathbf{R}_1 \qquad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

The last matrix corresponds to the system

$$\begin{aligned} x &= 3 \\ y &= 2, \end{aligned}$$

so we can read the solution directly from the last column of the final matrix. Check that (3, 2) is the solution by substitution in the equations of the original matrix.

Method 2 Fraction-Free

An alternate form of Gauss-Jordan is to first transform the matrix so that it contains zeros above and below the main diagonal. Then, use the second transformation to get the required 1's. When doing calculations by hand, this second method simplifies the calculations by avoiding fractions and decimals. We will use this method when doing calculations by hand throughout the remainder of this chapter.

To begin, we change the 5 in row 2 to 0.

$$\left[\begin{array}{cc|c} 3 & -4 & 1 \\ 5 & 2 & 19 \end{array}\right] \qquad \text{Augmented matrix}$$

$$5\mathbf{R}_1 + (-3)\mathbf{R}_2 \to \mathbf{R}_2 \qquad \left[\begin{array}{cc|c} 3 & -4 & 1 \\ 0 & -26 & -52 \end{array}\right]$$

We change the -4 in row 1 to 0.

$$-4\mathbf{R}_2 + 26\mathbf{R}_1 \to \mathbf{R}_1 \qquad \left[\begin{array}{cc|c} 78 & 0 & 234 \\ 0 & -26 & -52 \end{array}\right]$$

Then we change the first nonzero number in each row to 1.

$$\begin{aligned} \tfrac{1}{78}\mathbf{R}_1 &\to \mathbf{R}_1 \\ -\tfrac{1}{26}\mathbf{R}_2 &\to \mathbf{R}_2 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

The solution is read directly from this last matrix: $x = 3$ and $y = 2$, or (3, 2).

TRY YOUR TURN 1

YOUR TURN 1 Use the Gauss-Jordan method to solve the system

$4x + 5y = 10$
$7x + 8y = 19.$

NOTE If your solution does not check, the most efficient way to find the error is to substitute back through the equations that correspond to each matrix, starting with the last matrix. When you find a system that is not satisfied by your (incorrect) answers, you have probably reached the matrix just before the error occurred. Look for the error in the transformation to the next matrix. For example, if you erroneously wrote the 2 as -2 in the final matrix of the fraction-free method of Example 2, you would find that $(3, -2)$ was not a solution of the system represented by the previous matrix because $-26(-2) \neq -52$, telling you that your error occurred between this matrix and the final one.

When the Gauss-Jordan method is used to solve a system, the final matrix always will have zeros above and below the diagonal of 1's on the left of the vertical bar. To transform the matrix, it is best to work column by column from left to right. Such an orderly method avoids confusion and going around in circles. For each column, first perform the steps that give the zeros. When all columns have zeros in place, multiply each row by the reciprocal of the coefficient of the remaining nonzero number in that row to get the required 1's. With dependent equations or inconsistent systems, it will not be possible to get the complete diagonal of 1's.

We will demonstrate the method in an example with three variables and three equations, after which we will summarize the steps.

EXAMPLE 2 Gauss-Jordan Method

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} x + 5z &= -6 + y \\ 3x + 3y &= 10 + z \\ x + 3y + 2z &= 5. \end{aligned}$$

Method I
Calculating by Hand

SOLUTION

First, rewrite the system in proper form, as follows.

$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$

Begin to find the solution by writing the augmented matrix of the linear system.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$$

Row transformations will be used to rewrite this matrix in the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & m \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & p \end{array}\right],$$

where m, n, and p are real numbers (if this form is possible). From this final form of the matrix, the solution can be read: $x = m$, $y = n$, $z = p$, or (m, n, p).

In the first column, we need zeros in the second and third rows. Multiply the first row by -3 and add to the second row to get a zero there. Then multiply the first row by -1 and add to the third row to get that zero.

$$\begin{array}{r} \\ -3\mathrm{R}_1 + \mathrm{R}_2 \to \mathrm{R}_2 \\ -1\mathrm{R}_1 + \mathrm{R}_3 \to \mathrm{R}_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right]$$

Now get zeros in the second column in a similar way. We want zeros in the first and third rows. Row 2 will not change.

$$\begin{array}{r} \mathrm{R}_2 + 6\mathrm{R}_1 \to \mathrm{R}_1 \\ \\ 2\mathrm{R}_2 + (-3)\mathrm{R}_3 \to \mathrm{R}_3 \end{array} \quad \left[\begin{array}{ccc|c} 6 & 0 & 14 & -8 \\ 0 & 6 & -16 & 28 \\ 0 & 0 & -23 & 23 \end{array}\right]$$

In transforming the third row, you may have used the operation $4\mathrm{R}_2 + (-6)\mathrm{R}_3 \to \mathrm{R}_3$ instead of $2\mathrm{R}_2 + (-3)\mathrm{R}_3 \to \mathrm{R}_3$. This is perfectly fine; the last row would then have -46 and 46 in place of -23 and 23. To avoid errors, it helps to keep the numbers as small as possible. We observe at this point that all of the numbers can be reduced in size by multiplying each row by an appropriate constant. This next step is not essential, but it simplifies the arithmetic.

$$\begin{array}{r} \frac{1}{2}\mathrm{R}_1 \to \mathrm{R}_1 \\ \frac{1}{2}\mathrm{R}_2 \to \mathrm{R}_2 \\ -\frac{1}{23}\mathrm{R}_3 \to \mathrm{R}_3 \end{array} \quad \left[\begin{array}{ccc|c} 3 & 0 & 7 & -4 \\ 0 & 3 & -8 & 14 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Next, we want zeros in the first and second rows of the third column. Row 3 will not change.

$$\begin{array}{r} -7\mathrm{R}_3 + \mathrm{R}_1 \to \mathrm{R}_1 \\ 8\mathrm{R}_3 + \mathrm{R}_2 \to \mathrm{R}_2 \\ \\ \end{array} \quad \left[\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Finally, get 1's in each row by multiplying the row by the reciprocal of (or dividing the row by) the number in the diagonal position.

$$\begin{array}{l} \frac{1}{3}\mathbf{R}_1 \to \mathbf{R}_1 \\ \frac{1}{3}\mathbf{R}_2 \to \mathbf{R}_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The linear system associated with the final augmented matrix is

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= -1, \end{aligned}$$

and the solution is $(1, 2, -1)$. Verify that this is the solution to the original system of equations.

CAUTION Notice that we have performed two or three operations on the same matrix in one step. This is permissible as long as we do not use a row that we are changing as part of another row operation. For example, when we changed row 2 in the first step, we could not use row 2 to transform row 3 in the same step. To avoid difficulty, use *only* row 1 to get zeros in column 1, row 2 to get zeros in column 2, and so on.

Method 2 Graphing Calculator

The row operations of the Gauss-Jordan method can also be done on a graphing calculator. For example, Figure 5 shows the result when the augmented matrix is entered into a TI-84 Plus. Figures 6 and 7 show how row operations can be used to get zeros in rows 2 and 3 of the first column.

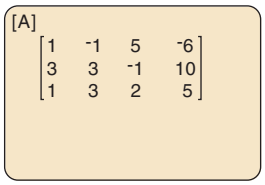


FIGURE 5

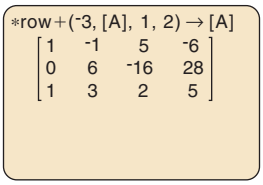


FIGURE 6

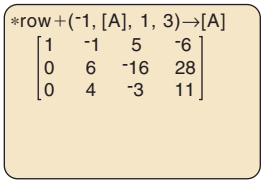


FIGURE 7

Calculators typically do not allow any multiple of a row to be added to any multiple of another row, such as in the operation $2\mathbf{R}_2 + 6\mathbf{R}_1 \to \mathbf{R}_1$. They normally allow a multiple of a row to be added only to another unmodified row. To get around this restriction, we can convert the diagonal element to a 1 before changing the other elements in the column to 0, as we did in the first method of Example 1. In this example, we change the 6 in row 2, column 2, to a 1 by dividing by 6. The result is shown in Figure 8. (The right side of the matrix is not visible but can be seen by pressing the right arrow key.) Notice that this operation introduces decimals. Converting to fractions is preferable on calculators that have that option; 1/3 is certainly more concise than 0.3333333333. Figure 9 shows such a conversion on the TI-84 Plus.

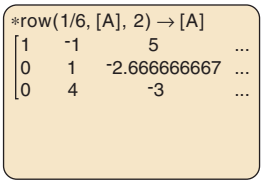


FIGURE 8

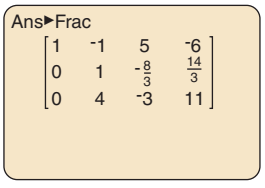


FIGURE 9

When performing row operations without a graphing calculator, it is best to avoid fractions and decimals, because these make the operations more difficult and more prone to error. A calculator, on the other hand, encounters no such difficulties.

Continuing in the same manner, the solution $(1, 2, -1)$ is found as shown in Figure 10.

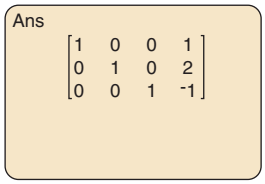


FIGURE 10

Some calculators can do the entire Gauss-Jordan process with a single command; on the TI-84 Plus, for example, this is done with the `rref` command. This is very useful in practice, although it does not show any of the intermediate steps. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

Method 3 Spreadsheet

The Gauss-Jordan method can be done using a spreadsheet either by using a macro or by developing the pivot steps using formulas with the copy and paste commands. However, spreadsheets also have built-in methods to solve systems of equations. Although these solvers do not usually employ the Gauss-Jordan method for solving systems of equations, they are, nonetheless, efficient and practical to use.

The Solver included with Excel can solve systems of equations that are both linear and nonlinear. The Solver is located in the Tools menu and requires that cells be identified ahead of time for each variable in the problem. It also requires that the left-hand side of each equation be placed in the spreadsheet as a formula. For example, to solve the above problem, we could identify cells A1, B1, and C1 for the variables x, y, and z, respectively. The Solver requires that we place a guess for the answer in these cells. It is convenient to place a zero in each of these cells. The left-hand side of each equation must be placed in a cell. We could choose A3, A4, and A5 to hold each of these formulas. Thus, in cell A3, we would type "`=A1 − B1 + 5*C1`" and put the other two equations in cells A4 and A5.

We now click on the Tools menu and choose Solver. (In some versions, it may be necessary to install the Solver. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.) Since this solver attempts to find a solution that is best in some way, we are required to identify a cell with a formula in it that we want to optimize. In this case, it is convenient to use the cell with the left-hand side of the first constraint in it, A3. Figure 11 illustrates the Solver box and the items placed in it.

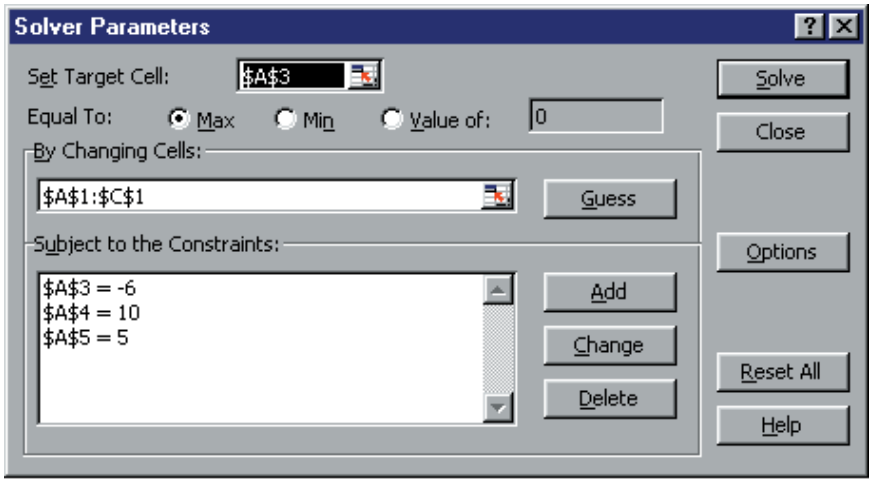


FIGURE 11

To obtain a solution, click on Solve. The approximate solution is located in cells A1, B1, and C1, and these correspond to x, y, and z, respectively.

YOUR TURN 2 Use the Gauss-Jordan method to solve the system

$$\begin{aligned} x + 2y + 3z &= 2 \\ 2x + 2y - 3z &= 27 \\ 3x + 2y + 5z &= 10. \end{aligned}$$

TRY YOUR TURN 2

In summary, the Gauss-Jordan method of solving a linear system requires the following steps.

Gauss-Jordan Method of Solving a Linear System

1. Write each equation so that variable terms are in the same order on the left side of the equal sign and constants are on the right.
2. Write the augmented matrix that corresponds to the system.
3. Use row operations to transform the first column so that all elements except the element in the first row are zero.
4. Use row operations to transform the second column so that all elements except the element in the second row are zero.
5. Use row operations to transform the third column so that all elements except the element in the third row are zero.
6. Continue in this way, when possible, until the last row is written in the form
$$[0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad j \mid k],$$
where j and k are constants with $j \neq 0$. When this is not possible, continue until every row has more zeros on the left than the previous row (except possibly for any rows of all zero at the bottom of the matrix), and the first nonzero entry in each row is the only nonzero entry in its column.
7. Multiply each row by the reciprocal of the nonzero element in that row.

Systems without a Unique Solution

In the previous examples, we were able to get the last row in the form $[0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad j \mid k]$, where j and k are constants with $j \neq 0$. We will now look at examples where this is not the case.

EXAMPLE 3 Solving a System of Equations with No Solution

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} x - 2y &= 2 \\ 3x - 6y &= 5. \end{aligned}$$

SOLUTION Begin by writing the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -6 & 5 \end{array}\right]$$

To get a zero for the second element in column 1, multiply the numbers in row 1 by -3 and add the results to the corresponding elements in row 2.

$$-3R_1 + R_2 \rightarrow R_2 \qquad \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 0 & -1 \end{array}\right]$$

This matrix corresponds to the system

$$\begin{aligned} x - 2y &= 2 \\ 0x + 0y &= -1. \end{aligned}$$

Since the second equation is $0 = -1$, the system is inconsistent and, therefore, has no solution. The row $[0 \quad 0 \mid k]$ for any nonzero k is a signal that the given system is inconsistent.

EXAMPLE 4 Solving a System of Equations with an Infinite Number of Solutions

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} x + 2y - z &= 0 \\ 3x - y + z &= 6 \\ -2x - 4y + 2z &= 0. \end{aligned}$$

SOLUTION The augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 3 & -1 & 1 & 6 \\ -2 & -4 & 2 & 0 \end{array}\right].$$

We first get zeros in the second and third rows of column 1.

$$\begin{array}{r} \\ -3\mathbf{R}_1 + \mathbf{R}_2 \to \mathbf{R}_2 \\ 2\mathbf{R}_1 + \mathbf{R}_3 \to \mathbf{R}_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -7 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To continue, we get a zero in the first row of column 2 using the second row, as usual.

$$2\mathbf{R}_2 + 7\mathbf{R}_1 \to \mathbf{R}_1 \quad \left[\begin{array}{rrr|r} 7 & 0 & 1 & 12 \\ 0 & -7 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We cannot get a zero for the first-row, third-column element without changing the form of the first two columns. We must multiply each of the first two rows by the reciprocal of the first nonzero number.

$$\begin{array}{r} \frac{1}{7}\mathbf{R}_1 \to \mathbf{R}_1 \\ -\frac{1}{7}\mathbf{R}_2 \to \mathbf{R}_2 \\ \\ \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{7} & \frac{12}{7} \\ 0 & 1 & -\frac{4}{7} & -\frac{6}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To complete the solution, write the equations that correspond to the first two rows of the matrix.

$$\begin{aligned} x + \frac{1}{7}z &= \frac{12}{7} \\ y - \frac{4}{7}z &= -\frac{6}{7} \end{aligned}$$

Because both equations involve z, let z be the parameter. There are an infinite number of solutions, corresponding to the infinite number of values of z. Solve the first equation for x and the second for y to get

$$x = \frac{12 - z}{7} \quad \text{and} \quad y = \frac{4z - 6}{7}.$$

As shown in the previous section, the general solution is written

$$\left(\frac{12 - z}{7}, \frac{4z - 6}{7}, z\right),$$

where z is any real number. For example, $z = 2$ and $z = 12$ lead to the solutions $(10/7, 2/7, 2)$ and $(0, 6, 12)$.

EXAMPLE 5 Solving a System of Equations with an Infinite Number of Solutions

Consider the following system of equations.

$$\begin{aligned} x + 2y + 3z - w &= 4 \\ 2x + 3y \qquad + w &= -3 \\ 3x + 5y + 3z \qquad &= 1 \end{aligned}$$

(a) Set this up as an augmented matrix, and verify that the result after the Gauss-Jordan method is

$$\left[\begin{array}{cccc|c} 1 & 0 & -9 & 5 & -18 \\ 0 & 1 & 6 & -3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) Find the solution to this system of equations.

SOLUTION To complete the solution, write the equations that correspond to the first two rows of the matrix.

$$\begin{aligned} x \quad - 9z + 5w &= -18 \\ y + 6z - 3w &= 11 \end{aligned}$$

Because both equations involve both z and w, let z and w be parameters. There are an infinite number of solutions, corresponding to the infinite number of values of z and w. Solve the first equation for x and the second for y to get

$$x = -18 + 9z - 5w \qquad \text{and} \qquad y = 11 - 6z + 3w.$$

In an analogous manner to problems with a single parameter, the general solution is written

$$(-18 + 9z - 5w, 11 - 6z + 3w, z, w),$$

where z and w are any real numbers. For example, $z = 1$ and $w = -2$ leads to the solution $(1, -1, 1, -2)$.

TRY YOUR TURN 3

YOUR TURN 3 Use the Gauss-Jordan method to solve the system

$$\begin{aligned} 2x - 2y + 3z - 4w &= 6 \\ 3x + 2y + 5z - 3w &= 7 \\ 4x + \ y + 2z - 2w &= 8. \end{aligned}$$

Although the examples have used only systems with two equations in two unknowns, three equations in three unknowns, or three equations in four unknowns, the Gauss-Jordan method can be used for any system with n equations and m unknowns. The method becomes tedious with more than three equations in three unknowns; on the other hand, it is very suitable for use with graphing calculators and computers, which can solve fairly large systems quickly. Sophisticated computer programs modify the method to reduce round-off error. Other methods used for special types of large matrices are studied in a course on numerical analysis.

EXAMPLE 6 Soda Sales

A convenience store sells 23 sodas one summer afternoon in 12-, 16-, and 20-oz cups (small, medium, and large). The total volume of soda sold was 376 oz.

(a) Suppose that the prices for a small, medium, and large soda are \$1, \$1.25, and \$1.40, respectively, and that the total sales were \$28.45. How many of each size did the store sell?

SOLUTION As in Example 6 of the previous section, we will organize the information in a table.

Soda Sales				
	Small	Medium	Large	Total
Number	x	y	z	23
Volume	12	16	20	376
Price	1.00	1.25	1.40	28.45

The three rows of the table lead to three equations: one for the total number of sodas, one for the volume, and one for the price.

$$\begin{aligned} x + \quad y + \quad z &= \quad 23 \\ 12x + \quad 16y + \quad 20z &= \quad 376 \\ 1.00x + 1.25y + 1.40z &= 28.45 \end{aligned}$$

Set this up as an augmented matrix, and verify that the result after the Gauss-Jordan method is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 8 \end{array}\right].$$

The store sold 6 small, 9 medium, and 8 large sodas.

(b) Suppose the prices for small, medium, and large sodas are changed to \$1, \$2, and \$3, respectively, but all other information is kept the same. How many of each size did the store sell?

SOLUTION Change the third equation to

$$x + 2y + 3z = 28.45$$

and go through the Gauss-Jordan method again. The result is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 25 \\ 0 & 0 & 0 & -19.55 \end{array}\right].$$

(If you do the row operations in a different order in this example, you will have different numbers in the last column.) The last row of this matrix says that $0 = -19.55$, so the system is inconsistent and has no solution. (In retrospect, this is clear, because each soda sells for a whole number of dollars, and the total amount of money is not a whole number of dollars. In general, however, it is not easy to tell whether a system of equations has a solution or not by just looking at it.)

(c) Suppose the prices are the same as in part (b), but the total revenue is \$48. Now how many of each size did the store sell?

SOLUTION The third equation becomes

$$x + 2y + 3z = 48,$$

and the Gauss-Jordan method leads to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 25 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is dependent, similar to Example 4. Let z be the parameter, and solve the first two equations for x and y, yielding

$$x = z - 2 \qquad \text{and} \qquad y = 25 - 2z.$$

Remember that in this problem, x, y, and z must be nonnegative integers. From the equation for x, we must have

$$z \geq 2,$$

and from the equation for y, we must have

$$25 - 2z \geq 0,$$

from which we find

$$z \leq 12.5.$$

Therefore, we have 11 solutions corresponding to $z = 2, 3, \ldots, 12$.

(d) Give the solutions from part (c) that have the smallest and largest numbers of large sodas.

SOLUTION For the smallest number of large sodas, let $z = 2$, giving $x = 2 - 2 = 0$ and $y = 25 - 2(2) = 21$. There are 0 small, 21 medium, and 2 large sodas.

For the largest number of large sodas, let $z = 12$, giving $x = 12 - 2 = 10$ and $y = 25 - 2(12) = 1$. There are 10 small, 1 medium, and 12 large sodas.

2.2 EXERCISES

Write the augmented matrix for each system. Do not solve.

1. $\begin{aligned} 3x + y &= 6 \\ 2x + 5y &= 15 \end{aligned}$

2. $\begin{aligned} 4x - 2y &= 8 \\ -7y &= -12 \end{aligned}$

3. $\begin{aligned} 2x + y + z &= 3 \\ 3x - 4y + 2z &= -7 \\ x + y + z &= 2 \end{aligned}$

4. $\begin{aligned} 2x - 5y + 3z &= 4 \\ -4x + 2y - 7z &= -5 \\ 3x - y &= 8 \end{aligned}$

Write the system of equations associated with each augmented matrix.

5. $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$

6. $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right]$

7. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

9. _____ on a matrix correspond to transformations of a system of equations.

10. Describe in your own words what $2R_1 + R_3 \rightarrow R_3$ means.

Use the indicated row operations to change each matrix.

11. Replace R_2 by $R_1 + (-3)R_2$.

$\left[\begin{array}{ccc|c} 3 & 7 & 4 & 10 \\ 1 & 2 & 3 & 6 \\ 0 & 4 & 5 & 11 \end{array}\right]$

12. Replace R_3 by $(-1)R_1 + 3R_3$.

$\left[\begin{array}{ccc|c} 3 & 2 & 6 & 18 \\ 2 & -2 & 5 & 7 \\ 1 & 0 & 5 & 20 \end{array}\right]$

13. Replace R_1 by $(-2)R_2 + R_1$.

$\left[\begin{array}{ccc|c} 1 & 6 & 4 & 7 \\ 0 & 3 & 2 & 5 \\ 0 & 5 & 3 & 7 \end{array}\right]$

14. Replace R_1 by $R_3 + (-3)R_1$.

$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 21 \\ 0 & 6 & 5 & 30 \\ 0 & 0 & 12 & 15 \end{array}\right]$

15. Replace R_1 by $\frac{1}{3}R_1$.

$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 18 \\ 0 & 5 & 0 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right]$

16. Replace R_3 by $\frac{1}{6}R_3$.

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 17 \\ 0 & 0 & 6 & 162 \end{array}\right]$

Use the Gauss-Jordan method to solve each system of equations.

17. $\begin{aligned} x + y &= 5 \\ 3x + 2y &= 12 \end{aligned}$

18. $\begin{aligned} x + 2y &= 5 \\ 2x + y &= -2 \end{aligned}$

19. $\begin{aligned} x + y &= 7 \\ 4x + 3y &= 22 \end{aligned}$

20. $\begin{aligned} 4x - 2y &= 3 \\ -2x + 3y &= 1 \end{aligned}$

21. $\begin{aligned} 2x - 3y &= 2 \\ 4x - 6y &= 1 \end{aligned}$

22. $\begin{aligned} 2x + 3y &= 9 \\ 4x + 6y &= 7 \end{aligned}$

23. $\begin{aligned} 6x - 3y &= 1 \\ -12x + 6y &= -2 \end{aligned}$

24. $\begin{aligned} x - y &= 1 \\ -x + y &= -1 \end{aligned}$

25. $\begin{aligned} y &= x - 3 \\ y &= 1 + z \\ z &= 4 - x \end{aligned}$

26. $\begin{aligned} x &= 1 - y \\ 2x &= z \\ 2z &= -2 - y \end{aligned}$

27. $\begin{aligned} 2x - 2y &= -5 \\ 2y + z &= 0 \\ 2x + z &= -7 \end{aligned}$

28. $\begin{aligned} x - z &= -3 \\ y + z &= 9 \\ -2x + 3y + 5z &= 33 \end{aligned}$

29. $\begin{aligned} 4x + 4y - 4z &= 24 \\ 2x - y + z &= -9 \\ x - 2y + 3z &= 1 \end{aligned}$

30. $\begin{aligned} x + 2y - 7z &= -2 \\ -2x - 5y + 2z &= 1 \\ 3x + 5y + 4z &= -9 \end{aligned}$

31. $\begin{aligned} 3x + 5y - z &= 0 \\ 4x - y + 2z &= 1 \\ 7x + 4y + z &= 1 \end{aligned}$

32. $\begin{aligned} 3x - 6y + 3z &= 11 \\ 2x + y - z &= 2 \\ 5x - 5y + 2z &= 6 \end{aligned}$

33.
$$\begin{aligned} 5x - 4y + 2z &= 6 \\ 5x + 3y - z &= 11 \\ 15x - 5y + 3z &= 23 \end{aligned}$$

34.
$$\begin{aligned} 3x + 2y - z &= -16 \\ 6x - 4y + 3z &= 12 \\ 5x - 2y + 2z &= 4 \end{aligned}$$

35.
$$\begin{aligned} 2x + 3y + z &= 9 \\ 4x + 6y + 2z &= 18 \\ -\frac{1}{2}x - \frac{3}{4}y - \frac{1}{4}z &= -\frac{9}{4} \end{aligned}$$

36.
$$\begin{aligned} 3x - 5y - 2z &= -9 \\ -4x + 3y + z &= 11 \\ 8x - 5y + 4z &= 6 \end{aligned}$$

37.
$$\begin{aligned} x + 2y - w &= 3 \\ 2x + 4z + 2w &= -6 \\ x + 2y - z &= 6 \\ 2x - y + z + w &= -3 \end{aligned}$$

38.
$$\begin{aligned} x + 3y - 2z - w &= 9 \\ 2x + 4y + 2w &= 10 \\ -3x - 5y + 2z - w &= -15 \\ x - y - 3z + 2w &= 6 \end{aligned}$$

39.
$$\begin{aligned} x + y - z + 2w &= -20 \\ 2x - y + z + w &= 11 \\ 3x - 2y + z - 2w &= 27 \end{aligned}$$

40.
$$\begin{aligned} 4x - 3y + z + w &= 21 \\ -2x - y + 2z + 7w &= 2 \\ 10x - 5z - 20w &= 15 \end{aligned}$$

41.
$$\begin{aligned} 10.47x + 3.52y + 2.58z - 6.42w &= 218.65 \\ 8.62x - 4.93y - 1.75z + 2.83w &= 157.03 \\ 4.92x + 6.83y - 2.97z + 2.65w &= 462.3 \\ 2.86x + 19.10y - 6.24z - 8.73w &= 398.4 \end{aligned}$$

42.
$$\begin{aligned} 28.6x + 94.5y + 16.0z - 2.94w &= 198.3 \\ 16.7x + 44.3y - 27.3z + 8.9w &= 254.7 \\ 12.5x - 38.7y + 92.5z + 22.4w &= 562.7 \\ 40.1x - 28.3y + 17.5z - 10.2w &= 375.4 \end{aligned}$$

43. On National Public Radio, the "Weekend Edition" program on Sunday, July 29, 2001, posed the following puzzle: Draw a three-by-three square (three boxes across by three boxes down). Put the fraction 3/8 in the first square in the first row. Put the fraction 1/4 in the last square in the second row. The object is to put a fraction in each of the remaining boxes, so the three numbers in each row, each column, and each of the long diagonals add up to 1. Solve this puzzle by letting seven variables represent the seven unknown fractions, writing eight equations for the eight sums, and solving by the Gauss-Jordan method.

APPLICATIONS

Business and Economics

44. Surveys The president of Sam's Supermarkets plans to hire two public relations firms to survey 500 customers by phone, 750 by mail, and 250 by in-person interviews. The Garcia firm has personnel to do 10 phone surveys, 30 mail surveys, and 5 interviews per hour. The Wong firm can handle 20 phone surveys, 10 mail surveys, and 10 interviews per hour. For how many hours should each firm be hired to produce the exact number of surveys needed?

45. Investments Katherine Chong invests $10,000 received from her grandmother in three ways. With one part, she buys U.S. savings bonds at an interest rate of 2.5% per year. She uses the second part, which amounts to twice the first, to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of $470. How much did she invest in each way?

46. Office Technology Pyro-Tech, Inc. is upgrading office technology by purchasing inkjet printers, LCD monitors, and additional memory chips. The total number of pieces of hardware purchased is 46. The cost of each inkjet printer is $109, the cost of each LD monitor is $129, and the cost of each memory chip is $89. The total amount of money spent on new hardware came to $4774. They purchased two times as many memory chips as they did LCD monitors. Determine the number of each that was purchased. ***Source: Nathan Borchelt.***

47. Manufacturing Fred's Furniture Factory has 1950 machine hours available each week in the cutting department, 1490 hours in the assembly department, and 2160 in the finishing department. Manufacturing a chair requires 0.2 hours of cutting, 0.3 hours of assembly, and 0.1 hours of finishing. A cabinet requires 0.5 hours of cutting, 0.4 hours of assembly, and 0.6 hours of finishing. A buffet requires 0.3 hours of cutting, 0.1 hours of assembly, and 0.4 hours of finishing. How many chairs, cabinets, and buffets should be produced in order to use all the available production capacity?

48. Manufacturing Nadir, Inc. produces three models of television sets: deluxe, super-deluxe, and ultra. Each deluxe set requires 2 hours of electronics work, 3 hours of assembly time, and 5 hours of finishing time. Each super-deluxe requires 1, 3, and 2 hours of electronics, assembly, and finishing time, respectively. Each ultra requires 2, 2, and 6 hours of the same work, respectively.

a. There are 54 hours available for electronics, 72 hours available for assembly, and 148 hours available for finishing per week. How many of each model should be produced each week if all available time is to be used?

b. Suppose everything is the same as in part a, but a super-deluxe set requires 1, rather than 2, hours of finishing time. How many solutions are there now?

c. Suppose everything is the same as in part b, but the total hours available for finishing changes from 148 hours to 144 hours. Now how many solutions are there?

49. Transportation An electronics company produces three models of stereo speakers, models A, B, and C, and can deliver them by truck, van, or SUV. A truck holds 2 boxes of model A, 2 of model B, and 3 of model C. A van holds 3 boxes of model A, 4 boxes of model B, and 2 boxes of model C. An SUV holds 3 boxes of model A, 5 boxes of model B, and 1 box of model C.

a. If 25 boxes of model A, 33 boxes of model B, and 22 boxes of model C are to be delivered, how many vehicles of each type should be used so that all operate at full capacity?

b. Model C has been discontinued. If 25 boxes of model A and 33 boxes of model B are to be delivered, how many vehicles of each type should be used so that all operate at full capacity?

50. Truck Rental The U-Drive Rent-A-Truck company plans to spend $7 million on 200 new vehicles. Each commercial van will cost $35,000, each small truck $30,000, and each large truck $50,000. Past experience shows that they need twice as many vans as small trucks. How many of each type of vehicle can they buy?

51. Loans To get the necessary funds for a planned expansion, a small company took out three loans totaling $25,000. Company owners were able to get interest rates of 8%, 9%, and 10%. They borrowed $1000 more at 9% than they borrowed at 10%. The total annual interest on the loans was $2190.

a. How much did they borrow at each rate?

b. Suppose we drop the condition that they borrowed $1000 more at 9% than at 10%. What can you say about the amount borrowed at 10%? What is the solution if the amount borrowed at 10% is $5000?

c. Suppose the bank sets a maximum of $10,000 at the lowest interest rate of 8%. Is a solution possible that still meets all of the original conditions?

d. Explain why $10,000 at 8%, $8000 at 9%, and $7000 at 10% is not a feasible solution for part c.

52. APPLY IT Transportation An auto manufacturer sends cars from two plants, I and II, to dealerships A and B located in a midwestern city. Plant I has a total of 28 cars to send, and plant II has 8. Dealer A needs 20 cars, and dealer B needs 16. Transportation costs per car, based on the distance of each dealership from each plant, are $220 from I to A, $300 from I to B, $400 from II to A, and $180 from II to B. The manufacturer wants to limit transportation costs to $10,640. How many cars should be sent from each plant to each of the two dealerships?

53. Transportation A manufacturer purchases a part for use at both of its plants — one at Roseville, California, the other at Akron, Ohio. The part is available in limited quantities from two suppliers. Each supplier has 75 units available. The Roseville plant needs 40 units, and the Akron plant requires 75 units. The first supplier charges $70 per unit delivered to Roseville and $90 per unit delivered to Akron. Corresponding costs from the second supplier are $80 and $120. The manufacturer wants to order a total of 75 units from the first, less expensive supplier, with the remaining 40 units to come from the second supplier. If the company spends $10,750 to purchase the required number of units for the two plants, find the number of units that should be sent from each supplier to each plant.

54. Packaging A company produces three combinations of mixed vegetables that sell in 1-kg packages. Italian style combines 0.3 kg of zucchini, 0.3 of broccoli, and 0.4 of carrots. French style combines 0.6 kg of broccoli and 0.4 of carrots. Oriental style combines 0.2 kg of zucchini, 0.5 of broccoli, and 0.3 of carrots. The company has a stock of 16,200 kg of zucchini, 41,400 kg of broccoli, and 29,400 kg of carrots. How many packages of each style should it prepare to use up existing supplies?

55. Tents L.L. Bean makes three sizes of Ultra Dome tents: two-person, four-person, and six-person models, which cost $129, $179, and $229, respectively. A two-person tent provides 40 ft^2 of floor space, while a four-person and a six-person model provide 64 ft^2 and 88 ft^2 of floor space, respectively. ***Source: L. L. Bean.*** A recent order by an organization that takes children camping ordered enough tents to hold 200 people and provide 3200 ft^2 of floor space. The total cost was $8950, and we wish to know how many tents of each size were ordered.

a. How many solutions are there to this problem?

b. What is the solution with the most four-person tents?

c. What is the solution with the most two-person tents?

d. Discuss the company's pricing strategy that led to a system of equations that is dependent. Do you think that this is a coincidence or an example of logical thinking?

Life Sciences

56. Animal Breeding An animal breeder can buy four types of food for Vietnamese pot-bellied pigs. Each case of Brand A contains 25 units of fiber, 30 units of protein, and 30 units of fat. Each case of Brand B contains 50 units of fiber, 30 units of protein, and 20 units of fat. Each case of Brand C contains 75 units of fiber, 30 units of protein, and 20 units of fat. Each case of Brand D contains 100 units of fiber, 60 units of protein, and 30 units of fat. How many cases of each should the breeder mix together to obtain a food that provides 1200 units of fiber, 600 units of protein, and 400 units of fat?

57. Dietetics A hospital dietician is planning a special diet for a certain patient. The total amount per meal of food groups A, B, and C must equal 400 grams. The diet should include one-third as much of group A as of group B, and the sum of the amounts of group A and group C should equal twice the amount of group B.

a. How many grams of each food group should be included?

b. Suppose we drop the requirement that the diet include one-third as much of group A as of group B. Describe the set of all possible solutions.

c. Suppose that, in addition to the conditions given in the original problem, foods A and B cost 2 cents per gram and food C costs 3 cents per gram, and that a meal must cost $8. Is a solution possible?

58. Bacterial Food Requirements Three species of bacteria are fed three foods, I, II, and III. A bacterium of the first species consumes 1.3 units each of foods I and II and 2.3 units of food III each day. A bacterium of the second species consumes 1.1 units of food I, 2.4 units of food II, and 3.7 units of food III each day. A bacterium of the third species consumes 8.1 units of I, 2.9 units of II, and 5.1 units of III each day. If 16,000 units of I, 28,000 units of II, and 44,000 units of III are supplied each day, how many of each species can be maintained in this environment?

59. Fish Food Requirements A lake is stocked each spring with three species of fish, A, B, and C. Three foods, I, II, and III, are available in the lake. Each fish of species A requires an average of 1.32 units of food I, 2.9 units of food II, and 1.75 units of food III each day. Species B fish each require 2.1 units of food I, 0.95 unit of food II, and 0.6 unit of food III daily. Species C fish require 0.86, 1.52, and 2.01 units of I, II, and III per day, respectively. If 490 units of food I, 897 units of food II, and

653 units of food III are available daily, how many of each species should be stocked?

60. Agriculture According to data from a Texas agricultural report, the amount of nitrogen (in lb/acre), phosphate (in lb/acre), and labor (in hr/acre) needed to grow honeydews, yellow onions, and lettuce is given by the following table. ***Source: The AMATYC Review.***

	Honeydews	Yellow Onions	Lettuce
Nitrogen	120	150	180
Phosphate	180	80	80
Labor	4.97	4.45	4.65

a. If the farmer has 220 acres, 29,100 lb of nitrogen, 32,600 lb of phosphate, and 480 hours of labor, is it possible to use all resources completely? If so, how many acres should he allot for each crop?

b. Suppose everything is the same as in part a, except that 1061 hours of labor are available. Is it possible to use all resources completely? If so, how many acres should he allot for each crop?

61. Archimedes' Problem Bovinum Archimedes is credited with the authorship of a famous problem involving the number of cattle of the sun god. A simplified version of the problem is stated as follows:

The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown.

Among the bulls, the number of white ones was one-half plus one-third the number of the black greater than the brown; the number of the black, one-quarter plus one-fifth the number of the spotted greater than the brown; the number of the spotted, one-sixth and one-seventh the number of the white greater than the brown.

Among the cows, the number of white ones was one-third plus one-quarter of the total black cattle; the number of the black, one-quarter plus one-fifth the total of the spotted cattle; the number of the spotted, one-fifth plus one-sixth the total of the brown cattle; the number of the brown, one-sixth plus one-seventh the total of the white cattle.

What was the composition of the herd?

Source: 100 Great Problems of Elementary Mathematics.

The problem can be solved by converting the statements into two systems of equations, using X, Y, Z, and T for the number of white, black, spotted, and brown bulls, respectively, and x, y, z, and t for the number of white, black, spotted, and brown cows, respectively. For example, the first statement can be written as $X = (1/2 + 1/3)Y + T$ and then reduced. The result is the following two systems of equations:

$$\begin{aligned} 6X - 5Y &= 6T \\ 20Y - 9Z &= 20T \\ 42Z - 13X &= 42T \end{aligned} \qquad \text{and} \qquad \begin{aligned} 12x - 7y &= 7Y \\ 20y - 9z &= 9Z \\ 30z - 11t &= 11T \\ -13x + 42t &= 13X \end{aligned}$$

a. Show that these two systems of equations represent Archimedes' Problem Bovinum.

b. If it is known that the number of brown bulls, T, is 4,149,387, use the Gauss-Jordan method to first find a solution to the 3×3 system and then use these values and the Gauss-Jordan method to find a solution to the 4×4 system of equations.

62. Health The U.S. National Center for Health Statistics tracks the major causes of death in the United States. After a steady increase, the death rate by cancer has decreased since the early 1990s. The table lists the age-adjusted death rate per 100,000 people for 4 years. ***Source: The New York Times 2010 Almanac.***

Year	Rate
1980	183.9
1990	203.3
2000	196.5
2006	187.0

a. If the relationship between the death rate R and the year t is expressed as $R = at^2 + bt + c$, where $t = 0$ corresponds to 1980, use data from 1980, 1990, and 2000 and a linear system of equations to determine the constants a, b, and c.

b. Use the equation from part a to predict the rate in 2006, and compare the result with the actual data.

c. If the relationship between the death rate R and the year t is expressed as $R = at^3 + bt^2 + ct + d$, where $t = 0$ corresponds to 1980, use all four data points and a linear system of equations to determine the constants a, b, c, and d.

d. Discuss the appropriateness of the functions used in parts a and c to model this data.

Social Sciences

63. Modeling War One of the factors that contribute to the success or failure of a particular army during war is its ability to get new troops ready for service. It is possible to analyze the rate of change in the number of troops of two hypothetical armies with the following simplified model,

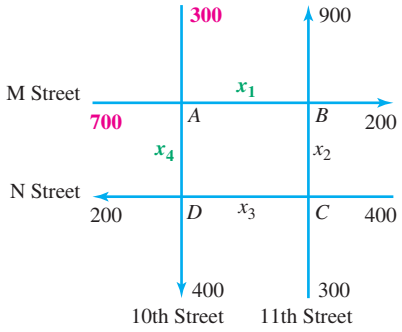
$$\text{Rate of increase (RED ARMY)} = 200{,}000 - 0.5r - 0.3b$$
$$\text{Rate of increase (BLUE ARMY)} = 350{,}000 - 0.5r - 0.7b,$$

where r is the number of soldiers in the Red Army at a given time and b is the number of soldiers in the Blue Army at a given time. The factors 0.5 and 0.7 represent each army's efficiency of bringing new soldiers to the fight. ***Source: Journal of Peace Research.***

a. Solve this system of equations to determine the number of soldiers in each army when the rate of increase for each is zero.

b. Describe what might be going on in a war when the rate of increase is zero.

64. Traffic Control At rush hours, substantial traffic congestion is encountered at the traffic intersections shown in the figure on the next page. (The streets are one-way, as shown by the arrows.)



The city wishes to improve the signals at these corners so as to speed the flow of traffic. The traffic engineers first gather data. As the figure shows, 700 cars per hour come down M Street to intersection A, and 300 cars per hour come down 10th Street to intersection A. A total of x_1 of these cars leave A on M Street, and x_4 cars leave A on 10th Street. The number of cars entering A must equal the number leaving, so that

$$x_1 + x_4 = 700 + 300$$

or

$$x_1 + x_4 = 1000.$$

For intersection B, x_1 cars enter on M Street and x_2 on 11th Street. The figure shows that 900 cars leave B on 11th and 200 on M. Thus,

$$x_1 + x_2 = 900 + 200$$
$$x_1 + x_2 = 1100.$$

a. Write two equations representing the traffic entering and leaving intersections C and D.

b. Use the four equations to set up an augmented matrix, and solve the system by the Gauss-Jordan method, using x_4 as the parameter.

c. Based on your solution to part b, what are the largest and smallest possible values for the number of cars leaving intersection A on 10th Street?

d. Answer the question in part c for the other three variables.

e. Verify that you could have discarded any one of the four original equations without changing the solution. What does this tell you about the original problem?

General Interest

65. Snack Food According to the nutrition labels on the package, a single serving of Oreos® has 10 g of fat, 36 g of carbohydrates, 2 g of protein, and 240 calories. The figures for a single serving of Twix® are 14 g, 37 g, 3 g, and 280 calories. The figures for a single serving of trail mix are 20 g, 23 g, 11 g, and 295 calories. How many calories are in each gram of fat, carbohydrate, and protein? ***Source: The Mathematics Teacher.****

*For a discussion of some complications that can come up in solving linear systems, read the original article: Szydlik, Stephen D., "The Problem with the Snack Food Problem," *The Mathematics Teacher,* Vol. 103, No. 1, Aug. 2009, p. 18–28.

66. Basketball Kobe Bryant has the second highest single game point total in the NBA. Bryant scored 81 points for the Los Angeles Lakers on January 22, 2006, against the Toronto Raptors. Bryant made a total of 46 baskets, including foul shots (worth one point), field goals (worth two points), and three-point shots (worth three points). The number of field goal shots he made is equal to three times the number of three pointers he made. Find the number of foul shots, field goals, and three pointers Bryant made. ***Source: ESPN.***

67. Toys One hundred toys are to be given out to a group of children. A ball costs \$2, a doll costs \$3, and a car costs \$4. A total of \$295 was spent on the toys.

a. A ball weighs 12 oz, a doll 16 oz, and a car 18 oz. The total weight of all the toys is 1542 oz. Find how many of each toy there are.

b. Now suppose the weight of a ball, doll, and car are 11, 15, and 19 oz, respectively. If the total weight is still 1542 oz, how many solutions are there now?

c. Keep the weights as in part b, but change the total weight to 1480 oz. How many solutions are there?

d. Give the solution to part c that has the smallest number of cars.

e. Give the solution to part c that has the largest number of cars.

68. Ice Cream Researchers have determined that the amount of sugar contained in ice cream helps to determine the overall "degree of like" that a consumer has toward that particular flavor. They have also determined that too much or too little sugar will have the same negative affect on the "degree of like" and that this relationship follows a quadratic function. In an experiment conducted at Pennsylvania State University, the following condensed table was obtained. ***Source: Journal of Food Science.***

Percentage of Sugar	Degree of Like
8	5.4
13	6.3
18	5.6

a. Use this information and the Gauss-Jordan method to determine the coefficients a, b, c, of the quadratic equation

$$y = ax^2 + bx + c,$$

where y is the "degree of like" and x is the percentage of sugar in the ice cream mix.

b. Repeat part a by using the quadratic regression feature on a graphing calculator. Compare your answers.

69. Lights Out The Tiger Electronics' game, Lights Out, consists of five rows of five lighted buttons. When a button is pushed, it changes the on/off status of it and the status of all of its vertical and horizontal neighbors. For any given situation where some of the lights are on and some are off, the goal of the game is to push buttons until all of the lights are turned off. It turns out that for any given array of lights, solving a system of equations can be used to develop a strategy for turning the lights out. The following system of equations can be used to solve the problem for a simplified version of

the game with 2 rows of 2 buttons where all of the lights are initially turned on:

$$x_{11} + x_{12} + x_{21} = 1$$
$$x_{11} + x_{12} + x_{22} = 1$$
$$x_{11} + x_{21} + x_{22} = 1$$
$$x_{12} + x_{21} + x_{22} = 1,$$

where $x_{ij} = 1$ if the light in row i, column j, is on and $x_{ij} = 0$ when it is off. The order in which the buttons are pushed does not matter, so we are only seeking which buttons should be pushed. ***Source: Mathematics Magazine.***

a. Solve this system of equations and determine a strategy to turn the lights out. (*Hint:* While doing row operations, if an odd number is found, immediately replace this value with a 1; if an even number is found, then immediately replace that number with a zero. This is called modulo 2 arithmetic, and it is necessary in problems dealing with on/off switches.)

b. Resolve the equation with the right side changed to (0, 1, 1, 0).

70. Baseball Ichiro Suzuki holds the American League record for the most hits in a single baseball season. In 2004, Suzuki had a total of 262 hits for the Seattle Mariners. He hit three fewer triples than home runs, and he hit three times as many doubles as home runs. Suzuki also hit 45 times as many singles as triples. Find the number of singles, doubles, triples, and home runs hit by Suzuki during the season. ***Source: Baseball Almanac.***

YOUR TURN ANSWERS

1. $(5, -2)$ **2.** $(7, 2, -3)$

3. $(17/9 + w/3, -4/9 - 2w/3, 4/9 + 2w/3, w)$

2.3 Addition and Subtraction of Matrices

APPLY IT **A company sends monthly shipments to its warehouses in several cities. How might the company keep track of the shipments to each warehouse most efficiently?**

In the previous section, matrices were used to store information about systems of linear equations. In this section, we begin to study calculations with matrices, which we will use in Examples 1, 5, and 7 to answer the question posed above.

The use of matrices has gained increasing importance in the fields of management, natural science, and social science because matrices provide a convenient way to organize data, as Example 1 demonstrates.

EXAMPLE 1 Furniture Shipments

The EZ Life Company manufactures sofas and armchairs in three models, A, B, and C. The company has regional warehouses in New York, Chicago, and San Francisco. In its August shipment, the company sends 10 model-A sofas, 12 model-B sofas, 5 model-C sofas, 15 model-A chairs, 20 model-B chairs, and 8 model-C chairs to each warehouse. Use a matrix to organize this information.

APPLY IT

SOLUTION To organize this data, we might tabulate the data in a chart.

Number of Furniture Pieces

		Model		
		A	B	C
Furniture Type	*Sofas*	10	12	5
	Chairs	15	20	8

With the understanding that the numbers in each row refer to the furniture type (sofa, chair) and the numbers in each column refer to the model (A, B, C), the same information can be given by a matrix, as follows.

$$M = \begin{bmatrix} 10 & 12 & 5 \\ 15 & 20 & 8 \end{bmatrix}$$

Matrices often are named with capital letters, as in Example 1. Matrices are classified by **size**; that is, by the number of rows and columns they contain. For example, matrix M above

has two rows and three columns. This matrix is a 2×3 (read "2 by 3") matrix. By definition, a matrix with m rows and n columns is an $m \times n$ matrix. The number of rows is always given first.

EXAMPLE 2 Matrix Size

(a) The matrix $\begin{bmatrix} -3 & 5 \\ 2 & 0 \\ 5 & -1 \end{bmatrix}$ is a 3×2 matrix.

(b) $\begin{bmatrix} 0.5 & 8 & 0.9 \\ 0 & 5.1 & -3 \\ -4 & 0 & 5 \end{bmatrix}$ is a 3×3 matrix.

(c) $\begin{bmatrix} 1 & 6 & 5 & -2 & 5 \end{bmatrix}$ is a 1×5 matrix.

(d) $\begin{bmatrix} 3 \\ -5 \\ 0 \\ 2 \end{bmatrix}$ is a 4×1 matrix.

A matrix with the same number of rows as columns is called a **square matrix**. The matrix in Example 2(b) is a square matrix.

A matrix containing only one row is called a **row matrix** or a **row vector**. The matrix in Example 2(c) is a row matrix, as are

$$\begin{bmatrix} 5 & 8 \end{bmatrix}, \quad \begin{bmatrix} 6 & -9 & 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -4 & 0 & 0 & 0 \end{bmatrix}.$$

A matrix of only one column, as in Example 2(d), is a **column matrix** or a **column vector**.

Equality for matrices is defined as follows.

Matrix Equality

Two matrices are equal if they are the same size and if each pair of corresponding elements is equal.

By this definition,

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$$

are not equal (even though they contain the same elements and are the same size) since the corresponding elements differ.

EXAMPLE 3 Matrix Equality

(a) From the definition of matrix equality given above, the only way that the statement

$$\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$$

can be true is if $2 = x$, $1 = y$, $p = -1$, and $q = 0$.

(b) The statement

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

can never be true, since the two matrices are different sizes. (One is 2×1 and the other is 3×1.)

Addition

The matrix given in Example 1,

$$M = \begin{bmatrix} 10 & 12 & 5 \\ 15 & 20 & 8 \end{bmatrix},$$

shows the August shipment from the EZ Life plant to each of its warehouses. If matrix N below gives the September shipment to the New York warehouse, what is the total shipment of each item of furniture to the New York warehouse for these two months?

$$N = \begin{bmatrix} 45 & 35 & 20 \\ 65 & 40 & 35 \end{bmatrix}$$

If 10 model-A sofas were shipped in August and 45 in September, then altogether $10 + 45 = 55$ model-A sofas were shipped in the two months. The other corresponding entries can be added in a similar way to get a new matrix Q, which represents the total shipment for the two months.

$$Q = \begin{bmatrix} 55 & 47 & 25 \\ 80 & 60 & 43 \end{bmatrix}$$

It is convenient to refer to Q as the sum of M and N.

The way these two matrices were added illustrates the following definition of addition of matrices.

Adding Matrices

The sum of two $m \times n$ matrices X and Y is the $m \times n$ matrix $\boldsymbol{X + Y}$ in which each element is the sum of the corresponding elements of X and Y.

CAUTION It is important to remember that only matrices that are the same size can be added.

YOUR TURN 1 Find each sum, if possible.

(a) $\begin{bmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 4 \\ -2 & -4 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 4 \\ -2 & -4 & 8 \end{bmatrix}$

EXAMPLE 4 Adding Matrices

Find each sum, if possible.

SOLUTION

(a) $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5 + (-4) & -6 + 6 \\ 8 + 8 & 9 + (-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$

(b) The matrices

$$A = \begin{bmatrix} 5 & -8 \\ 6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -9 & 1 \\ 4 & 2 & -5 \end{bmatrix}$$

are different sizes. Therefore, the sum $A + B$ does not exist. TRY YOUR TURN 1

EXAMPLE 5 Furniture Shipments

The September shipments from the EZ Life Company to the New York, San Francisco, and Chicago warehouses are given in matrices N, S, and C below.

$$N = \begin{bmatrix} 45 & 35 & 20 \\ 65 & 40 & 35 \end{bmatrix} \quad S = \begin{bmatrix} 30 & 32 & 28 \\ 43 & 47 & 30 \end{bmatrix} \quad C = \begin{bmatrix} 22 & 25 & 38 \\ 31 & 34 & 35 \end{bmatrix}$$

What was the total amount shipped to the three warehouses in September?

APPLY IT

SOLUTION The total of the September shipments is represented by the sum of the three matrices N, S, and C.

$$N + S + C = \begin{bmatrix} 45 & 35 & 20 \\ 65 & 40 & 35 \end{bmatrix} + \begin{bmatrix} 30 & 32 & 28 \\ 43 & 47 & 30 \end{bmatrix} + \begin{bmatrix} 22 & 25 & 38 \\ 31 & 34 & 35 \end{bmatrix}$$
$$= \begin{bmatrix} 97 & 92 & 86 \\ 139 & 121 & 100 \end{bmatrix}$$

For example, this sum shows that the total number of model-C sofas shipped to the three warehouses in September was 86.

Subtraction

Subtraction of matrices is defined similarly to addition.

Subtracting Matrices

The difference of two $m \times n$ matrices X and Y is the $m \times n$ matrix $X - Y$ in which each element is found by subtracting the corresponding elements of X and Y.

EXAMPLE 6 Subtracting Matrices

Subtract each pair of matrices, if possible.

SOLUTION

(a) $$\begin{bmatrix} 8 & 6 & -4 \\ -2 & 7 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \\ -4 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 8-3 & 6-5 & -4-(-8) \\ -2-(-4) & 7-2 & 5-9 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \\ 2 & 5 & -4 \end{bmatrix}.$$

(b) The matrices

$$\begin{bmatrix} -2 & 5 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

are different sizes and cannot be subtracted.

YOUR TURN 2 Calculate $\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 4 \\ -2 & -4 & 8 \end{bmatrix}$.

TRY YOUR TURN 2

EXAMPLE 7 Furniture Shipments

APPLY IT

During September the Chicago warehouse of the EZ Life Company shipped out the following numbers of each model.

$$K = \begin{bmatrix} 5 & 10 & 8 \\ 11 & 14 & 15 \end{bmatrix}$$

What was the Chicago warehouse inventory on October 1, taking into account only the number of items received and sent out during the month?

Method 1 Calculating by Hand

SOLUTION The number of each kind of item received during September is given by matrix C from Example 5; the number of each model sent out during September is given by matrix K. The October 1 inventory will be represented by the matrix $C - K$:

$$\begin{bmatrix} 22 & 25 & 38 \\ 31 & 34 & 35 \end{bmatrix} - \begin{bmatrix} 5 & 10 & 8 \\ 11 & 14 & 15 \end{bmatrix} = \begin{bmatrix} 17 & 15 & 30 \\ 20 & 20 & 20 \end{bmatrix}.$$

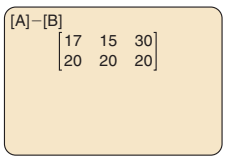


FIGURE 12

Method 2 Graphing Calculator

Matrix operations are easily performed on a graphing calculator. Figure 12 shows the previous operation; the matrices A and B were already entered into the calculator.

TECHNOLOGY NOTE

Spreadsheet programs are designed to effectively organize data that can be represented in rows and columns. Accordingly, matrix operations are also easily performed on spreadsheets. See the *Graphing Calculator and Excel Spreadsheet Manual* available with this book for details.

2.3 EXERCISES

Decide whether each statement is true or false. If false, tell why.

1. $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 \quad 2 \quad 3]$

3. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ if $x = -2$ and $y = 8$.

4. $\begin{bmatrix} 3 & 5 & 2 & 8 \\ 1 & -1 & 4 & 0 \end{bmatrix}$ is a 4×2 matrix.

5. $\begin{bmatrix} 1 & 9 & -4 \\ 3 & 7 & 2 \\ -1 & 1 & 0 \end{bmatrix}$ is a square matrix.

6. $\begin{bmatrix} 2 & 4 & -1 \\ 3 & 7 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 7 & 5 \end{bmatrix}$

Find the size of each matrix. Identify any square, column, or row matrices.

7. $\begin{bmatrix} -4 & 8 \\ 2 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -3 & 7 \\ 1 & 0 & 4 \end{bmatrix}$

9. $\begin{bmatrix} -6 & 8 & 0 & 0 \\ 4 & 1 & 9 & 2 \\ 3 & -5 & 7 & 1 \end{bmatrix}$

10. $[8 \quad -2 \quad 4 \quad 6 \quad 3]$

11. $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$

12. $[-9]$

13. The sum of an $n \times m$ matrix and an $m \times n$ matrix, where $m \neq n$, is _____.

14. If A is a 5×2 matrix and $A + K = A$, what do you know about K?

Find the values of the variables in each equation.

15. $\begin{bmatrix} 3 & 4 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} 3 & x \\ y & z \end{bmatrix}$

16. $\begin{bmatrix} -5 \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

17. $\begin{bmatrix} s-4 & t+2 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -5 & r \end{bmatrix}$

18. $\begin{bmatrix} 9 & 7 \\ r & 0 \end{bmatrix} = \begin{bmatrix} m-3 & n+5 \\ 8 & 0 \end{bmatrix}$

19. $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

20. $\begin{bmatrix} a+2 & 3z+1 & 5m \\ 4k & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2z & 5m \\ 2k & 5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$

Perform the indicated operations, where possible.

21. $\begin{bmatrix} 2 & 4 & 5 & -7 \\ 6 & -3 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -10 & 1 \\ -2 & 8 & -9 & 11 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 5 \\ 2 & -3 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 8 & 5 \\ -1 & 9 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 6 & 4 \\ -5 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 8 & 0 & -3 \\ 1 & 19 & -5 \end{bmatrix} - \begin{bmatrix} 1 & -5 & 2 \\ 3 & 9 & -8 \end{bmatrix}$

25. $\begin{bmatrix} 2 & 8 & 12 & 0 \\ 7 & 4 & -1 & 5 \\ 1 & 2 & 0 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 6 & 9 \\ 2 & -3 & -3 & 4 \\ 8 & 0 & -2 & 17 \end{bmatrix}$

26. $\begin{bmatrix} 2 & 1 \\ 5 & -3 \\ -7 & 2 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -8 & 0 \\ 5 & 3 & 2 \\ -6 & 7 & -5 \\ 2 & -1 & 0 \end{bmatrix}$

27. $\begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

28. $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$

29. $\begin{bmatrix} 2 & -1 \\ 0 & 13 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ -5 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 7 \\ 5 & 3 \end{bmatrix}$

30. $\begin{bmatrix} 5 & 8 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} -5 & -8 \\ 6 & 1 \end{bmatrix}$

31. $\begin{bmatrix} -4x+2y & -3x+y \\ 6x-3y & 2x-5y \end{bmatrix} + \begin{bmatrix} -8x+6y & 2x \\ 3y-5x & 6x+4y \end{bmatrix}$

32. $\begin{bmatrix} 4k-8y \\ 6z-3x \\ 2k+5a \\ -4m+2n \end{bmatrix} - \begin{bmatrix} 5k+6y \\ 2z+5x \\ 4k+6a \\ 4m-2n \end{bmatrix}$

33. For matrices $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, find the matrix $0 - X$.

Using matrices $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$, $T = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$, and $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, verify the statements in Exercises 34–37.

34. $X + T = T + X$ (commutative property of addition of matrices)

35. $X + (T + P) = (X + T) + P$ (associative property of addition of matrices)

36. $X - X = O$ (inverse property of addition of matrices)

37. $P + O = P$ (identity property of addition of matrices)

38. Which of the above properties are valid for matrices that are not square?

APPLICATIONS

Business and Economics

39. Management A toy company has plants in Boston, Chicago, and Seattle that manufacture toy phones and calculators. The following matrix gives the production costs (in dollars) for each item at the Boston plant:

	Phones	Calculators
Material	4.27	6.94
Labor	3.45	3.65

a. In Chicago, a phone costs \$4.05 for material and \$3.27 for labor; a calculator costs \$7.01 for material and \$3.51 for labor. In Seattle, material costs are \$4.40 for a phone and \$6.90 for a calculator; labor costs are \$3.54 for a phone and \$3.76 for a calculator. Write the production cost matrices for Chicago and Seattle.

b. Suppose labor costs increase by \$0.11 per item in Chicago and material costs there increase by \$0.37 for a phone and \$0.42 for a calculator. What is the new production cost matrix for Chicago?

40. Management There are three convenience stores in Folsom. This week, store I sold 88 loaves of bread, 48 qt of milk, 16 jars of peanut butter, and 112 lb of cold cuts. Store II sold 105 loaves of bread, 72 qt of milk, 21 jars of peanut butter, and 147 lb of cold cuts. Store III sold 60 loaves of bread, 40 qt of milk, no peanut butter, and 50 lb of cold cuts.

a. Use a 4×3 matrix to express the sales information for the three stores.

b. During the following week, sales on these products at store I increased by 25%; sales at store II increased by 1/3; and sales at store III increased by 10%. Write the sales matrix for that week.

c. Write a matrix that represents total sales over the two-week period.

Life Sciences

41. Dietetics A dietician prepares a diet specifying the amounts a patient should eat of four basic food groups: group I, meats; group II, fruits and vegetables; group III, breads and starches; group IV, milk products. Amounts are given in "exchanges" that represent 1 oz (meat), 1/2 cup (fruits and vegetables), 1 slice (bread), 8 oz (milk), or other suitable measurements.

a. The number of "exchanges" for breakfast for each of the four food groups, respectively, are 2, 1, 2, and 1; for lunch, 3, 2, 2, and 1; and for dinner, 4, 3, 2, and 1. Write a 3×4 matrix using this information.

b. The amounts of fat, carbohydrates, and protein (in appropriate units) in each food group, respectively, are as follows.

Fat: 5, 0, 0, 10

Carbohydrates: 0, 10, 15, 12

Protein: 7, 1, 2, 8

Use this information to write a 4×3 matrix.

c. There are 8 calories per exchange of fat, 4 calories per exchange of carbohydrates, and 5 calories per exchange of protein. Summarize this data in a 3×1 matrix.

42. Animal Growth At the beginning of a laboratory experiment, five baby rats measured 5.6, 6.4, 6.9, 7.6, and 6.1 cm in length, and weighed 144, 138, 149, 152, and 146 g, respectively.

a. Write a 2×5 matrix using this information.

b. At the end of two weeks, their lengths (in centimeters) were 10.2, 11.4, 11.4, 12.7, and 10.8 and their weights (in grams) were 196, 196, 225, 250, and 230. Write a 2×5 matrix with this information.

c. Use matrix subtraction and the matrices found in parts a and b to write a matrix that gives the amount of change in length and weight for each rat.

d. During the third week, the rats grew by the amounts shown in the matrix below.

$$\begin{matrix}\text{Length}\\ \text{Weight}\end{matrix}\begin{bmatrix}1.8 & 1.5 & 2.3 & 1.8 & 2.0\\ 25 & 22 & 29 & 33 & 20\end{bmatrix}$$

What were their lengths and weights at the end of this week?

43. Testing Medication A drug company is testing 200 patients to see if Painfree (a new headache medicine) is effective. Half the patients receive Painfree and half receive a placebo. The data on the first 50 patients is summarized in this matrix:

$$\begin{matrix}\\ \text{Painfree}\\ \text{Placebo}\end{matrix}\begin{matrix}\text{Pain Relief Obtained}\\ \begin{matrix}\text{Yes} & \text{No}\end{matrix}\end{matrix}\begin{bmatrix}22 & 3\\ 8 & 17\end{bmatrix}.$$

a. Of those who took the placebo, how many got relief?

b. Of those who took the new medication, how many got no relief?

c. The test was repeated on three more groups of 50 patients each, with the results summarized by these matrices.

$$\begin{bmatrix}21 & 4\\ 6 & 19\end{bmatrix} \quad \begin{bmatrix}19 & 6\\ 10 & 15\end{bmatrix} \quad \begin{bmatrix}23 & 2\\ 3 & 22\end{bmatrix}$$

Find the total results for all 200 patients.

d. On the basis of these results, does it appear that Painfree is effective?

44. Motorcycle Helmets The following table shows the percentage of motorcyclists in various regions of the country who used helmets compliant with federal safety regulations and the percentage who used helmets that were noncompliant in two recent years. ***Source: NHTSA.***

2008	Compliant	Noncompliant
Northeast	45	8
Midwest	67	16
South	61	14
West	71	5

2009	Compliant	Noncompliant
Northeast	61	15
Midwest	67	8
South	65	6
West	83	4

a. Write two matrices for the 2008 and 2009 helmet usage.

b. Use the two matrices from part a to write a matrix showing the change in helmet usage from 2008 to 2009.

c. Analyze the results from part b and discuss the extent to which changes from 2008 to 2009 differ from one region to another.

45. Life Expectancy The following table gives the life expectancy of African American males and females and white American males and females at the beginning of each decade since 1970. *Source: The New York Times 2010 Almanac.*

	African American		White American	
Year	Male	Female	Male	Female
1970	60.0	68.3	68.0	75.6
1980	63.8	72.5	70.7	78.1
1990	64.5	73.6	72.7	79.4
2000	68.3	75.2	74.9	80.1

a. Write a matrix for the life expectancy of African Americans.

b. Write a matrix for the life expectancy of white Americans.

c. Use the matrices from parts a and b to write a matrix showing the difference between the two groups.

d. Analyze the results from part c and discuss any noticeable trends.

46. Educational Attainment The table below gives the educational attainment of the U.S. population 25 years and older since 1970. *Source: U. S. Census Bureau.*

	Male		Female	
Year	Percentage with 4 Years of High School or More	Percentage with 4 Years of College or More	Percentage with 4 Years of High School or More	Percentage with 4 Years of College or More
1970	55.0	14.1	55.4	8.2
1980	69.2	20.9	68.1	13.6
1990	77.7	24.4	77.5	18.4
2000	84.2	27.8	84.0	23.6
2008	85.9	30.1	87.2	28.8

a. Write a matrix for the educational attainment of males.

b. Write a matrix for the educational attainment of females.

c. Use the matrices from parts a and b to write a matrix showing the difference in educational attainment between males and females since 1970.

47. Educational Attainment The following table gives the educational attainment of African Americans and Hispanic Americans 25 years and older since 1980. *Source: U. S. Census Bureau.*

	African American		Hispanic American	
Year	Percentage with 4 Years of High School or More	Percentage with 4 Years of College or More	Percentage with 4 Years of High School or More	Percentage with 4 Years of College or More
1980	51.2	7.9	45.3	7.9
1985	59.8	11.1	47.9	8.5
1990	66.2	11.3	50.8	9.2
1995	73.8	13.2	53.4	9.3
2000	78.5	16.5	57.0	10.6
2008	83.0	19.6	62.3	13.3

a. Write a matrix for the educational attainment of African Americans.

b. Write a matrix for the educational attainment of Hispanic Americans.

c. Use the matrices from parts a and b to write a matrix showing the difference in educational attainment between African and Hispanic Americans.

General Interest

48. Animal Interactions When two kittens named Cauchy and Cliché were introduced into a household with Jamie (an older cat) and Musk (a dog), the interactions among animals were complicated. The two kittens liked each other and Jamie, but didn't like Musk. Musk liked everybody, but Jamie didn't like any of the other animals.

a. Write a 4×4 matrix in which rows (and columns) 1, 2, 3, and 4 refer to Musk, Jamie, Cauchy, and Cliché. Make an element a 1 if the animal for that row likes the animal for that column, and otherwise make the element a 0. Assume every animal likes herself.

b. Within a few days, Cauchy and Cliché decided that they liked Musk after all. Write a 4×4 matrix, as you did in part a, representing the new situation.

YOUR TURN ANSWERS

1. (a) Does not exist **(b)** $\begin{bmatrix} 4 & 2 & 9 \\ -1 & -2 & 11 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 6 & 1 \\ 3 & 6 & -5 \end{bmatrix}$

2.4 Multiplication of Matrices

APPLY IT **What is a contractor's total cost for materials required for various types of model homes?**

Matrix multiplication will be used to answer this question in Example 6.

We begin by defining the product of a real number and a matrix. In work with matrices, a real number is called a **scalar**.

Product of a Matrix and a Scalar

The product of a scalar k and a matrix X is the matrix kX, each of whose elements is k times the corresponding element of X.

EXAMPLE 1 **Scalar Product of a Matrix**

Calculate $-5A$, where $A = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix}$.

SOLUTION

$$(-5)\begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -15 & -20 \\ 0 & 5 \end{bmatrix}.$$

Finding the product of two matrices is more involved, but such multiplication is important in solving practical problems. To understand the reasoning behind matrix multiplication, it may be helpful to consider another example concerning EZ Life Company discussed in the previous section. Suppose sofas and chairs of the same model are often sold as sets. Matrix W shows the number of sets of each model in each warehouse.

$$\begin{array}{l} \\ \text{New York} \\ \text{Chicago} \\ \text{San Francisco} \end{array} \begin{array}{c} \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{bmatrix} 10 & 7 & 3 \\ 5 & 9 & 6 \\ 4 & 8 & 2 \end{bmatrix} \end{array} = W$$

If the selling price of a model-A set is \$1000, of a model-B set \$1200, and of a model-C set \$1400, the total value of the sets in the New York warehouse is found as follows.

	Value of Furniture				
Type	Number of Sets		Price of Set		Total
A	10	×	\$1000	=	\$10,000
B	7	×	\$1200	=	\$8400
C	3	×	\$1400	=	\$4200
	(Total for New York)				\$22,600

The total value of the three kinds of sets in New York is \$22,600.

The work done in the table above is summarized as follows:

$$10(\$1000) + 7(\$1200) + 3(\$1400) = \$22{,}600.$$

In the same way, we find that the Chicago sets have a total value of

$$5(\$1000) + 9(\$1200) + 6(\$1400) = \$24{,}200,$$

and in San Francisco, the total value of the sets is

$$4(\$1000) + 8(\$1200) + 2(\$1400) = \$16{,}400.$$

The selling prices can be written as a column matrix P, and the total value in each location as another column matrix, V.

$$\begin{bmatrix} 1000 \\ 1200 \\ 1400 \end{bmatrix} = P \qquad \begin{bmatrix} 22{,}600 \\ 24{,}200 \\ 16{,}400 \end{bmatrix} = V$$

Look at the elements of W and P below; multiplying the first, second, and third elements of the first row of W by the first, second, and third elements, respectively, of the column matrix P and then adding these products gives the first element in V. Doing the same thing with the second row of W gives the second element of V; the third row of W leads to the third element of V, suggesting that it is reasonable to write the product of matrices

$$W = \begin{bmatrix} 10 & 7 & 3 \\ 5 & 9 & 6 \\ 4 & 8 & 2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1000 \\ 1200 \\ 1400 \end{bmatrix}$$

as

$$WP = \begin{bmatrix} 10 & 7 & 3 \\ 5 & 9 & 6 \\ 4 & 8 & 2 \end{bmatrix} \begin{bmatrix} 1000 \\ 1200 \\ 1400 \end{bmatrix} = \begin{bmatrix} 22{,}600 \\ 24{,}200 \\ 16{,}400 \end{bmatrix} = V.$$

The product was found by multiplying the elements of *rows* of the matrix on the left and the corresponding elements of the *column* of the matrix on the right, and then finding the sum of these separate products. Notice that the product of a 3×3 matrix and a 3×1 matrix is a 3×1 matrix. Notice also that each element of the product of two matrices is a sum of products. This is exactly what you do when you go to the store and buy 8 candy bars at \$0.80 each, 4 bottles of water at \$1.25 each, and so forth, and calculate the total as $8 \times 0.80 + 4 \times 1.25 + \cdots$.

The product AB of an $m \times n$ matrix A and an $n \times k$ matrix B is found as follows. Multiply each element of the first row of A by the corresponding element of the *first column* of B. The sum of these n products is the *first-row*, *first-column* element of AB. Similarly, the sum of the products found by multiplying the elements of the *first row* of A by the corresponding elements of the *second column* of B gives the *first-row*, *second-column* element of AB, and so on.

Product of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the ith row and jth column of the **product matrix** AB, multiply each element in the ith row of A by the corresponding element in the jth column of B, and then add these products. The product matrix AB is an $m \times k$ matrix.

EXAMPLE 2 Matrix Product

Find the product AB of matrices

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}.$$

SOLUTION Since A is $2 \times \mathbf{3}$ and B is $\mathbf{3} \times 1$, we can find the product matrix AB.

Step 1 Multiply the elements of the first row of A and the corresponding elements of the column of B.

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{8} \\ \mathbf{6} \end{bmatrix} \quad \mathbf{2} \cdot \mathbf{1} + \mathbf{3} \cdot \mathbf{8} + \mathbf{(-1)} \cdot \mathbf{6} = 20$$

Thus, 20 is the first-row entry of the product matrix AB.

Step 2 Multiply the elements of the second row of A and the corresponding elements of B.

$$\begin{bmatrix} 2 & 3 & -1 \\ \mathbf{4} & \mathbf{2} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{8} \\ \mathbf{6} \end{bmatrix} \quad \mathbf{4} \cdot \mathbf{1} + \mathbf{2} \cdot \mathbf{8} + \mathbf{2} \cdot \mathbf{6} = 32$$

The second-row entry of the product matrix AB is 32.

Step 3 Write the product as a column matrix using the two entries found above.

$$AB = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 20 \\ 32 \end{bmatrix}$$

Note that the product of a 2×3 matrix and a 3×1 matrix is a 2×1 matrix.

EXAMPLE 3 Matrix Product

Find the product CD of matrices

$$C = \begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}.$$

SOLUTION Since C is $2 \times \mathbf{3}$ and D is $\mathbf{3} \times 2$, we can find the product matrix CD.

Step 1
$$\begin{bmatrix} \mathbf{-3} & \mathbf{4} & \mathbf{2} \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} \mathbf{-6} & 4 \\ \mathbf{2} & 3 \\ \mathbf{3} & -2 \end{bmatrix} \quad \mathbf{(-3)} \cdot \mathbf{(-6)} + \mathbf{4} \cdot \mathbf{2} + \mathbf{2} \cdot \mathbf{3} = 32$$

Step 2
$$\begin{bmatrix} \mathbf{-3} & \mathbf{4} & \mathbf{2} \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & \mathbf{4} \\ 2 & \mathbf{3} \\ 3 & \mathbf{-2} \end{bmatrix} \quad \mathbf{(-3)} \cdot \mathbf{4} + \mathbf{4} \cdot \mathbf{3} + \mathbf{2} \cdot \mathbf{(-2)} = -4$$

Step 3
$$\begin{bmatrix} -3 & 4 & 2 \\ \mathbf{5} & \mathbf{0} & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{-6} & 4 \\ \mathbf{2} & 3 \\ \mathbf{3} & -2 \end{bmatrix} \quad \mathbf{5} \cdot \mathbf{(-6)} + \mathbf{0} \cdot \mathbf{2} + \mathbf{4} \cdot \mathbf{3} = -18$$

Step 4
$$\begin{bmatrix} -3 & 4 & 2 \\ \mathbf{5} & \mathbf{0} & \mathbf{4} \end{bmatrix} \begin{bmatrix} -6 & \mathbf{4} \\ 2 & \mathbf{3} \\ 3 & \mathbf{-2} \end{bmatrix} \quad \mathbf{5} \cdot \mathbf{4} + \mathbf{0} \cdot \mathbf{3} + \mathbf{4} \cdot \mathbf{(-2)} = 12$$

Step 5 The product is

$$CD = \begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}.$$

Here the product of a 2×3 matrix and a 3×2 matrix is a 2×2 matrix.

TRY YOUR TURN 1

YOUR TURN 1 Calculate the product AB where $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -2 & -4 \end{bmatrix}$.

NOTE One way to avoid errors in matrix multiplication is to lower the first matrix so it is below and to the left of the second matrix, and then write the product in the space between the two matrices. For example, to multiply the matrices in Example 2, we could rewrite the product as shown on the following page.

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} \quad \begin{matrix} \rightarrow \\ \\ \\ \end{matrix} \quad \begin{matrix} \\ \downarrow \\ \\ \end{matrix}$$

$$\begin{bmatrix} & \\ * & \\ & \end{bmatrix}$$

To find the entry where the * is, for example, multiply the row and the column indicated by the arrows: $5 \cdot (-6) + 0 \cdot 2 + 4 \cdot 3 = -18$.

As the definition of matrix multiplication shows,

the product AB of two matrices A and B can be found only if the number of columns of A is the same as the number of rows of B.

The final product will have as many rows as A and as many columns as B.

EXAMPLE 4 Matrix Product

Suppose matrix A is 2×2 and matrix B is 2×4. Can the products AB and BA be calculated? If so, what is the size of each product?

SOLUTION The following diagram helps decide the answers to these questions.

Matrix A size 2×2 — must match — Matrix B size 2×4

Size of AB is 2×4

The product of A and B can be found because A has two columns and B has two rows. The size of the product is 2×4.

Matrix B size 2×4 — do not match — Matrix A size 2×2

The product BA cannot be found because B has 4 columns and A has 2 rows.

EXAMPLE 5 Comparing Matrix Products AB and BA

Find AB and BA, given

$$A = \begin{bmatrix} 1 & -3 \\ 7 & 2 \\ -2 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix}.$$

SOLUTION

Method 1 Calculating by Hand

$$AB = \begin{bmatrix} 1 & -3 \\ 7 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -3 & -13 \\ 13 & 2 & 1 \\ 13 & 5 & 22 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 7 & 2 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -8 \\ 2 & 13 \end{bmatrix}$$

YOUR TURN 2 Calculate each product AB and BA, if possible, where $A = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -5 & -3 \end{bmatrix}$.

Method 2 Graphing Calculator

Matrix multiplication is easily performed on a graphing calculator. Figure 13 in the margin below shows the results. The matrices A and B were already entered into the calculator.

TRY YOUR TURN 2

TECHNOLOGY NOTE Matrix multiplication can also be easily done with a spreadsheet. See the *Graphing Calculator and Excel Spreadsheet Manual* available with this textbook for details.

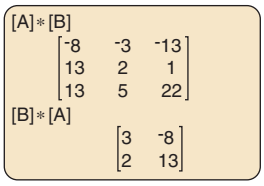


FIGURE 13

Notice in Example 5 that $AB \neq BA$; matrices AB and BA aren't even the same size. In Example 4, we showed that they may not both exist. This means that matrix multiplication is *not* commutative. Even if both A and B are square matrices, in general, matrices AB and BA are not equal. (See Exercise 31.) Of course, there may be special cases in which they are equal, but this is not true in general.

CAUTION Since matrix multiplication is not commutative, always be careful to multiply matrices in the correct order.

Matrix multiplication *is* associative, however. For example, if

$$C = \begin{bmatrix} 3 & 2 \\ 0 & -4 \\ -1 & 1 \end{bmatrix},$$

then $(AB)C = A(BC)$, where A and B are the matrices given in Example 5. (Verify this.) Also, there is a distributive property of matrices such that, for appropriate matrices A, B, and C,

$$A(B + C) = AB + AC.$$

(See Exercises 32 and 33.) Other properties of matrix multiplication involving scalars are included in the exercises. Multiplicative inverses and multiplicative identities are defined in the next section.

EXAMPLE 6 Home Construction

A contractor builds three kinds of houses, models A, B, and C, with a choice of two styles, Spanish and contemporary. Matrix P shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix Q. (Concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix R gives the cost in dollars for each kind of material.

$$\begin{array}{c} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array} \begin{array}{c} \begin{array}{cc} \text{Spanish} & \text{Contemporary} \end{array} \\ \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \end{array} = P$$

$$\begin{array}{c} \\ \text{Spanish} \\ \text{Contemporary} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Concrete} & \text{Lumber} & \text{Brick} & \text{Shingles} \end{array} \\ \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \end{array} = Q$$

$$\begin{array}{c} \\ \text{Concrete} \\ \text{Lumber} \\ \text{Brick} \\ \text{Shingles} \end{array} \begin{array}{c} \begin{array}{c} \text{Cost per Unit} \end{array} \\ \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} \end{array} = R$$

(a) What is the total cost of these materials for each model?

APPLY IT

SOLUTION To find the cost for each model, first find PQ, which shows the amount of each material needed for each model.

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cccc} \text{Concrete} & \text{Lumber} & \text{Brick} & \text{Shingles} \end{array} \\ \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

Now multiply PQ and R, the cost matrix, to get the total cost of the exterior materials for each model.

$$\begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{array}{c} \text{Cost} \\ \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix} \end{array} \begin{array}{l} \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

The total cost of materials is \$72,900 for model A, \$54,700 for model B, and \$60,800 for model C.

(b) How much of each of the four kinds of material must be ordered?

SOLUTION The totals of the columns of matrix PQ will give a matrix whose elements represent the total amounts of each material needed for the subdivision. Call this matrix T, and write it as a row matrix.

$$T = [3800 \quad 130 \quad 1400 \quad 200]$$

Thus, 3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

(c) What is the total cost for exterior materials?

SOLUTION For the total cost of all the exterior materials, find the product of matrix T, the matrix showing the total amount of each material, and matrix R, the cost matrix. (To multiply these and get a 1×1 matrix representing total cost, we need a 1×4 matrix multiplied by a 4×1 matrix. This is why T was written as a row matrix in (b) above.)

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{bmatrix} 188{,}400 \end{bmatrix}$$

The total cost for exterior materials is $188,400.

(d) Suppose the contractor builds the same number of homes in five subdivisions. Calculate the total amount of each exterior material for each model for all five subdivisions.

SOLUTION Multiply PQ by the scalar 5, as follows.

$$5(PQ) = 5\begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} = \begin{bmatrix} 7500 & 150 & 3000 & 300 \\ 5500 & 200 & 2000 & 300 \\ 6000 & 300 & 2000 & 400 \end{bmatrix}$$

The total amount of concrete needed for model A homes, for example, is 7500 yd^3.

Meaning of a Matrix Product

It is helpful to use a notation that keeps track of the quantities a matrix represents. We will use the notation

meaning of the rows/meaning of the columns,

that is, writing the meaning of the rows first, followed by the meaning of the columns. In Example 6, we would use the notation models/styles for matrix P, styles/materials for matrix Q, and materials/cost for matrix R. In multiplying PQ, we are multiplying models/styles by styles/materials. The result is models/materials. Notice that styles, the common quantity in both P and Q, was eliminated in the product PQ. By this method, the product $(PQ)R$ represents models/cost.

In practical problems this notation helps us decide in which order to multiply matrices so that the results are meaningful. In Example 6(c) either RT or TR can be calculated. Since T represents subdivisions/materials and R represents materials/cost, the product TR gives subdivisions/cost, while the product RT is meaningless.

2.4 EXERCISES

Let $A = \begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$. **Find each value.**

1. $2A$ **2.** $-3B$ **3.** $-6A$

4. $5B$ **5.** $-4A + 5B$ **6.** $7B - 3A$

In Exercises 7–12, the sizes of two matrices A and B are given. Find the sizes of the product AB and the product BA, whenever these products exist.

7. A is 2×2, and B is 2×2. **8.** A is 3×3, and B is 3×3.

9. A is 3×4, and B is 4×4. **10.** A is 4×3, and B is 3×6.

11. A is 4×2, and B is 3×4. **12.** A is 3×2, and B is 1×3.

13. To find the product matrix AB, the number of _____ of A must be the same as the number of _____ of B.

14. The product matrix AB has the same number of _____ as A and the same number of _____ as B.

Find each matrix product, if possible.

15. $\begin{bmatrix} 2 & -1 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ **16.** $\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

17. $\begin{bmatrix} 2 & -1 & 7 \\ -3 & 0 & -4 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix}$ **18.** $\begin{bmatrix} 5 & 2 \\ 7 & 6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -1 & 0 & 4 \\ 5 & -2 & 0 \end{bmatrix}$ **20.** $\begin{bmatrix} 6 & 0 & -4 \\ 1 & 2 & 5 \\ 10 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}$ **22.** $\begin{bmatrix} -3 & 1 & 0 \\ 6 & 0 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 8 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

25. $\begin{bmatrix} -2 & -3 & 7 \\ 1 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

26. $\begin{bmatrix} 2 \\ -9 \\ 12 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$

27. $\left(\begin{bmatrix} 2 & 1 \\ -3 & -6 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

28. $\begin{bmatrix} 2 & 1 \\ -3 & -6 \\ 4 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$

29. $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ -1 & 5 \end{bmatrix}\right)$

30. $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ -1 & 5 \end{bmatrix}$

31. Let $A = \begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 6 \end{bmatrix}$.

a. Find AB.

b. Find BA.

c. Did you get the same answer in parts a and b?

d. In general, for matrices A and B such that AB and BA both exist, does AB always equal BA?

Given matrices $P = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$, $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, and $T = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$, verify that the statements in Exercises 32–35 are true. The statements are valid for any matrices whenever matrix multiplication and addition can be carried out. This, of course, depends on the size of the matrices.

32. $(PX)T = P(XT)$ (associative property: see Exercises 27 and 28)

33. $P(X + T) = PX + PT$ (distributive property: see Exercises 29 and 30)

34. $k(X + T) = kX + kT$ for any real number k.

35. $(k + h)P = kP + hP$ for any real numbers k and h.

36. Let I be the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and let matrices P, X, and T be defined as for Exercises 32–35.

a. Find IP, PI, and IX.

b. Without calculating, guess what the matrix IT might be.

c. Suggest a reason for naming a matrix such as I an *identity matrix.*

37. Show that the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 - 4x_2 + 5x_3 &= 8 \end{aligned}$$

can be written as the matrix equation

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

38. Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $B = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$. Show that the equation $AX = B$ represents a linear system of two equations in two unknowns. Solve the system and substitute into the matrix equation to check your results.

Use a computer or graphing calculator and the following matrices to find the matrix products and sums in Exercises 39–41.

$$A = \begin{bmatrix} 2 & 3 & -1 & 5 & 10 \\ 2 & 8 & 7 & 4 & 3 \\ -1 & -4 & -12 & 6 & 8 \\ 2 & 5 & 7 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 3 & 7 & -6 \\ -1 & 0 & 4 & 2 \\ -10 & -7 & 6 & 9 \\ 8 & 4 & 2 & -1 \\ 2 & -5 & 3 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} -6 & 8 & 2 & 4 & -3 \\ 1 & 9 & 7 & -12 & 5 \\ 15 & 2 & -8 & 10 & 11 \\ 4 & 7 & 9 & 6 & -2 \\ 1 & 3 & 8 & 23 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & -3 & 7 & 9 & 2 \\ 6 & 8 & -5 & 2 & 1 \\ 3 & 7 & -4 & 2 & 11 \\ 5 & -3 & 9 & 4 & -1 \\ 0 & 3 & 2 & 5 & 1 \end{bmatrix}$$

39. **a.** Find AC. **b.** Find CA. **c.** Does $AC = CA$?

40. **a.** Find CD. **b.** Find DC. **c.** Does $CD = DC$?

41. **a.** Find $C + D$. **b.** Find $(C + D)B$. **c.** Find CB.

d. Find DB. **e.** Find $CB + DB$.

f. Does $(C + D)B = CB + DB$?

42. Which property of matrices does Exercise 41 illustrate?

APPLICATIONS

Business and Economics

43. **Cost Analysis** The four departments of Spangler Enterprises need to order the following amounts of the same products.

	Paper	Tape	Binders	Memo Pads	Pens
Department 1	10	4	3	5	6
Department 2	7	2	2	3	8
Department 3	4	5	1	0	10
Department 4	0	3	4	5	5

The unit price (in dollars) of each product is given below for two suppliers.

	Supplier A	Supplier B
Paper	2	3
Tape	1	1
Binders	4	3
Memo Pads	3	3
Pens	1	2

a. Use matrix multiplication to get a matrix showing the comparative costs for each department for the products from the two suppliers.

b. Find the total cost over all departments to buy products from each supplier. From which supplier should the company make the purchase?

44. Cost Analysis The Mundo Candy Company makes three types of chocolate candy: Cheery Cherry, Mucho Mocha, and Almond Delight. The company produces its products in San Diego, Mexico City, and Managua using two main ingredients: chocolate and sugar.

a. Each kilogram of Cheery Cherry requires 0.5 kg of sugar and 0.2 kg of chocolate; each kilogram of Mucho Mocha requires 0.4 kg of sugar and 0.3 kg of chocolate; and each kilogram of Almond Delight requires 0.3 kg of sugar and 0.3 kg of chocolate. Put this information into a 2×3 matrix called A, labeling the rows and columns.

b. The cost of 1 kg of sugar is \$4 in San Diego, \$2 in Mexico City, and \$1 in Managua. The cost of 1 kg of chocolate is \$3 in San Diego, \$5 in Mexico City, and \$7 in Managua. Put this information into a matrix called C in such a way that when you multiply it with your matrix from part a, you get a matrix representing the ingredient cost of producing each type of candy in each city.

c. Only one of the two products AC and CA is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

d. From your answer to part c, what is the combined sugar-and-chocolate cost to produce 1 kg of Mucho Mocha in Managua?

e. Mundo Candy needs to quickly produce a special shipment of 100 kg of Cheery Cherry, 200 kg of Mucho Mocha, and 500 kg of Almond Delight, and it decides to select one factory to fill the entire order. Use matrix multiplication to determine in which city the total sugar-and-chocolate cost to produce the order is the smallest.

45. Management In Exercise 39 from Section 2.3, consider the matrices $\begin{bmatrix} 4.27 & 6.94 \\ 3.45 & 3.65 \end{bmatrix}$, $\begin{bmatrix} 4.05 & 7.01 \\ 3.27 & 3.51 \end{bmatrix}$, and $\begin{bmatrix} 4.40 & 6.90 \\ 3.54 & 3.76 \end{bmatrix}$ for the production costs at the Boston, Chicago, and Seattle plants, respectively.

a. Assume each plant makes the same number of each item. Write a matrix that expresses the average production costs for all three plants.

b. In part b of Exercise 39 in Section 2.3, cost increases for the Chicago plant resulted in a new production cost matrix $\begin{bmatrix} 4.42 & 7.43 \\ 3.38 & 3.62 \end{bmatrix}$. Following those cost increases the Boston plant was closed and production divided evenly between the Chicago and Seattle plants. What is the matrix that now expresses the average production cost for the entire country?

46. House Construction Consider the matrices P, Q, and R given in Example 6.

a. Find and interpret the matrix product QR.

b. Verify that $P(QR)$ is equal to $(PQ)R$ calculated in Example 5.

47. Shoe Sales Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals, and boots, respectively. Half of all sales in California stores are shoes, $1/4$ are sandals, and $1/4$ are boots. In Arizona the fractions are $1/5$ shoes, $1/5$ sandals, and $3/5$ boots.

a. Write a 2×3 matrix called P representing prices for the two stores and three types of footwear.

b. Write a 3×2 matrix called F representing the fraction of each type of footwear sold in each state.

c. Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

d. From your answer to part c, what is the average price for a pair of footwear at an outlet of Fred's in Arizona?

48. Management In Exercise 40 from Section 2.3, consider the matrix

$$\begin{bmatrix} 88 & 105 & 60 \\ 48 & 72 & 40 \\ 16 & 21 & 0 \\ 112 & 147 & 50 \end{bmatrix}$$

expressing the sales information for the three stores.

a. Write a 3×1 matrix expressing the factors by which sales in each store should be multiplied to reflect the fact that sales increased during the following week by 25%, $1/3$, and 10% in stores I, II, and III, respectively, as described in part b of Exercise 40 from Section 2.3.

b. Multiply the matrix expressing sales information by the matrix found in part a of this exercise to find the sales for all three stores in the second week.

Life Sciences

49. Dietetics In Exercise 41 from Section 2.3, label the matrices

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 7 \\ 0 & 10 & 1 \\ 0 & 15 & 2 \\ 10 & 12 & 8 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

found in parts a, b, and c, respectively, X, Y, and Z.

a. Find the product matrix XY. What do the entries of this matrix represent?

b. Find the product matrix YZ. What do the entries represent?

c. Find the products $(XY)Z$ and $X(YZ)$ and verify that they are equal. What do the entries represent?

50. Motorcycle Helmets In Exercise 44 from Section 2.3, you constructed matrices that represented usage of motorcycle helmets for 2008 and 2009. Use matrix operations to combine these two matrices to form one matrix that represents the average of the two years.

51. Life Expectancy In Exercise 45 from Section 2.3, you constructed matrices that represent the life expectancy of African American and white American males and females. Use matrix operations to combine these two matrices to form one matrix that represents the combined life expectancy of both races at

the beginning of each decade since 1970. Use the fact that of the combined African and white American population, African Americans are about one-sixth of the total and white Americans about five-sixths. (*Hint:* Multiply the matrix for African Americans by $1/6$ and the matrix for the white Americans by $5/6$, and then add the results.)

52. Northern Spotted Owl Population In an attempt to save the endangered northern spotted owl, the U.S. Fish and Wildlife Service imposed strict guidelines for the use of 12 million acres of Pacific Northwest forest. This decision led to a national debate between the logging industry and environmentalists. Mathematical ecologists have created a mathematical model to analyze population dynamics of the northern spotted owl by dividing the female owl population into three categories: juvenile (up to 1 year old), subadult (1 to 2 years), and adult (over 2 years old). By analyzing these three subgroups, it is possible to use the number of females in each subgroup at time n to estimate the number of females in each group at any time $n + 1$ with the following matrix equation:

$$\begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix},$$

where j_n is the number of juveniles, s_n is the number of subadults, and a_n is the number of adults at time n. ***Source: Conservation Biology.***

a. If there are currently 4000 female northern spotted owls made up of 900 juveniles, 500 subadults, and 2600 adults, use a graphing calculator or spreadsheet and matrix operations to determine the total number of female owls for each of the next 5 years. (*Hint:* Round each answer to the nearest whole number after each matrix multiplication.)

b. With advanced techniques from linear algebra, it is possible to show that in the long run, the following holds.

$$\begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix} \approx 0.98359 \begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix}$$

What can we conclude about the long-term survival of the northern spotted owl?

c. Notice that only 18 percent of the juveniles become subadults. Assuming that, through better habitat management, this number could be increased to 40 percent, rework part a. Discuss possible reasons why only 18 percent of the juveniles become subadults. Under the new assumption, what can you conclude about the long-term survival of the northern spotted owl?

Social Sciences

53. World Population The 2010 birth and death rates per million for several regions and the world population (in millions) by region are given in the following tables. ***Source: U. S. Census Bureau.***

	Births	Deaths
Africa	0.0346	0.0118
Asia	0.0174	0.0073
Latin America	0.0189	0.0059
North America	0.0135	0.0083
Europe	0.0099	0.0103

	Population (millions)				
Year	Africa	Asia	Latin America	North America	Europe
1970	361	2038	286	227	460
1980	473	2494	362	252	484
1990	627	2978	443	278	499
2000	803	3435	524	314	511
2010	1013	3824	591	344	522

a. Write the information in each table as a matrix.

b. Use the matrices from part a to find the total number (in millions) of births and deaths in each year (assuming birth and death rates for all years are close to these in 2010).

c. Using the results of part b, compare the number of births in 1970 and in 2010. Also compare the birth rates from part a. Which gives better information?

d. Using the results of part b, compare the number of deaths in 1980 and in 2010. Discuss how this comparison differs from a comparison of death rates from part a.

YOUR TURN ANSWERS

1. $\begin{bmatrix} -5 & -22 \\ -3 & -10 \end{bmatrix}$ **2.** AB does not exist; $BA = \begin{bmatrix} 1 & -1 & 5 \\ -21 & -37 & 11 \end{bmatrix}$

2.5 Matrix Inverses

APPLY IT **One top leader needs to get an important message to one of her agents. How can she encrypt the message to ensure secrecy?**
This question is answered in Example 7.

In this section, we introduce the idea of a matrix inverse, which is comparable to the reciprocal of a real number. This will allow us to solve a matrix equation.

The real number 1 is the *multiplicative* identity for real numbers: for any real number a, we have $a \cdot 1 = 1 \cdot a = a$. In this section, we define a *multiplicative identity matrix* I that has properties similar to those of the number 1. We then use the definition of matrix I to find the *multiplicative inverse* of any square matrix that has an inverse.

If I is to be the identity matrix, both of the products AI and IA must equal A. This means that an identity matrix exists only for square matrices. The 2×2 **identity matrix** that satisfies these conditions is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To check that I, as defined above, is really the 2×2 identity, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then AI and IA should both equal A.

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a(1) + b(0) & a(0) + b(1) \\ c(1) + d(0) & c(0) + d(1) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1(a) + 0(c) & 1(b) + 0(d) \\ 0(a) + 1(c) & 0(b) + 1(d) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

This verifies that I has been defined correctly.

It is easy to verify that the identity matrix I is unique. Suppose there is another identity; call it J. Then IJ must equal I, because J is an identity, and IJ must also equal J, because I is an identity. Thus $I = J$.

The identity matrices for 3×3 matrices and 4×4 matrices, respectively, are

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By generalizing, we can find an $n \times n$ identity matrix for any value of n.

Recall that the multiplicative inverse of the nonzero real number a is $1/a$. The product of a and its multiplicative inverse $1/a$ is 1. Given a matrix A, can a **multiplicative inverse matrix A^{-1}** (read "A-inverse") that will satisfy both

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

be found? For a given matrix, we often can find an inverse matrix by using the row operations of Section 2.2.

NOTE A^{-1} does not mean $1/A$; here, A^{-1} is just the notation for the multiplicative inverse of matrix A. Also, only square matrices can have inverses because both $A^{-1}A$ and AA^{-1} must exist and be equal to an identity matrix of the same size.

EXAMPLE 1 Inverse Matrices

Verify that the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ are inverses of each other.

SOLUTION Multiply A times B as in the previous section:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly,

$$BA = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $AB = BA = I$, A and B are inverses of each other.

If an inverse exists, it is unique. That is, any given square matrix has no more than one inverse. The proof of this is left to Exercise 50 in this section.

As an example, let us find the inverse of

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}.$$

Let the unknown inverse matrix be

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

By the definition of matrix inverse, $AA^{-1} = I$, or

$$AA^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By matrix multiplication,

$$\begin{bmatrix} x + 3z & y + 3w \\ -x + 2z & -y + 2w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Setting corresponding elements equal gives the system of equations

$$\begin{aligned} x \quad + 3z \quad &= 1 && (1)\\ y \quad + 3w &= 0 && (2)\\ -x \quad + 2z \quad &= 0 && (3)\\ -y \quad + 2w &= 1. && (4) \end{aligned}$$

Since equations (1) and (3) involve only x and z, while equations (2) and (4) involve only y and w, these four equations lead to two systems of equations,

$$\begin{aligned} x + 3z &= 1 \\ -x + 2z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} y + 3w &= 0 \\ -y + 2w &= 1. \end{aligned}$$

Writing the two systems as augmented matrices gives

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ -1 & 2 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & 2 & 1 \end{array}\right].$$

Each of these systems can be solved by the Gauss-Jordan method. Notice, however, that the elements to the left of the vertical bar are identical. The two systems can be combined into the single matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right].$$

This is of the form $[A|I]$. It is solved simultaneously as follows.

$$\begin{array}{rl} & \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right] \quad \text{Get 0 in the second-row, first-column position.} \\ \mathbf{R_1 + R_2 \to R_2} & \\ \mathbf{-3R_2 + 5R_1 \to R_1} & \left[\begin{array}{cc|cc} 5 & 0 & 2 & -3 \\ 0 & 5 & 1 & 1 \end{array}\right] \quad \text{Get 0 in the first-row, second-column position.} \\ \mathbf{\frac{1}{5}R_1 \to R_1} & \\ \mathbf{\frac{1}{5}R_2 \to R_2} & \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] \quad \text{Get 1's down the diagonal.} \end{array}$$

The numbers in the first column to the right of the vertical bar give the values of x and z. The second column gives the values of y and w. That is,

$$\left[\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

so that

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}.$$

To check, multiply A by A^{-1}. The result should be I.

$$AA^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{5}+\frac{3}{5} & -\frac{3}{5}+\frac{3}{5} \\ -\frac{2}{5}+\frac{2}{5} & \frac{3}{5}+\frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Verify that $A^{-1}A = I$, also.

Finding a Multiplicative Inverse Matrix

To obtain A^{-1} for any $n \times n$ matrix A for which A^{-1} exists, follow these steps.

1. Form the augmented matrix $[A|I]$, where I is the $n \times n$ identity matrix.
2. Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$, if this is possible.
3. Matrix B is A^{-1}.

EXAMPLE 2 Inverse Matrix

Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$.

Method I
Calculating by Hand

SOLUTION Write the augmented matrix $[A \mid I]$.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Begin by selecting the row operation that produces a zero for the first element in row 2.

$$\begin{array}{rl} & \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \quad \text{Get 0's in the first column.} \\ \mathbf{-2R_1 + R_2 \to R_2} & \\ \mathbf{-3R_1 + R_3 \to R_3} & \end{array}$$

Column 2 already has zeros in the required positions, so work on column 3.

$$\begin{array}{r} \\ R_3 + 3R_1 \to R_1 \\ R_3 + (-1)R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 & -1 & 1 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \quad \text{Get 0's in the third column.}$$

Now get 1's down the main diagonal.

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ -\frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \quad \text{Get 1's down the diagonal.}$$

From the last transformation, the desired inverse is

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}.$$

Confirm this by forming the products $A^{-1}A$ and AA^{-1}, both of which should equal I.

YOUR TURN 1 Find A^{-1} if $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & -1 \\ 3 & 3 & 2 \end{bmatrix}$.

Method 2 Graphing Calculator

The inverse of A can also be found with a graphing calculator, as shown in Figure 14 in the margin below. (The matrix A had previously been entered into the calculator.) The entire answer can be viewed by pressing the right and left arrow keys on the calculator.

TRY YOUR TURN 1

TECHNOLOGY NOTE

Spreadsheets also have the capability of calculating the inverse of a matrix with a simple command. See the *Graphing Calculator and Excel Spreadsheet Manual* available with this book for details.

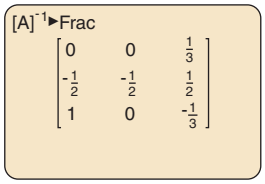


FIGURE 14

EXAMPLE 3 Inverse Matrix

Find A^{-1} if $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$.

SOLUTION Using row operations to transform the first column of the augmented matrix

$$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

gives the following results.

$$\begin{array}{r} R_1 + (-2)R_2 \to R_2 \end{array} \left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Because the last row has all zeros to the left of the vertical bar, there is no way to complete the process of finding the inverse matrix. What is wrong? Just as the real number 0 has no multiplicative inverse, some matrices do not have inverses. Matrix A is an example of a matrix that has no inverse: there is no matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

Solving Systems of Equations with Inverses

We used matrices to solve systems of linear equations by the Gauss-Jordan method in Section 2.2. Another way to use matrices to solve linear systems is to write the system as a matrix equation $AX = B$, where A is the matrix of the coefficients of the variables of the system, X is the matrix of the variables, and B is the matrix of the constants. Matrix A is called the **coefficient matrix**.

To solve the matrix equation $AX = B$, first see if A^{-1} exists. Assuming A^{-1} exists and using the facts that $A^{-1}A = I$ and $IX = X$ gives

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \text{Multiply both sides by } A^{-1}. \\ (A^{-1}A)X &= A^{-1}B && \text{Associative property} \\ IX &= A^{-1}B && \text{Multiplicative inverse property} \\ X &= A^{-1}B. && \text{Identity property} \end{aligned}$$

CAUTION When multiplying by matrices on both sides of a matrix equation, be careful to multiply in the same order on both sides of the equation, since multiplication of matrices is not commutative (unlike multiplication of real numbers).

The work thus far leads to the following method of solving a system of equations written as a matrix equation.

Solving a System $AX = B$ Using Matrix Inverses

To solve a system of equations $AX = B$, where A is the square matrix of coefficients and A^{-1} exists, X is the matrix of variables, and B is the matrix of constants, first find A^{-1}. Then $X = A^{-1}B$.

This method is most practical in solving several systems that have the same coefficient matrix but different constants, as in Example 5 in this section. Then just one inverse matrix must be found.

EXAMPLE 4 Inverse Matrices and Systems of Equations

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 2. \end{aligned}$$

SOLUTION To represent the system as a matrix equation, use the coefficient matrix of the system together with the matrix of variables and the matrix of constants:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

The system can now be written in matrix form as the equation $AX = B$ since

$$AX = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ x + 5y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = B.$$

To solve the system, first find A^{-1}. Do this by using row operations on matrix $[A|I]$ to get

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{13} & \frac{3}{13} \\ 0 & 1 & -\frac{1}{13} & \frac{2}{13} \end{array}\right].$$

From this result,

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}.$$

Next, find the product $A^{-1}B$.

$$A^{-1}B = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The solution of the system is $(2, 0)$.

YOUR TURN 2 Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 5x + 4y &= 23 \\ 4x - 3y &= 6. \end{aligned}$$

TRY YOUR TURN 2

EXAMPLE 5 Fertilizer

Three brands of fertilizer are available that provide nitrogen, phosphoric acid, and soluble potash to the soil. One bag of each brand provides the following units of each nutrient.

		Brand		
		Fertifun	Big Grow	Soakem
Nutrient	*Nitrogen*	1	2	3
	Phosphoric Acid	3	1	2
	Potash	2	0	1

For ideal growth, the soil on a Michigan farm needs 18 units of nitrogen, 23 units of phosphoric acid, and 13 units of potash per acre. The corresponding numbers for a California farm are 31, 24, and 11, and for a Kansas farm are 20, 19, and 15. How many bags of each brand of fertilizer should be used per acre for ideal growth on each farm?

SOLUTION Rather than solve three separate systems, we consider the single system

$$\begin{aligned} x + 2y + 3z &= a \\ 3x + y + 2z &= b \\ 2x + z &= c, \end{aligned}$$

where a, b, and c represent the units of nitrogen, phosphoric acid, and potash needed for the different farms. The system of equations is then of the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

B has different values for the different farms. We find A^{-1} first, then use it to solve all three systems.

To find A^{-1}, we start with the matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

and use row operations to get $[I|A^{-1}]$. The result is

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{bmatrix}.$$

Now we can solve each of the three systems by using $X = A^{-1}B$.

For the Michigan farm, $B = \begin{bmatrix} 18 \\ 23 \\ 13 \end{bmatrix}$, and

$$X = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 18 \\ 23 \\ 13 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.$$

Therefore, $x = 5$, $y = 2$, and $z = 3$. Buy 5 bags of Fertifun, 2 bags of Big Grow, and 3 bags of Soakem.

For the California farm, $B = \begin{bmatrix} 31 \\ 24 \\ 11 \end{bmatrix}$, and

$$X = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 31 \\ 24 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}.$$

Buy 2 bags of Fertifun, 4 bags of Big Grow, and 7 bags of Soakem.

For the Kansas farm, $B = \begin{bmatrix} 20 \\ 19 \\ 15 \end{bmatrix}$. Verify that this leads to $x = 1$, $y = -10$, and $z = 13$. We cannot have a negative number of bags, so this solution is impossible. In buying enough bags to meet all of the nutrient requirements, the farmer must purchase an excess of some nutrients. In the next two chapters, we will study a method of solving such problems at a minimum cost.

In Example 5, using the matrix inverse method of solving the systems involved considerably less work than using row operations for each of the three systems.

EXAMPLE 6 Solving an Inconsistent System of Equations

Use the inverse of the coefficient matrix to solve the system

$$\begin{aligned} 2x - 4y &= 13 \\ x - 2y &= 1. \end{aligned}$$

SOLUTION We saw in Example 3 that the coefficient matrix $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ does not have an inverse. This means that the given system either has no solution or has an infinite number of solutions. Verify that this system is inconsistent and has no solution.

NOTE If a matrix has no inverse, then a corresponding system of equations might have either no solutions or an infinite number of solutions, and we instead use the echelon or Gauss-Jordan method. In Example 3, we saw that the matrix $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ had no inverse. The reason is that the first row of this matrix is double the second row. The system

$$\begin{aligned} 2x - 4y &= 2 \\ x - 2y &= 1 \end{aligned}$$

has an infinite number of solutions because the first equation is double the second equation. The system

$$\begin{aligned} 2x - 4y &= 13 \\ x - 2y &= 1 \end{aligned}$$

has no solutions because the left side of the first equation is double the left side of the second equation, but the right side is not.

EXAMPLE 7 Cryptography

Throughout the Cold War and as the Internet has grown and developed, the need for sophisticated methods of coding and decoding messages has increased. Although there are many methods of encrypting messages, one fairly sophisticated method uses matrix operations. This method first assigns a number to each letter of the alphabet. The simplest way to do this is to assign the number 1 to A, 2 to B, and so on, with the number 27 used to represent a space between words.

APPLY IT

For example, the message *math is cool* can be divided into groups of three letters and spaces each and then converted into numbers as follows

$$\begin{bmatrix} m \\ a \\ t \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ 20 \end{bmatrix}.$$

The entire message would then consist of four 3×1 columns of numbers:

$$\begin{bmatrix} 13 \\ 1 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} 8 \\ 27 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} 19 \\ 27 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 15 \\ 15 \\ 12 \end{bmatrix}.$$

This code is easy to break, so we further complicate the code by choosing a matrix that has an inverse (in this case a 3×3 matrix) and calculate the products of the matrix and each of the column vectors above.

If we choose the coding matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix},$$

then the products of A with each of the column vectors above produce a new set of vectors

$$\begin{bmatrix} 96 \\ 87 \\ 74 \end{bmatrix}, \quad \begin{bmatrix} 125 \\ 70 \\ 95 \end{bmatrix}, \quad \begin{bmatrix} 112 \\ 74 \\ 133 \end{bmatrix}, \quad \begin{bmatrix} 108 \\ 81 \\ 102 \end{bmatrix}.$$

This set of vectors represents our coded message, and it will be transmitted as 96, 87, 74, 125 and so on.

When the intended person receives the message, it is divided into groups of three numbers, and each group is formed into a column matrix. The message is easily decoded if the receiver knows the inverse of the original matrix. The inverse of matrix A is

$$A^{-1} = \begin{bmatrix} -0.2 & 0.2 & 0.2 \\ 0.4 & -0.6 & 0.2 \\ 0 & 0.4 & -0.2 \end{bmatrix}.$$

Thus, the message is decoded by taking the product of the inverse matrix with each column vector of the received message. For example,

$$A^{-1} \begin{bmatrix} 96 \\ 87 \\ 74 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ 20 \end{bmatrix}.$$

Unless the original matrix or its inverse is known, this type of code can be difficult to break. In fact, very large matrices can be used to encrypt data. It is interesting to note that many mathematicians are employed by the National Security Agency to develop encryption methods that are virtually unbreakable.

YOUR TURN 3 Use the matrix A and its inverse A^{-1} in Example 7 to do the following.

a) Encode the message "Behold".

b) Decode the message 96, 87, 74, 141, 117, 114.

TRY YOUR TURN 3

2.5 EXERCISES

Decide whether the given matrices are inverses of each other. (Check to see if their product is the identity matrix I.)

1. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -4 \\ 2 & -7 \end{bmatrix}$ and $\begin{bmatrix} -7 & 4 \\ -2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 6 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$

4. $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$ and $\begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

9. Does a matrix with a row of all zeros have an inverse? Why?

10. Matrix A has A^{-1} as its inverse. What does $(A^{-1})^{-1}$ equal? (*Hint:* Experiment with a few matrices to see what you get.)

Find the inverse, if it exists, for each matrix.

11. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

14. $\begin{bmatrix} -3 & -8 \\ 1 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$

16. $\begin{bmatrix} 5 & 10 \\ -3 & -6 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

19. $\begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 4 & -1 & -3 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 2 & 0 & 4 \\ 1 & 0 & -1 \\ 3 & 0 & -2 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -3 \\ 3 & 8 & -5 \end{bmatrix}$

24. $\begin{bmatrix} 4 & 1 & -4 \\ 2 & 1 & -1 \\ -2 & -4 & 5 \end{bmatrix}$

25. $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ -2 & 2 & -2 & 4 \\ 0 & 2 & -3 & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & -1 \\ 3 & 3 & 2 & -2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

Solve each system of equations by using the inverse of the coefficient matrix if it exists and by the echelon method if the inverse doesn't exist.

27. $\begin{aligned} 2x + 5y &= 15 \\ x + 4y &= 9 \end{aligned}$

28. $\begin{aligned} -x + 2y &= 15 \\ -2x - y &= 20 \end{aligned}$

29. $\begin{aligned} 2x + y &= 5 \\ 5x + 3y &= 13 \end{aligned}$

30. $\begin{aligned} -x - 2y &= 8 \\ 3x + 4y &= 24 \end{aligned}$

31. $\begin{aligned} 3x - 2y &= 3 \\ 7x - 5y &= 0 \end{aligned}$

32. $\begin{aligned} 3x - 6y &= 1 \\ -5x + 9y &= -1 \end{aligned}$

33. $\begin{aligned} -x - 8y &= 12 \\ 3x + 24y &= -36 \end{aligned}$

34. $\begin{aligned} 2x + 7y &= 14 \\ 4x + 14y &= 28 \end{aligned}$

Solve each system of equations by using the inverse of the coefficient matrix if it exists and by the Gauss-Jordan method if the inverse doesn't exist. (The inverses for the first four problems were found in Exercises 19, 20, 23, and 24.)

35. $\begin{aligned} -x - y - z &= 1 \\ 4x + 5y &= -2 \\ y - 3z &= 3 \end{aligned}$

36. $\begin{aligned} 2x + y &= 1 \\ 3y + z &= 8 \\ 4x - y - 3z &= 8 \end{aligned}$

37. $\begin{aligned} x + 3y - 2z &= 4 \\ 2x + 7y - 3z &= 8 \\ 3x + 8y - 5z &= -4 \end{aligned}$

38. $\begin{aligned} 4x + y - 4z &= 17 \\ 2x + y - z &= 12 \\ -2x - 4y + 5z &= 17 \end{aligned}$

39. $\begin{aligned} 2x - 2y &= 5 \\ 4y + 8z &= 7 \\ x + 2z &= 1 \end{aligned}$

40. $\begin{aligned} x + 2z &= -1 \\ y - z &= 5 \\ x + 2y &= 7 \end{aligned}$

Solve each system of equations by using the inverse of the coefficient matrix. (The inverses were found in Exercises 25 and 26.)

41. $\begin{aligned} x - 2y + 3z &= 4 \\ y - z + w &= -8 \\ -2x + 2y - 2z + 4w &= 12 \\ 2y - 3z + w &= -4 \end{aligned}$

42. $\begin{aligned} x + y + 2w &= 3 \\ 2x - y + z - w &= 3 \\ 3x + 3y + 2z - 2w &= 5 \\ x + 2y + z &= 3 \end{aligned}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in Exercises 43–48.

43. Show that $IA = A$.

44. Show that $AI = A$.

45. Show that $A \cdot O = O$.

46. Find A^{-1}. (Assume $ad - bc \neq 0$.)

47. Show that $A^{-1}A = I$.

48. Show that $AA^{-1} = I$.

49. Using the definition and properties listed in this section, show that for square matrices A and B of the same size, if $AB = O$ and if A^{-1} exists, then $B = O$, where O is a matrix whose elements are all zeros.

50. Prove that, if it exists, the inverse of a matrix is unique. (*Hint:* Assume there are two inverses B and C for some matrix A, so that $AB = BA = I$ and $AC = CA = I$. Multiply the first equation by C and the second by B.)

Use matrices C and D in Exercises 51–55.

$$C = \begin{bmatrix} -6 & 8 & 2 & 4 & -3 \\ 1 & 9 & 7 & -12 & 5 \\ 15 & 2 & -8 & 10 & 11 \\ 4 & 7 & 9 & 6 & -2 \\ 1 & 3 & 8 & 23 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & -3 & 7 & 9 & 2 \\ 6 & 8 & -5 & 2 & 1 \\ 3 & 7 & -4 & 2 & 11 \\ 5 & -3 & 9 & 4 & -1 \\ 0 & 3 & 2 & 5 & 1 \end{bmatrix}$$

51. Find C^{-1}.

52. Find $(CD)^{-1}$.

53. Find D^{-1}.

54. Is $C^{-1}D^{-1} = (CD)^{-1}$?

55. Is $D^{-1}C^{-1} = (CD)^{-1}$?

Solve the matrix equation $AX = B$ for X by finding A^{-1}, given A and B as follows.

56. $A = \begin{bmatrix} 2 & -5 & 7 \\ 4 & -3 & 2 \\ 15 & 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ 5 \\ 8 \end{bmatrix}$

57. $A = \begin{bmatrix} 2 & 5 & 7 & 9 \\ 1 & 3 & -4 & 6 \\ -1 & 0 & 5 & 8 \\ 2 & -2 & 4 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 7 \\ -1 \\ 5 \end{bmatrix}$

58. $A = \begin{bmatrix} 3 & 2 & -1 & -2 & 6 \\ -5 & 17 & 4 & 3 & 15 \\ 7 & 9 & -3 & -7 & 12 \\ 9 & -2 & 1 & 4 & 8 \\ 1 & 21 & 9 & -7 & 25 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ 5 \\ 3 \\ -8 \\ 25 \end{bmatrix}$

APPLICATIONS

Business and Economics

Solve each exercise by using the inverse of the coefficient matrix to solve a system of equations.

59. Analysis of Orders The Bread Box Bakery sells three types of cakes, each requiring the amounts of the basic ingredients shown in the following matrix.

		Type of Cake		
		I	II	III
Ingredient	Flour (in cups)	2	4	2
	Sugar (in cups)	2	1	2
	Eggs	2	1	3

To fill its daily orders for these three kinds of cake, the bakery uses 72 cups of flour, 48 cups of sugar, and 60 eggs.

a. Write a 3×1 matrix for the amounts used daily.

b. Let the number of daily orders for cakes be a 3×1 matrix X with entries x_1, x_2, and x_3. Write a matrix equation that can be solved for X, using the given matrix and the matrix from part a.

c. Solve the equation from part b to find the number of daily orders for each type of cake.

60. Production Requirements An electronics company produces transistors, resistors, and computer chips. Each transistor requires 3 units of copper, 1 unit of zinc, and 2 units of glass. Each resistor requires 3, 2, and 1 units of the three materials, and each computer chip requires 2, 1, and 2 units of these materials, respectively. How many of each product can be made with the following amounts of materials?

a. 810 units of copper, 410 units of zinc, and 490 units of glass

b. 765 units of copper, 385 units of zinc, and 470 units of glass

c. 1010 units of copper, 500 units of zinc, and 610 units of glass

61. Investments An investment firm recommends that a client invest in AAA-, A-, and B-rated bonds. The average yield on AAA bonds is 6%, on A bonds 6.5%, and on B bonds 8%. The client wants to invest twice as much in AAA bonds as in B bonds. How much should be invested in each type of bond under the following conditions?

a. The total investment is \$25,000, and the investor wants an annual return of \$1650 on the three investments.

b. The values in part a are changed to \$30,000 and \$1985, respectively.

c. The values in part a are changed to \$40,000 and \$2660, respectively.

62. Production Pretzels cost \$4 per lb, dried fruit \$5 per lb, and nuts \$9 per lb. The three ingredients are to be combined in a trail mix containing twice the weight of pretzels as dried fruit. How many pounds of each should be used to produce the following amounts at the given cost?

a. 140 lb at \$6 per lb

b. 100 lb at \$7.60 per lb

c. 125 lb at \$6.20 per lb

Life Sciences

63. Vitamins Greg Tobin mixes together three types of vitamin tablets. Each Super Vim tablet contains, among other things, 15 mg of niacin and 12 I.U. of vitamin E. The figures for a Multitab tablet are 20 mg and 15 I.U., and for a Mighty Mix are 25 mg and 35 I.U. How many of each tablet are there if the total number of tablets, total amount of niacin, and total amount of vitamin E are as follows?

a. 225 tablets, 4750 mg of niacin, and 5225 I.U. of vitamin E

b. 185 tablets, 3625 mg of niacin, and 3750 I.U. of vitamin E

c. 230 tablets, 4450 mg of niacin, and 4210 I.U. of vitamin E

General Interest

64. Encryption Use the matrices presented in Example 7 of this section to do the following:

a. Encode the message, "All is fair in love and war."

b. Decode the message 138, 81, 102, 101, 67, 109, 162, 124, 173, 210, 150, 165.

65. Encryption Use the methods presented in Example 7 along with the given matrix B to do the following.

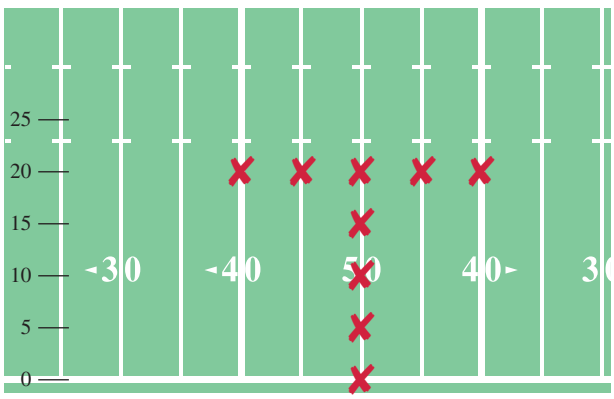
$$B = \begin{bmatrix} 2 & 4 & 6 \\ -1 & -4 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

a. Encode the message, "To be or not to be."

b. Find the inverse of B.

c. Use the inverse of B to decode the message 116, -60, -15, 294, -197, -2, 148, -92, -9, 96, -64, 4, 264, -182, -2.

66. Music During a marching band's half-time show, the band members generally line up in such a way that a common shape is recognized by the fans. For example, as illustrated in the figure, a band might form a letter T, where an x represents a member of the band. As the music is played, the band will either create a new shape or rotate the original shape. In doing this, each member of the band will need to move from one point on the field to another. For larger bands, keeping track of who goes where can be a daunting task. However, it is possible to use matrix inverses to make the process a bit easier. The entire process is calculated by knowing how three band members, all of whom cannot be in a straight line, will move from the current position to a new position. For example, in the figure, we can see that there are band members at $(50, 0)$, $(50, 15)$, and $(45, 20)$. We will assume that these three band members move to $(40, 10)$, $(55, 10)$, and $(60, 15)$, respectively. ***Source: The College Mathematics Journal.***



a. Find the inverse of $B = \begin{bmatrix} 50 & 50 & 45 \\ 0 & 15 & 20 \\ 1 & 1 & 1 \end{bmatrix}$.

b. Find $A = \begin{bmatrix} 40 & 55 & 60 \\ 10 & 10 & 15 \\ 1 & 1 & 1 \end{bmatrix} B^{-1}$.

c. Use the result of part b to find the new position of the other band members. What is the shape of the new position? (*Hint:* Multiply the matrix A by a 3×1 column vector with the first two components equal to the original position of each band member and the third component equal to 1. The new position of the band member is in the first two components of the product.)

YOUR TURN ANSWERS

1. $\begin{bmatrix} 1/8 & 3/8 & 1/8 \\ 5/8 & -1/8 & -3/8 \\ -9/8 & -3/8 & 7/8 \end{bmatrix}$ **2.** (3, 2) **3. a.** 49, 33, 26, 67, 54, 88 **b.** matrix

2.6 Input-Output Models

APPLY IT

What production levels are needed to keep an economy going and to supply demands from outside the economy?

A method for solving such questions is developed in this section and applied in Examples 3 and 4.

Wassily Leontief (1906–1999) developed an interesting and powerful application of matrix theory to economics and was recognized for this contribution with the Nobel prize in economics in 1973. His matrix models for studying the interdependencies in an economy are called *input-output* models. In practice these models are very complicated, with many variables. Only simple examples with a few variables are discussed here.

Input-output models are concerned with the production and flow of goods (and perhaps services). In an economy with n basic commodities, or sectors, the production of each commodity uses some (perhaps all) of the commodities in the economy as inputs. For example, oil is needed to run the machinery that plants and harvests the wheat, and wheat is used to feed the people who drill and refine the oil. The amounts of each commodity used in the production of one unit of each commodity can be written as an $n \times n$ matrix A, called the **technological matrix** or **input-output matrix** of the economy.

EXAMPLE 1 Input-Output Matrix

Suppose a simplified economy involves just three commodity categories: agriculture, manufacturing, and transportation, all in appropriate units. Production of 1 unit of agriculture requires $1/2$ unit of manufacturing and $1/4$ unit of transportation; production of 1 unit of manufacturing requires $1/4$ unit of agriculture and $1/4$ unit of transportation; and production of 1 unit of transportation requires $1/3$ unit of agriculture and $1/4$ unit of manufacturing. Give the input-output matrix for this economy.

SOLUTION

	Agriculture	Manufacturing	Transportation
Agriculture	0	$\frac{1}{4}$	$\frac{1}{3}$
Manufacturing	$\frac{1}{2}$	0	$\frac{1}{4}$
Transportation	$\frac{1}{4}$	$\frac{1}{4}$	0

$= A$

The first column of the input-output matrix represents the amount of each of the three commodities consumed in the production of 1 unit of agriculture. The second column gives the amounts required to produce 1 unit of manufacturing, and the last column gives the amounts required to produce 1 unit of transportation. (Although it is perhaps unrealistic that production of a unit of each commodity requires none of that commodity, the simpler matrix involved is useful for our purposes.)

NOTE Notice that for each commodity produced, the various units needed are put in a column. Each column corresponds to a commodity produced, and the rows correspond to what is needed to produce the commodity.

Another matrix used with the input-output matrix is the matrix giving the amount of each commodity produced, called the **production matrix**, or the matrix of gross output. In an economy producing n commodities, the production matrix can be represented by a column matrix X with entries $x_1, x_2, x_3, \ldots, x_n$.

EXAMPLE 2 Production Matrix

In Example 1, suppose the production matrix is

$$X = \begin{bmatrix} 60 \\ 52 \\ 48 \end{bmatrix}.$$

Then 60 units of agriculture, 52 units of manufacturing, and 48 units of transportation are produced. Because $1/4$ unit of agriculture is used for each unit of manufacturing produced, $1/4 \times 52 = 13$ units of agriculture must be used in the "production" of manufacturing. Similarly, $1/3 \times 48 = 16$ units of agriculture will be used in the "production" of transportation. Thus, $13 + 16 = 29$ units of agriculture are used for production in the economy. Look again at the matrices A and X. Since X gives the number of units of each commodity produced and A gives the amount (in units) of each commodity used to produce 1 unit of each of the various commodities, the matrix product AX gives the amount of each commodity used in the production process.

$$AX = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 60 \\ 52 \\ 48 \end{bmatrix} = \begin{bmatrix} 29 \\ 42 \\ 28 \end{bmatrix}$$

From this result, 29 units of agriculture, 42 units of manufacturing, and 28 units of transportation are used to produce 60 units of agriculture, 52 units of manufacturing, and 48 units of transportation.

The matrix product AX represents the amount of each commodity used in the production process. The remainder (if any) must be enough to satisfy the demand for the various commodities from outside the production system. In an n-commodity economy, this demand can be represented by a **demand matrix** D with entries $d_1, d_2, \ldots, d_n$. If no production is to remain unused, the difference between the production matrix X and the amount AX used in the production process must equal the demand D, or

$$D = X - AX.$$

In Example 2,

$$D = \begin{bmatrix} 60 \\ 52 \\ 48 \end{bmatrix} - \begin{bmatrix} 29 \\ 42 \\ 28 \end{bmatrix} = \begin{bmatrix} 31 \\ 10 \\ 20 \end{bmatrix},$$

so production of 60 units of agriculture, 52 units of manufacturing, and 48 units of transportation would satisfy a demand of 31, 10, and 20 units of each commodity, respectively.

In practice, A and D usually are known and X must be found. That is, we need to decide what amounts of production are needed to satisfy the required demands. Matrix algebra can be used to solve the equation $D = X - AX$ for X.

FOR REVIEW

Recall that I is the identity matrix, a square matrix in which each element on the main diagonal is 1 and all other elements are 0.

$$\begin{aligned} D &= X - AX \\ D &= IX - AX && \text{Identity property} \\ D &= (I - A)X && \text{Distributive property} \end{aligned}$$

If the matrix $I - A$ has an inverse, then

$$X = (I - A)^{-1}D.$$

TECHNOLOGY NOTE If the production matrix is large or complicated, we could use a graphing calculator. On the TI-84 Plus, for example, we would enter the command `(identity(3) − [A])^-1*[D]` for a 3×3 matrix A.

TECHNOLOGY NOTE It is also practical to do these calculations on a spreadsheet. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

EXAMPLE 3 Demand Matrix

Suppose, in the three-commodity economy from Examples 1 and 2, there is a demand for 516 units of agriculture, 258 units of manufacturing, and 129 units of transportation. What should production of each commodity be?

APPLY IT

SOLUTION The demand matrix is

$$D = \begin{bmatrix} 516 \\ 258 \\ 129 \end{bmatrix}.$$

To find the production matrix X, first calculate $I - A$.

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{3} \\ -\frac{1}{2} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

Use row operations to find the inverse of $I - A$ (the entries are rounded to three decimal places).

$$(I - A)^{-1} = \begin{bmatrix} 1.395 & 0.496 & 0.589 \\ 0.837 & 1.364 & 0.620 \\ 0.558 & 0.465 & 1.302 \end{bmatrix}$$

Since $X = (I - A)^{-1}D$,

$$X = \begin{bmatrix} 1.395 & 0.496 & 0.589 \\ 0.837 & 1.364 & 0.620 \\ 0.558 & 0.465 & 1.302 \end{bmatrix} \begin{bmatrix} 516 \\ 258 \\ 129 \end{bmatrix} = \begin{bmatrix} 924 \\ 864 \\ 576 \end{bmatrix}.$$

(Each entry in X has been rounded to the nearest whole number.)

The last result shows that production of 924 units of agriculture, 864 units of manufacturing, and 576 units of transportation are required to satisfy demands of 516, 258, and 129 units, respectively.

The entries in the matrix $(I - A)^{-1}$ are often called *multipliers*, and they have important economic interpretations. For example, every \$1 increase in total agricultural demand will result in an increase in agricultural production by about \$1.40, an increase in manufacturing production by about \$0.84, and an increase in transportation production by about \$0.56. Similarly, every \$3 increase in total manufacturing demand will result in an increase of about $3(0.50) = 1.50$, $3(1.36) = 4.08$, and $3(0.47) = 1.41$ dollars in agricultural production, manufacturing production, and transportation production, respectively.

YOUR TURN 1 Find the production of each commodity for the economy in Example 3 if there is a demand for 322 units of agriculture, 447 units of manufacturing, and 133 units of transportation.

TRY YOUR TURN 1

EXAMPLE 4 Wheat and Oil Production

An economy depends on two basic products, wheat and oil. To produce 1 metric ton of wheat requires 0.25 metric tons of wheat and 0.33 metric tons of oil. Production of 1 metric ton of oil consumes 0.08 metric tons of wheat and 0.11 metric tons of oil.

(a) Find the production that will satisfy a demand for 500 metric tons of wheat and 1000 metric tons of oil.

APPLY IT

SOLUTION The input-output matrix is

$$A = \begin{bmatrix} 0.25 & 0.08 \\ 0.33 & 0.11 \end{bmatrix}.$$

Also,

$$I - A = \begin{bmatrix} 0.75 & -0.08 \\ -0.33 & 0.89 \end{bmatrix}.$$

Next, calculate $(I - A)^{-1}$.

$$(I - A)^{-1} = \begin{bmatrix} 1.3882 & 0.1248 \\ 0.5147 & 1.1699 \end{bmatrix} \quad \text{(rounded)}$$

To find the production matrix X, use the equation $X = (I - A)^{-1}D$, with

$$D = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}.$$

The production matrix is

$$X = \begin{bmatrix} 1.3882 & 0.1248 \\ 0.5147 & 1.1699 \end{bmatrix} \begin{bmatrix} 500 \\ 1000 \end{bmatrix} \approx \begin{bmatrix} 819 \\ 1427 \end{bmatrix}.$$

Production of 819 metric tons of wheat and 1427 metric tons of oil is required to satisfy the indicated demand.

(b) Suppose the demand for wheat goes up from 500 to 600 metric tons. Find the increased production in wheat and oil that will be required to meet the new demand.

SOLUTION One way to solve this problem is using the multipliers for wheat, found in the first column of $(I - A)^{-1}$ from part (a). The element in the first row, 1.3882, is used to find the increased production in wheat, while the item in the second row, 0.5147, is used to find the increased production in oil. Since the increase in demand for wheat is 100 metric tons, the increased production in wheat must be $100(1.3882) \approx 139$ metric tons. Similarly, the increased production in oil is $100(0.5147) \approx 51$ metric tons.

Alternatively, we could have found the new production in wheat and oil with the equation $X = (I - A)^{-1}D$, giving

$$X = \begin{bmatrix} 1.3882 & 0.1248 \\ 0.5147 & 1.1699 \end{bmatrix} \begin{bmatrix} 600 \\ 1000 \end{bmatrix} \approx \begin{bmatrix} 958 \\ 1479 \end{bmatrix}.$$

We find the increased production by subtracting the answers found in part (a) from these answers. The increased production in wheat is $958 - 819 = 139$ metric tons, and the increased production in oil is $1479 - 1427 = 52$ metric tons. The slight difference here from the previous answer of 51 metric tons is due to rounding.

Closed Models

The input-output model discussed above is referred to as an **open model**, since it allows for a surplus from the production equal to D. In the **closed model**, all the production is consumed internally in the production process, so that $X = AX$. There is nothing left over to satisfy any outside demands from other parts of the economy or from other economies. In this case, the sum of each column in the input-output matrix equals 1.

To solve the closed model, set $D = O$ in the equation derived earlier, where O is a **zero matrix,** a matrix whose elements are all zeros.

$$(I - A)X = D = O$$

FOR REVIEW

Parameters were discussed in the first section of this chapter. As mentioned there, parameters are required when a system has infinitely many solutions.

The system of equations that corresponds to $(I - A)X = O$ does not have a single unique solution, but it can be solved in terms of a parameter. (It can be shown that if the columns of a matrix A sum to 1, then the equation $(I - A)X = O$ has an infinite number of solutions.)

EXAMPLE 5 Closed Input-Output Model

Use matrix A below to find the production of each commodity in a closed model.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

SOLUTION Find the value of $I - A$, then set $(I - A)X = O$ to find X.

$$I - A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{3} \\ 0 & \frac{3}{4} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

$$(I - A)X = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{3} \\ 0 & \frac{3}{4} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiply to get

$$\begin{bmatrix} \frac{1}{2}x_1 - \frac{1}{4}x_2 - \frac{1}{3}x_3 \\ 0x_1 + \frac{3}{4}x_2 - \frac{1}{3}x_3 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{2}{3}x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The last matrix equation corresponds to the following system.

$$\begin{aligned} \tfrac{1}{2}x_1 - \tfrac{1}{4}x_2 - \tfrac{1}{3}x_3 &= 0 \\ \tfrac{3}{4}x_2 - \tfrac{1}{3}x_3 &= 0 \\ -\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 + \tfrac{2}{3}x_3 &= 0 \end{aligned}$$

Solving the system with x_3 as the parameter gives the solution of the system

$$(\tfrac{8}{9}x_3, \tfrac{4}{9}x_3, x_3).$$

For example, if $x_3 = 9$ (a choice that eliminates fractions in the answer), then $x_1 = 8$ and $x_2 = 4$, so the production of the three commodities should be in the ratio 8:4:9.

TRY YOUR TURN 2

YOUR TURN 2 Change the last column of matrix A in Example 5 to $\begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{bmatrix}$ and find the solution corresponding to $x_3 = 9$.

Finding a Production Matrix

To obtain the production matrix, X, for an open input-output model, follow these steps:

1. Form the $n \times n$ input-output matrix, A, by placing in each column the amount of the various commodities required to produce 1 unit of a particular commodity.
2. Calculate $I - A$, where I is the $n \times n$ identity matrix.
3. Find the inverse, $(I - A)^{-1}$.
4. Multiply the inverse on the right by the demand matrix, D, to obtain $X = (I - A)^{-1}D$.

To obtain a production matrix, X, for a closed input-output model, solve the system $(I - A)X = O$.

Production matrices for actual economies are much larger than those shown in this section. An analysis of the U.S. economy in 1997 has close to 500 commodity categories. ***Source: U.S. Bureau of Economic Analysis.*** Such matrices require large human and computer resources for their analysis. Some of the exercises at the end of this section use actual data in which categories have been combined to simplify the work.

2.6 EXERCISES

Find the production matrix for the following input-output and demand matrices using the open model.

1. $A = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.7 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 0.2 & 0.04 \\ 0.6 & 0.05 \end{bmatrix}, \quad D = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$

3. $A = \begin{bmatrix} 0.1 & 0.03 \\ 0.07 & 0.6 \end{bmatrix}, \quad D = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

4. $A = \begin{bmatrix} 0.02 & 0.03 \\ 0.06 & 0.08 \end{bmatrix}, \quad D = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$

5. $A = \begin{bmatrix} 0.8 & 0 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0 & 0 & 0.7 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$

Find the ratios of products A, B, and C using a closed model.

7.
$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{bmatrix} 0.3 & 0.1 & 0.8 \\ 0.5 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \end{array}$$

8.
$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.3 \end{bmatrix} \end{array}$$

Use a graphing calculator or computer to find the production matrix X, given the following input-output and demand matrices.

9. $A = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.05 \\ 0.01 & 0.02 & 0.01 & 0.1 \\ 0.3 & 0.3 & 0.01 & 0.1 \\ 0.2 & 0.01 & 0.3 & 0.01 \end{bmatrix}, \quad D = \begin{bmatrix} 2930 \\ 3570 \\ 2300 \\ 580 \end{bmatrix}$

10. $A = \begin{bmatrix} 0.01 & 0.2 & 0.01 & 0.2 \\ 0.5 & 0.02 & 0.03 & 0.02 \\ 0.09 & 0.05 & 0.02 & 0.03 \\ 0.3 & 0.2 & 0.2 & 0.01 \end{bmatrix}, \quad D = \begin{bmatrix} 5000 \\ 1000 \\ 4000 \\ 500 \end{bmatrix}$

APPLICATIONS

Business and Economics

Input-Output Open Model **In Exercises 11 and 12, refer to Example 4.**

11. If the demand is changed to 925 metric tons of wheat and 1250 metric tons of oil, how many units of each commodity should be produced?

12. Change the technological matrix so that production of 1 metric ton of wheat requires 1/5 metric ton of oil (and no wheat), and production of 1 metric ton of oil requires 1/3 metric ton of wheat (and no oil). To satisfy the same demand matrix, how many units of each commodity should be produced?

Input-Output Open Model **In Exercises 13–16, refer to Example 3.**

13. If the demand is changed to 607 units of each commodity, how many units of each commodity should be produced?

14. Suppose 1/3 unit of manufacturing (no agriculture or transportation) is required to produce 1 unit of agriculture, 1/4 unit of transportation is required to produce 1 unit of manufacturing, and 1/2 unit of agriculture is required to produce 1 unit of transportation. How many units of each commodity should be produced to satisfy a demand of 1000 units of each commodity?

15. Suppose 1/4 unit of manufacturing and 1/2 unit of transportation are required to produce 1 unit of agriculture, 1/2 unit of agriculture and 1/4 unit of transportation to produce 1 unit of manufacturing, and 1/4 unit of agriculture and 1/4 unit of manufacturing to produce 1 unit of transportation. How many units of each commodity should be produced to satisfy a demand of 1000 units for each commodity?

16. If the input-output matrix is changed so that 1/4 unit of manufacturing and 1/2 unit of transportation are required to produce 1 unit of agriculture, 1/2 unit of agriculture and 1/4 unit of transportation are required to produce 1 unit of manufacturing, and 1/4 unit each of agriculture and manufacturing are required to produce 1 unit of transportation, find the number of units of each commodity that should be produced to satisfy a demand for 500 units of each commodity.

Input-Output Open Model

17. A primitive economy depends on two basic goods, yams and pork. Production of 1 bushel of yams requires 1/4 bushel of yams and 1/2 of a pig. To produce 1 pig requires 1/6 bushel of yams. Find the amount of each commodity that should be produced to get the following.

a. 1 bushel of yams and 1 pig

b. 100 bushels of yams and 70 pigs

18. A simple economy depends on three commodities: oil, corn, and coffee. Production of 1 unit of oil requires 0.2 unit of oil, 0.4 unit of corn, and no units of coffee. To produce 1 unit of corn requires 0.4 unit of oil, 0.2 unit of corn, and 0.1 unit of coffee. To produce 1 unit of coffee requires 0.2 unit of oil, 0.1 unit of corn, and 0.2 unit of coffee. Find the production required to meet a demand of 1000 units each of oil, corn, and coffee.

19. In his work *Input-Output Economics*, Leontief provides an example of a simplified economy with just three sectors: agriculture, manufacturing, and households (i.e., the sector of the economy that produces labor). It has the following input-output matrix: ***Source: Input-Output Economics.***

	Agriculture	Manufacturing	Households
Agriculture	0.25	0.40	0.133
Manufacturing	0.14	0.12	0.100
Households	0.80	3.60	0.133

He also gives the demand matrix

$$D = \begin{bmatrix} 35 \\ 38 \\ 40 \end{bmatrix}.$$

Find the amount of each commodity that should be produced.

20. A much-simplified version of Leontief's 42-sector analysis of the 1947 American economy has the following input-output matrix. ***Source: Input-Output Economics.***

	Agriculture	Manufacturing	Households
Agriculture	0.245	0.102	0.051
Manufacturing	0.099	0.291	0.279
Households	0.433	0.372	0.011

The demand matrix (in billions of dollars) is

$$D = \begin{bmatrix} 2.88 \\ 31.45 \\ 30.91 \end{bmatrix}.$$

Find the amount of each commodity that should be produced.

21. An analysis of the 1958 Israeli economy is simplified here by grouping the economy into three sectors, with the following input-output matrix. ***Source: Input-Output Economics.***

	Agriculture	Manufacturing	Energy
Agriculture	0.293	0	0
Manufacturing	0.014	0.207	0.017
Energy	0.044	0.010	0.216

The demand (in thousands of Israeli pounds) as measured by exports is

$$D = \begin{bmatrix} 138,213 \\ 17,597 \\ 1786 \end{bmatrix}.$$

Find the amount of each commodity that should be produced.

22. The 1981 Chinese economy can be simplified to three sectors: agriculture, industry and construction, and transportation and commerce. The input-output matrix is given below. ***Source: Input-Output Tables of China, 1981.***

	Agriculture	Industry/ Constr.	Trans./ Commerce
Agriculture	0.158	0.156	0.009
Industry/Constr.	0.136	0.432	0.071
Trans./Commerce	0.013	0.041	0.011

The demand (in 100,000 RMB, the unit of money in China) is

$$D = \begin{bmatrix} 106,674 \\ 144,739 \\ 26,725 \end{bmatrix}.$$

a. Find the amount of each commodity that should be produced.

b. Interpret the economic value of an increase in demand of 1 RMB in agricultural exports.

23. Washington The 1987 economy of the state of Washington has been simplified to four sectors: natural resources, manufacturing, trade and services, and personal consumption. The input-output matrix is given below. ***Source: University of Washington.***

	Natural Resources	Manufacturing	Trade & Services	Personal Consumption
Natural Resources	0.1045	0.0428	0.0029	0.0031
Manufacturing	0.0826	0.1087	0.0584	0.0321
Trade & Services	0.0867	0.1019	0.2032	0.3555
Personal Consumption	0.6253	0.3448	0.6106	0.0798

Suppose the demand (in millions of dollars) is

$$D = \begin{bmatrix} 450 \\ 300 \\ 125 \\ 100 \end{bmatrix}.$$

Find the amount of each commodity that should be produced.

24. Washington In addition to solving the previous input-output model, most models of this nature also include an employment equation. For the previous model, the employment equation is added and a new system of equations is obtained as follows. ***Source: University of Washington.***

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ N \end{bmatrix} = (I - B)^{-1}C,$$

where x_1, x_2, x_3, x_4 represent the amount, in millions of dollars, that must be produced to satisfy internal and external demands of the four sectors; N is the total workforce required for a particular set of demands; and

$$B = \begin{bmatrix} 0.1045 & 0.0428 & 0.0029 & 0.0031 & 0 \\ 0.0826 & 0.1087 & 0.0584 & 0.0321 & 0 \\ 0.0867 & 0.1019 & 0.2032 & 0.3555 & 0 \\ 0.6253 & 0.3448 & 0.6106 & 0.0798 & 0 \\ 21.6 & 6.6 & 20.2 & 0 & 0 \end{bmatrix}.$$

a. Suppose that a \$50 million change in manufacturing occurs. How will this increase in demand affect the economy?

(*Hint:* Find $(I - B)^{-1}C$, where $C = \begin{bmatrix} 0 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.)

b. Interpret the meaning of the bottom row in the matrix $(I - B)^{-1}$.

25. Community Links The use of input-output analysis can also be used to model how changes in one city can affect cities that are connected with it in some way. For example, if a large manufacturing company shuts down in one city, it is very likely that the economic welfare of all of the cities around it will suffer. Consider three Pennsylvania communities: Sharon, Farrell, and Hermitage. Due to their proximity to each other, residents of these three communities regularly spend time and money in the other communities. Suppose that we have gathered information in the form of an input-output matrix. ***Source: Thayer Watkins.***

$$A = \begin{array}{c} \\ \text{S} \\ \text{F} \\ \text{H} \end{array} \begin{array}{c} \begin{array}{ccc} \text{S} & \text{F} & \text{H} \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 \\ 0.5 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

This matrix can be thought of as the likelihood that a person from a particular community will spend money in each of the communities.

a. Treat this matrix like an input-output matrix and calculate $(I - A)^{-1}$.

b. Interpret the entries of this inverse matrix.

Input-Output Closed Model

26. Use the input-output matrix

$$\begin{array}{c} \\ \text{Yams} \\ \text{Pigs} \end{array} \begin{array}{c} \begin{array}{cc} \text{Yams} & \text{Pigs} \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

and the closed model to find the ratio of yams to pigs produced.

27. Use the input-output matrix

$$\begin{array}{c} \\ \text{Steel} \\ \text{Coal} \end{array} \begin{array}{c} \begin{array}{cc} \text{Steel} & \text{Coal} \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{bmatrix} \end{array}$$

and the closed model to find the ratio of coal to steel produced.

28. Suppose that production of 1 unit of agriculture requires 1/3 unit of agriculture, 1/3 unit of manufacturing, and 1/3 unit of transportation. To produce 1 unit of manufacturing requires 1/2 unit of agriculture, 1/4 unit of manufacturing, and 1/4 unit of transportation. To produce 1 unit of transportation requires 0 units of agriculture, 1/4 unit of manufacturing, and 3/4 unit of transportation. Find the ratio of the three commodities in the closed model.

29. Suppose that production of 1 unit of mining requires 1/5 unit of mining, 2/5 unit of manufacturing, and 2/5 unit of communication. To produce 1 unit of manufacturing requires 3/5 unit of mining, 1/5 unit of manufacturing, and 1/5 unit of communication. To produce 1 unit of communication requires 0 units of mining, 4/5 unit of manufacturing, and 1/5 unit of communication. Find the ratio of the three commodities in the closed model.

YOUR TURN ANSWERS

1. 749 units of agriculture, 962 units of manufacturing, and 561 units of transportation

2. (4, 2, 9)

2 CHAPTER REVIEW

SUMMARY

In this chapter we extended our study of linear functions to include finding solutions of systems of linear equations. Techniques such as the echelon method and the Gauss-Jordan method were developed and used to solve systems of linear equations. We introduced matrices, which are used to store mathematical information. We saw that matrices can be combined using addition, subtraction, scalar multiplication, and matrix multiplication. Two special matrices, the zero matrix and the identity matrix, were also introduced.

- The zero matrix O is a matrix whose elements are all zero.
- The identity matrix I is an $n \times n$ matrix consisting of 1's along the diagonal and 0's elsewhere.

We then developed the concept of a multiplicative inverse of a matrix and used such inverses to solve systems of equations. We concluded the chapter by introducing the Leontief input-output models, which are used to study interdependencies in an economy.

Row Operations

For any augmented matrix of a system of equations, the following operations produce the augmented matrix of an equivalent system:

1. interchanging any two rows;
2. multiplying the elements of a row by a nonzero real number;
3. adding a nonzero multiple of the elements of one row to the corresponding elements of a nonzero multiple of some other row.

The Gauss-Jordan Method

1. Write each equation so that variable terms are in the same order on the left side of the equal sign and constants are on the right.
2. Write the augmented matrix that corresponds to the system.
3. Use row operations to transform the first column so that all elements except the element in the first row are zero.
4. Use row operations to transform the second column so that all elements except the element in the second row are zero.
5. Use row operations to transform the third column so that all elements except the element in the third row are zero.
6. Continue in this way, when possible, until the last row is written in the form

$$[0\,0\,0\,\cdots\,0\,j\,|k],$$

where j and k are constants with $j \neq 0$. When this is not possible, continue until every row has more zeros on the left than the previous row (except possibly for any rows of all zero at the bottom of the matrix), and the first nonzero entry in each row is the only nonzero entry in its column.

7. Multiply each row by the reciprocal of the nonzero element in that row.

Adding Matrices

The sum of two $m \times n$ matrices X and Y is the $m \times n$ matrix $X + Y$ in which each element is the sum of the corresponding elements of X and Y.

Subtracting Matrices

The difference of two $m \times n$ matrices X and Y is the $m \times n$ matrix $X - Y$ in which each element is found by subtracting the corresponding elements of X and Y.

Product of a Matrix and a Scalar

The product of a scalar k and a matrix X is the matrix kX, each of whose elements is k times the corresponding element of X.

Product of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the ith row and jth column of the product AB, multiply each element in the ith row of A by the corresponding element in the jth column of B, and then add these products. The product matrix AB is an $m \times k$ matrix.

Solving a System $AX = B$ Using Matrix Inverses

To solve a system of equations $AX = B$, where A is a square matrix of coefficients, X is the matrix of variables, and B is the matrix of constants, first find A^{-1}. Then, $X = A^{-1}B$.

Finding a Production Matrix

1. Form the input-output matrix, A.
2. Calculate $I - A$, where I is the $n \times n$ identity matrix.
3. Find the inverse, $(I - A)^{-1}$.
4. Multiply the inverse on the right by the demand matrix, D, to obtain $X = (I - A)^{-1}D$.

To obtain a production matrix, X, for a closed input-output model, solve the system $(I - A)X = O$.

KEY TERMS

To understand the concepts presented in this chapter, you should know the meaning and use of the following terms. For easy reference, the section in the chapter where a word (or expression) was first used is provided.

system of equations

2.1

first-degree equation in n unknowns
unique solution
inconsistent system
dependent equations
equivalent system
echelon method
back-substitution
parameter

2.2

matrix (matrices)
element (entry)
augmented matrix

row operations
Gauss-Jordan method

2.3
size
square matrix
row matrix (row vector)
column matrix (column vector)

2.4
scalar
product matrix

2.5
identity matrix
multiplicative inverse matrix
coefficient matrix

2.6
input-output (technological) matrix
production matrix
demand matrix
open model
closed model
zero matrix

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. If a system of equations has three equations and four unknowns, then it could have a unique solution.

2. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 7 & 4 \\ 1 & 0 \end{bmatrix}$, then $A + B = \begin{bmatrix} 5 & 7 \\ 8 & 3 \\ 1 & 0 \end{bmatrix}$.

3. If a system of equations has three equations and three unknowns, then it may have a unique solution, an infinite number of solutions, or no solutions.

4. The only solution to the system of equations

$$2x + 3y = 7$$
$$5x - 4y = 6$$

is $x = 2$ and $y = 1$.

5. If A is a 2×3 matrix and B is a 3×4 matrix, then $A + B$ is a 2×4 matrix.

6. If A is an $n \times k$ matrix and B is a $k \times m$ matrix, then AB is an $n \times m$ matrix.

7. If A is a 4×4 matrix and B is a 4×4 matrix, then $AB = BA$.

8. A 3×4 matrix could have an inverse.

9. It is not possible to find a matrix A such that $OA = AO = I$, where O is a 5×5 zero matrix and I is a 5×5 identity matrix.

10. When solving a system of equations by the Gauss-Jordan method, we can add a nonzero multiple of the elements of one column to the corresponding elements of some nonzero multiple of some other column.

11. Every square matrix has an inverse.

12. If A, B, and C are matrices such that $AB = C$, then $B = \dfrac{C}{A}$.

13. A system of three equations in three unknowns might have exactly five positive integer solutions.

14. If A and B are matrices such that $A = B^{-1}$, then $AB = BA$.

15. If A, B, and C are matrices such that $AB = CB$, then $A = C$.

16. The difference between an open and a closed input-output model is that in a closed model, the demand matrix D is a zero matrix.

PRACTICE AND EXPLORATIONS

17. What is true about the number of solutions to a system of m linear equations in n unknowns if $m = n$? If $m < n$? If $m > n$?

18. Suppose someone says that a more reasonable way to multiply two matrices than the method presented in the text is to multiply corresponding elements. For example, the result of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix} \text{ should be } \begin{bmatrix} 3 & 10 \\ 21 & 44 \end{bmatrix},$$

according to this person. How would you respond?

Solve each system by the echelon method.

19. $2x - 3y = 14$
$3x + 2y = -5$

20. $\dfrac{x}{2} + \dfrac{y}{4} = 3$
$\dfrac{x}{4} - \dfrac{y}{2} = 4$

21. $2x - 3y + z = -5$
$5x + 5y + 3z = 14$

22. $2x + 3y + 4z = 5$
$3x + 4y + 5z = 6$

Solve each system by the Gauss-Jordan method.

23. $2x + 4y = -6$
$-3x - 5y = 12$

24. $x - 4y = 10$
$5x + 3y = 119$

25. $x - y + 3z = 13$
$4x + y + 2z = 17$
$3x + 2y + 2z = 1$

26. $x + 2y + 3z = 9$
$x - 2y = 4$
$3x + 2z = 12$

27. $3x - 6y + 9z = 12$
$-x + 2y - 3z = -4$
$x + y + 2z = 7$

28. $x - 2z = 5$
$3x + 2y = 8$
$-x + 2z = 10$

Find the size of each matrix, find the values of any variables, and identify any square, row, or column matrices.

29. $\begin{bmatrix} 2 & 3 \\ 5 & q \end{bmatrix} = \begin{bmatrix} a & b \\ c & 9 \end{bmatrix}$

30. $\begin{bmatrix} 2 & x \\ y & 6 \\ 5 & z \end{bmatrix} = \begin{bmatrix} a & -1 \\ 4 & 6 \\ p & 7 \end{bmatrix}$

31. $[2m \quad 4 \quad 3z \quad -12] = [12 \quad k+1 \quad -9 \quad r-3]$

32. $\begin{bmatrix} a+5 & 3b & 6 \\ 4c & 2+d & -3 \\ -1 & 4p & q-1 \end{bmatrix} = \begin{bmatrix} -7 & b+2 & 2k-3 \\ 3 & 2d-1 & 4l \\ m & 12 & 8 \end{bmatrix}$

Given the matrices

$$A = \begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix},$$

$$D = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}, \quad E = [1 \quad 3 \quad -4],$$

$$F = \begin{bmatrix} -1 & 4 \\ 3 & 7 \end{bmatrix}, \quad G = \begin{bmatrix} -2 & 0 \\ 1 & 5 \end{bmatrix},$$

find each of the following, if it exists.

33. $A + C$ **34.** $2G - 4F$ **35.** $3C + 2A$

36. $B - C$ **37.** $2A - 5C$ **38.** AG

39. AC **40.** DE **41.** ED

42. BD **43.** EC **44.** F^{-1}

45. B^{-1} **46.** $(A + C)^{-1}$

Find the inverse of each matrix that has an inverse.

47. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ **48.** $\begin{bmatrix} -4 & 2 \\ 0 & 3 \end{bmatrix}$

49. $\begin{bmatrix} 3 & -6 \\ -4 & 8 \end{bmatrix}$ **50.** $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$

51. $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$ **52.** $\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

53. $\begin{bmatrix} 1 & 3 & 6 \\ 4 & 0 & 9 \\ 5 & 15 & 30 \end{bmatrix}$ **54.** $\begin{bmatrix} 2 & -3 & 4 \\ 1 & 5 & 7 \\ -4 & 6 & -8 \end{bmatrix}$

Solve the matrix equation $AX = B$ for X using the given matrices.

55. $A = \begin{bmatrix} 5 & 1 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -8 \\ 24 \end{bmatrix}$

56. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

57. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 4 \\ -6 \end{bmatrix}$

58. $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 72 \\ -24 \\ 48 \end{bmatrix}$

Solve each system of equations by inverses.

59.
$$x + 2y = 4$$
$$2x - 3y = 1$$

60.
$$5x + 10y = 80$$
$$3x - 2y = 120$$

61.
$$x + y + z = 1$$
$$2x + y = -2$$
$$+ 3y + z = 2$$

62.
$$x - 4y + 2z = -1$$
$$-2x + y - 3z = -9$$
$$3x + 5y - 2z = 7$$

Find each production matrix, given the following input-output and demand matrices.

63. $A = \begin{bmatrix} 0.01 & 0.05 \\ 0.04 & 0.03 \end{bmatrix}, \quad D = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$

64. $A = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.1 & 0 & 0.2 \\ 0 & 0 & 0.4 \end{bmatrix}, \quad D = \begin{bmatrix} 500 \\ 200 \\ 100 \end{bmatrix}$

65. The following system of equations is given.

$$x + 2y + z = 7$$
$$2x - y - z = 2$$
$$3x - 3y + 2z = -5$$

a. Solve by the echelon method.

b. Solve by the Gauss-Jordan method. Compare with the echelon method.

c. Write the system as a matrix equation, $AX = B$.

d. Find the inverse of matrix A from part c.

e. Solve the system using A^{-1} from part d.

APPLICATIONS

Business and Economics

In Exercises 66–69, write a system of equations and solve.

66. Scheduling Production An office supply manufacturer makes two kinds of paper clips, standard and extra large. To make 1000 standard paper clips requires 1/4 hour on a cutting machine and 1/2 hour on a machine that shapes the clips. One thousand extra large paper clips require 1/3 hour on each machine. The manager of paper clip production has 4 hours per day available on the cutting machine and 6 hours per day on the shaping machine. How many of each kind of clip can he make?

67. Production Requirements The Waputi Indians make woven blankets, rugs, and skirts. Each blanket requires 24 hours for spinning the yarn, 4 hours for dyeing the yarn, and 15 hours for weaving. Rugs require 30, 5, and 18 hours and skirts 12, 3, and 9 hours, respectively. If there are 306, 59, and 201 hours available for spinning, dyeing, and weaving, respectively, how many of each item can be made? (*Hint:* Simplify the equations you write, if possible, before solving the system.)

68. Distribution An oil refinery in Tulsa sells 50% of its production to a Chicago distributor, 20% to a Dallas distributor, and 30% to an Atlanta distributor. Another refinery in New Orleans sells 40% of its production to the Chicago distributor, 40% to the Dallas distributor, and 20% to the Atlanta distributor. A third refinery in Ardmore sells the same distributors 30%, 40%, and 30% of its production. The three distributors received 219,000, 192,000, and 144,000 gal of oil, respectively. How many gallons of oil were produced at each of the three plants?

69. Stock Reports The New York Stock Exchange reports in daily newspapers give the dividend, price-to-earnings ratio, sales (in hundreds of shares), last price, and change in price for each company. Write the following stock reports as a 4×5 matrix: American Telephone & Telegraph: 1.33, 17.6, 152,000, 26.75,

+1.88; General Electric: 1.00, 20.0, 238,200, 32.36, −1.50; Sara Lee: 0.79, 25.4, 39,110, 16.51, −0.89; Walt Disney Company: 0.27, 21.2, 122,500, 28.60, +0.75.

70. Filling Orders A printer has three orders for pamphlets that require three kinds of paper, as shown in the following matrix.

		Order I	Order II	Order III
Paper	High-grade	10	5	8
	Medium-grade	12	0	4
	Coated	0	10	5

The printer has on hand 3170 sheets of high-grade paper, 2360 sheets of medium-grade paper, and 1800 sheets of coated paper. All the paper must be used in preparing the order.

a. Write a 3×1 matrix for the amounts of paper on hand.

b. Write a matrix of variables to represent the number of pamphlets that must be printed in each of the three orders.

c. Write a matrix equation using the given matrix and your matrices from parts a and b.

d. Solve the equation from part c.

71. Input-Output An economy depends on two commodities, goats and cheese. It takes $2/3$ of a unit of goats to produce 1 unit of cheese and $1/2$ unit of cheese to produce 1 unit of goats.

a. Write the input-output matrix for this economy.

b. Find the production required to satisfy a demand of 400 units of cheese and 800 units of goats.

72. Nebraska The 1970 economy of the state of Nebraska has been condensed to six sectors: livestock, crops, food products, mining and manufacturing, households, and other. The input-output matrix is given below. ***Source: University of Nebraska Lincoln.***

$$\begin{bmatrix} 0.178 & 0.018 & 0.411 & 0 & 0.005 & 0 \\ 0.143 & 0.018 & 0.088 & 0 & 0.001 & 0 \\ 0.089 & 0 & 0.035 & 0 & 0.060 & 0.003 \\ 0.001 & 0.010 & 0.012 & 0.063 & 0.007 & 0.014 \\ 0.141 & 0.252 & 0.088 & 0.089 & 0.402 & 0.124 \\ 0.188 & 0.156 & 0.103 & 0.255 & 0.008 & 0.474 \end{bmatrix}$$

a. Find the matrix $(I - A)^{-1}$ and interpret the value in row 2, column 1 of this matrix.

b. Suppose the demand (in millions of dollars) is

$$D = \begin{bmatrix} 1980 \\ 650 \\ 1750 \\ 1000 \\ 2500 \\ 3750 \end{bmatrix}.$$

Find the dollar amount of each commodity that should be produced.

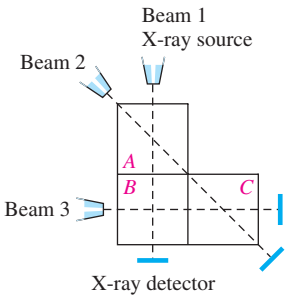
Life Sciences

73. Animal Activity The activities of a grazing animal can be classified roughly into three categories: grazing, moving, and resting. Suppose horses spend 8 hours grazing, 8 moving, and 8 resting; cattle spend 10 grazing, 5 moving, and 9 resting; sheep spend 7 grazing, 10 moving, and 7 resting; and goats spend 8 grazing, 9 moving, and 7 resting. Write this information as a 4×3 matrix.

74. CAT Scans Computer Aided Tomography (CAT) scanners take X-rays of a part of the body from different directions, and put the information together to create a picture of a cross section of the body. The amount by which the energy of the X-ray decreases, measured in linear-attenuation units, tells whether the X-ray has passed through healthy tissue, tumorous tissue, or bone, based on the following table. ***Source: The Mathematics Teacher.***

Type of Tissue	Linear-Attenuation Values
Healthy tissue	0.1625–0.2977
Tumorous tissue	0.2679–0.3930
Bone	0.3857–0.5108

The part of the body to be scanned is divided into cells. If an X-ray passes through more than one cell, the total linear-attenuation value is the sum of the values for the cells. For example, in the figure, let a, b, and c be the values for cells A, B, and C. The attenuation value for beam 1 is $a + b$ and for beam 2 is $a + c$.



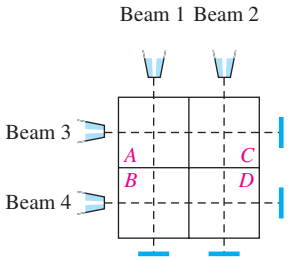
a. Find the attenuation value for beam 3.

b. Suppose that the attenuation values are 0.8, 0.55, and 0.65 for beams 1, 2, and 3, respectively. Set up and solve the system of three equations for a, b, and c. What can you conclude about cells A, B, and C?

c. Find the inverse of the coefficient matrix from part b to find a, b, and c for the following three cases, and make conclusions about cells A, B, and C for each.

Patient	Linear-Attenuation Values Beam 1	Beam 2	Beam 3
X	0.54	0.40	0.52
Y	0.65	0.80	0.75
Z	0.51	0.49	0.44

75. CAT Scans (Refer to Exercise 74.) Four X-ray beams are aimed at four cells, as shown in the following figure. ***Source: The Mathematics Teacher.***

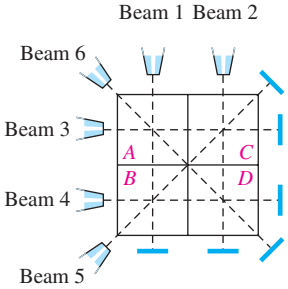


a. Suppose the attenuation values for beams 1, 2, 3, and 4 are 0.60, 0.75, 0.65, and 0.70, respectively. Do we have enough information to determine the values of a, b, c, and d? Explain.

b. Suppose we have the data from part a, as well as the following values for d. Find the values for a, b, and c, and draw conclusions about cells A, B, C, and D in each case.

(i) 0.33 **(ii)** 0.43

c. Two X-ray beams are added, as shown in the figure. In addition to the data in part a, we now have attenuation values for beams 5 and 6 of 0.85 and 0.50. Find the values for a, b, c, and d, and make conclusions about cells A, B, C, and D. 109



d. Six X-ray beams are not necessary because four appropriately chosen beams are sufficient. Give two examples of four beams (chosen from beams 1–6 in part c) that will give the solution. (*Note:* There are 12 possible solutions.)

e. Discuss what properties the four beams selected in part d must have in order to provide a unique solution.

76. Hockey In a recent study, the number of head and neck injuries among hockey players wearing full face shields and half face shields were compared. The following table provides the rates per 1000 athlete-exposures for specific injuries that caused a player wearing either shield to miss one or more events. ***Source: JAMA.***

	Half Shield	Full Shield
Head and Face Injuries (excluding Concussions)	3.54	1.41
Concussions	1.53	1.57
Neck Injuries	0.34	0.29
Other	7.53	6.21

If an equal number of players in a large league wear each type of shield and the total number of athlete-exposures for the league in a season is 8000, use matrix operations to estimate the total number of injuries of each type.

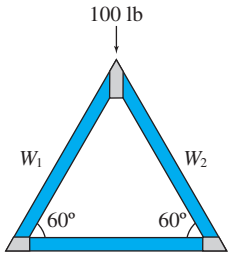
Physical Sciences

77. Roof Trusses Linear systems occur in the design of roof trusses for new homes and buildings. The simplest type of roof truss is a triangle. The truss shown in the figure below is used to frame roofs of small buildings. If a 100-lb force is applied at the peak of the truss, then the forces or weights W_1 and W_2 exerted parallel to each rafter of the truss are determined by the following linear system of equations.

$$\frac{\sqrt{3}}{2}(W_1 + W_2) = 100$$

$$W_1 - W_2 = 0$$

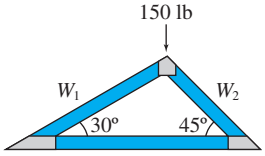
Solve the system to find W_1 and W_2. ***Source: Structural Analysis.***



78. Roof Trusses (Refer to Exercise 77.) Use the following system of equations to determine the force or weights W_1 and W_2 exerted on each rafter for the truss shown in the figure.

$$\frac{1}{2}W_1 + \frac{\sqrt{2}}{2}W_2 = 150$$

$$\frac{\sqrt{3}}{2}W_1 - \frac{\sqrt{2}}{2}W_2 = 0$$



79. Carbon Dioxide Determining the amount of carbon dioxide in the atmosphere is important because carbon dioxide is known to be a greenhouse gas. Carbon dioxide concentrations (in parts per million) have been measured at Mauna Loa, Hawaii, for more than 40 years. The concentrations have increased quadratically. The table lists readings for 3 years. *Source: Scripps Institution of Oceanography.*

Year	CO_2
1960	317
1980	339
2004	377

a. If the relationship between the carbon dioxide concentration C and the year t is expressed as $C = at^2 + bt + c$, where $t = 0$ corresponds to 1960, use a linear system of equations to determine the constants a, b, and c.

b. Predict the year when the amount of carbon dioxide in the atmosphere will double from its 1960 level. (*Hint:* This requires solving a quadratic equation. For review on how to do this, see Section R.4.)

80. Chemistry When carbon monoxide (CO) reacts with oxygen (O_2), carbon dioxide (CO_2) is formed. This can be written as $CO + (1/2)O_2 = CO_2$ and as a matrix equation. If we form a 2×1 column matrix by letting the first element be the number of carbon atoms and the second element be the number of oxygen atoms, then CO would have the column matrix *Source: Journal of Chemical Education.*

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Similarly, O_2 and CO_2 would have the column matrices $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively.

a. Use the Gauss-Jordan method to find numbers x and y (known as *stoichiometric numbers*) that solve the system of equations

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Compare your answers to the equation written above.

b. Repeat the process for $xCO_2 + yH_2 + zCO = H_2O$, where H_2 is hydrogen, and H_2O is water. In words, what does this mean?

General Interest

81. Students Suppose 20% of the boys and 30% of the girls in a high school like tennis, and 60% of the boys and 90% of the girls like math. If 500 students like tennis and 1500 like math, how many boys and girls are in the school? Find all possible solutions.

82. Baseball In the 2009 Major League Baseball season, slugger Ichiro Suzuki had a total of 225 hits. The number of singles he hit was 11 more than four times the combined total of doubles and home runs. The number of doubles he hit was 1 more than twice the combined total of triples and home runs. The number of singles and home runs together was 15 more than five times the combined total of doubles and triples. Find the number of singles, doubles, triples, and home runs that Suzuki hit during the season. *Source: Baseball-Reference.com.*

83. Cookies Regular Nabisco Oreo cookies are made of two chocolate cookie wafers surrounding a single layer of vanilla cream. The claim on the package states that a single serving is 34 g, which is three cookies. Nabisco Double Stuf cookies are made of the same two chocolate cookie wafers surrounding a double layer of vanilla cream. The claim on this package states that a single serving is 29 g, which is two Double Stuf cookies. If the Double Stuf cookies truly have a double layer of vanilla cream, find the weight of a single chocolate wafer and the weight of a single layer of vanilla cream.

EXTENDED APPLICATION

CONTAGION

Suppose that three people have contracted a contagious disease. *Source: Finite Mathematics with Applications to Business, Life Sciences, and Social Sciences.* A second group of five people may have been in contact with the three infected persons. A third group of six people may have been in contact with the second group. We can form a 3×5 matrix P with rows representing the first group of three and columns representing the second group of five. We enter a one in the corresponding position if a person in the first group has contact with a person in the second group. These direct contacts are called *first-order contacts.* Similarly, we form a 5×6 matrix Q representing the first-order contacts between the second and third group. For example, suppose

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$Q = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

From matrix P we see that the first person in the first group had contact with the first and fourth persons in the second group. Also, none of the first group had contact with the last person in the second group.

A *second-order contact* is an indirect contact between persons in the first and third groups through some person in the second group. The product matrix PQ indicates these contacts. Verify that the second-row, fourth-column entry of PQ is 1. That is, there is one second-order contact between the second person in group one and the fourth person in group three. Let a_{ij} denote the element in the ith row and jth column of the matrix PQ. By looking at the products that form a_{24} below, we see that the common contact was with the fourth individual in group two. (The p_{ij} are entries in P, and the q_{ij} are entries in Q.)

$$\begin{aligned} a_{24} &= p_{21}q_{14} + p_{22}q_{24} + p_{23}q_{34} + p_{24}q_{44} + p_{25}q_{54} \\ &= 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \\ &= 1 \end{aligned}$$

The second person in group 1 and the fourth person in group 3 both had contact with the fourth person in group 2.

This idea could be extended to third-, fourth-, and larger-order contacts. It indicates a way to use matrices to trace the spread of a contagious disease. It could also pertain to the dispersal of ideas or anything that might pass from one individual to another.

EXERCISES

1. Find the second-order contact matrix PQ mentioned in the text.
2. How many second-order contacts were there between the second contagious person and the third person in the third group?
3. Is there anyone in the third group who has had no contacts at all with the first group?
4. The totals of the columns in PQ give the total number of second-order contacts per person, while the column totals in P and Q give the total number of first-order contacts per person. Which person(s) in the third group had the most contacts, counting first- and second-order contacts?
5. Go to the website WolframAlpha.com and enter: "multiply matrices." Study how matrix multiplication can be performed by Wolfram|Alpha. Try Exercise 1 with Wolfram|Alpha and discuss how it compares with Microsoft Excel and with your graphing calculator.

DIRECTIONS FOR GROUP PROJECT

Assume that your group (3–5 students) is trying to map the spread of a new disease. Suppose also that the information given above has been obtained from interviews with the first three people that were hospitalized with symptoms of the disease and their contacts. Using the questions above as a guide, prepare a presentation for a public meeting that describes the method of obtaining the data, the data itself, and addresses the spirit of each question. Formulate a strategy for how to handle the spread of this disease to other people. The presentation should be mathematically sound, grammatically correct, and professionally crafted. Use presentation software, such as Microsoft PowerPoint, to present your findings.

3 Linear Programming: The Graphical Method

3.1 Graphing Linear Inequalities

3.2 Solving Linear Programming Problems Graphically

3.3 Applications of Linear Programming

Chapter 3 Review

Extended Application: Sensitivity Analysis

An oil refinery turns crude oil into many different products, including gasoline and fuel oil. Efficient management requires matching the output of each product to the demand and the available shipping capacity. In an exercise in Section 3, we explore the use of linear programming to allocate refinery production for maximum profit.

Many realistic problems involve inequalities—a factory can manufacture *no more* than 12 items on a shift, or a medical researcher must interview *at least* a hundred patients to be sure that a new treatment for a disease is better than the old treatment. *Linear inequalities* of the form $ax + by \le c$ (or with $\ge$, $<$, or $>$ instead of $\le$) can be used in a process called *linear programming* to *optimize* (find the maximum or minimum value of a quantity) for a given situation.

In this chapter we introduce some *linear programming* problems that can be solved by graphical methods. Then, in Chapter 4, we discuss the simplex method, a general method for solving linear programming problems with many variables.

3.1 Graphing Linear Inequalities

APPLY IT **How can a company determine the feasible number of units of each product to manufacture in order to meet all production requirements?**
We will answer this question in Example 6 by graphing a set of inequalities.

As mentioned earlier, a linear inequality is defined as follows.

Linear Inequality

A **linear inequality** in two variables has the form

$$ax + by \le c$$
$$ax + by < c,$$
$$ax + by \ge c,$$
or $$ax + by > c,$$

for real numbers a, b, and c, with a and b not both 0.

EXAMPLE 1 Graphing an Inequality

Graph the linear inequality $2x - 3y \le 12$.

SOLUTION Because of the "=" portion of $\le$, the points of the line $2x - 3y = 12$ satisfy the linear inequality $2x - 3y \le 12$ and are part of its graph. As in Chapter 1, find the intercepts by first letting $x = 0$ and then letting $y = 0$; use these points to get the graph of $2x - 3y = 12$ shown in Figure 1.

FOR REVIEW

Recall from Chapter 1 that one way to sketch a line is to first let $x = 0$ to find the y-intercept, then let $y = 0$ to find the x-intercept. For example, given $2x - 3y = 12$, letting $x = 0$ yields $-3y = 12$, so $y = -4$, and the corresponding point is $(0, -4)$. Letting $y = 0$ yields $2x = 12$, so $x = 6$ and the point is (6, 0). Plot these two points, as in Figure 1, then use a straightedge to draw a line through them.

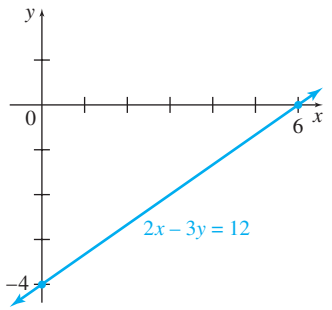


FIGURE 1

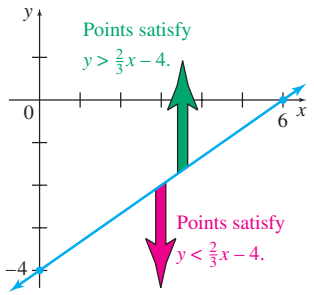


FIGURE 2

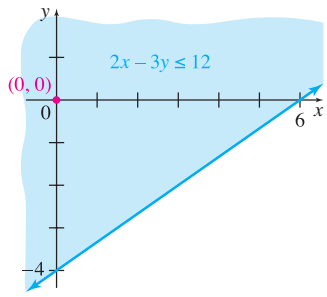


FIGURE 3

The points on the line satisfy "$2x - 3y$ *equals* 12." To locate the points satisfying "$2x - 3y$ *is less than* or equal to 12," first solve $2x - 3y \le 12$ for y.

$$\begin{aligned} 2x - 3y &\le 12 \\ -3y &\le -2x + 12 \qquad \text{Subtract } 2x. \\ y &\ge \frac{2}{3}x - 4 \qquad \text{Multiply by } -\frac{1}{3}. \end{aligned}$$

(Recall that multiplying both sides of an inequality by a negative number reverses the direction of the inequality symbol.)

As shown in Figure 2, the points *above* the line $2x - 3y = 12$ satisfy

$$y > \frac{2}{3}x - 4,$$

while those below the line satisfy

$$y < \frac{2}{3}x - 4.$$

In summary, the inequality $2x - 3y \le 12$ is satisfied by all points *on or above* the line $2x - 3y = 12$. Indicate the points above the line by shading, as in Figure 3. The line and shaded region in Figure 3 make up the graph of the linear inequality $2x - 3y \le 12$.

CAUTION In this chapter, be sure to use a straightedge to draw lines and to plot the points with care. A sloppily drawn line could give a deceptive picture of the region being considered.

In Example 1, the line $2x - 3y = 12$, which separates the points in the solution from the points that are not in the solution, is called the **boundary**.

There is an alternative way to find the correct region to shade or to check the method shown above. Choose as a test point any point not on the boundary line. For example, in Example 1 we could choose the point (0, 0), which is not on the line $2x - 3y = 12$. Substitute 0 for x and 0 for y in the given inequality.

$$\begin{aligned} 2x - 3y &\le 12 \\ 2(0) - 3(0) &\le 12 \\ 0 &\le 12 \qquad \text{True} \end{aligned}$$

Since the result $0 \le 12$ is true, the test point (0, 0) belongs on the side of the boundary where all points satisfy $2x - 3y < 12$. For this reason, we shade the side containing (0, 0), as in Figure 3. Choosing a point on the other side of the line, such as (4, −3), would produce a false result when the values $x = 4$ and $y = -3$ were substituted into the given inequality. In such a case, we would shade the side of the line *not including* the test point.

NOTE Many of the inequalities in this chapter are of the form $ax + by \le c$ or $ax + by \ge c$. These are most easily graphed by changing the $\le$ or $\ge$ to $=$ and then plotting the two intercepts by letting $x = 0$ and $y = 0$, as in Example 1. Shade the appropriate side of the line as described above. If both $a > 0$ and $b > 0$ the shaded region for $ax + by \le c$ is *below* the line, while the shaded region for $ax + by \ge c$ is *above* the line.

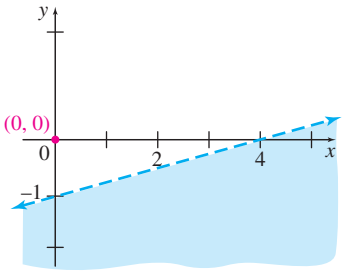


FIGURE 4

EXAMPLE 2 Graphing an Inequality

Graph $x - 4y > 4$.

SOLUTION The boundary here is the line $x - 4y = 4$. Since the points on this line do not satisfy $x - 4y > 4$, the line is drawn dashed, as in Figure 4. To decide whether to shade the

region above the line or the region below the line, we will choose a test point not on the boundary line. Choosing (0, 0), we replace x with 0 and y with 0:

$$x - 4y > 4$$
$$0 - 4(0) > 4$$
$$0 > 4. \quad \text{False}$$

The correct half-plane is the one that does *not* contain (0, 0); the region below the boundary line is shaded, as shown in Figure 4.

YOUR TURN 1
Graph $3x + 2y \le 18$.

TRY YOUR TURN 1

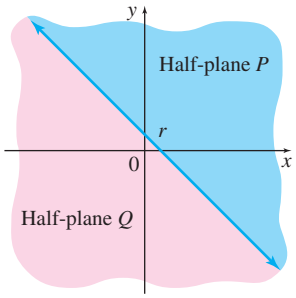


FIGURE 5

CAUTION Be careful. If the point $(0, 0)$ is on the boundary line, it cannot be used as a test point.

As the examples above suggest, the graph of a linear inequality is represented by a shaded region in the plane, perhaps including the line that is the boundary of the region. Each shaded region is an example of a **half-plane**, a region on one side of a line. For example, in Figure 5 line r divides the plane into half-planes P and Q. The points on r belong neither to P nor to Q. Line r is the boundary of each half-plane.

TECHNOLOGY NOTE

Graphing calculators can shade regions on the plane. TI calculators have a DRAW menu that includes an option to shade above or below a line. For instance, to graph the inequality in Example 2, first solve the equation for y, then use your calculator to graph the line $y = (1/4)x - 1$. Then the command `Shade(-10, (1/4)X - 1)` produces Figure 6(a).

The TI-84 Plus calculator offers another way to graph the region above or below a line. Press the Y= key. Note the slanted line to the left of Y_1, Y_2, and so on. Use the left arrow key to move the cursor to that position for Y_1. Press ENTER until you see the symbol ◣. This indicates that the calculator will shade below the line whose equation is entered in Y_1. (The symbol ◥ operates similarly to shade above a line.) We used this method to get the graph in Figure 6(b). For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

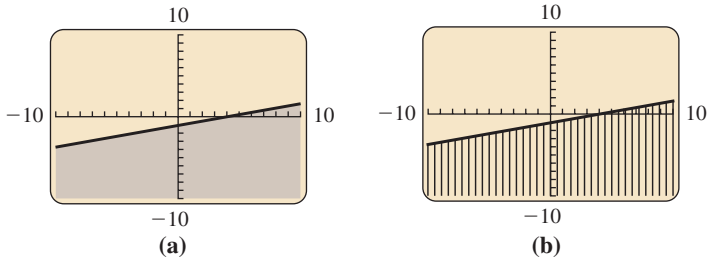


FIGURE 6

The steps in graphing a linear inequality are summarized below.

Graphing a Linear Inequality

1. Draw the graph of the boundary line. Make the line solid if the inequality involves $\le$ or $\ge$; make the line dashed if the inequality involves $<$ or $>$.
2. Decide which half-plane to shade. Use either of the following methods.
   a. Solve the inequality for y; shade the region above the line if the inequality is of the form $y >$ or $y \ge$; shade the region below the line if the inequality is of the form $y <$ or $y \le$.
   b. Choose any point not on the line as a test point. Shade the half-plane that includes the test point if the test point satisfies the original inequality; otherwise, shade the half-plane on the other side of the boundary line.

Systems of Inequalities

Realistic problems often involve many inequalities. For example, a manufacturing problem might produce inequalities resulting from production requirements as well as inequalities about cost requirements. A collection of at least two inequalities is called a **system of inequalities**. The solution of a system of inequalities is made up of all those points that satisfy all the inequalities of the system at the same time. To graph the solution of a system of inequalities, graph all the inequalities on the same axes and identify, by heavy shading or direction arrows, the region common to all graphs. The next example shows how this is done.

EXAMPLE 3 Graphing a System of Inequalities

Graph the system

$$y < -3x + 12$$
$$x < 2y.$$

SOLUTION The graph of the first inequality has the line $y = -3x + 12$ as its boundary. Because of the $<$ symbol, we use a dotted line and shade *below* the line. The second inequality should first be solved for y to get $y > (1/2)x$ to see that the graph is the region *above* the dotted boundary line $y = (1/2)x$.

The heavily shaded region in Figure 7(a) shows all the points that satisfy both inequalities of the system. Since the points on the boundary lines are not in the solution, the boundary lines are dashed.

NOTE
When shading regions by hand, it may be difficult to tell what is shaded heavily and what is shaded only lightly, particularly when more than two inequalities are involved. In such cases, an alternative technique, called *reverse shading,* is to shade the region *opposite* that of the inequality. In other words, the region that is *not* wanted can be shaded. Then, when the various regions are shaded, whatever is not shaded is the desired region. We will not use this technique in this text, but you may wish to try it on your own.

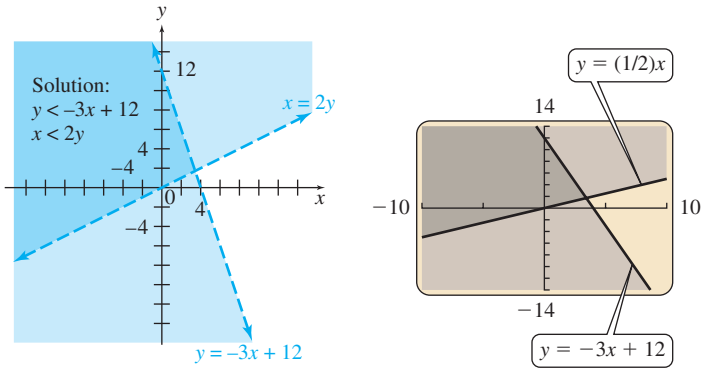


FIGURE 7

TECHNOLOGY NOTE
A calculator graph of the system in Example 3 is shown in Figure 7(b). You can graph this system on a TI-84 Plus using `Shade(Y2, Y1)`.

A region consisting of the overlapping parts of two or more graphs of inequalities in a system, such as the heavily shaded region in Figure 7, is sometimes called the **region of feasible solutions** or the **feasible region**, since it is made up of all the points that satisfy (are feasible for) all inequalities of the system.

EXAMPLE 4 Graphing a Feasible Region

Graph the feasible region for the system

$$y \le -2x + 8$$
$$-2 \le x \le 1.$$

SOLUTION The boundary line of the first inequality is $y = -2x + 8$. Because of the $\le$ symbol, we use a solid line and shade *below* the line.

YOUR TURN 2 Graph the feasible region for the system

$$3x - 4y \geq 12$$
$$x + y \geq 0.$$

The second inequality is a compound inequality, indicating $-2 \leq x$ *and* $x \leq 1$. Recall that the graph $x = -2$ is the vertical line through $(-2, 0)$ and that the graph $x = 1$ is the vertical line through $(1, 0)$. For $-2 \leq x$, we draw a vertical line and shade the region to the right. For $x \leq 1$, we draw a vertical line and shade the region to the left.

The shaded region in Figure 8 shows all the points that satisfy the system of inequalities. TRY YOUR TURN 2

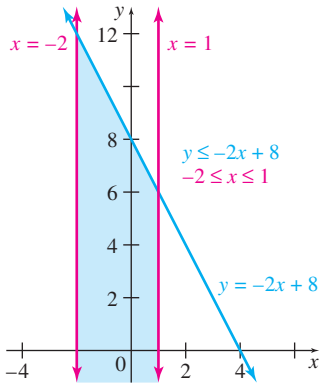


FIGURE 8

EXAMPLE 5 Graphing a Feasible Region

Graph the feasible region for the system

$$2x - 5y \leq 10$$
$$x + 2y \leq 8$$
$$x \geq 0$$
$$y \geq 0.$$

SOLUTION On the same axes, graph each inequality by graphing the boundary and choosing the appropriate half-plane. Then find the feasible region by locating the overlap of all the half-planes. This feasible region is shaded in Figure 9. The inequalities $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant.

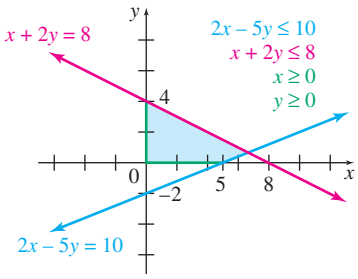


FIGURE 9

The feasible region in Example 5 is **bounded**, since the region is enclosed by boundary lines on all sides. On the other hand, the feasible regions in Examples 3 and 4 are **unbounded.**

Applications

As shown in Section 3 of this chapter, many realistic problems lead to systems of linear inequalities. The next example is typical of such problems.

EXAMPLE 6 Manufacturing

APPLY IT

Happy Ice Cream Cone Company makes cake cones and sugar cones, both of which must be processed in the mixing department and the baking department. Manufacturing one batch of cake cones requires 1 hour in the mixing department and 2 hours in the baking department, and producing one batch of sugar cones requires 2 hours in the mixing department and 1 hour in the baking department. Each department is operated for at most 12 hours per day.

(a) Write a system of inequalities that expresses these restrictions.

SOLUTION Let x represent the number of batches of cake cones made and y represent the number of batches of sugar cones made. Then, make a table that summarizes the given information.

Production Requirements for Ice Cream Cones

	Cake	Sugar		Total
Number of Units Made	x	y		
Hours in Mixing Dept.	1	2	$\leq$	12
Hours in Baking Dept.	2	1	$\leq$	12

Since the departments operate at most 12 hours per day, we put the total number of hours as ≤ 12. Putting the inequality ($\leq$ or $\geq$) next to the number in the chart may help you remember which way to write the inequality.

In the mixing department, x batches of cake cones require a total of $1 \cdot x = x$ hours, and y batches of sugar cones require $2 \cdot y = 2y$ hours. Since the mixing department can operate no more than 12 hours per day,

$$x + 2y \leq 12. \quad \text{Mixing department}$$

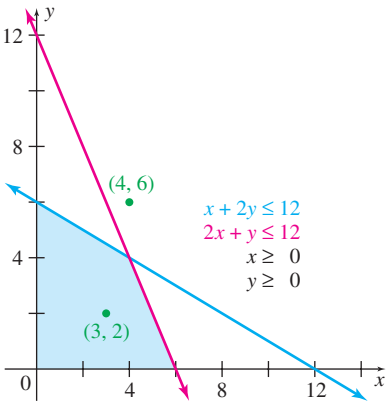


FIGURE 10

We translated "no more than" as "less than or equal to." Notice how this inequality corresponds to the row in the table for the mixing department. Similarly, the row corresponding to the baking department gives

$$2x + y \le 12. \qquad \text{Baking department}$$

Since it is not possible to produce a negative number of cake cones or sugar cones,

$$x \ge 0 \qquad \text{and} \qquad y \ge 0.$$

(b) Graph the feasible region.

SOLUTION The feasible region for this system of inequalities is shown in Figure 10.

(c) Using the graph from part (b), can 3 batches of cake cones and 2 batches of sugar cones be manufactured in one day? Can 4 batches of cake cones and 6 batches of sugar cones be manufactured in one day?

SOLUTION Three batches of cake cones and two batches of sugar cones correspond to the point (3, 2). Since (3, 2) is in the feasible region in Figure 10, it is possible to manufacture these quantities in one day. However, since (4, 6) is *not* in the feasible region in Figure 10, it is *not* possible to manufacture 4 batches of cake cones and 6 batches of sugar cones in one day.

The following steps summarize the process of finding the feasible region.

1. Form a table that summarizes the information.

2. Convert the table into a set of linear inequalities.

3. Graph each linear inequality.

4. Graph the region that is common to all the regions graphed in step 3.

3.1 EXERCISES

Graph each linear inequality.

1. $x + y \le 2$

2. $y \le x + 1$

3. $x \ge 2 - y$

4. $y \ge x - 3$

5. $4x - y < 6$

6. $4y + x > 6$

7. $4x + y < 8$

8. $2x - y > 2$

9. $x + 3y \ge -2$

10. $2x + 3y \le 6$

11. $x \le 3y$

12. $2x \ge y$

13. $x + y \le 0$

14. $3x + 2y \ge 0$

15. $y < x$

16. $y > 5x$

17. $x < 4$

18. $y > 5$

19. $y \le -2$

20. $x \ge -4$

Graph the feasible region for each system of inequalities. Tell whether each region is bounded or unbounded.

21.
$$x + y \le 1$$
$$x - y \ge 2$$

22.
$$4x - y < 6$$
$$3x + y < 9$$

23.
$$x + 3y \le 6$$
$$2x + 4y \ge 7$$

24.
$$-x - y < 5$$
$$2x - y < 4$$

25.
$$x + y \le 7$$
$$x - y \le -4$$
$$4x + y \ge 0$$

26.
$$3x - 2y \ge 6$$
$$x + y \le -5$$
$$y \le -6$$

27.
$$-2 < x < 3$$
$$-1 \le y \le 5$$
$$2x + y < 6$$

28.
$$1 < x < 4$$
$$y > 2$$
$$x > y$$

29.
$$y - 2x \le 4$$
$$y \ge 2 - x$$
$$x \ge 0$$
$$y \ge 0$$

30.
$$2x + 3y \le 12$$
$$2x + 3y > 3$$
$$3x + y < 4$$
$$x \ge 0$$
$$y \ge 0$$

31.
$$3x + 4y > 12$$
$$2x - 3y < 6$$
$$0 \le y \le 2$$
$$x \ge 0$$

32.
$$0 \le x \le 9$$
$$x - 2y \ge 4$$
$$3x + 5y \le 30$$
$$y \ge 0$$

Use a graphing calculator to graph the following.

33. $2x - 6y > 12$

34. $4x - 3y < 12$

35.
$$3x - 4y < 6$$
$$2x + 5y > 15$$

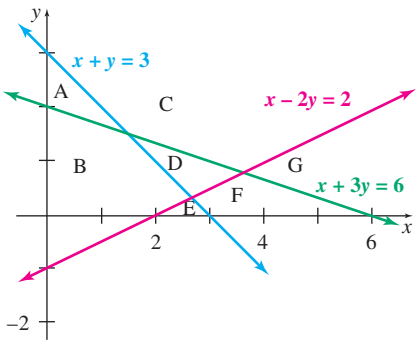
36.
$$6x - 4y > 8$$
$$2x + 5y < 5$$

37. The regions A through G in the figure can be described by the inequalities

$$x + 3y \;?\; 6$$
$$x + y \;?\; 3$$
$$x - 2y \;?\; 2$$
$$x \geq 0$$
$$y \geq 0,$$

where ? can be either $\leq$ or $\geq$. For each region, tell what the ? should be in the three inequalities. For example, for region A, the ? should be $\geq$, $\leq$, and $\leq$, because region A is described by the inequalities

$$x + 3y \geq 6$$
$$x + y \leq 3$$
$$x - 2y \leq 2$$
$$x \geq 0$$
$$y \geq 0.$$



APPLICATIONS

Business and Economics

38. Production Scheduling A small pottery shop makes two kinds of planters, glazed and unglazed. The glazed type requires 1/2 hour to throw on the wheel and 1 hour in the kiln. The unglazed type takes 1 hour to throw on the wheel and 6 hours in the kiln. The wheel is available for at most 8 hours per day, and the kiln for at most 20 hours per day.

a. Complete the following table.

	Glazed	Unglazed	Total
Number Made	x	y	
Time on Wheel			
Time in Kiln			

b. Set up a system of inequalities and graph the feasible region.

c. Using your graph from part b, can 5 glazed and 2 unglazed planters be made? Can 10 glazed and 2 unglazed planters be made?

39. Time Management Carmella and Walt produce handmade shawls and afghans. They spin the yarn, dye it, and then weave it. A shawl requires 1 hour of spinning, 1 hour of dyeing, and 1 hour of weaving. An afghan needs 2 hours of spinning, 1 hour of dyeing, and 4 hours of weaving. Together, they spend at most 8 hours spinning, 6 hours dyeing, and 14 hours weaving.

a. Complete the following table.

	Shawls	Afghans	Total
Number Made	x	y	
Spinning Time			
Dyeing Time			
Weaving Time			

b. Set up a system of inequalities and graph the feasible region.

c. Using your graph from part b, can 3 shawls and 2 afghans be made? Can 4 shawls and 3 afghans be made?

For Exercises 40–45, perform the following steps.

a. Write a system of inequalities to express the conditions of the problem.

b. Graph the feasible region of the system.

40. Transportation Southwestern Oil supplies two distributors located in the Northwest. One distributor needs at least 3000 barrels of oil, and the other needs at least 5000 barrels. Southwestern can send out at most 10,000 barrels. Let x = the number of barrels of oil sent to distributor 1 and y = the number sent to distributor 2.

41. Finance The loan department in a bank will use at most $30 million for commercial and home loans. The bank's policy is to allocate at least four times as much money to home loans as to commercial loans. The bank's return is 6% on a home loan and 8% on a commercial loan. The manager of the loan department wants to earn a return of at least $1.6 million on these loans. Let x = the amount (in millions) for home loans and y = the amount (in millions) for commercial loans.

42. Transportation The California Almond Growers have at most 2400 boxes of almonds to be shipped from their plant in Sacramento to Des Moines and San Antonio. The Des Moines market needs at least 1000 boxes, while the San Antonio market must have at least 800 boxes. Let x = the number of boxes to be shipped to Des Moines and y = the number of boxes to be shipped to San Antonio.

43. Management The Gillette Company produces two popular battery-operated razors, the M3Power™ and the Fusion Power™. Because of demand, the number of M3Power™ razors is never more than one-half the number of Fusion Power™ razors. The factory's production cannot exceed 800 razors per day. Let x = the number of M3Power™ razors and y = the number of Fusion Power™ razors produced per day.

44. Production Scheduling A cement manufacturer produces at least 3.2 million barrels of cement annually. He is told by the Environmental Protection Agency (EPA) that his operation emits 2.5 lb of dust for each barrel produced. The EPA has ruled that annual emissions must be reduced to no more than 1.8 million lb. To do this, the manufacturer plans to replace the

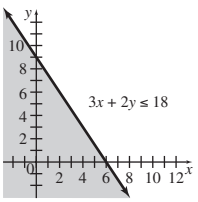
present dust collectors with two types of electronic precipitators. One type would reduce emissions to 0.5 lb per barrel and operating costs would be 16¢ per barrel. The other would reduce the dust to 0.3 lb per barrel and operating costs would be 20¢ per barrel. The manufacturer does not want to spend more than 0.8 million dollars in operating costs on the precipitators. He needs to know how many barrels he could produce with each type. Let x = the number of barrels (in millions) produced with the first type and y = the number of barrels (in millions) produced with the second type.

Life Sciences

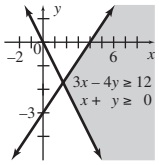
45. Nutrition A dietician is planning a snack package of fruit and nuts. Each ounce of fruit will supply 1 unit of protein, 2 units of carbohydrates, and 1 unit of fat. Each ounce of nuts will supply 1 unit of protein, 1 unit of carbohydrates, and 1 unit of fat. Every package must provide at least 7 units of protein, at least 10 units of carbohydrates, and no more than 9 units of fat. Let x = the ounces of fruit and y = the ounces of nuts to be used in each package.

YOUR TURN ANSWERS

1.



2.



3.2 Solving Linear Programming Problems Graphically

Many mathematical models designed to solve problems in business, biology, and economics involve finding an optimum value (maximum or minimum) of a function, subject to certain restrictions. In a **linear programming** problem, we must find the maximum or minimum value of a function, called the **objective function**, and satisfy a set of restrictions, or **constraints**, given by linear inequalities. When only two variables are involved, the solution to a linear programming problem can be found by first graphing the set of constraints, then finding the feasible region as discussed in the previous section. This method is explained in the following example.

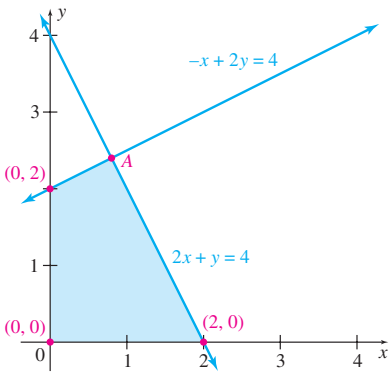


FIGURE 11

EXAMPLE 1 Maximization

Find the maximum value of the objective function $z = 3x + 4y$, subject to the following constraints.

$$\begin{aligned} 2x + y &\le 4 \\ -x + 2y &\le 4 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

SOLUTION The feasible region is graphed in Figure 11. We can find the coordinates of point A, $(4/5, 12/5)$, by solving the system

$$\begin{aligned} 2x + y &= 4 \\ -x + 2y &= 4. \end{aligned}$$

Every point in the feasible region satisfies all the constraints; however, we want to find those points that produce the maximum possible value of the objective function. To see how to find this maximum value, change the graph of Figure 11 by adding lines that represent the objective function $z = 3x + 4y$ for various sample values of z. By choosing the values 0, 5, 10, and 15 for z, the objective function becomes (in turn)

$$0 = 3x + 4y, \quad 5 = 3x + 4y, \quad 10 = 3x + 4y, \quad \text{and} \quad 15 = 3x + 4y.$$

FOR REVIEW

Recall from Chapter 2 that two equations in two unknowns can be solved by using row operations to eliminate one variable. For example, to solve the system

$$\begin{aligned} 2x + y &= 4 \\ -x + 2y &= 4, \end{aligned}$$

we could take the first equation plus 2 times the second to eliminate x. (This is equivalent to $R_1 + 2R_2 \to R_2$ in the Gauss-Jordan method.) The result is $5y = 12$, so $y = 12/5$. We can then substitute this value of y into either equation and solve for x. For example, substitution into the first equation yields

$$\begin{aligned} 2x + \frac{12}{5} &= 4 \\ 2x &= \frac{8}{5} \\ x &= \frac{4}{5}. \end{aligned}$$

We instead could have subtracted the original second equation from twice the first equation to eliminate y, yielding $5x = 4$, or $x = 4/5$, and then substitute into either equation and solve for y.

These four lines (known as **isoprofit lines**) are graphed in Figure 12. (Why are the lines parallel?) The figure shows that z cannot take on the value 15 because the graph for $z = 15$ is entirely outside the feasible region. The maximum possible value of z will be obtained from a line parallel to the others and between the lines representing the objective function when $z = 10$ and $z = 15$. The value of z will be as large as possible and all constraints will be satisfied if this line just touches the feasible region. This occurs at point A. We find that A has coordinates $(4/5, 12/5)$. (See the review in the margin.) The value of z at this point is

$$z = 3x + 4y = 3\left(\frac{4}{5}\right) + 4\left(\frac{12}{5}\right) = \frac{60}{5} = 12.$$

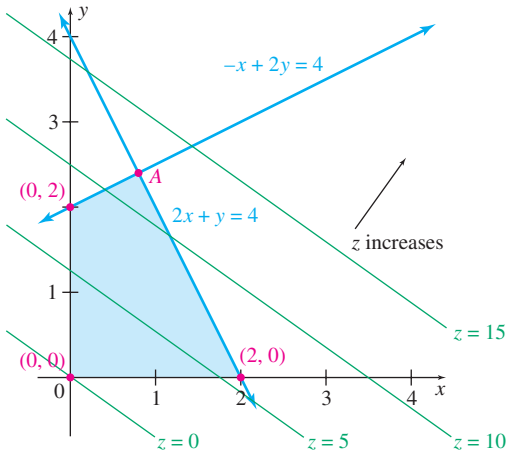


FIGURE 12

The maximum possible value of z is 12. Of all the points in the feasible region, A leads to the largest possible value of z.

TECHNOLOGY NOTE

A graphing calculator can be useful for finding the coordinates of intersection points such as point A. We do this by solving each equation for y, graphing each line, and then using the capability of the calculator to find the coordinates of the point of intersection.

Points such as A in Example 1 are called corner points. A **corner point** is a point in the feasible region where the boundary lines of two constraints cross. Since corner points occur where two straight lines cross, the coordinates of a corner point are the solution of a system of two linear equations. As we saw in Example 1, corner points play a key role in the solution of linear programming problems. We will make this explicit after the following example.

EXAMPLE 2 Minimization

Solve the following linear programming problem.

$$\begin{aligned} \text{Minimize} \quad & z = 2x + 4y \\ \text{subject to:} \quad & x + 2y \geq 10 \\ & 3x + y \geq 10 \\ & x \geq 0 \\ & y \geq 0. \end{aligned}$$

SOLUTION Figure 13 on the next page shows the feasible region and the lines that result when z in the objective function is replaced by 0, 10, 20, 40, and 50. The line representing the objective function touches the region of feasible solutions when $z = 20$. Two corner

points, $(2, 4)$ and $(10, 0)$, lie on this line; both $(2, 4)$ and $(10, 0)$, as well as all the points on the boundary line between them, give the same optimum value of z. There are infinitely many equally good values of x and y that will give the same minimum value of the objective function $z = 2x + 4y$. This minimum value is 20.

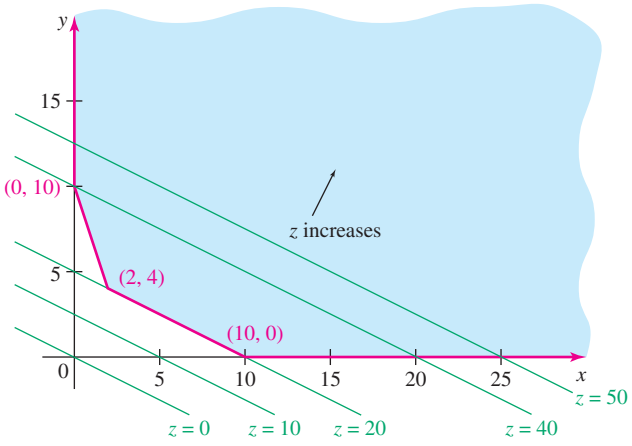
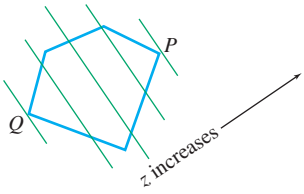
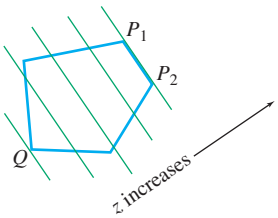


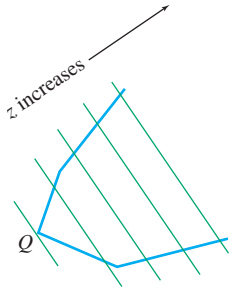
FIGURE 13



(a)



(b)



(c)

FIGURE 14

As long as the feasible region is not empty, linear programming problems with bounded regions always have solutions. On the other hand, the feasible region in Example 2 is unbounded, and no solution will *maximize* the value of the objective function because z can be made as large as you like.

Some general conclusions can be drawn from the method of solution used in Examples 1 and 2. Imagine a line sliding across a region. Figure 14 shows various feasible regions and the position of the line $ax + by = z$ for various values of z. We assume here that as the line slides from the lower left to the upper right, the value of z increases. In Figure 14(a), the objective function takes on its minimum value at corner point Q and its maximum value at P. The minimum is again at Q in part (b), but the maximum occurs at P_1 or P_2, or any point on the line segment connecting them. Finally, in part (c), the minimum value occurs at Q, but the objective function has no maximum value because the feasible region is unbounded. As long as the objective function increases as x and y increase, the objective function will have no maximum over an unbounded region.

The preceding discussion suggests the truth of the **corner point theorem**.

Corner Point Theorem

If an optimum value (either a maximum or a minimum) of the objective function exists, it will occur at one or more of the corner points of the feasible region.

This theorem simplifies the job of finding an optimum value. First, we graph the feasible region and find all corner points. Then we test each corner point in the objective function. Finally, we identify the corner point producing the optimum solution. For unbounded regions, we must decide whether the required optimum can be found (see Example 2).

With the theorem, we can solve the problem in Example 1 by first identifying the four corner points in Figure 11: $(0, 0)$, $(0, 2)$, $(4/5, 12/5)$, and $(2, 0)$. Then we substitute each of the four points into the objective function $z = 3x + 4y$ to identify the corner point that produces the maximum value of z.

Values of the Objective Function at Corner Points		
Corner Point	Value of $z = 3x + 4y$	
$(0, 0)$	$3(0) + 4(0) = 0$	
$(0, 2)$	$3(0) + 4(2) = 8$	
$\left(\frac{4}{5}, \frac{12}{5}\right)$	$3\left(\frac{4}{5}\right) + 4\left(\frac{12}{5}\right) = 12$	Maximum
$(2, 0)$	$3(2) + 4(0) = 6$	

From these results, the corner point $(4/5, 12/5)$ yields the maximum value of 12. This is the same as the result found earlier.

The following summary gives the steps to use in solving a linear programming problem by the graphical method.

NOTE
As the corner point theorem states and Example 2 illustrates, the optimal value of a linear programming problem may occur at more than one corner point. When the optimal solution occurs at two corner points, every point on the line segment between the two points is also an optimal solution.

Solving a Linear Programming Problem

1. Write the objective function and all necessary constraints.
2. Graph the feasible region.
3. Identify all corner points.
4. Find the value of the objective function at each corner point.
5. For a bounded region, the solution is given by the corner point producing the optimum value of the objective function.
6. For an unbounded region, check that a solution actually exists. If it does, it will occur at a corner point.

When asked to solve a linear programming problem, give the maximum or minimum value, as well any points where that value occurs.

EXAMPLE 3 Maximization and Minimization

Sketch the feasible region for the following set of constraints, and then find the maximum and minimum values of the objective function $z = x + 10y$.

$$x + 4y \geq 12$$
$$x - 2y \leq 0$$
$$2y - x \leq 6$$
$$x \leq 6$$

SOLUTION The graph in Figure 15 shows that the feasible region is bounded. Use the corner points from the graph to find the maximum and minimum values of the objective function.

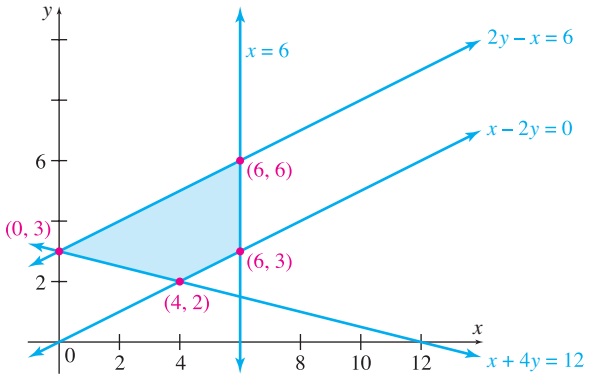


FIGURE 15

Values of the Objective Function at Corner Points		
Corner Point	Value of $z = x + 10y$	
$(0, 3)$	$0 + 10(3) = 30$	
$(4, 2)$	$4 + 10(2) = 24$	Minimum
$(6, 3)$	$6 + 10(3) = 36$	
$(6, 6)$	$6 + 10(6) = 66$	Maximum

YOUR TURN 1 Find the maximum and minimum values of the objective function $z = 3x + 4y$ on the region bounded by

$$\begin{aligned} 2x + y &\geq 5 \\ x + 5y &\geq 16 \\ 2x + y &\leq 14 \\ -x + 4y &\leq 20 \end{aligned}$$

The minimum value of $z = x + 10y$ is 24 at the corner point $(4, 2)$. The maximum value is 66 at $(6, 6)$. TRY YOUR TURN 1

To verify that the minimum or maximum is correct in a linear programming problem, you might want to add the graph of the line $z = 0$ to the graph of the feasible region. For instance, in Example 3, the result of adding the line $x + 10y = 0$ is shown in Figure 16. Now imagine moving a straightedge through the feasible region parallel to this line. It appears that the first place the line touches the feasible region is at $(4, 2)$, where we found the minimum. Similarly, the last place the line touches is at $(6, 6)$, where we found the maximum. In Figure 16, these parallel lines, labeled $z = 24$ and $z = 66$, are also shown.

NOTE
The graphical method is very difficult for a problem with three variables (where the feasible region is three dimensional) and impossible with four or more variables. A method for solving such problems will be shown in the next chapter.

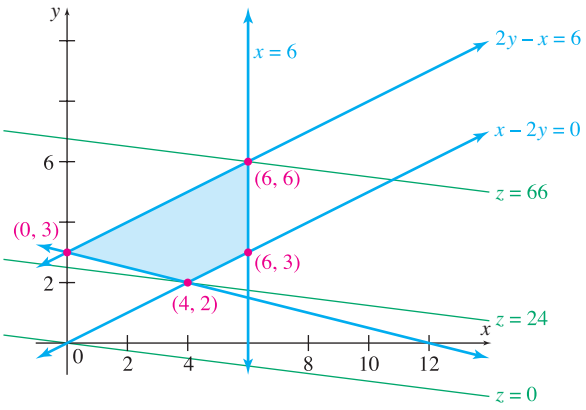


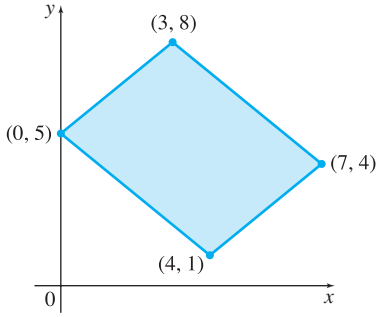
FIGURE 16

3.2 EXERCISES

The following graphs show regions of feasible solutions. Use these regions to find maximum and minimum values of the given objective functions.

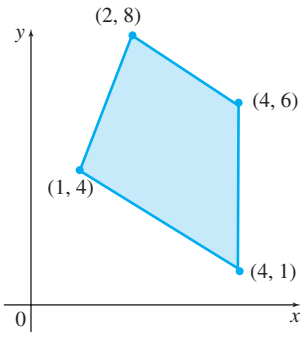
1. a. $z = 3x + 2y$

b. $z = x + 4y$



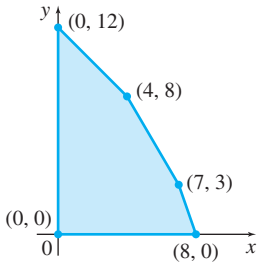
2. a. $z = x + 4y$

b. $z = 5x + 2y$



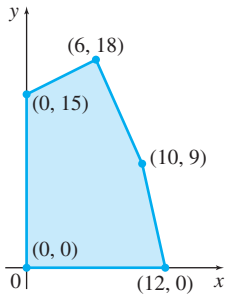
3. a. $z = 0.40x + 0.75y$

b. $z = 1.50x + 0.25y$



4. a. $z = 0.35x + 1.25y$

b. $z = 1.5x + 0.5y$

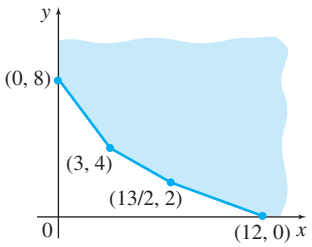


5. a. $z = 4x + 2y$

b. $z = 2x + 3y$

c. $z = 2x + 4y$

d. $z = x + 4y$

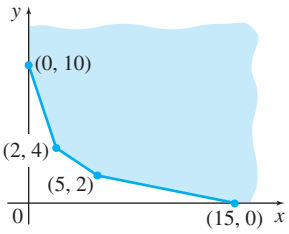


6. a. $z = 4x + y$

b. $z = 5x + 6y$

c. $z = x + 2y$

d. $z = x + 6y$



Use graphical methods to solve each linear programming problem.

7. Minimize $z = 4x + 7y$

subject to:
$$x - y \geq 1$$
$$3x + 2y \geq 18$$
$$x \geq 0$$
$$y \geq 0.$$

8. Minimize $z = x + 3y$

subject to:
$$x + y \leq 10$$
$$5x + 2y \geq 20$$
$$-x + 2y \geq 0$$
$$x \geq 0$$
$$y \geq 0.$$

9. Maximize $z = 5x + 2y$

subject to:
$$4x - y \leq 16$$
$$2x + y \geq 11$$
$$x \geq 3$$
$$y \leq 8.$$

10. Maximize $z = 10x + 8y$

subject to:
$$2x + 3y \leq 100$$
$$5x + 4y \leq 200$$
$$x \geq 10$$
$$0 \leq y \leq 20.$$

11. Maximize $z = 10x + 10y$

subject to:
$$5x + 8y \geq 200$$
$$25x - 10y \geq 250$$
$$x + y \leq 150$$
$$x \geq 0$$
$$y \geq 0.$$

12. Maximize $z = 4x + 5y$

subject to:
$$10x - 5y \leq 100$$
$$20x + 10y \geq 150$$
$$x + y \geq 12$$
$$x \geq 0$$
$$y \geq 0.$$

13. Maximize $z = 3x + 6y$

subject to:
$$2x - 3y \leq 12$$
$$x + y \leq 5$$
$$3x + 4y \geq 24$$
$$x \geq 0$$
$$y \geq 0.$$

14. Maximize $z = 4x + 6y$

subject to:
$$3 \leq x + y \leq 10$$
$$x - y \geq 3$$
$$x \geq 0$$
$$y \geq 0.$$

15. Find values of $x \geq 0$ and $y \geq 0$ that maximize $z = 10x + 12y$ subject to each set of constraints.

a. $x + y \leq 20$, $x + 3y \leq 24$

b. $3x + y \leq 15$, $x + 2y \leq 18$

c. $2x + 5y \geq 22$, $4x + 3y \leq 28$, $2x + 2y \leq 17$

16. Find values of $x \geq 0$ and $y \geq 0$ that minimize $z = 3x + 2y$ subject to each set of constraints.

a. $10x + 7y \leq 42$, $4x + 10y \geq 35$

b. $6x + 5y \geq 25$, $2x + 6y \geq 15$

c. $x + 2y \geq 10$, $2x + y \geq 12$, $x - y \leq 8$

17. You are given the following linear programming problem:*

$$\begin{aligned} \text{Maximize} \quad & z = c_1x_1 + c_2x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \leq 11 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

If $c_2 > 0$, determine the range of c_1/c_2 for which $(x_1, x_2) = (4, 3)$ is an optimal solution. (Choose one of the following.) ***Source: Society of Actuaries.***

a. $[-2, 1/2]$ **b.** $[-1/2, 2]$ **c.** $[-11, -1]$

d. $[1, 11]$ **e.** $[-11, 11]$

*The notation x_1 and x_2 used in this exercise is the same as x and y.

YOUR TURN ANSWERS

1. Maximum of 36 at (4,6); minimum of 15 at (1,3)

3.3 Applications of Linear Programming

APPLY IT **How many canoes and kayaks should a business purchase, given a limited budget and limited storage?**

We will use linear programming to answer this question in Example 1.

EXAMPLE 1 Canoe Rentals

Andrew Crowley plans to start a new business called River Explorers, which will rent canoes and kayaks to people to travel 10 miles down the Clarion River in Cook Forest State Park. He has $45,000 to purchase new boats. He can buy the canoes for $600 each and the kayaks for $750 each. His facility can hold up to 65 boats. The canoes will rent for $25 a day, and the kayaks will rent for $30 a day. How many canoes and how many kayaks should he buy to earn the most revenue if all boats can be rented each day?

APPLY IT

SOLUTION Let x represent the number of canoes and let y represent the number of kayaks. Summarize the given information in a table.

Rental Information				
	Canoes	Kayaks		Total
Number of Boats	x	y	$\leq$	65
Cost of Each	$600	$750	$\leq$	$45,000
Revenue	$25	$30		

The constraints, imposed by the number of boats and the cost, correspond to the rows in the table as follows.

$$x + y \leq 65$$

$$600x + 750y \leq 45{,}000$$

Dividing both sides of the second constraint by 150 gives the equivalent inequality

$$4x + 5y \leq 300.$$

Since the number of boats cannot be negative, $x \geq 0$ and $y \geq 0$. The objective function to be maximized gives the amount of revenue. If the variable z represents the total revenue, the objective function is

$$z = 25x + 30y.$$

In summary, the mathematical model for the given linear programming problem is as follows:

$$\begin{aligned} \text{Maximize} \quad & z = 25x + 30y && (1)\\ \text{subject to:} \quad & x + y \le 65 && (2)\\ & 4x + 5y \le 300 && (3)\\ & x \ge 0 && (4)\\ & y \ge 0. && (5) \end{aligned}$$

Using the methods described in the previous section, graph the feasible region for the system of inequalities (2)–(5), as in Figure 17. Three of the corner points can be identified from the graph as (0, 0), (65, 0), and (0, 60). The fourth corner point, labeled Q in the figure, can be found by solving the system of equations

$$\begin{aligned} x + y &= 65\\ 4x + 5y &= 300. \end{aligned}$$

Solve this system to find that Q is the point (25, 40). Now test these four points in the objective function to determine the maximum value of z. The results are shown in the table.

YOUR TURN 1 Solve Example 1 with everything the same except the revenue for a kayak changed to $35 a day.

YOUR TURN 2 Suppose that in Example 6 in Sec. 3.1, the company earns $20 in profit for each batch of cake cones and $30 for each batch of sugar cones. How many batches of each should the company make to maximize profit?

Values of the Objective Function at Corner Points

Corner Point	Value of $z = 25x + 30y$	
$(0, 0)$	$25(0) + 30(0) = 0$	
$(65, 0)$	$25(65) + 30(0) = 1625$	
$(0, 60)$	$25(0) + 30(60) = 1800$	
$(25, 40)$	$25(25) + 30(40) = 1825$	Maximum

The objective function, which represents revenue, is maximized when $x = 25$ and $y = 40$. He should buy 25 canoes and 40 kayaks for a maximum revenue of $1825 a day.

TRY YOUR TURN 1 AND 2

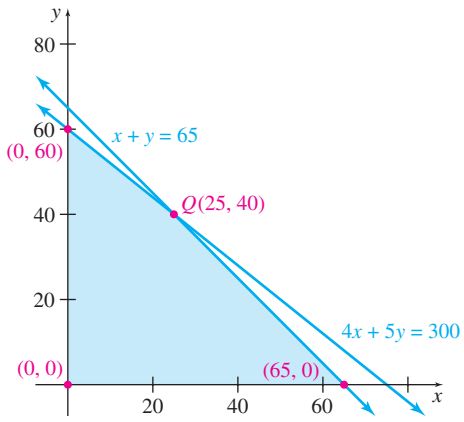


FIGURE 17

Fortunately, the answer to the linear programming problem in Example 1 is a point with integer coordinates, as the number of each type of boat must be an integer. Unfortunately, there is no guarantee that this will always happen at a corner point. When the solution to a linear programming problem is restricted to integers, it is an *integer programming* problem, which is more difficult to solve than a linear programming problem. The feasible region for an integer programming problem consists only of those points with integer coordinates that satisfy the constraints. In this text, all problems in which fractional solutions are meaningless are contrived to have integer solutions.

EXAMPLE 2 Farm Animals

A 4-H member raises only goats and pigs. She wants to raise no more than 16 animals, including no more than 10 goats. She spends $25 to raise a goat and $75 to raise a pig, and she has $900 available for this project. Each goat produces $12 in profit and each pig $40 in profit. How many goats and how many pigs should she raise to maximize total profit?

SOLUTION First, set up a table that shows the information given in the problem.

4-H Animal Information				
	Goats	**Pigs**		**Total**
Number Raised	x	y	$\leq$	16
Goat Limit	x		$\leq$	10
Cost to Raise	$25	$75	$\leq$	$900
Profit (each)	$12	$40		

Use the table to write the necessary constraints. Since the total number of animals cannot exceed 16, the first constraint is

$$x + y \leq 16.$$

"No more than 10 goats" means

$$x \leq 10.$$

The cost to raise x goats at $25 per goat is $25x$ dollars, while the cost for y pigs at $75 each is $75y$ dollars. Since only $900 is available,

$$25x + 75y \leq 900.$$

Dividing both sides by 25 gives the equivalent inequality

$$x + 3y \leq 36.$$

The number of goats and pigs cannot be negative, so

$$x \geq 0 \quad \text{and} \quad y \geq 0.$$

The 4-H member wants to know how many goats and pigs to raise in order to produce maximum profit. Each goat yields $12 profit and each pig $40. If z represents total profit, then

$$z = 12x + 40y.$$

In summary, we have the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & z = 12x + 40y \\ \text{subject to:} \quad & x + y \leq 16 \\ & x + 3y \leq 36 \\ & x \leq 10 \\ & x \geq 0 \\ & y \geq 0. \end{aligned}$$

A graph of the feasible region is shown in Figure 18. The corner points $(0, 12)$, $(0, 0)$, and $(10, 0)$ can be read directly from the graph. The coordinates of each of the other corner points can be found by solving a system of linear equations.

Test each corner point in the objective function to find the maximum profit.

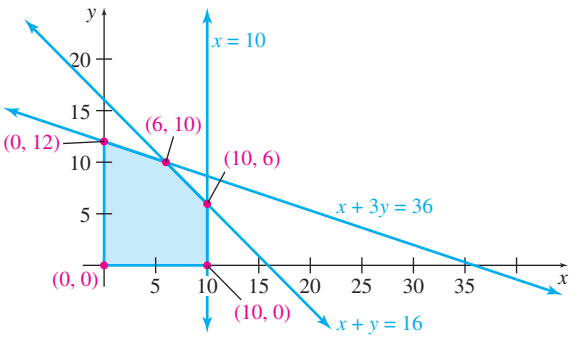


FIGURE 18

YOUR TURN 3 Solve Example 2 with everything the same except the total amount available for the project changed to \$1050.

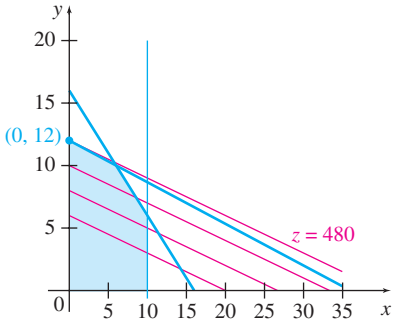


FIGURE 19

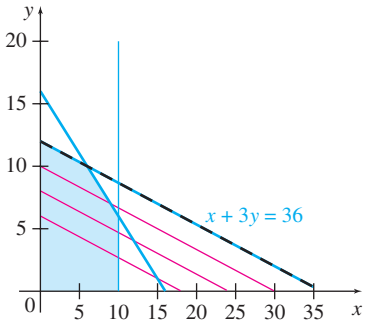


FIGURE 20

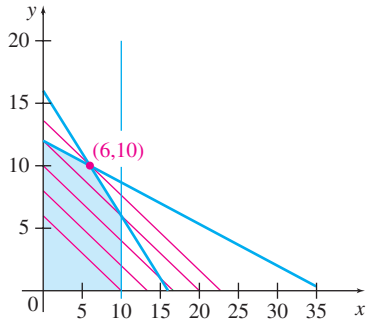


FIGURE 21

Values of the Objective Function at Corner Points

Corner Point	Value of $z = 12x + 40y$	
$(0, 12)$	$12(0) + 40(12) = 480$	Maximum
$(6, 10)$	$12(6) + 40(10) = 472$	
$(10, 6)$	$12(10) + 40(6) = 360$	
$(10, 0)$	$12(10) + 40(0) = 120$	
$(0, 0)$	$12(0) + 40(0) = 0$	

The maximum of 480 occurs at $(0, 12)$. Thus, 12 pigs and no goats will produce a maximum profit of \$480. **TRY YOUR TURN 3**

In the maximization problem in Example 2, since the profit for a single pig is \$40 and the profit for a single goat is only \$12, it is more profitable to raise only pigs and no goats. However, if the profit from raising pigs begins to decrease (or the profit from goats begins to increase), it will eventually be more profitable to raise both goats and pigs. In fact, if the profit from raising pigs decreases to a number below \$36, then the previous solution is no longer optimal.

To see why this is true, in Figure 19 we have graphed the original objective function $(z = 12x + 40y)$ for various values of z, as we did in Example 1 of the previous section. Notice that each of these objective lines has slope $m = -12/40 = -3/10$. When $z = 480$, the line touches only one feasible point, (0, 12), which is where the maximum profit occurs.

If the profit from raising pigs decreases from \$40 to \$$p$, where p is a value slightly below 40, the objective function lines will have the equation $z = 12x + py$ for various values of z, and the slope of the lines becomes $m = -12/p$. Eventually, as p becomes smaller, the slope of these objective lines will be equal to the slope of the line $x + 3y = 36$ (that is, $-1/3$), corresponding to the second constraint. This occurs when $-12/p = -1/3$, or $p = 36$, as illustrated by the overlapping blue and dotted lines in Figure 20. In this case, the optimal solution occurs at every point on the line segment that joins (0, 12) and (6, 10).

Once the profit from raising pigs decreases to below \$36, the slopes of the sample objective function lines become more negative (steeper) and the optimal solution changes, as indicated in Figure 21. As z increases, the last feasible point that the lines touch is (6, 10). For profits from raising pigs that are slightly below \$36, the optimal solution will occur when $x = 6$ and $y = 10$. In other words, the maximum profit will occur when she raises both goats and pigs.

EXAMPLE 3 Nutrition

Certain animals in a rescue shelter must have at least 30 g of protein and at least 20 g of fat per feeding period. These nutrients come from food A, which costs 18 cents per unit and supplies 2 g of protein and 4 g of fat; and food B, which costs 12 cents per unit and has 6 g of protein and 2 g of fat. Food B is bought under a long-term contract requiring that at least 2 units of B be used per serving. Another contract requires that the amount of food B used be no more than 3 times the amount of food A used.

(a) How much of each food must be bought to produce the minimum cost per serving?

SOLUTION Let x represent the required amount of food A and y the amount of food B. Use the given information to prepare the following table.

Rescue Animal Nutrition Information				
	Food A	**Food B**		**Total**
Number of Units	x	y		
Grams of Protein	2	6	$\geq$	30
Grams of Fat	4	2	$\geq$	20
Long-Term Contract		y	$\geq$	2
Cost	18¢	12¢		

Since the animals must have *at least* 30 g of protein and 20 g of fat, we use $\geq$ in the inequality. If the animals needed *at most* a certain amount of some nutrient, we would use $\leq$. The long-term contract requires that $y \geq 2$.

In addition to the information in the table, we also have the requirement that the amount of food B used be no more than 3 times the amount of food A used. We can write this as $y \leq 3x$.

The linear programming problem can be stated as follows.

$$\begin{aligned} \text{Minimize} \quad & z = 0.18x + 0.12y \\ \text{subject to:} \quad & 2x + 6y \geq 30 \quad \text{Protein} \\ & 4x + 2y \geq 20 \quad \text{Fat} \\ & 2 \leq y \leq 3x \quad \text{Contracts} \\ & x \geq 0. \end{aligned}$$

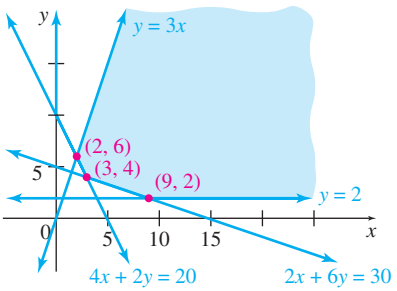


FIGURE 22

(The usual constraint $y \geq 0$ is redundant because of the constraint $y \geq 2$.) A graph of the feasible region is shown in Figure 22. The corner points are $(2, 6)$, $(3, 4)$, and $(9, 2)$. Test each corner point in the objective function to find the minimum cost.

Values of the Objective Function at Corner Points		
Corner Point	**Value of $z = 0.18x + 0.12y$**	
$(2, 6)$	$0.18(2) + 0.12(6) = 1.08$	
$(3, 4)$	$0.18(3) + 0.12(4) = 1.02$	Minimum
$(9, 2)$	$0.18(9) + 0.12(2) = 1.86$	

The minimum of 1.02 occurs at $(3, 4)$. Thus, 3 units of food A and 4 units of food B will produce a minimum cost of \$1.02 per serving.

(b) The rescue shelter manager notices that although the long-term contract states that at least 2 units of food B be used per serving, the solution uses 4 units of food B, which is 2 units more than the minimum amount required. Can a more economical solution be found that only uses 2 units of food B?

SOLUTION The solution found in part (a) is the most economical solution, even though it exceeds the requirement for using at least 2 units of food B. Notice from Figure 22 that the four lines representing the four constraints do not meet at a single point, so any solution in the feasible region will have to exceed at least one constraint. The rescue shelter manager might use this information to negotiate a better deal with the distributor of food B by making a guarantee to use at least 4 units of food B per serving in the future.

The notion that some constraints are not met exactly is related to the concepts of *surplus* and *slack variables*, which will be explored in the next chapter.

The feasible region in Figure 22 is an *unbounded* feasible region—the region extends indefinitely to the upper right. With this region it would not be possible to *maximize* the objective function, because the total cost of the food could always be increased by encouraging the animals to eat more.

3.3 EXERCISES

Write Exercises 1–6 as linear inequalities. Identify all variables used. (Note: Not all of the given information is used in Exercises 5 and 6.)

1. Product A requires 3 hours on a machine, while product B needs 5 hours on the same machine. The machine is available for at most 60 hours per week.
2. A cow requires a third of an acre of pasture and a sheep needs a quarter acre. A rancher wants to use at least 120 acres of pasture.
3. Jessica Corpo needs at least 1500 units of calcium supplements per day. Her calcium carbonate supplement provides 600 units, and her calcium citrate supplement supplies 250 units.
4. Pauline Wong spends 3 hours selling a small computer and 5 hours selling a larger model. She works no more than 45 hours per week.
5. Coffee costing \$8 per lb is to be mixed with coffee costing \$10 per lb to get at least 40 lb of a blended coffee.
6. A tank in an oil refinery holds 120 gal. The tank contains a mixture of light oil worth \$1.25 per gal and heavy oil worth \$0.80 per gal.

APPLICATIONS

Business and Economics

7. **Transportation** The Miers Company produces small engines for several manufacturers. The company receives orders from two assembly plants for their Top-flight engine. Plant I needs at least 45 engines, and plant II needs at least 32 engines. The company can send at most 120 engines to these two assembly plants. It costs \$30 per engine to ship to plant I and \$40 per engine to ship to plant II. Plant I gives Miers \$20 in rebates toward its products for each engine they buy, while plant II gives similar \$15 rebates. Miers estimates that they need at least \$1500 in rebates to cover products they plan to buy from the two plants. How many engines should be shipped to each plant to minimize shipping costs? What is the minimum cost?
8. **Transportation** A manufacturer of refrigerators must ship at least 100 refrigerators to its two West Coast warehouses. Each warehouse holds a maximum of 100 refrigerators. Warehouse A holds 25 refrigerators already, and warehouse B has 20 on hand. It costs \$12 to ship a refrigerator to warehouse A and \$10 to ship one to warehouse B. Union rules require that at least 300 workers be hired. Shipping a refrigerator to warehouse A requires 4 workers, while shipping a refrigerator to warehouse B requires 2 workers. How many refrigerators should be shipped to each warehouse to minimize costs? What is the minimum cost?
9. **Insurance Premiums** A company is considering two insurance plans with the types of coverage and premiums shown in the following table.

	Policy A	Policy B
Fire/Theft	\$10,000	\$15,000
Liability	\$180,000	\$120,000
Premium	\$50	\$40

(For example, this means that \$50 buys one unit of plan A, consisting of \$10,000 fire and theft insurance and \$180,000 of liability insurance.)

a. The company wants at least \$300,000 fire/theft insurance and at least \$3,000,000 liability insurance from these plans. How

many units should be purchased from each plan to minimize the cost of the premiums? What is the minimum premium?

b. Suppose the premium for policy A is reduced to $25. Now how many units should be purchased from each plan to minimize the cost of the premiums? What is the minimum premium?

10. Profit The Muro Manufacturing Company makes two kinds of plasma screen television sets. It produces the Flexscan set that sells for $350 profit and the Panoramic I that sells for $500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I. Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hours on the assembly line, 900 work-hours in the cabinet shop, and 2600 work-hours in the testing and packing department.

a. How many sets of each type should it produce to make a maximum profit? What is the maximum profit?

b. Suppose the profit on the Flexscan goes up to $450. Now how many sets of each type should it produce to make a maximum profit? What is the maximum profit?

c. The solutions from parts a and b leave some unused time in either the assembly line, the cabinet shop, or the testing and packing department. Identify any unused time in each solution. Is it possible to have a solution that leaves no excess time? Explain.

11. Revenue A machine shop manufactures two types of bolts. The bolts require time on each of the three groups of machines, but the time required on each group differs, as shown in the table below.

	Type I	Type II
Machine 1	0.2 min	0.2 min
Machine 2	0.6 min	0.2 min
Machine 3	0.04 min	0.08 min

Production schedules are made up one day at a time. In a day, 300, 720, and 100 minutes are available, respectively, on these machines. Type I bolts sell for 15¢ and type II bolts for 20¢.

a. How many of each type of bolt should be manufactured per day to maximize revenue?

b. What is the maximum revenue?

c. Suppose the selling price of type I bolts began to increase. Beyond what amount would this price have to increase before a different number of each type of bolts should be produced to maximize revenue?

12. Revenue The manufacturing process requires that oil refineries must manufacture at least 2 gal of gasoline for every gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gal a day must be produced. The demand for gasoline is no more than 6.4 million gal per day. It takes 15 minutes to ship each million gal of gasoline and 1 hour to ship each million gal of fuel oil out of the warehouse. No more than 4 hours and 39 minutes are available for shipping.

a. If the refinery sells gasoline for $2.50 per gal and fuel oil for $2 per gal, how much of each should be produced to maximize revenue?

b. Find the maximum revenue.

c. Suppose the price for fuel oil begins to increase. Beyond what amount would this price have to increase before a different amount of gasoline and fuel oil should be produced to maximize revenue?

13. Revenue A candy company has 150 kg of chocolate-covered nuts and 90 kg of chocolate-covered raisins to be sold as two different mixes. One mix will contain half nuts and half raisins and will sell for $7 per kg. The other mix will contain 3/4 nuts and 1/4 raisins and will sell for $9.50 per kg.

a. How many kilograms of each mix should the company prepare for the maximum revenue? Find the maximum revenue.

b. The company raises the price of the second mix to $11 per kg. Now how many kilograms of each mix should the company prepare for the maximum revenue? Find the maximum revenue.

14. Profit A small country can grow only two crops for export, coffee and cocoa. The country has 500,000 hectares of land available for the crops. Long-term contracts require that at least 100,000 hectares be devoted to coffee and at least 200,000 hectares to cocoa. Cocoa must be processed locally, and production bottlenecks limit cocoa to 270,000 hectares. Coffee requires two workers per hectare, with cocoa requiring five. No more than 1,750,000 people are available for working with these crops. Coffee produces a profit of $220 per hectare and cocoa a profit of $550 per hectare. How many hectares should the country devote to each crop in order to maximize profit? Find the maximum profit.

15. Blending The Mostpure Milk Company gets milk from two dairies and then blends the milk to get the desired amount of butterfat for the company's premier product. Milk from dairy I costs $2.40 per gal, and milk from dairy II costs $0.80 per gal. At most $144 is available for purchasing milk. Dairy I can supply at most 50 gal of milk averaging 3.7% butterfat. Dairy II can supply at most 80 gal of milk averaging 3.2% butterfat.

a. How much milk from each dairy should Mostpure use to get at most 100 gal of milk with the maximum total amount of butterfat? What is the maximum amount of butterfat?

b. The solution from part a leaves both dairy I and dairy II with excess capacity. Calculate the amount of additional milk each dairy could produce. Is there any way all this capacity could be used while still meeting the other constraints? Explain.

16. Transportation A flash drive manufacturer has 370 boxes of a particular drive in warehouse I and 290 boxes of the same drive in warehouse II. A computer store in San Jose orders 350 boxes of the drive, and another store in Memphis orders 300 boxes. The shipping costs per box to these stores from the two warehouses are shown in the following table.

		Destination	
		San Jose	Memphis
Warehouse	*I*	$2.50	$2.20
	II	$2.30	$2.10

How many boxes should be shipped to each city from each warehouse to minimize shipping costs? What is the minimum cost? (*Hint:* Use x, $350 - x$, y, and $300 - y$ as the variables.)

17. Finance A pension fund manager decides to invest a total of at most $30 million in U.S. Treasury bonds paying 4% annual interest and in mutual funds paying 8% annual interest. He plans to invest at least $5 million in bonds and at least $10 million in mutual funds. Bonds have an initial fee of $100 per million dollars, while the fee for mutual funds is $200 per million. The fund manager is allowed to spend no more than $5000 on fees. How much should be invested in each to maximize annual interest? What is the maximum annual interest?

Manufacturing (Note: Exercises 18–20 are from qualification examinations for Certified Public Accountants.) ***Source: American Institute of Certified Public Accountants.*** **The Random Company manufactures two products, Zeta and Beta. Each product must pass through two processing operations. All materials are introduced at the start of Process No. 1. There are no work-in-process inventories. Random may produce either one product exclusively or various combinations of both products subject to the following constraints:**

	Process No. 1	Process No. 2	Contribution Margin (per unit)
Hours Required to Produce One Unit:			
Zeta	1 hr	1 hr	$4.00
Beta	2 hr	3 hr	$5.25
Total Capacity (in hours per day)	1000 hr	1275 hr	

A shortage of technical labor has limited Beta production to 400 units per day. There are no constraints on the production of Zeta other than the hour constraints in the above schedule. Assume that all relationships between capacity and production are linear.

18. Given the objective to maximize total contribution margin, what is the production constraint for Process No. 1? (Choose one of the following.)

a. Zeta + Beta $\leq$ 1000 **b.** Zeta + 2 Beta $\leq$ 1000

c. Zeta + Beta $\geq$ 1000 **d.** Zeta + 2 Beta $\geq$ 1000

19. Given the objective to maximize total contribution margin, what is the labor constraint for production of Beta? (Choose one of the following.)

a. Beta $\leq$ 400 **b.** Beta $\geq$ 400

c. Beta $\leq$ 425 **d.** Beta $\geq$ 425

20. What is the objective function of the data presented? (Choose one of the following.)

a. Zeta + 2 Beta = $9.25

b. $4.00 Zeta + 3($5.25)Beta = Total Contribution Margin

c. $4.00 Zeta + $5.25 Beta = Total Contribution Margin

d. 2($4.00) Zeta + 3($5.25) Beta = Total Contribution Margin

Life Sciences

21. Health Care Glen Spencer takes vitamin pills. Each day he must have at least 480 IU of vitamin A, 5 mg of vitamin B_1, and 100 mg of vitamin C. He can choose between pill 1, which contains 240 IU of vitamin A, 1 mg of vitamin B_1, and 10 mg of vitamin C, and pill 2, which contains 60 IU of vitamin A, 1 mg of vitamin B_1, and 35 mg of vitamin C. Pill 1 costs 15¢, and pill 2 costs 30¢.

a. How many of each pill should he buy in order to minimize his cost? What is the minimum cost?

b. For the solution in part a, Glen is receiving more than he needs of at least one vitamin. Identify that vitamin, and tell how much surplus he is receiving. Is there any way he can avoid receiving that surplus while still meeting the other constraints and minimizing the cost? Explain.

22. Predator Food Requirements A certain predator requires at least 10 units of protein and 8 units of fat per day. One prey of species I provides 5 units of protein and 2 units of fat; one prey of species II provides 3 units of protein and 4 units of fat. Capturing and digesting each species-II prey requires 3 units of energy, and capturing and digesting each species-I prey requires 2 units of energy. How many of each prey would meet the predator's daily food requirements with the least expenditure of energy? Are the answers reasonable? How could they be interpreted?

23. Nutrition A dietician is planning a snack package of fruit and nuts. Each ounce of fruit will supply zero units of protein, 2 units of carbohydrates, and 1 unit of fat, and will contain 20 calories. Each ounce of nuts will supply 3 units of protein, 1 unit of carbohydrate, and 2 units of fat, and will contain 30 calories. Every package must provide at least 6 units of protein, at least 10 units of carbohydrates, and no more than 9 units of fat. Find the number of ounces of fruit and number of ounces of nuts that will meet the requirement with the least number of calories. What is the least number of calories?

24. Health Care Ms. Oliveras was given the following advice. She should supplement her daily diet with at least 6000 USP units of vitamin A, at least 195 mg of vitamin C, and at least 600 USP units of vitamin D. Ms. Oliveras finds that Mason's Pharmacy carries Brand X vitamin pills at 5¢ each and Brand Y vitamins at 4¢ each. Each Brand X pill contains 3000 USP units of A, 45 mg of C, and 75 USP units of D, while Brand Y pills contain 1000 USP units of A, 50 mg of C, and 200 USP units of D.

a. What combination of vitamin pills should she buy to obtain the least possible cost? What is the least possible cost per day?

b. For the solution in part a, Ms. Oliveras is receiving more than she needs of at least one vitamin. Identify that vitamin, and tell how much surplus she is receiving. Is there any way she can avoid receiving that surplus while still meeting the other constraints and minimizing the cost? Explain.

Social Sciences

25. Anthropology An anthropology article presents a hypothetical situation that could be described by a linear programming model. Suppose a population gathers plants and animals for

survival. They need at least 360 units of energy, 300 units of protein, and 8 hides during some time period. One unit of plants provides 30 units of energy, 10 units of protein, and no hides. One animal provides 20 units of energy, 25 units of protein, and 1 hide. Only 25 units of plants and 25 animals are available. It costs the population 30 hours of labor to gather one unit of a plant and 15 hours for an animal. Find how many units of plants and how many animals should be gathered to meet the requirements with a minimum number of hours of labor. ***Source: Annual Review of Anthropology.***

General Interest

26. Construction In a small town in South Carolina, zoning rules require that the window space (in square feet) in a house be at least one-sixth of the space used up by solid walls. The cost to build windows is \$10 per ft^2, while the cost to build solid walls is \$20 per ft^2. The total amount available for building walls and windows is no more than \$12,000. The estimated monthly cost to heat the house is \$0.32 for each square foot of windows and \$0.20 for each square foot of solid walls. Find the maximum total area (windows plus walls) if no more than \$160 per month is available to pay for heat.

27. Farming An agricultural advisor looks at the results of Example 2 and claims that it cannot possibly be correct. After all, the 4-H member is able to raise 16 animals, and she is only raising 12 animals. Surely she can earn more profit by raising all 16 animals. How would you respond?

YOUR TURN ANSWERS

1. He should buy 60 kayaks and no canoes for a maximum revenue of \$2100 a day.
2. Make 4 batches of each for a maximum profit of \$200.
3. 14 pigs and no goats produces a maximum profit of \$560.

3 CHAPTER REVIEW

SUMMARY

In this chapter, we introduced linear programming, which attempts to solve maximization and minimization problems with linear constraints. Linear programming models can be used to analyze a wide range of applications from many disciplines. The corner point theorem assures us that the optimal solution to a linear program, if it exists, must occur at one or more of the corner points of the feasible region. Linear programs can be solved using the graphical method, which graphs the region described by the linear constraints and then locates the corner point corresponding to the optimal solution value. The graphical method, however, is restricted to problems with two or three variables. In the next chapter, we will study a method that does not have this restriction.

Graphing a Linear Inequality

1. Draw the graph of the boundary line. Make the line solid if the inequality involves $\leq$ or $\geq$; make the line dashed if the inequality involves $<$ or $>$.
2. Decide which half-plane to shade. Use either of the following methods.
   - **a.** Solve the inequality for y; shade the region above the line if the inequality is of the form of $y >$ or $y \geq$; shade the region below the line if the inequality is of the form of $y <$ or $y \leq$.
   - **b.** Choose any point not on the line as a test point. Shade the half-plane that includes the test point if the test point satisfies the original inequality; otherwise, shade the half-plane on the other side of the boundary line.

Corner Point Theorem

If an optimum value (either a maximum or a minimum) of the objective function exists, it will occur at one or more of the corner points of the feasible region.

Solving a Linear Programming Problem

1. Write the objective function and all necessary constraints.
2. Graph the feasible region.
3. Identify all corner points.
4. Find the value of the objective function at each corner point.
5. For a bounded region, the solution is given by the corner point(s) producing the optimum value of the objective function.
6. For an unbounded region, check that a solution actually exists. If it does, it will occur at one or more corner points.

KEY TERMS

To understand the concepts presented in this chapter, you should know the meaning and use of the following terms.

3.1

linear inequality
boundary
half-plane
system of inequalities
region of feasible solutions
feasible region
bounded
unbounded

3.2

linear programming
objective function
constraints
isoprofit line
corner point

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The graphical method can be used to solve a linear programming problem with four variables.

2. For the inequality $5x + 4y \geq 20$, the test point (3, 4) suggests that the correct half-plane to shade includes this point.

3. Let x represent the number of acres of wheat planted and y represent the number of acres of corn planted. The inequality $x \leq 2y$ implies that the number of acres of wheat planted will be at least twice the number of acres of corn planted.

4. For the variables in Exercise 3, assume that we have a total of 60 hours to plant the wheat and corn and that it takes 2 hours per acre to prepare a wheat field and 1 hour per acre to prepare a corn field. The inequality $2x + y \geq 60$ represents the constraint on the amount of time available for planting.

5. For the variables in Exercise 3, assume that we make a profit of \$14 for each acre of corn and \$10 for each acre of wheat. The objective function that can be used to maximize profit is $14x + 10y$.

6. The point (2, 3) is a corner point of the linear programming problem

$$\begin{aligned} \text{Maximize} \quad & z = 7x + 4y \\ \text{subject to:} \quad & 3x + 8y \leq 30 \\ & 4x + 2y \leq 15 \\ & x \geq 0, y \geq 0. \end{aligned}$$

7. The point (2, 3) is in the feasible region of the linear programming problem in Exercise 6.

8. The optimal solution to the linear programming problem in Exercise 6 occurs at point (2, 3).

9. It is possible to find a point that lies on both sides of a linear inequality.

10. Every linear programming problem with a feasible region that is not empty either has a solution or is unbounded.

11. Solutions to linear programming problems may include fractions.

12. The inequality $4^2x + 5^2y \leq 7^2$ is a linear constraint.

13. The optimal solution to a linear programming problem can occur at a point that is not a corner point.

PRACTICE AND EXPLORATIONS

14. How many constraints are we limited to in the graphical method?

Graph each linear inequality.

15. $y \geq 2x + 3$

16. $5x - 2y \leq 10$

17. $2x + 6y \leq 8$

18. $2x - 6y \geq 18$

19. $y \geq x$

20. $y \geq -2$

Graph the solution of each system of inequalities. Find all corner points and tell whether each region is bounded or unbounded.

21.
$$\begin{aligned} x + y &\leq 6 \\ 2x - y &\geq 3 \end{aligned}$$

22.
$$\begin{aligned} 3x + 2y &\geq 12 \\ 4x - 5y &\leq 20 \end{aligned}$$

23.
$$\begin{aligned} -4 \leq x &\leq 2 \\ -1 \leq y &\leq 3 \\ x + y &\leq 4 \end{aligned}$$

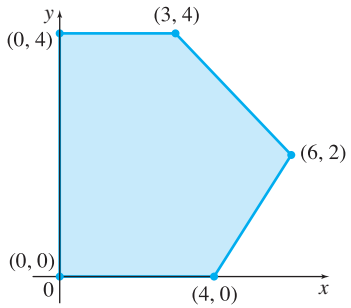
24.
$$\begin{aligned} 2 \leq x &\leq 5 \\ 1 \leq y &\leq 7 \\ x - y &\leq 3 \end{aligned}$$

25.
$$\begin{aligned} x + 2y &\leq 4 \\ 5x - 6y &\leq 12 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

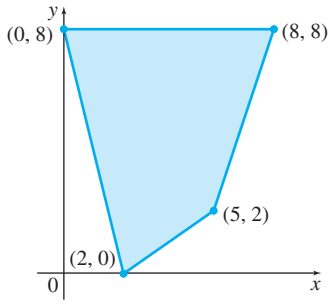
26.
$$\begin{aligned} x + 2y &\leq 4 \\ 2x - 3y &\leq 6 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Use the given regions to find the maximum and minimum values of the objective function $z = 2x + 4y$.

27.



28.



Use the graphical method to solve each linear programming problem.

29. Maximize $z = 2x + 4y$
subject to: $3x + 2y \le 12$
$5x + y \ge 5$
$x \ge 0$
$y \ge 0.$

30. Minimize $z = 5x + 3y$
subject to: $8x + 5y \ge 40$
$4x + 10y \ge 40$
$x \ge 0$
$y \ge 0.$

31. Minimize $z = 4x + 2y$
subject to: $x + y \le 50$
$2x + y \ge 20$
$x + 2y \ge 30$
$x \ge 0$
$y \ge 0.$

32. Maximize $z = 8x + 4y$
subject to: $3x + 12y \le 36$
$x + y \le 4$
$x \ge 0$
$y \ge 0.$

33. Why must the solution to a linear programming problem always occur at a corner point of the feasible region?

34. Is there necessarily a unique point in the feasible region where the maximum or minimum occurs? Why or why not?

35. It is not necessary to check all corner points in a linear programming problem. This exercise illustrates an alternative procedure, which is essentially an expansion of the ideas illustrated in Example 1 of Section 3.2.

Maximize $z = 3x + 4y$
subject to: $2x + y \le 4$
$-x + 2y \le 4$
$x \ge 0$
$y \ge 0.$

a. Sketch the feasible region, and add the line $z = 8$. (*Note:* 8 is chosen because the numbers work out simply, but the chosen value of z is arbitrary.)

b. Draw a line parallel to the line $z = 8$ that is as far from the origin as possible but still touches the feasible region.

c. The line you drew in part b should go through the point $(4/5, 12/5)$. Explain how you know the maximum must be located at this point.

36. Use the method described in the previous exercise to solve Exercise 32.

APPLICATIONS

Business and Economics

37. Time Management A bakery makes both cakes and cookies. Each batch of cakes requires 2 hours in the oven and 3 hours in the decorating room. Each batch of cookies needs $1\frac{1}{2}$ hours in the oven and $\frac{2}{3}$ hour in the decorating room. The oven is available no more than 15 hours per day, and the decorating room can be used no more than 13 hours per day. Set up a system of inequalities, and then graph the solution of the system.

38. Cost Analysis DeMarco's pizza shop makes two specialty pizzas, the Mighty Meaty and the Very Veggie. The Mighty Meaty is topped with 5 different meat toppings and 2 different cheeses. The Very Veggie has 6 different vegetable toppings and 4 different cheeses. The shop sells at least 4 Mighty Meaty and 6 Very Veggie pizzas every day. The cost of the toppings for each Mighty Meaty is \$3, and the cost of the vegetable toppings is \$2 for each Very Veggie. No more than \$60 per day can be spent on these toppings. The cheese used for the Mighty Meaty is \$2 per pizza, and the cheese for the Very Veggie is \$4 per pizza. No more than \$80 per day can be spent on cheese. Set up a system of inequalities, and then graph the solution of the system.

39. Profit Refer to Exercise 37.

a. How many batches of cakes and cookies should the bakery in Exercise 37 make in order to maximize profits if cookies produce a profit of \$20 per batch and cakes produce a profit of \$30 per batch?

b. How much would the profit from selling cookies have to increase before it becomes more profitable to sell only cookies?

40. Revenue How many pizzas of each kind should the pizza shop in Exercise 38 make in order to maximize revenue if the Mighty Meaty sells for \$15 and the Very Veggie sells for \$12?

41. Planting In Karla's garden shop, she makes two kinds of mixtures for planting. A package of gardening mixture requires 2 lb of soil, 1 lb of peat moss, and 1 lb of fertilizer. A package of potting mixture requires 1 lb of soil, 2 lb of peat moss, and 3 lb of fertilizer. She has 16 lb of soil, 11 lb of peat moss, and 15 lb of fertilizer. If a package of gardening mixture sells for \$3 and a package of potting mixture for \$5, how many of each should she make in order to maximize her income? What is the maximum income?

42. Construction A contractor builds boathouses in two basic models, the Atlantic and the Pacific. Each Atlantic model requires 1000 ft of framing lumber, 3000 ft^3 of concrete, and \$2000 for advertising. Each Pacific model requires 2000 ft of framing lumber, 3000 ft^3 of concrete, and \$3000 for advertising. Contracts call for using at least 8000 ft of framing lumber, 18,000 ft^3 of concrete, and \$15,000 worth of advertising. If the construction cost for each Atlantic model is \$30,000 and the construction cost for each Pacific model is \$40,000, how many of each model should be built to minimize construction costs?

43. Steel A steel company produces two types of alloys. A run of type I requires 3000 lb of molybdenum and 2000 tons of iron ore pellets as well as \$2000 in advertising. A run of type II requires 3000 lb of molybdenum and 1000 tons of iron ore

pellets as well as $3000 in advertising. Total costs are $15,000 on a run of type I and $6000 on a run of type II. Because of various contracts, the company must use at least 18,000 lb of molybdenum and 7000 tons of iron ore pellets and spend at least $14,000 on advertising. How much of each type should be produced to minimize costs?

Life Sciences

44. Nutrition A dietician in a hospital is to arrange a special diet containing two foods, Health Trough and Power Gunk. Each ounce of Health Trough contains 30 mg of calcium, 10 mg of iron, 10 IU of vitamin A, and 8 mg of cholesterol. Each ounce of Power Gunk contains 10 mg of calcium, 10 mg of iron, 30 IU of vitamin A, and 4 mg of cholesterol. If the minimum daily requirements are 360 mg of calcium, 160 mg of iron, and 240 IU of vitamin A, how many ounces of each food should be used to meet the minimum requirements and at the same time minimize the cholesterol intake? Also, what is the minimum cholesterol intake?

Social Sciences

45. Anthropology A simplified model of the Mountain Fur economy of central Africa has been proposed. In this model, two crops can be grown, millet and wheat, which produce 400 lb and 800 lb per acre, respectively. Millet requires 36 days to harvest one acre, while wheat requires only 8 days. There are 2 acres of land and 48 days of harvest labor available. How many acres should be devoted to each crop to maximize the pounds of grain harvested? ***Source: Themes in Economic Anthropology.***

General Interest

46. Studying Jim Pringle is trying to allocate his study time this weekend. He can spend time working with either his math tutor or his accounting tutor to prepare for exams in both classes the following Monday. His math tutor charges $20 per hour, and his accounting tutor charges $40 per hour. He has $220 to spend on tutoring. Each hour that he spends working with his math tutor requires 1 aspirin and 1 hour of sleep to recover, while each hour he spends with his accounting tutor requires 1/2 aspirin and 3 hours of sleep. The maximum dosage of aspirin that he can safely take during his study time is 8 tablets, and he can only afford 15 hours of sleep this weekend. He expects that each hour with his math tutor will increase his score on the math exam by 3 points, while each hour with his accounting tutor will increase his score on the accounting exam by 5 points. How many hours should he spend with each tutor in order to maximize the number of points he will get on the two tests combined?

EXTENDED APPLICATION

SENSITIVITY ANALYSIS

In Section 3 of this chapter we used the graphical method to solve linear programming problems. A quick analysis of the strengths and weaknesses of the process of modeling real-life problems using these techniques reveals two apparent problems. First, most real-life problems require far more than two variables to adequately capture the essence of the problem. Second, our world is so dynamic that the values used to model a problem today can quickly change, leaving the current solution irrelevant or even infeasible.

In Chapter 4, we show how the simplex method can be used to solve linear programming problems with thousands or even millions of variables and constraints. In this extended application, we introduce the idea of *sensitivity analysis*, which provides a way to efficiently handle the constantly changing inputs to a linear programming problem, often without having to resolve the problem from scratch. We will limit our study to linear programs with only two variables and begin by analyzing changes in objective function coefficients.

Recall that in Example 2 of Section 3.3, a linear programming problem was developed to determine the optimal number of goats and pigs to raise for a 4-H project. The linear program was written as

$$\begin{aligned} \text{Maximize} \quad & z = 12x + 40y \\ \text{Subject to:} \quad & x + y \le 16 \\ & x + 3y \le 36 \\ & x \le 10 \\ & x \ge 0,\ y \ge 0, \end{aligned}$$

where x is the number of goats and y is the number of pigs.

A graph of the feasible region, with corner points illustrated, is shown in Figure 23 on the next page. We determined that a maximum profit of $480 occurs when 12 pigs and no goats are raised. This is no surprise given that the profit from a pig is $40 and the profit for a goat is just $12. In fact, we saw in that example that if the profit from raising a pig decreases to $36, then it would become profitable to raise goats, too.

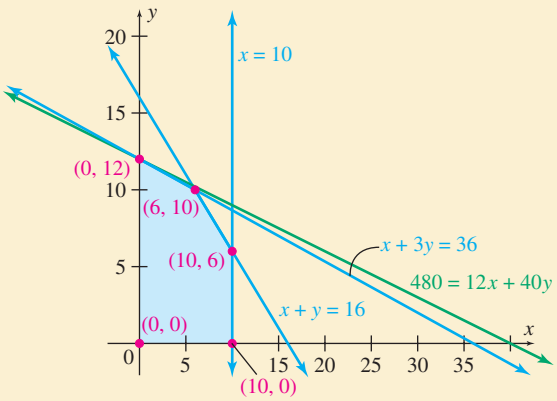


FIGURE 23

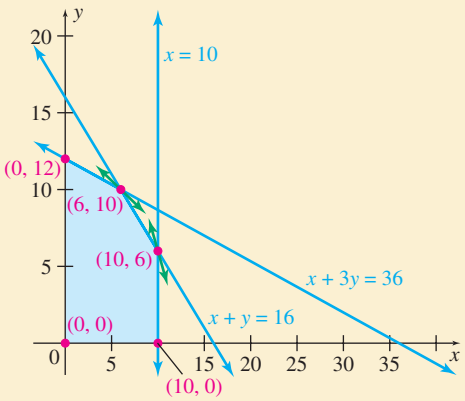


FIGURE 24

CHANGES IN OBJECTIVE FUNCTION COEFFICIENTS

Suppose that the 4-H member, whom we will call Sarah, now wonders what the profit from raising a goat would have to become so that it would become profitable for her to raise both pigs and goats.

Mathematically, we replace the \$12 in the objective function with the variable $12 + c$, where c is the change in profit for raising a goat. That is, $z = (12 + c)x + 40y$. Now, for any particular profit z, we can think of this equation as a line, which we called an *isoprofit line* in Section 3.2. With $c = 0$ and $z = 480$, the isoprofit line is $480 = 12x + 40y$, which is plotted in green in Figure 23. Solving for y helps us to see that the line represented by this equation has slope $-\dfrac{12 + c}{40}$. That is,

$$z = (12 + c)x + 40y,$$

or

$$40y = -(12 + c)x + z,$$

which simplifies to

$$y = -\frac{12 + c}{40}x + \frac{z}{40}.$$

The line that represents the boundary of the constraint $x + 3y = 36$ has slope $-\frac{1}{3}$. If c begins to increase, forcing the slope of the isoprofit line to become more negative, eventually the slope of the isoprofit line corresponding to the optimal profit of \$480 will have slope $-\frac{1}{3}$. If c continues to increase beyond this level, the optimal profit will occur at the corner point (6, 10). That is, as long as the slope of the isoprofit line exceeds $-\frac{1}{3}$, the optimal solution will continue to be to raise 12 pigs and no goats. If, however, the slope of the isoprofit line becomes less than $-\frac{1}{3}$, then it will become profitable to raise goats as well. That is, if c increases to the point at which

$$-\frac{12 + c}{40} < -\frac{1}{3},$$

which simplifies to,

$$\frac{12 + c}{40} > \frac{1}{3},$$

or

$$c > \frac{40}{3} - 12 = \frac{4}{3},$$

the optimal solution changes from the corner point (0, 12) to (6, 10). Further analysis, as illustrated by Figure 24, shows that the optimal solution will remain at the corner point (6, 10) as long as the slope of the isoprofit line stays between

$$-1 \le -\frac{12 + c}{40} \le -\frac{1}{3},$$

or when

$$\frac{4}{3} \le c \le 28.$$

Then the optimal solution will be to raise 6 goats and 10 pigs.

Beyond c = \$28 or a profit of \$40 per goat, it will be optimal to raise 10 goats and 6 pigs. Note that this analysis can be used to completely determine the optimal solution for changes in a single cost coefficient for any two variable linear programming problem, without having to resolve the linear program.

CHANGES IN THE RIGHT HAND SIDE OF A CONSTRAINT

Sarah begins to wonder if the right-hand sides of any of the constraints might be changed so that the profit would increase. For example, the easiest thing to do would be for Sarah to increase the third constraint, corresponding to the restriction that no more than 10 goats are raised. Sarah considers the possibility of lifting this constraint. Notice, however, that if we increase the right-hand side of the third constraint, it has no affect on the optimal corner point. As you can see graphically with the red lines in Figure 25, increasing the right-hand side will not change the optimal solution. In fact, decreasing the third constraint even to zero will not change the optimal solution.

Now suppose that Sarah realizes that she has overestimated the total number of animals she can raise by 3 and wants to see how this will affect the optimal solution. The boundary of the first

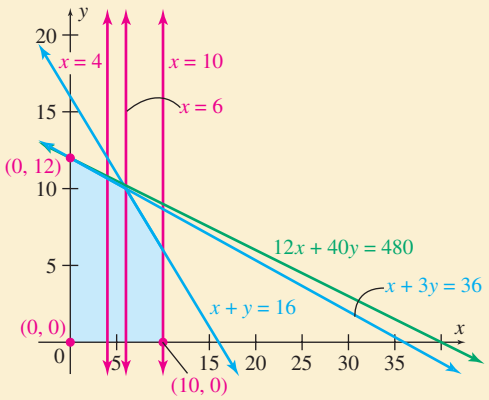


FIGURE 25

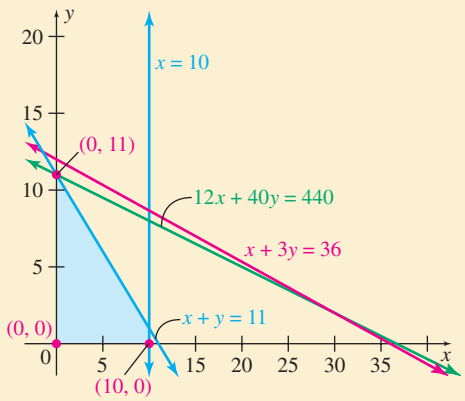


FIGURE 27

constraint can be written as the line $y = -x + 16$, which has y-intercept at the point (0,16). Since Sarah has overestimated the total number of animals by three, the boundary line becomes $y = -x + 13$, which has y-intercept at the point (0,13), as illustrated in Figure 26.

This change will not have an effect on the solution because raising 13 pigs is not feasible for the second constraint. That is, (0, 13) is not a feasible corner point and cannot be a solution to the problem. In fact, as long as the total number of animals is at least 12 in the first constraint, the optimal solution is for Sarah to raise 12 pigs and no goats. In this case, we would say that the right-hand side of the first constraint is not sensitive to small changes. If, on the other hand, Sarah has overestimated the total number of animals she can raise and that, in fact, the total number is actually 11, then the optimal solution changes. As you can see in Figure 27, the feasible region has diminished to the point that the second constraint is now redundant and has no influence on the optimal solution. In fact, the optimal solution is to raise 11 pigs and no goats. This makes sense since the profit from raising pigs is so high that it does not become profitable to raise goats when a smaller total number of animals are raised. The same type of analysis can be performed on the right-hand side of the second constraint.

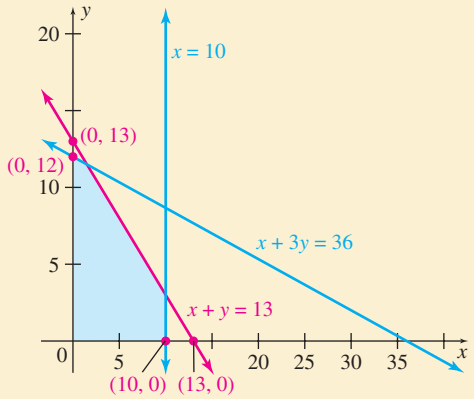


FIGURE 26

SHADOW PROFITS

Changes in the resources available in a linear programming problem usually result in a change in the value of the objective function in the optimal solution. For example, if an optimal solution uses all of the material available, then a sudden increase/decrease in the amount of that material available can change the objective function value. In the analysis above, decreasing the right-hand side of the first constraint from 16 to 12 did not change the optimal solution. In this case, we would say that the shadow profit, or the increase in the profit for changing the right-hand side of the first constraint by one unit is \$0. On the other hand, if we increase the amount of money available for the project by \$75, the optimal solution would change from raising 12 pigs to raising 13 pigs and thereby increase profit by \$40. This implies that for each dollar increase in the amount of money available, the profit will increase by \$40/75 or by about \$0.53. In this case, the shadow profit would be \$0.53.

CHANGES IN CONSTRAINT COEFFICIENTS

Sarah continues to explore reasons why the optimal policy is to only raise pigs. She wonders if any of the coefficients in the constraints are having an effect on the optimal solution. In the process, she realizes that it may cost more than \$75 to raise a pig. In the original statement of the problem, Sarah had up to \$900 to spend on her project, with each goat costing \$25 to raise and each pig costing \$75 to raise. This information was translated into the constraint $25x + 75y \le 900$, which was then simplified by dividing the inequality by 25. Sarah wonders how much the \$75 would have to increase before the optimal solution would include raising goats. Letting b represent the change in the cost of raising a pig, the constraint becomes $25x + (75 + b)y \le 900$, and the corresponding boundary line can then be simplified and written as

$$y = -\frac{25}{75 + b}x + \frac{900}{75 + b}.$$

To be consistent, every dollar increase in the cost of raising a pig also forces the profit to decrease by a dollar. That is, the 40 in the objective function is replaced with 40 – b, becoming

$$z = 12x + (40 - b)y.$$

The isoprofit lines for this objective function will each have slope of $-\dfrac{12}{40 - b}$.

When the slope of the boundary line corresponding to the altered second constraint increases to the point that it is larger than the slope of the isoprofit lines, the optimal solution will change and move to the intersection point of the first and second constraints. This occurs when

$$-\frac{25}{75 + b} > -\frac{12}{40 - b},$$

which simplifies to

$$b > \frac{100}{37} \approx 2.70.$$

In other words, as long as the cost of raising a pig is at or below \$75 + 2.70 = \$77.70, the optimal policy is to only raise pigs. When the cost exceeds this amount the optimal solution will change, and it may become profitable to raise goats, too. For example, suppose the cost of raising a pig increases to \$100. The new feasible region has corner points at (0, 0), (0, 9), (10, 0), (10, 6), and (28/3, 20/3), as indicated by Figure 28. It can be easily verified that the optimal solution now occurs at (28/3, 20/3), with objective function value of $z = 12(\frac{28}{3}) + 15(\frac{20}{3}) = 212$. Of course, it is impossible to raise a fractional number of animals so some additional work is required to determine the optimal solution. Unfortunately, simply rounding the fractional solution downward to obtain $x = 9$ and $y = 6$ does not give the optimal solution. In fact, it can be shown that the optimal integer solution is $x = 10$ and $y = 6$ with an objective function value of \$210. Thus, if the cost of raising a pig is \$100 then the optimal policy is to raise 6 pigs and 10 goats.

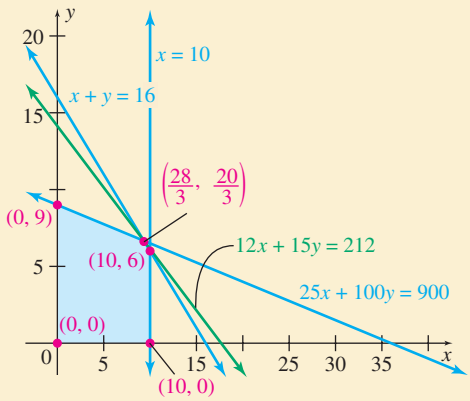


FIGURE 28

EXERCISES

1. Use sensitivity analysis to find the optimal solution to the original 4-H problem, if the profit from raising a goat increases from \$12 to \$25.

2. Use sensitivity analysis to find the optimal solution to the original 4-H problem, if the profit from raising a pig decreases from \$40 to \$35.

3. Use sensitivity analysis to determine the smallest decrease in the cost of raising a goat, so that the optimal policy of raising 12 pigs and no goats changes. Note that changing this cost also changes the profit level for goats.

4. Katie's Landscaping Services weeds flower beds and mows lawns. Katie makes a profit of \$8 for every hour she weeds and \$25 for every hour she mows. Because Katie is also in college, she can only take on 18 jobs each week for her business. Katie has found that the flower beds in her neighborhood are very poorly maintained and that weeding a flower bed must be considered as two jobs. Each week, Katie sets aside \$72 for her brother Nate to help her and feels that a fair price to pay Nate is \$3 per hour for weeding and \$9 per hour for mowing a lawn.

a. Show that this problem can be set up as follows:

$$\begin{aligned} \text{Maximize} \quad & z = 8x + 25y \\ \text{Subject to:} \quad & 2x + y \le 18 \\ & 3x + 9y \le 72 \\ & x \ge 0, y \ge 0 \end{aligned}$$

b. Use the graphical method to solve this linear program.

c. Alter the graph from part b to determine the new optimal solution if the right hand side of the second constraint increases to 81. (*Hint:* You shouldn't have to completely resolve this problem. Simply change the intercepts of one of the constraints and use isoprofit lines to identify the corner point where the new optimal solution occurs.)

d. What is the shadow profit associated with the change from 72 to 81 in part c?

e. Using the original problem, determine the smallest increase (to the nearest penny) in the objective function coefficient for variable x that will cause the value of x to increase in the optimal solution. In this case, we would say

that the cost coefficient associated with variable x is sensitive to change.

f. If the y-coefficient in the second constraint of the original problem begins to increase, forcing an equal decrease in the y-coefficient in the objective function, determine the smallest amount (to the nearest penny) that this coefficient can increase before it becomes optimal to produce positive quantities of variable x. For simplicity, assume that it is possible to have fractional amounts of each variable.

5. Go to the website WolframAlpha.com, and enter "minimize $8x + 25y$ with $2x + y <= 18$, $3x + 9y <= 72$, $x >= 0$, $y >= 0$". Study the solution provided by Wolfram|Alpha. Notice that the solution given is the same as the solution you found in Exercise 4b. Use Wolfram|Alpha to verify your answers to Exercises 4c–4f.

DIRECTIONS FOR GROUP PROJECT

Suppose that you are the manager of a company that manufactures two types of products. Develop a simple linear program with a solution that gives how many of each type of product the company should produce. Then perform sensitivity analysis on various aspects of the model. Use presentation software to report your model and your findings to the class.

4 Linear Programming: The Simplex Method

4.1 Slack Variables and the Pivot

4.2 Maximization Problems

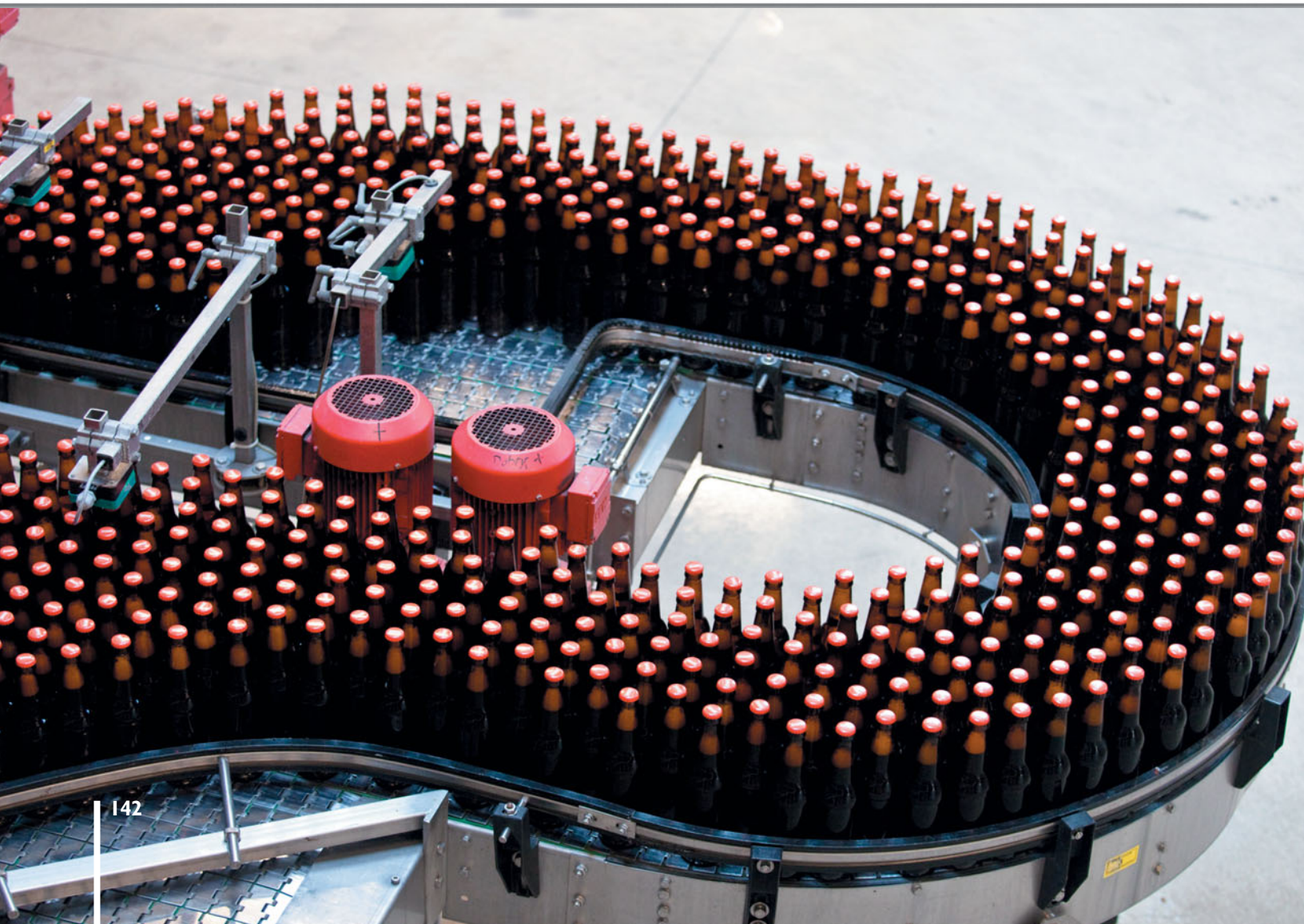
4.3 Minimization Problems; Duality

4.4 Nonstandard Problems

Chapter 4 Review

Extended Application: Using Integer Programming in the Stock-Cutting Problem

Each type of beer has its own recipe and an associated cost per unit and brings in a specific revenue per unit. The brewery manager must meet a revenue target with minimum production costs. An exercise in Section 3 formulates the manager's goal as a linear programming problem and solves for the optimum production schedule when there are two beer varieties.



In the previous chapter we discussed solving linear programming problems by the graphical method. This method illustrates the basic ideas of linear programming, but it is practical only for problems with two variables. For problems with more than two variables, or problems with two variables and many constraints, the *simplex method* is used. This method grew out of a practical problem faced by George B. Dantzig in 1947. Dantzig was concerned with finding the least expensive way to allocate supplies for the United States Air Force.

The **simplex method** starts with the selection of one corner point (often the origin) from the feasible region. Then, in a systematic way, another corner point is found that attempts to improve the value of the objective function. Finally, an optimum solution is reached, or it can be seen that no such solution exists.

The simplex method requires a number of steps. In this chapter we divide the presentation of these steps into two parts. First, a problem is set up in Section 4.1 and the method started; then, in Section 4.2, the method is completed. Special situations are discussed in the remainder of the chapter.

4.1 Slack Variables and the Pivot

Because the simplex method is used for problems with many variables, it usually is not convenient to use letters such as x, y, z, or w as variable names. Instead, the symbols x_1 (read "x-sub-one"), x_2, x_3, and so on, are used. These variable names lend themselves easily to use on the computer.

In this section we will use the simplex method only for problems such as the following:

$$\begin{aligned} &\text{Maximize} && z = 2x_1 - 3x_2 \\ &\text{subject to:} && 2x_1 + x_2 \le 10 \\ &&& x_1 - 3x_2 \le 5 \\ &\text{with} && x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

This type of problem is said to be in *standard maximum form.* All constraints must be expressed in the linear form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \le b,$$

where $x_1, x_2, x_3, \ldots, x_n$ are variables and $a_1, a_2, \ldots, a_n$ and b are constants, with $b \ge 0$.

Standard Maximum Form

A linear programming problem is in **standard maximum form** if the following conditions are satisfied.

1. The objective function is to be maximized.
2. All variables are nonnegative $(x_i \ge 0)$.
3. All remaining constraints are stated in the form

$$\boldsymbol{a_1x_1 + a_2x_2 + \cdots + a_nx_n \le b} \qquad \text{with } \boldsymbol{b \ge 0}.$$

(Problems that do not meet all of these conditions are discussed in Sections 4.3 and 4.4.)

To use the simplex method, we start by converting the constraints, which are linear inequalities, into linear equations by adding a nonnegative variable, called a **slack variable**,

to each constraint. For example, the inequality $x_1 + x_2 \le 10$ is converted into an equation by adding the slack variable s_1 to get

$$x_1 + x_2 + s_1 = 10, \qquad \text{where } s_1 \ge 0.$$

The inequality $x_1 + x_2 \le 10$ says that the sum $x_1 + x_2$ is less than or perhaps equal to 10. The variable s_1 "takes up any slack" and represents the amount by which $x_1 + x_2$ fails to equal 10. For example, if $x_1 + x_2$ equals 8, then s_1 is 2. If $x_1 + x_2 = 10$, then s_1 is 0.

CAUTION A different slack variable must be used for each constraint.

EXAMPLE 1 Slack Variables

Restate the following linear programming problem by introducing slack variables.

$$\begin{aligned} &\text{Maximize} && z = 3x_1 + 2x_2 + x_3 \\ &\text{subject to:} && 2x_1 + x_2 + x_3 \le 150 \\ &&& 2x_1 + 2x_2 + 8x_3 \le 200 \\ &&& 2x_1 + 3x_2 + x_3 \le 320 \\ &\text{with} && x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

SOLUTION Rewrite the three constraints as equations by adding slack variables s_1, s_2, and s_3, one for each constraint. Then the problem can be restated as follows.

$$\begin{aligned} &\text{Maximize} && z = 3x_1 + 2x_2 + x_3 \\ &\text{subject to:} && 2x_1 + x_2 + x_3 + s_1 = 150 \\ &&& 2x_1 + 2x_2 + 8x_3 + s_2 = 200 \\ &&& 2x_1 + 3x_2 + x_3 + s_3 = 320 \\ &\text{with} && x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad s_1 \ge 0, \quad s_2 \ge 0, \quad s_3 \ge 0. \end{aligned}$$

In Example 4, we will take another step toward solving this linear programming problem.

Adding slack variables to the constraints converts a linear programming problem into a system of linear equations. In each of these equations, all variables should be on the left side of the equal sign and all constants on the right. All the equations in Example 1 satisfy this condition except for the objective function, $z = 3x_1 + 2x_2 + x_3$, which may be written with all variables on the left as

$$-3x_1 - 2x_2 - x_3 + z = 0.$$

Now the equations in Example 1 can be written as the following augmented matrix.

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 2 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Indicators

This matrix is called the initial **simplex tableau**. The numbers in the bottom row, which are from the objective function, are called **indicators** (except for the 1 and 0 at the far right).

EXAMPLE 2 Initial Simplex Tableau

Set up the initial simplex tableau for the following problem.

A farmer has 100 acres of available land on which he wishes to plant a mixture of potatoes, corn, and cabbage. It costs him $400 to produce an acre of potatoes, $160 to produce

an acre of corn, and \$280 to produce an acre of cabbage. He has a maximum of \$20,000 to spend. He makes a profit of \$120 per acre of potatoes, \$40 per acre of corn, and \$60 per acre of cabbage. How many acres of each crop should he plant to maximize his profit?

SOLUTION Begin by summarizing the given information as follows.

Profits and Constraints for Crops					
	Potatoes	Corn	Cabbage		Total
Number of Acres	x_1	x_2	x_3	$\leq$	100
Cost (per acre)	\$400	\$160	\$280	$\leq$	\$20,000
Profit (per acre)	\$120	\$40	\$60		

If the number of acres allotted to each of the three crops is represented by x_1, x_2, and x_3, respectively, then the constraint pertaining to the number of acres can be expressed as

$$x_1 + x_2 + x_3 \leq 100 \qquad \text{Number of acres}$$

where x_1, x_2, and x_3 are all nonnegative. This constraint says that $x_1 + x_2 + x_3$ is less than or perhaps equal to 100. Use s_1 as the slack variable, giving the equation

$$x_1 + x_2 + x_3 + s_1 = 100.$$

Here s_1 represents the amount of the farmer's 100 acres that will not be used (s_1 may be 0 or any value up to 100).

The constraint pertaining to the production cost can be expressed as

$$400x_1 + 160x_2 + 280x_3 \leq 20{,}000, \qquad \text{Production costs}$$

or if we divide both sides by 40, as

$$10x_1 + 4x_2 + 7x_3 \leq 500.$$

This inequality can also be converted into an equation by adding a slack variable, s_2.

$$10x_1 + 4x_2 + 7x_3 + s_2 = 500$$

If we had not divided by 40, the slack variable would have represented any unused portion of the farmer's \$20,000 capital. Instead, the slack variable represents $1/40$ of that unused portion. (Note that s_2 may be any value from 0 to 500.)

The objective function represents the profit. The farmer wants to maximize

$$z = 120x_1 + 40x_2 + 60x_3.$$

The linear programming problem can now be stated as follows:

$$\begin{array}{ll} \text{Maximize} & z = 120x_1 + 40x_2 + 60x_3 \\ \text{subject to:} & x_1 + x_2 + x_3 + s_1 = 100 \\ & 10x_1 + 4x_2 + 7x_3 + s_2 = 500 \\ \text{with} & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0. \end{array}$$

Rewrite the objective function as $-120x_1 - 40x_2 - 60x_3 + z = 0$, and complete the initial simplex tableau as follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

YOUR TURN 1 Set up the initial simplex tableau for Example 2 with the following two modifications. All the costs per acre are \$100 less. Also, there are taxes of \$5, \$10, and \$15 per acre of potatoes, corn, and cabbage, respectively, and the farmer does not want to pay more than \$900 in taxes.

TRY YOUR TURN 1

In Example 2 (which we will finish solving in the next section), the feasible region consists of the points (x_1, x_2, x_3) that satisfy the two inequalities for the number of acres and

the production costs. It is difficult to draw such a region in three dimensions by hand. If there were a fourth crop, and hence a fourth dimension, plotting the region would be impossible. Nevertheless, as with the graphical method, the objective function is maximized at one of the corner points (or two corner points and all points on the line segment between the two points). With the simplex method, which we develop in this section and the next, we find that corner point using matrix methods that don't require drawing a graph of the feasible region.

After we have converted the two inequalities into equations by the addition of slack variables, the maximization problem in Example 2 consists of a system of two equations in five variables, together with the objective function. Because there are more variables than equations, the system will have an infinite number of solutions. To see this, solve the system for s_1 and s_2.

$$\begin{aligned} s_1 &= 100 - x_1 - x_2 - x_3 \\ s_2 &= 500 - 10x_1 - 4x_2 - 7x_3 \end{aligned}$$

Each choice of values for x_1, x_2, and x_3 gives corresponding values for s_1 and s_2 that produce a solution of the system. But only some of these solutions are feasible.

In a feasible solution, all variables must be nonnegative. To get a unique feasible solution, we set three of the five variables equal to 0. In general, if there are m equations, then m variables can be nonzero. These m nonzero variables are called **basic variables**, and the corresponding solutions are called **basic feasible solutions**. Each basic feasible solution corresponds to a corner point. In particular, if we choose the solution with $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, then $s_1 = 100$ and $s_2 = 500$ are the basic variables. This solution, which corresponds to the corner point at the origin, is hardly optimal. It produces a profit of \$0 for the farmer, since the equation that corresponds to the objective function becomes

$$-120(0) - 40(0) - 60(0) + 0s_1 + 0s_2 + z = 0.$$

In the next section we will use the simplex method to start with this solution and improve it to find the maximum possible profit.

Each step of the simplex method produces a solution that corresponds to a corner point of the region of feasible solutions. These solutions can be read directly from the matrix, as shown in the next example.

EXAMPLE 3 Basic Variables

Read a solution from the following simplex tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 2 & \mathbf{0} & 8 & 5 & \mathbf{2} & 0 & 17 \\ 9 & \mathbf{5} & 3 & 12 & \mathbf{0} & 0 & 45 \\ \hline -2 & \mathbf{0} & -4 & 0 & \mathbf{0} & 3 & 90 \end{array}\right] \end{array}$$

SOLUTION In this solution, the variables x_2 and s_2 are basic variables. They can be identified quickly because the columns for these variables have all zeros except for one nonzero entry. All variables that are not basic variables have the value 0. This means that in the tableau just shown, x_2 and s_2 are the basic variables, while x_1, x_3, and s_1 have the value 0. The nonzero entry for x_2 is 5 in the second row. Since x_1, x_3, and s_1 are zero, the second row of the tableau represents the equation $5x_2 = 45$, so $x_2 = 9$. Similarly, from the top row, $2s_2 = 17$, so $s_2 = 17/2$. From the bottom row, $3z = 90$, so $z = 30$. The solution is thus $x_1 = 0$, $x_2 = 9$, $x_3 = 0$, $s_1 = 0$, and $s_2 = 17/2$, with $z = 30$.

Pivots

Solutions read directly from the initial simplex tableau are seldom optimal. It is necessary to proceed to other solutions (corresponding to other corner points of the feasible region) until an optimum solution is found. To get these other solutions, we use restricted

FOR REVIEW

We discussed three row operations in Chapter 2:

1. interchanging any two rows;
2. multiplying the elements of a row by any nonzero real number; and
3. adding a multiple of the elements of one row to the corresponding elements of a multiple of any other row.

In this chapter we will only use operation 2 and a restricted version of operation 3; we will never interchange two rows.

versions of the row operations from Chapter 2 to change the tableau by using one of the nonzero entries of the tableau as a **pivot**. The row operations are performed to change to 0 all entries in the column containing the pivot (except for the pivot itself, which remains unchanged). Pivoting, explained in the next example, produces a new tableau leading to another solution of the system of equations obtained from the original problem.

CAUTION In this chapter, when adding a multiple of one row to a multiple of another, we will never take a negative multiple of the row being changed. For example, when changing row 2, we might use $-2R_1 + 3R_2 \to R_2$, but we will never use $2R_1 - 3R_2 \to R_2$. If you get a negative number in the rightmost column, you will know immediately that you have made an error. The reason for this restriction is that violating it turns negative numbers into positive, and vice versa. This is disastrous in the bottom row, where we will seek negative numbers when we choose our pivot column. It will also cause problems with choosing pivots, particularly in the algorithm for solving nonstandard problems in Section 4.4.

When we are performing row operations by hand, as we did in Chapter 2, we will postpone getting a 1 in each basic variable column until the final step. This will avoid fractions and decimals, which can make the operations more difficult and more prone to error. When using a graphing calculator, however, we must change the pivot to a 1 before performing row operations. The next example illustrates both of these methods.

EXAMPLE 4 Pivot

Pivot about the indicated 2 of the following initial simplex tableau.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{matrix} \\ \left[\begin{array}{ccccccc|c} \boxed{2} & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

SOLUTION

Method 1 Calculating by Hand

Using the row operations indicated in color to get zeros in the column with the pivot, we arrive at the following tableau.

$$\begin{array}{r} \\ \\ -R_1 + 2R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ 3R_1 + 2R_4 \to R_4 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 0 & 3 & 15 & -1 & 2 & 0 & 0 & 250 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline 0 & -1 & 1 & 3 & 0 & 0 & 2 & 450 \end{array}\right] \end{array}$$

In this simplex tableau, the variables x_1, s_2, and s_3 are basic variables. The solution is $x_1 = 75$, $x_2 = 0$, $x_3 = 0$, $s_1 = 0$, $s_2 = 125$, and $s_3 = 170$. Substituting these results into the objective function gives

$$0(75) - 1(0) + 1(0) + 3(0) + 0(125) + 0(170) + 2z = 450,$$

or $z = 225$. (This shows that the value of z can always be found using the number in the bottom row of the z column and the number in the lower right-hand corner.)

Finally, to be able to read the solution directly from the tableau, we multiply rows 1, 2, and 4 by $1/2$, getting the following tableau.

$$\begin{array}{r} \tfrac{1}{2}R_1 \to R_1 \\ \tfrac{1}{2}R_2 \to R_2 \\ \\ \tfrac{1}{2}R_4 \to R_4 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 75 \\ 0 & \frac{3}{2} & \frac{15}{2} & -\frac{1}{2} & 1 & 0 & 0 & 125 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline 0 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 0 & 0 & 1 & 225 \end{array}\right] \end{array}$$

Method 2 Graphing Calculator

The row operations of the simplex method can also be done on a graphing calculator, as we saw in Chapter 2. Figure 1 shows the result when the tableau in this example is entered into a TI-84 Plus. The right side of the tableau is not visible but can be seen by pressing the right arrow key.

Recall that we must change the pivot to 1 before performing row operations with a graphing calculator. Figure 2 shows the result of multiplying row 1 of matrix A by $1/2$. In Figure 3 we show the same result with the decimal numbers changed to fractions.

We can now modify column 1, using the commands described in Chapter 2, to agree with the tableau under Method 1. The result is shown in Figure 4.

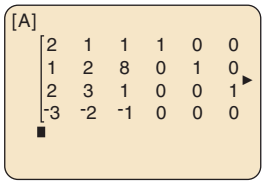


FIGURE 1

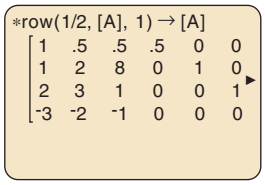


FIGURE 2

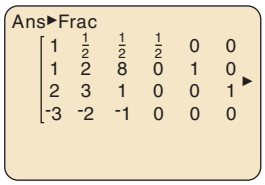


FIGURE 3

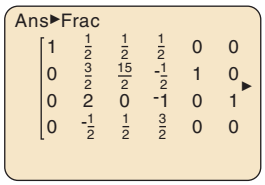


FIGURE 4

YOUR TURN 2 Pivot about the indicated 6 of the following initial simplex tableau, and then read a solution from the tableau.

x_1	x_2	x_3	s_1	s_2	s_3	z	
3	6	2	1	0	0	0	60
8	5	4	0	1	0	0	80
3	6	7	0	0	1	0	120
−30	−50	−15	0	0	0	1	0

TRY YOUR TURN 2

In the simplex method, the pivoting process (without the final step of getting a 1 in each basic variable column when using Method 1) is repeated until an optimum solution is found, if one exists. In the next section we will see how to decide where to pivot to improve the value of the objective function and how to tell when an optimum solution either has been reached or does not exist.

4.1 EXERCISES

Convert each inequality into an equation by adding a slack variable.

1. $x_1 + 2x_2 \le 6$

2. $6x_1 + 2x_2 \le 50$

3. $2.3x_1 + 5.7x_2 + 1.8x_3 \le 17$

4. $8x_1 + 6x_2 + 5x_3 \le 250$

For Exercises 5–8, (a) determine the number of slack variables needed, (b) name them, and (c) use slack variables to convert each constraint into a linear equation.

5. Maximize $z = 5x_1 + 7x_2$
subject to: $2x_1 + 3x_2 \le 15$
$4x_1 + 5x_2 \le 35$
$x_1 + 6x_2 \le 20$
with $x_1 \ge 0, \quad x_2 \ge 0.$

6. Maximize $z = 1.2x_1 + 3.5x_2$
subject to: $2.4x_1 + 1.5x_2 \le 10$
$1.7x_1 + 1.9x_2 \le 15$
with $x_1 \ge 0, \quad x_2 \ge 0.$

7. Maximize $z = 8x_1 + 3x_2 + x_3$
subject to: $7x_1 + 6x_2 + 8x_3 \le 118$
$4x_1 + 5x_2 + 10x_3 \le 220$
with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

8. Maximize $z = 12x_1 + 15x_2 + 10x_3$
subject to: $2x_1 + 2x_2 + x_3 \le 8$
$x_1 + 4x_2 + 3x_3 \le 12$
with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

Introduce slack variables as necessary, then write the initial simplex tableau for each linear programming problem.

9. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$4x_1 + 2x_2 \leq 5$$
$$x_1 + 2x_2 \leq 4$$

and $z = 7x_1 + x_2$ is maximized.

10. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$2x_1 + 3x_2 \leq 100$$
$$5x_1 + 4x_2 \leq 200$$

and $z = x_1 + 3x_2$ is maximized.

11. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$x_1 + x_2 \leq 10$$
$$5x_1 + 2x_2 \leq 20$$
$$x_1 + 2x_2 \leq 36$$

and $z = x_1 + 3x_2$ is maximized.

12. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$x_1 + x_2 \leq 25$$
$$4x_1 + 3x_2 \leq 48$$

and $z = 5x_1 + 3x_2$ is maximized.

13. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$3x_1 + x_2 \leq 12$$
$$x_1 + x_2 \leq 15$$

and $z = 2x_1 + x_2$ is maximized.

14. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$10x_1 + 4x_2 \leq 100$$
$$20x_1 + 10x_2 \leq 150$$

and $z = 4x_1 + 5x_2$ is maximized.

Write the solutions that can be read from each simplex tableau.

15.

x_1	x_2	x_3	s_1	s_2	z	
1	0	4	5	1	0	8
3	1	1	2	0	0	4
−2	0	2	3	0	1	28

16.

x_1	x_2	x_3	s_1	s_2	z	
1	5	0	1	2	0	6
0	2	1	2	3	0	15
0	4	0	1	−2	1	64

17.

x_1	x_2	x_3	s_1	s_2	s_3	z	
6	2	2	3	0	0	0	16
2	2	0	1	0	5	0	35
2	1	0	3	1	0	0	6
−3	−2	0	2	0	0	3	36

18.

x_1	x_2	x_3	s_1	s_2	s_3	z	
0	2	0	5	2	2	0	15
0	3	1	0	1	2	0	2
7	4	0	0	3	5	0	35
0	−4	0	0	4	3	2	40

Pivot once as indicated in each simplex tableau. Read the solution from the result.

19.

x_1	x_2	x_3	s_1	s_2	z	
1	2	4	1	0	0	56
2	**2**	1	0	1	0	40
−1	−3	−2	0	0	1	0

20.

x_1	x_2	x_3	s_1	s_2	z	
2	3	4	1	0	0	18
6	**3**	2	0	1	0	15
−1	−6	−2	0	0	1	0

21.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	2	**1**	1	0	0	0	12
1	2	3	0	1	0	0	45
3	1	1	0	0	1	0	20
−2	−1	−3	0	0	0	1	0

22.

x_1	x_2	x_3	s_1	s_2	s_3	z	
4	2	3	1	0	0	0	22
2	2	**5**	0	1	0	0	28
1	3	2	0	0	1	0	45
−3	−2	−4	0	0	0	1	0

23.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	**2**	3	1	0	0	0	500
4	1	1	0	1	0	0	300
7	2	4	0	0	1	0	700
−3	−4	−2	0	0	0	1	0

24.

x_1	x_2	x_3	x_4	s_1	s_2	s_3	z	
1	2	3	1	1	0	0	0	115
2	1	8	5	0	1	0	0	200
1	0	1	0	0	0	1	0	50
−2	−1	−1	−1	0	0	0	1	0

25. Explain the purpose of a slack variable.

26. How can you tell by looking at a linear programming problem how many slack variables will be needed?

APPLICATIONS

Set up Exercises 27–31 for solution by the simplex method. First express the linear constraints and objective function, then add slack variables to convert each constraint into a linear equation, and then set up the initial simplex tableau. The solutions of some of these problems will be completed in the exercises for the next section.

Business and Economics

27. Royalties The authors of a best-selling textbook in finite mathematics are told that, for the next edition of their book, each simple figure would cost the project $20, each figure with additions would cost $35, and each computer-drawn sketch would cost $60. They are limited to 400 figures, for which they are allowed to spend up to $2200. The number of computer-drawn sketches must be no more than the number of the other two types combined, and there must be at least twice as many

simple figures as there are figures with additions. If each simple figure increases the royalties by $95, each figure with additions increases royalties by $200, and each computer-drawn figure increases royalties by $325, how many of each type of figure should be included to maximize royalties, assuming that all art costs are borne by the publisher?

28. Manufacturing Bicycles A manufacturer of bicycles builds racing, touring, and mountain models. The bicycles are made of both aluminum and steel. The company has available 91,800 units of steel and 42,000 units of aluminum. The racing, touring, and mountain models need 17, 27, and 34 units of steel, and 12, 21, and 15 units of aluminum, respectively. How many of each type of bicycle should be made in order to maximize profit if the company makes $8 per racing bike, $12 per touring bike, and $22 per mountain bike? What is the maximum possible profit?

29. Production—Picnic Tables The manager of a large park has received many complaints about the insufficient number of picnic tables available. At the end of the park season, she has surplus cash and labor resources available and decides to make as many tables as possible. She considers three possible models: redwood, stained Douglas fir, and stained white spruce (all of which last equally well). She has carpenters available for assembly work for a maximum of 90 eight-hour days, while laborers for staining work are available for no more than 60 eight-hour days. Each redwood table requires 8 hours to assemble but no staining, and it costs $159 (including all labor and materials). Each Douglas fir table requires 7 hours to assemble and 2 hours to stain, and it costs $138.85. Each white spruce table requires 8 hours to assemble and 2 hours to stain, and it costs $129.35. If no more than $15,000 is available for this project, what is the maximum number of tables which can be made, and how many of each type should be made? ***Source: Karl K. Norton.***

30. Production—Knives The Cut-Right Company sells sets of kitchen knives. The Basic Set consists of 2 utility knives and 1 chef's knife. The Regular Set consists of 2 utility knives, 1 chef's knife, and 1 slicer. The Deluxe Set consists of 3 utility knives, 1 chef's knife, and 1 slicer. Their profit is $30 on a Basic Set, $40 on a Regular Set, and $60 on a Deluxe Set. The factory has on hand 800 utility knives, 400 chef's knives, and 200 slicers. Assuming that all sets will be sold, how many of each type should be produced in order to maximize profit? What is the maximum profit?

31. Advertising The Fancy Fashions, an independent, local boutique, has $8000 available each month for advertising. Newspaper ads cost $400 each, and no more than 30 can run per month. Internet banner ads cost $20 each, and no more than 60 can run per month. TV ads cost $2000 each, with a maximum of 10 available each month. Approximately 4000 women will see each newspaper ad, 3000 will see each Internet banner, and 10,000 will see each TV ad. How much of each type of advertising should be used if the store wants to maximize its ad exposure?

YOUR TURN ANSWERS

1.

x_1	x_2	x_3	s_1	s_2	s_3	z	
1	1	1	1	0	0	0	100
15	3	9	0	1	0	0	1000
1	2	3	0	0	1	0	180
−120	−40	−60	0	0	0	1	0

2.

x_1	x_2	x_3	s_1	s_2	s_3	z	
1/2	1	1/3	1/6	0	0	0	10
11/2	0	7/3	−5/6	1	0	0	30
0	0	5	−1	0	1	0	60
−5	0	5/3	25/3	0	0	1	500

$x_1 = 0, x_2 = 10, x_3 = 0, s_1 = 0, s_2 = 30, s_3 = 60, z = 500$

4.2 Maximization Problems

APPLY IT **How many racing, touring, and mountain bicycles should a bicycle manufacturer make to maximize profit?**

We will answer this question in Exercise 27 of this section using an algorithm called the simplex method.

In the previous section we showed how to prepare a linear programming problem for solution. First, we converted the constraints to linear equations with slack variables; then we used the coefficients of the variables from the linear equation to write an augmented matrix. Finally, we used the pivot to go from one corner point to another corner point in the region of feasible solutions.

Now we are ready to put all this together and produce an optimum value for the objective function. To see how this is done, let us complete Example 2 from Section 4.1. In this example, we were trying to determine, under certain constraints, the number of acres of

potatoes (x_1), corn (x_2), and cabbage (x_3) the farmer should plant in order to optimize his profit (z). In the previous section, we set up the following simplex tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

This tableau leads to the solution $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $s_1 = 100$, and $s_2 = 500$, with s_1 and s_2 as the basic variables. These values produce a value of 0 for z. In this solution, the farmer is planting 0 acres and earning \$0 profit. We can easily see that there are other combinations of potatoes, corn, and cabbage that produce a nonzero profit, and thus we know that the farmer has better alternatives than planting nothing.

To decide which crops he should plant, we look at the original objective function representing profit,

$$z = 120x_1 + 40x_2 + 60x_3.$$

The coefficient of x_1 is the largest, which indicates that he will make the most profit per acre planting potatoes. It makes sense, then, to first try increasing x_1 to improve the profit.

To determine how much we can increase x_1, we look at the constraint equations:

$$\begin{aligned} x_1 + x_2 + x_3 + s_1 &= 100 \\ 10x_1 + 4x_2 + 7x_3 + s_2 &= 500. \end{aligned}$$

Because there are two equations, only two of the five variables can be basic (and nonzero). If x_1 is nonzero in the solution, then x_1 will be a basic variable. Therefore, x_2 and x_3 will stay at 0, and the equations simplify to

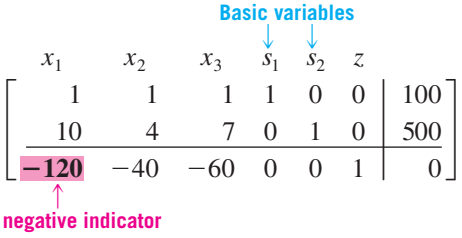
$$\begin{aligned} x_1 + s_1 &= 100 \\ 10x_1 + s_2 &= 500. \end{aligned}$$

Since s_1 and s_2 are both nonnegative, the first equation implies that x_1 cannot exceed 100, and the second implies that $10x_1$ cannot exceed 500, so x_1 cannot exceed $500/10$, or 50. To satisfy both of these conditions, x_1 cannot exceed 50, the smaller of 50 and 100. If we let x_1 take the value of 50, then $x_1 = 50$, $x_2 = 0$, $x_3 = 0$, and $s_2 = 0$. Since $x_1 + s_1 = 100$, then $s_1 = 100 - x_1 = 100 - 50 = 50$. Therefore, s_1 is still a basic variable, while s_2 is no longer a basic variable, having been replaced in the set of basic variables by x_1. This solution gives a profit of

$$\begin{aligned} z &= 120x_1 + 40x_2 + 60x_3 + 0s_1 + 0s_2 \\ &= 120(50) + 40(0) + 60(0) + 0(50) + 0(0) = 6000, \end{aligned}$$

or \$6000, when 50 acres of potatoes are planted.

The same result could have been found from the initial simplex tableau given below. Recall that the indicators are the numbers in the bottom row in the columns labeled with real or slack variables. To use the tableau, we select the variable with the most negative indicator. (If no indicator is negative, then the value of the objective function cannot be improved.) In this example, the variable with the most negative indicator is x_1.



The most negative indicator identifies the variable whose value is to be made nonzero, if possible, because it indicates the variable with the largest coefficient in the objective

function. To find the variable that is now basic and will become nonbasic, calculate the quotients that were found above. Do this by dividing each number from the right side of the tableau by the corresponding number from the column with the most negative indicator.

Basic variables (s_1, s_2)

Quotients	x_1	x_2	x_3	s_1	s_2	z	
100/1 = 100	1	1	1	1	0	0	100
Smaller → 500/10 = 50	**10**	4	7	0	1	0	500
	−120	−40	−60	0	0	1	0

Notice that we do not form a quotient for the bottom row. Of the two quotients found, the smallest is 50 (from the second row). This indicates that x_1 cannot exceed $500/10 = 50$, so 10 is the pivot. Using 10 as the pivot, perform the appropriate row operations to get zeros in the rest of the column. We will use Method 1 from Section 4.1 (calculating by hand) to perform the pivoting, but Method 2 (graphing calculator) could be used just as well. The new tableau is as follows.

Basic variables (x_1, s_1)

	x_1	x_2	x_3	s_1	s_2	z	
$-R_2 + 10R_1 \to R_1$	0	6	3	10	−1	0	500
	10	4	7	0	1	0	500
$12R_2 + R_3 \to R_3$	0	8	24	0	12	1	6000

The solution read from this tableau is

$$x_1 = 50, \quad x_2 = 0, \quad x_3 = 0, \quad s_1 = 50, \quad s_2 = 0,$$

with $z = 6000$, the same as the result found above.

None of the indicators in the final simplex tableau are negative, which means that the value of z cannot be improved beyond \$6000. To see why, recall that the last row gives the coefficients of the objective function so that

$$0x_1 + 8x_2 + 24x_3 + 0s_1 + 12s_2 + z = 6000,$$

or

$$z = 6000 - 0x_1 - 8x_2 - 24x_3 - 0s_1 - 12s_2.$$

Since x_2, x_3, and s_2 are zero, $z = 6000$, but if any of these three variables were to increase, z would decrease.

This result suggests that the optimal solution has been found as soon as no indicators are negative. As long as an indicator is negative, the value of the objective function may be improved. If any indicators are negative, we just find a new pivot and use row operations, repeating the process until no negative indicators remain.

Once there are no longer any negative numbers in the final row, create a 1 in the columns corresponding to the basic variables and z. In the previous example, this is accomplished by dividing rows 1 and 2 by 10.

	x_1	x_2	x_3	s_1	s_2	z	
$R_1/10 \to R_1$	0	$\frac{6}{10}$	$\frac{3}{10}$	1	$-\frac{1}{10}$	0	50
$R_2/10 \to R_2$	1	$\frac{4}{10}$	$\frac{7}{10}$	0	$\frac{1}{10}$	0	50
	0	8	24	0	12	1	6000

It is now easy to read the solution from this tableau:

$$x_1 = 50, \quad x_2 = 0, \quad x_3 = 0, \quad s_1 = 50, \quad s_2 = 0,$$

with $z = 6000$.

We can finally state the solution to the problem about the farmer. The farmer will make a maximum profit of \$6000 by planting 50 acres of potatoes, no acres of corn, and no acres of cabbage. The value $s_1 = 50$ indicates that of the 100 acres of land available, 50 acres should be left unplanted. It may seem strange that leaving assets unused can produce a maximum profit, but such results actually occur often.

Note that since each variable can be increased by a different amount, the most negative indicator is not always the best choice. On average, though, it has been found that the most negative indicator is the best choice.

In summary, the following steps are involved in solving a standard maximum linear programming problem by the simplex method.

Simplex Method For Standard Maximization Problems

1. Determine the objective function.
2. Write all the necessary constraints.
3. Convert each constraint into an equation by adding a slack variable in each.
4. Set up the initial simplex tableau.
5. Locate the most negative indicator. If there are two such indicators, choose the one farther to the left.
6. Form the necessary quotients to find the pivot. Disregard any quotients with 0 or a negative number in the denominator. The smallest nonnegative quotient gives the location of the pivot. If all quotients must be disregarded, no maximum solution exists. If two quotients are both equal and smallest, choose the pivot in the row nearest the top of the matrix.
7. Use row operations to change all other numbers in the pivot column to zero by adding a suitable multiple of the pivot row to a positive multiple of each row.
8. If the indicators are all positive or 0, this is the final tableau. If not, go back to Step 5 and repeat the process until a tableau with no negative indicators is obtained.
9. Read the solution from the final tableau.

In Steps 5 and 6, the choice of the column farthest to the left or the row closest to the top is arbitrary. You may choose another row or column in case of a tie, and you will get the same final answer, but your intermediate results will be different.

CAUTION In performing the simplex method, a negative number in the right-hand column signals that a mistake has been made. One possible error is using a negative value for c_2 in the operation $c_1R_i + c_2R_j \rightarrow R_j$.

EXAMPLE 1 Using the Simplex Method

To compare the simplex method with the graphical method, we use the simplex method to solve the problem in Example 1, Section 3.3. The graph is shown again in Figure 5. The objective function to be maximized was

$$z = 25x_1 + 30x_2. \quad \text{Revenue}$$

(Since we are using the simplex method, we use x_1 and x_2 instead of x and y as variables.) The constraints were as follows:

$$\begin{aligned} x_1 + x_2 &\le 65 && \text{Number} \\ 4x_1 + 5x_2 &\le 300 && \text{Cost} \end{aligned}$$

with $x_1 \ge 0, \quad x_2 \ge 0.$

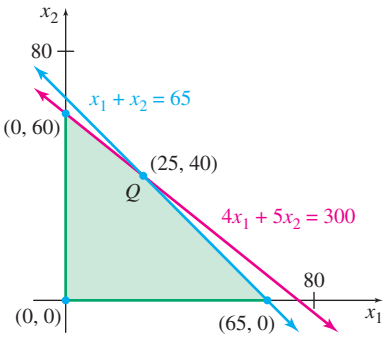


FIGURE 5

Add a slack variable to each constraint:

$$\begin{aligned} x_1 + x_2 + s_1 &= 65 \\ 4x_1 + 5x_2 + s_2 &= 300 \end{aligned}$$

with $x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0.$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 65 \\ 4 & 5 & 0 & 1 & 0 & 300 \\ \hline -25 & -30 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

This tableau leads to the solution $x_1 = 0$, $x_2 = 0$, $s_1 = 65$, and $s_2 = 300$, with $z = 0$, which corresponds to the origin in Figure 5. The most negative indicator is -30, which is in column 2 of row 3. The quotients of the numbers in the right-hand column and in column 2 are

$$\frac{65}{1} = 65 \quad \text{and} \quad \frac{300}{5} = 60.$$

The smaller quotient is 60, giving 5 as the pivot. Use row operations to get the new tableau. For clarity, we will continue to label the columns with x_1, x_2, and so on, although this is not necessary in practice.

$$\begin{array}{r} \\ \mathbf{-R_2 + 5R_1 \to R_1} \\ \\ \mathbf{6R_2 + R_3 \to R_3} \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -1 & 0 & 25 \\ 4 & 5 & 0 & 1 & 0 & 300 \\ \hline -1 & 0 & 0 & 6 & 1 & 1800 \end{array}\right] \end{array}$$

The solution from this tableau is $x_1 = 0$ and $x_2 = 60$, with $z = 1800$. (From now on, we will list only the original variables when giving the solution.) This corresponds to the corner point (0, 60) in Figure 5. Verify that if we instead pivoted on the column with the indicator of -25, we would arrive at the corner point (65, 0), where the objective function has the value 1625, which is smaller than its value at (0, 60).

Because of the indicator –1, the value of z might be improved. We compare quotients and choose the 1 in row 1, column 1, as pivot to get the final tableau.

$$\begin{array}{r} \\ \mathbf{-4R_1 + R_2 \to R_2} \\ \mathbf{R_1 + R_3 \to R_3} \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -1 & 0 & 25 \\ 0 & 5 & -20 & 5 & 0 & 200 \\ \hline 0 & 0 & 5 & 5 & 1 & 1825 \end{array}\right] \end{array}$$

There are no more negative indicators, so the optimum solution has been achieved. Create a 1 in column 2 by multiplying row 2 by $1/5$.

$$\begin{array}{r} \\ \mathbf{(1/5)R_2 \to R_2} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -1 & 0 & 25 \\ 0 & 1 & -4 & 1 & 0 & 40 \\ \hline 0 & 0 & 5 & 5 & 1 & 1825 \end{array}\right] \end{array}$$

Here the solution is $x_1 = 25$ and $x_2 = 40$, with $z = 1825$. This solution, which corresponds to the corner point (25, 40) in Figure 5, is the same as the solution found earlier.

Each simplex tableau above gave a solution corresponding to one of the corner points of the feasible region. As shown in Figure 6, the first solution corresponded to the origin, with $z = 0$. By choosing the appropriate pivot, we moved systematically to a new corner point, (0, 60), which improved the value of z to 1800. The next tableau took us to (25, 40), producing the optimum value of $z = 1825$. There was no reason to test the last corner point, (65, 0), since the optimum value z was found before that point was reached. TRY YOUR TURN 1

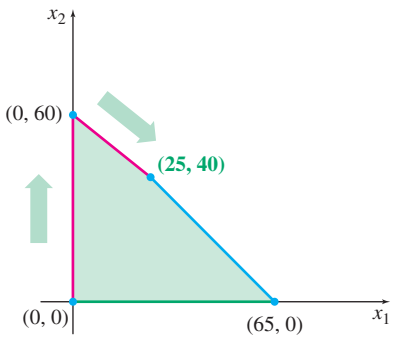


FIGURE 6

YOUR TURN 1 Use the simplex method to solve the problem in Example 2, Section 3.3.

It is good practice to verify your intermediate answers after each new tableau is calculated. You can check your answers by substituting these values for the original variables and the slack variables in the constraint equations and in the objective function.

CAUTION Never choose a zero or a negative number as the pivot. The reason for this is explained in the next example.

EXAMPLE 2 Finding the Pivot

Find the pivot for the following initial simplex tableau.

$$\begin{array}{c|cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline 1 & -2 & 1 & 0 & 0 & 0 & 100 \\ 3 & 4 & 0 & 1 & 0 & 0 & 200 \\ 5 & 0 & 0 & 0 & 1 & 0 & 150 \\ \hline -10 & -25 & 0 & 0 & 0 & 1 & 0 \end{array}$$

SOLUTION The most negative indicator is -25. To find the pivot, we find the quotients formed by the entries in the rightmost column and in the x_2 column: $100/(-2)$, $200/4$, and $150/0$. The quotients predict the value of a variable in the solution. Thus, since we want all variables to be nonnegative, we must reject a negative quotient. Furthermore, we cannot choose 0 as the pivot, because no multiple of the row with 0, when added to the other rows, will cause the other entries in the x_2 column to become 0.

The only usable quotient is $200/4 = 50$, making 4 the pivot. If all the quotients either are negative or have zero denominators, no unique optimum solution will be found. Such a situation indicates an unbounded feasible region because the variable corresponding to that column can be made as large as you like. The quotients, then, determine whether an optimum solution exists. TRY YOUR TURN 2

YOUR TURN 2 Pivot on the 4 in Example 2 and write the solution.

CAUTION If there is a 0 in the right-hand column, do not disregard that row, unless the corresponding number in the pivot column is negative or zero. In fact, such a row gives a quotient of 0, so it will automatically have the smallest ratio. It will not cause an increase in z, but it may lead to another tableau in which z can be further increased.

TECHNOLOGY NOTE We saw earlier that graphing calculators can be used to perform row operations. A program to solve a linear programming problem with a graphing calculator is given in the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

TECHNOLOGY NOTE Spreadsheets often have a program for solving linear programming problems built in. Figure 7 on the next page shows the Solver feature of Microsoft Excel (under the Tools menu) for Example 1. (On some versions of Excel, the Solver must be installed from an outside source.) For details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this text.

In addition, Solver provides a **sensitivity analysis**, which allows us to see how much the constraints could be varied without changing the solution. Under the Solver options, make sure that "Assume Linear Model" and "Assume Non-Negative" are selected. Figure 8 on the next page shows a sensitivity analysis for Example 1. Notice that the value of the first coefficient in the objective function is 25, with an allowable increase of 5 and an allowable decrease of 1. This means that, while keeping the second coefficient at 30, the first coefficient of 25 could be increased by 5 (to 30) or decreased by 1 (to 24), and (25, 40) would still be a solution to the maximization problem. Similarly, for the second coefficient of 30, increasing it by 1.25 (to 31.25) or decreasing it by 5 (to 25) would still leave (25, 40) as a solution to the maximization problem. This would be useful to the owner who decides on the solution of (25, 40) (25 canoes and 40 kayaks) and wonders how much the objective function would have to change before the solution would no longer be optimal. The original revenue for a canoe was \$25, which is the source of the first coefficient in the objective function. Assuming that everything else stays the same, the revenue could change to anything from \$24 to \$30, and the original decision would still be optimal.

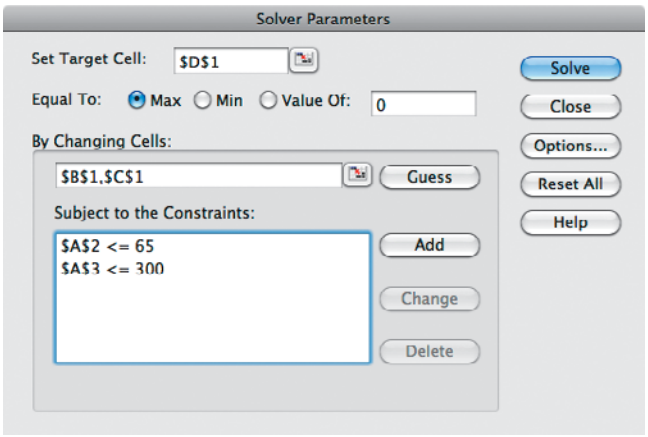


FIGURE 7

Notice, however, that any change in one of the revenues will change the total revenue in the optimal solution. For example, if the first coefficient of 25 is increased by 5 to 30, then the optimal objective value will increase by $5 \times 25 = 125$. One can perform similar changes to other parameters of the problem, but that is beyond the scope of this text.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B1		25	0	25	5	1
C1		40	0	30	1.25	5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
A2		65	5	65	10	5
A3		300	5	300	25	40

FIGURE 8

In many real-life problems, the number of variables and constraints may be in the hundreds, if not the thousands, in which case a computer is used to implement the simplex algorithm. Computer programs for the simplex algorithm differ in some ways from the algorithm we have shown. For example, it is not necessary for a computer to divide common factors out of inequalities to simplify the arithmetic. In fact, computer versions of the algorithm do not necessarily keep all the numbers as integers. As we saw in the previous section, dividing a row by a number may introduce decimals, which makes the arithmetic more difficult to do by hand, but creates no problem for a computer other than round-off error. Several linear programming models in actual use are presented on the website for this textbook.

If you use a graphing calculator to perform the simplex algorithm, we suggest that you review the pivoting procedure described in Method 2 of the previous section. It differs slightly from Method 1, because it converts each pivot element into a 1, but it works nicely with a calculator to keep track of the arithmetic details.

On the other hand, if you carry out the steps of the simplex method by hand, we suggest that you first eliminate fractions and decimals when setting up the initial tableau. For example, we would rewrite the constraint

$$\frac{2}{3}x_1 + \frac{5}{2}x_2 \leq 7$$

as

$$4x_1 + 15x_2 \leq 42,$$

by multiplying both sides of the equation by 6. Similarly, we would write

$$5.2x_1 + 4.4x_2 \le 8.5$$

as

$$52x_1 + 44x_2 \le 85$$

by multiplying both sides of the equation by 10. We must be cautious, however, in remembering that the value of the slack and surplus variables in the optimal solution must be adjusted by this factor to represent the original constraint.

NOTE Sometimes the simplex method cycles and returns to a previously visited solution, rather than making progress. Methods are available for handling cycling. In this text, we will avoid examples with this behavior. For more details, see Alan Sultan's *Linear Programming: An Introduction with Applications*, Academic Press, 1993. In real applications, cycling is rare and tends not to come up because of computer rounding.

4.2 EXERCISES

In Exercises 1–6, the initial tableau of a linear programming problem is given. Use the simplex method to solve each problem.

1.

x_1	x_2	x_3	s_1	s_2	z	
1	4	4	1	0	0	16
2	1	5	0	1	0	20
−3	−1	−2	0	0	1	0

2.

x_1	x_2	x_3	s_1	s_2	z	
3	3	2	1	0	0	18
2	2	3	0	1	0	16
−4	−6	−2	0	0	1	0

3.

x_1	x_2	s_1	s_2	s_3	z	
1	3	1	0	0	0	12
2	1	0	1	0	0	10
1	1	0	0	1	0	4
−2	−1	0	0	0	1	0

4.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	1	2	1	0	0	0	25
4	3	2	0	1	0	0	40
3	1	6	0	0	1	0	60
−4	−2	−3	0	0	0	1	0

5.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	2	8	1	0	0	0	40
4	−5	6	0	1	0	0	60
2	−2	6	0	0	1	0	24
−14	−10	−12	0	0	0	1	0

6.

x_1	x_2	x_3	s_1	s_2	z	
3	2	4	1	0	0	18
2	1	5	0	1	0	8
−1	−4	−2	0	0	1	0

Use the simplex method to solve each linear programming problem.

7. Maximize $z = 3x_1 + 5x_2$

subject to: $4x_1 + x_2 \le 25$

$2x_1 + 3x_2 \le 15$

with $x_1 \ge 0, \quad x_2 \ge 0.$

8. Maximize $z = 5x_1 + 2x_2$

subject to: $2x_1 + 4x_2 \le 15$

$3x_1 + x_2 \le 10$

with $x_1 \ge 0, \quad x_2 \ge 0.$

9. Maximize $z = 10x_1 + 12x_2$

subject to: $4x_1 + 2x_2 \le 20$

$5x_1 + x_2 \le 50$

$2x_1 + 2x_2 \le 24$

with $x_1 \ge 0, \quad x_2 \ge 0.$

10. Maximize $z = 1.5x_1 + 4.2x_2$

subject to: $2.8x_1 + 3.4x_2 \le 21$

$1.4x_1 + 2.2x_2 \le 11$

with $x_1 \ge 0, \quad x_2 \ge 0.$

11. Maximize $z = 8x_1 + 3x_2 + x_3$

subject to: $x_1 + 6x_2 + 8x_3 \le 118$

$x_1 + 5x_2 + 10x_3 \le 220$

with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

12. Maximize $z = 8x_1 + 10x_2 + 7x_3$

subject to: $x_1 + 3x_2 + 2x_3 \le 10$

$x_1 + 5x_2 + x_3 \le 8$

with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

13. Maximize $z = 10x_1 + 15x_2 + 10x_3 + 5x_4$

subject to: $x_1 + x_2 + x_3 + x_4 \le 300$

$x_1 + 2x_2 + 3x_3 + x_4 \le 360$

with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0.$

14. Maximize $z = x_1 + x_2 + 4x_3 + 5x_4$

subject to: $x_1 + 2x_2 + 3x_3 + x_4 \le 115$

$2x_1 + x_2 + 8x_3 + 5x_4 \le 200$

$x_1 + x_3 \le 50$

with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0.$

15. Maximize $z = 4x_1 + 6x_2$
subject to: $x_1 - 5x_2 \le 25$
$4x_1 - 3x_2 \le 12$
with $x_1 \ge 0, \quad x_2 \ge 0.$

16. Maximize $z = 2x_1 + 5x_2 + x_3$
subject to: $x_1 - 5x_2 + 2x_3 \le 30$
$4x_1 - 3x_2 + 6x_3 \le 72$
with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

Use a graphing calculator, Excel, or other technology to solve the following linear programming problems.

17. Maximize $z = 37x_1 + 34x_2 + 36x_3 + 30x_4 + 35x_5$
subject to: $16x_1 + 19x_2 + 23x_3 + 15x_4 + 21x_5 \le 42{,}000$
$15x_1 + 10x_2 + 19x_3 + 23x_4 + 10x_5 \le 25{,}000$
$9x_1 + 16x_2 + 14x_3 + 12x_4 + 11x_5 \le 23{,}000$
$18x_1 + 20x_2 + 15x_3 + 17x_4 + 19x_5 \le 36{,}000$
with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0, \quad x_5 \ge 0.$

18. Maximize $z = 2.0x_1 + 1.7x_2 + 2.1x_3 + 2.4x_4 + 2.2x_5$
subject to: $12x_1 + 10x_2 + 11x_3 + 12x_4 + 13x_5 \le 4250$
$8x_1 + 8x_2 + 7x_3 + 18x_4 + 5x_5 \le 4130$
$9x_1 + 10x_2 + 12x_3 + 11x_4 + 8x_5 \le 3500$
$5x_1 + 3x_2 + 4x_3 + 5x_4 + 4x_5 \le 1600$
with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0, \quad x_5 \ge 0.$

19. The simplex algorithm still works if an indicator other than the most negative one is chosen. (Try it!) List the disadvantages that might occur if this is done.

20. What goes wrong if a quotient other than the smallest nonnegative quotient is chosen in the simplex algorithm? (Try it!)

21. Add lines corresponding to $z = 0$, $z = 1800$, and $z = 1825$ to Figure 6. Then explain how the increase in the objective function as we move from (0, 0) to (0, 60) to (25, 40) relates to the discussion of the Corner Point Theorem in Section 3.2.

22. Explain why the objective function can be made larger as long as there are negative numbers in the bottom row of the simplex tableau. (*Hint:* Consider an example with negative numbers in the last row, and rewrite the last row as an equation.)

APPLICATIONS

Set up and solve Exercises 23–29 by the simplex method.

Business and Economics

23. Charitable Contributions Carrie Green is working to raise money for the homeless by sending information letters and making follow-up calls to local labor organizations and church groups. She discovers that each church group requires 2 hours of letter writing and 1 hour of follow-up, while for each labor union she needs 2 hours of letter writing and 3 hours of follow-up. Carrie can raise $100 from each church group and $200 from each union local, and she has a maximum of 16 hours of letter-writing time and a maximum of 12 hours of follow-up time available per month. Determine the most profitable mixture of groups she should contact and the most money she can raise in a month.

24. Profit The Muro Manufacturing Company makes two kinds of plasma screen television sets. It produces the Flexscan set that sells for $350 profit and the Panoramic I that sells for $500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I. Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hours on the assembly line, 900 work-hours in the cabinet shop, and 2600 work-hours in the testing and packing department. (See Exercise 10 in Section 3.3.)

a. How many sets of each type should it produce to make a maximum profit? What is the maximum profit?

b. Find the values of any nonzero slack variables and describe what they tell you about any unused time.

25. Poker The Texas Poker Company assembles three different poker sets. Each Royal Flush poker set contains 1000 poker chips, 4 decks of cards, 10 dice, and 2 dealer buttons. Each Deluxe Diamond poker set contains 600 poker chips, 2 decks of cards, 5 dice, and one dealer button. The Full House poker set contains 300 poker chips, 2 decks of cards, 5 dice, and one dealer button. The Texas Poker Company has 2,800,000 poker chips, 10,000 decks of cards, 25,000 dice, and 6000 dealer buttons in stock. They earn a profit of $38 for each Royal Flush poker set, $22 for each Deluxe Diamond poker set, and $12 for each Full House poker set.

a. How many of each type of poker set should they assemble to maximize profit? What is the maximum profit?

b. Find the values of any nonzero slack variables and describe what they tell you about any unused components.

26. Income A baker has 150 units of flour, 90 of sugar, and 150 of raisins. A loaf of raisin bread requires 1 unit of flour, 1 of sugar, and 2 of raisins, while a raisin cake needs 5, 2, and 1 units, respectively.

a. If raisin bread sells for $1.75 a loaf and raisin cake for $4.00 each, how many of each should be baked so that gross income is maximized?

b. What is the maximum gross income?

c. Does it require all of the available units of flour, sugar, and raisins to produce the number of loaves of raisin bread and raisin cakes that produce the maximum profit? If not, how much of each ingredient is left over? Compare any leftover to the value of the relevant slack variable.

27. APPLY IT Manufacturing Bicycles A manufacturer of bicycles builds racing, touring, and mountain models. The bicycles are made of both aluminum and steel. The company has available 91,800 units of steel and 42,000 units of aluminum. The racing, touring, and mountain models need 17, 27, and 34 units of steel, and 12, 21, and 15 units of aluminum, respectively. (See Exercise 28 in Section 4.1.)

a. How many of each type of bicycle should be made in order to maximize profit if the company makes $8 per racing bike, $12 per touring bike, and $22 per mountain bike?

b. What is the maximum possible profit?

c. Does it require all of the available units of steel and aluminum to build the bicycles that produce the maximum profit? If not, how much of each material is left over? Compare any leftover to the value of the relevant slack variable.

d. There are many unstated assumptions in the problem given above. Even if the mathematical solution is to make only one or two types of the bicycles, there may be demand for the type(s) not being made, which would create problems for the company. Discuss this and other difficulties that would arise in a real situation.

28. Production The Cut-Right Company sells sets of kitchen knives. The Basic Set consists of 2 utility knives and 1 chef's knife. The Regular Set consists of 2 utility knives, 1 chef's knife, and 1 slicer. The Deluxe Set consists of 3 utility knives, 1 chef's knife, and 1 slicer. Their profit is $30 on a Basic Set, $40 on a Regular Set, and $60 on a Deluxe Set. The factory has on hand 800 utility knives, 400 chef's knives, and 200 slicers. (See Exercise 30 in Section 4.1.)

a. Assuming that all sets will be sold, how many of each type should be made up in order to maximize profit? What is the maximum profit?

b. A consultant for the Cut-Right Company notes that more profit is made on a Regular Set of knives than on a Basic Set, yet the result from part a recommends making up 100 Basic Sets but no Regular Sets. She is puzzled how this can be the best solution. How would you respond?

29. Advertising The Fancy Fashions, an independent, local boutique, has $8000 available each month for advertising. Newspaper ads cost $400 each, and no more than 30 can run per month. Internet banner ads cost $20 each, and no more than 60 can run per month. TV ads cost $2000 each, with a maximum of 10 available each month. Approximately 4000 women will see each newspaper ad, 3000 will see each Internet banner, and 10,000 will see each TV ad. (See Exercise 31 in Section 4.1.)

a. How much of each type of advertising should be used if the store wants to maximize its ad exposure?

b. A marketing analyst is puzzled by the results of part a. More women see each TV ad than each newspaper ad or Internet banner, he reasons, so it makes no sense to use the newspaper ads and Internet banners and no TV ads. How would you respond?

30. Profit A manufacturer makes two products, toy trucks and toy fire engines. Both are processed in four different departments, each of which has a limited capacity. The sheet metal department requires 2 hours for each truck and 3 hours for each fire engine and has a total of 24,000 hours available. The truck assembly department can handle at most 6600 trucks per week; and the fire engine assembly department assembles at most 5500 fire engines weekly. The painting department, which finishes both toys, has a maximum capacity of 10,000 per week.

a. If the profit is $8.50 for a toy truck and $12.10 for a toy fire engine, how many of each item should the company produce to maximize profit?

b. Keeping the profit for a toy truck at $8.50, use the graphical method to find the largest possible profit less than $12.10 for a toy fire engine that would result in an additional solution besides the one found in part a. What is the additional solution? Verify this solution using the simplex method.

c. Keeping the profit for a toy truck at $8.50, use the graphical method to find the smallest possible profit greater than $12.10 for a toy fire engine that would result in an additional solution besides the one found in part a. What is the additional solution? Verify this solution using the simplex method.

Exercises 31 and 32 come from past CPA examinations. *Source: American Institute of Certified Public Accountants, Inc.* Select the appropriate answer for each question.

31. Profit The Ball Company manufactures three types of lamps, labeled A, B, and C. Each lamp is processed in two departments, I and II. Total available work-hours per day for departments I and II are 400 and 600, respectively. No additional labor is available. Time requirements and profit per unit for each lamp type are as follows:

	A	B	C
Work-hours in I	2	3	1
Work-hours in II	4	2	3
Profit per Unit	$5	$4	$3

The company has assigned you as the accounting member of its profit planning committee to determine the numbers of types of A, B, and C lamps that it should produce in order to maximize its total profit from the sale of lamps. The following questions relate to a linear programming model that your group has developed. (For each part, choose one of the four answers.)

a. The coefficients of the objective function would be

(1) 4, 2, 3. **(2)** 2, 3, 1.

(3) 5, 4, 3. **(4)** 400, 600.

b. The constraints in the model would be

(1) 2, 3, 1. **(2)** 5, 4, 3.

(3) 4, 2, 3. **(4)** 400, 600.

c. The constraint imposed by the available work-hours in department I could be expressed as

(1) $4X_1 + 2X_2 + 3X_3 \leq 400$.

(2) $4X_1 + 2X_2 + 3X_3 \geq 400$.

(3) $2X_1 + 3X_2 + 1X_3 \leq 400$.

(4) $2X_1 + 3X_2 + 1X_3 \geq 400$.

32. Profit The Golden Hawk Manufacturing Company wants to maximize the profits on products A, B, and C. The contribution margin for each product follows:

Product	Contribution Margin
A	$2
B	$5
C	$4

The production requirements and departmental capacities, by departments, are as follows:

Department	Production Requirements by Product (hours)			Departmental Capacity (total hours)
	A	*B*	*C*	
Assembling	2	3	2	30,000
Painting	1	2	2	38,000
Finishing	2	3	1	28,000

a. What is the profit-maximization formula for the Golden Hawk Company? (Choose one of the following.)

(1) $\$2A + \$5B + \$4C = X$ (where X = profit)

(2) $5A + 8B + 5C \leq 96{,}000$

(3) $\$2A + \$5B + \$4C \leq X$

(4) $\$2A + \$5B + \$4C = 96{,}000$

b. What is the constraint for the painting department of the Golden Hawk Company? (Choose one of the following.)

(1) $1A + 2B + 2C \geq 38{,}000$

(2) $\$2A + \$5B + \$4C \geq 38{,}000$

(3) $1A + 2B + 2C \leq 38{,}000$

(4) $2A + 3B + 2C \leq 30{,}000$

33. Sensitivity Analysis Using a computer spreadsheet, perform a sensitivity analysis for the objective function in Exercise 23. What are the highest and lowest possible values for the amount raised from each church group that would yield the same solution as the original problem? Answer the same question for the amount raised from each union local.

34. Sensitivity Analysis Using a computer spreadsheet, perform a sensitivity analysis for the objective function in Exercise 24. What are the highest and lowest possible values for profit on a Flexscan set that would yield the same solution as the original problem? Answer the same question for a Panoramic I set.

Set up and solve Exercises 35–40 by the simplex method.

Life Sciences

35. Calorie Expenditure Rachel Reeve, a fitness trainer, has an exercise regimen that includes running, biking, and walking. She has no more than 15 hours per week to devote to exercise, including at most 3 hours running. She wants to walk at least twice as many hours as she bikes. According to a website, a 130-pound person like Rachel will burn on average 531 calories per hour running, 472 calories per hour biking, and 354 calories per hour walking. How many hours per week should Rachel spend on each exercise to maximize the number of calories she burns? What is the maximum number of calories she will burn? (*Hint:* Write the constraint involving walking and biking in the form ≤ 0.) ***Source: NutriStrategy.com.***

36. Calorie Expenditure Joe Vetere's exercise regimen includes light calisthenics, swimming, and playing the drums. He has at most 10 hours per week to devote to these activities. He wants the total time he does calisthenics and plays the drums to be at least twice as long as he swims. His neighbors, however, will tolerate no more than 4 hours per week on the drums. According to a website, a 190-pound person like Joe will burn an average of 388 calories per hour doing calisthenics, 518 calories per hour swimming, and 345 calories per hour playing the drums. ***Source: NutriStrategy.com.***

a. How many hours per week should Joe spend on each exercise to maximize the number of calories he burns? What is the maximum number of calories he will burn?

b. What conclusions can you draw about Joe's selection of activities?

37. Resource Management The average weights of the three species stocked in the lake referred to in Section 2.2, Exercise 59, are 1.62, 2.14, and 3.01 kg for species A, B, and C, respectively.

a. If the largest amounts of food that can be supplied each day are as given in Exercise 59, how should the lake be stocked to maximize the weight of the fish supported by the lake?

b. Does it require all of the available food to produce the maximum weight of fish? If not, how much of each type of food is left over?

c. Find a value for each of the average weights of the three species that would result in none of species B or C being stocked to maximize the weight of the fish supported by the lake, given the constraints in part a.

d. Find a value for each of the average weights of the three species that would result in none of species A or B being stocked to maximize the weight of the fish supported by the lake, given the constraints in part a.

38. Blending Nutrients A biologist has 500 kg of nutrient A, 600 kg of nutrient B, and 300 kg of nutrient C. These nutrients will be used to make four types of food, whose contents (in percent of nutrient per kilogram of food) and whose "growth values" are as shown in the table.

	P	Q	R	S
A	0	0	37.5	62.5
B	0	75	50	37.5
C	100	25	12.5	0
Growth Value	90	70	60	50

a. How many kilograms of each food should be produced in order to maximize total growth value?

b. Find the maximum growth value.

c. Does it require all of the available nutrients to produce the four types of food that maximizes the total growth value? If not, how much of each nutrient is left over?

Social Sciences

39. Politics A political party is planning a half-hour television show. The show will have at least 3 minutes of direct requests for money from viewers. Three of the party's politicians will be on the show—a senator, a congresswoman, and a governor. The senator, a party "elder statesman," demands that he be on screen

at least twice as long as the governor. The total time taken by the senator and the governor must be at least twice the time taken by the congresswoman. Based on a pre-show survey, it is believed that 35, 40, and 45 (in thousands) viewers will watch the program for each minute the senator, congresswoman, and governor, respectively, are on the air. Find the time that should be allotted to each politician in order to get the maximum number of viewers. Find the maximum number of viewers.

40. Fund Raising The political party in Exercise 39 is planning its fund-raising activities for a coming election. It plans to raise money through large fund-raising parties, letters requesting funds, and dinner parties where people can meet the candidate personally. Each large fund-raising party costs $3000, each mailing costs $1000, and each dinner party costs $12,000. The party can spend up to $102,000 for these activities. From experience, the planners know that each large party will raise $200,000, each letter campaign will raise $100,000, and each dinner party will raise $600,000. They are able to carry out a total of 25 of these activities.

a. How many of each should the party plan in order to raise the maximum amount of money? What is the maximum amount?

b. Dinner parties are more expensive than letter campaigns, yet the optimum solution found in part a includes dinner parties but no letter campaigns. Explain how this is possible.

YOUR TURN ANSWERS

1. $x_1 = 0$, $x_2 = 12$, for a maximum value of $z = 480$.
2. $x_1 = 0$, $x_2 = 50$, $s_1 = 200$, $s_2 = 0$, $s_3 = 150$, $z = 1250$

4.3 Minimization Problems; Duality

APPLY IT

How many units of different types of feed should a dog breeder purchase to meet the nutrient requirements of her golden retrievers at a minimum cost?
Using the method of duals, we will answer this question in Example 5.

Minimization Problems

The definition of a problem in standard maximum form was given earlier in this chapter. Now we can define a linear programming problem in *standard minimum form*, as follows.

Standard Minimum Form

A linear programming problem is in **standard minimum form** if the following conditions are satisfied.

1. The objective function is to be minimized.
2. All variables are nonnegative.
3. All remaining constraints are stated in the form

$$a_1y_1 + a_2y_2 + \cdots + a_ny_n \geq b, \qquad \text{with } b \geq 0.$$

NOTE
In this section, we require that all coefficients in the objective function be positive, so $c_1 \geq 0$, $c_2 \geq 0, \ldots, c_n \geq 0$.

The difference between maximization and minimization problems is in conditions 1 and 3: In problems stated in standard minimum form, the objective function is to be *minimized*, rather than maximized, and all constraints must have $\geq$ instead of $\leq$.

We use y_1, y_2, etc., for the variables and w for the objective function as a reminder that these are minimizing problems. Thus, $w = c_1y_1 + c_2y_2 + \cdots + c_ny_n$.

Duality

An interesting connection exists between standard maximization and standard minimization problems: any solution of a standard maximization problem produces the solution of an associated standard minimization problem, and vice versa. Each of these associated problems is called the **dual** of the other. One advantage of duals is that standard minimization problems can be solved by the simplex method discussed in the first two sections of this chapter. Let us explain the idea of a dual with an example.

EXAMPLE 1 Duality

Minimize $w = 8y_1 + 16y_2$

subject to: $y_1 + 5y_2 \geq 9$

$2y_1 + 2y_2 \geq 10$

with $y_1 \geq 0, \quad y_2 \geq 0.$

SOLUTION Without considering slack variables just yet, write the augmented matrix of the system of inequalities and include the coefficients of the objective function (not their negatives) as the last row in the matrix.

Constants

$$\begin{bmatrix} 1 & 5 & 9 \\ 2 & 2 & 10 \\ 8 & 16 & 0 \end{bmatrix}$$

Objective function →

Now look at the following matrix, which we obtain from the one above by interchanging rows and columns.

Constants

$$\begin{bmatrix} 1 & 2 & 8 \\ 5 & 2 & 16 \\ 9 & 10 & 0 \end{bmatrix}$$

Objective function →

The *rows* of the first matrix (for the minimization problem) are the *columns* of the second matrix.

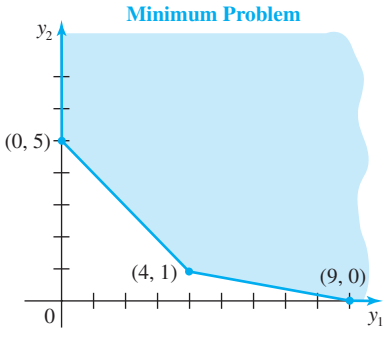
The entries in this second matrix could be used to write the following maximization problem in standard form (again ignoring the fact that the numbers in the last row are not negative):

Maximize $z = 9x_1 + 10x_2$

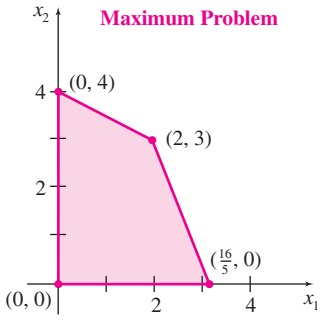
subject to: $x_1 + 2x_2 \leq 8$

$5x_1 + 2x_2 \leq 16$

with all variables nonnegative.



(a)



(b)

FIGURE 9

Figure 9(a) shows the region of feasible solutions for the minimization problem just given, while Figure 9(b) shows the region of feasible solutions for the maximization problem produced by exchanging rows and columns. The solutions of the two problems are given below.

Minimum Problem		
Corner Point	$w = 8y_1 + 16y_2$	
(0, 5)	80	
(4, 1)	**48**	Minimum
(9, 0)	72	

The minimum is 48 when $y_1 = 4$ and $y_2 = 1$.

Maximum Problem		
Corner Point	$z = 9x_1 + 10x_2$	
(0, 0)	0	
(0, 4)	40	
(2, 3)	**48**	Maximum
(16/5, 0)	28.8	

The maximum is 48 when $x_1 = 2$ and $x_2 = 3$.

The two feasible regions in Figure 9 are different and the corner points are different, but the values of the objective functions are equal—both are 48. An even closer connection between the two problems is shown by using the simplex method to solve this maximization problem.

Maximization Problem

$$\begin{array}{c} \\ \\ \\ \\ -\mathbf{R}_1 + \mathbf{R}_2 \to \mathbf{R}_2 \\ 5\mathbf{R}_1 + \mathbf{R}_3 \to \mathbf{R}_3 \\ \\ -\mathbf{R}_2 + 4\mathbf{R}_1 \to \mathbf{R}_1 \\ \\ \mathbf{R}_2 + \mathbf{R}_3 \to \mathbf{R}_3 \\ \\ \mathbf{R}_1/8 \to \mathbf{R}_1 \\ \mathbf{R}_2/4 \to \mathbf{R}_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 1 & \mathbf{2} & 1 & 0 & 0 & 8 \\ 5 & 2 & 0 & 1 & 0 & 16 \\ \hline -9 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 8 \\ \mathbf{4} & 0 & -1 & 1 & 0 & 8 \\ \hline -4 & 0 & 5 & 0 & 1 & 40 \end{array}\right] \\ \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 8 & 5 & -1 & 0 & 24 \\ 4 & 0 & -1 & 1 & 0 & 8 \\ \hline 0 & 0 & 4 & 1 & 1 & 48 \end{array}\right] \\ \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{5}{8} & -\frac{1}{8} & 0 & 3 \\ 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 2 \\ \hline 0 & 0 & \mathbf{4} & \mathbf{1} & 1 & 48 \end{array}\right] \end{array}$$

The maximum is 48 when $x_1 = 2$ and $x_2 = 3$.

Notice that the solution to the *minimization problem* is found in the bottom row and slack variable columns of the final simplex tableau for the maximization problem. This result suggests that standard minimization problems can be solved by forming the dual standard maximization problem, solving it by the simplex method, and then reading the solution for the minimization problem from the bottom row of the final simplex tableau.

Before using this method to actually solve a minimization problem, let us find the duals of some typical linear programming problems. The process of exchanging the rows and columns of a matrix, which is used to find the dual, is called *transposing* the matrix, and each of the two matrices is the **transpose** of the other. The transpose of an $m \times n$ matrix A, written A^T, is an $n \times m$ matrix.

EXAMPLE 2 Transposes

Find the transpose of each matrix.

(a) $A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & 8 & 0 \\ -3 & 7 & -1 \end{bmatrix}$

SOLUTION Both matrix A and its transpose are 3×3 matrices. Write the rows of matrix A as the columns of the transpose.

$$A^T = \begin{bmatrix} 2 & 6 & -3 \\ -1 & 8 & 7 \\ 5 & 0 & -1 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 2 & 1 & 7 & 6 \end{bmatrix}$

SOLUTION The matrix B is 2×4, so B^T is the 4×2 matrix

$$B^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 7 \\ 0 & 6 \end{bmatrix}.$$

EXAMPLE 3 Duals

Write the dual of each standard linear programming problem.

(a) Maximize $z = 2x_1 + 5x_2$

subject to: $x_1 + x_2 \le 10$

$2x_1 + x_2 \le 8$

with $x_1 \ge 0, \quad x_2 \ge 0.$

SOLUTION Begin by writing the augmented matrix for the given problem.

$$\left[\begin{array}{cc|c} 1 & 1 & 10 \\ 2 & 1 & 8 \\ \hline 2 & 5 & 0 \end{array}\right]$$

Form the transpose of the matrix as follows:

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 1 & 5 \\ \hline 10 & 8 & 0 \end{array}\right].$$

The dual problem is stated from this second matrix as follows (using y instead of x):

Minimize $w = 10y_1 + 8y_2$

subject to: $y_1 + 2y_2 \ge 2$

$y_1 + y_2 \ge 5$

with $y_1 \ge 0, \quad y_2 \ge 0.$

(b) Minimize $w = 7y_1 + 5y_2 + 8y_3$

subject to: $3y_1 + 2y_2 + y_3 \ge 10$

$4y_1 + 5y_2 \ge 25$

with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

SOLUTION The dual problem is stated as follows.

Maximize $z = 10x_1 + 25x_2$

subject to: $3x_1 + 4x_2 \le 7$

$2x_1 + 5x_2 \le 5$

$x_1 \le 8$

with $x_1 \ge 0, \quad x_2 \ge 0.$

YOUR TURN 1 Write the dual of the following linear programming problem.

Minimize $w = 25y_1 + 12y_2 + 27y_3$

subject to: $3y_1 + 3y_2 + 4y_3 \ge 24$

$5y_1 + y_2 + 3y_3 \ge 27$

with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0.$

TRY YOUR TURN 1

NOTE You might find it easier to set up the dual if you put the objective function *after* the constraints in the original problem, and then line up the variables, as in the following rewriting of Example 3 (b):

$$\begin{aligned} 3y_1 + 2y_2 + y_3 &\ge 10 \\ 4y_1 + 5y_2 &\ge 25 \end{aligned}$$

$$\text{Minimize } 7y_1 + 5y_2 + 8y_3.$$

Notice that you can read down the first column to get the coefficients of the first constraint in the dual $(3x_1 + 4x_2 \le 7)$. Reading down the second and third columns gives the coefficients of the next two constraints, and the last column gives the coefficients of the objective function.

In Example 3, all the constraints of the given standard maximization problems were $\le$ inequalities, while all those in the dual minimization problems were $\ge$ inequalities. This is generally the case; inequalities are reversed when the dual problem is stated.

NOTE
To solve a minimization problem with duals, all of the coefficients in the objective function must be positive. (To investigate what would happen without this requirement, see Exercise 18). For a method that does not have this restriction, see the next section.

The following table shows the close connection between a problem and its dual.

Duality	
Given Problem	**Dual Problem**
m variables	n variables
n constraints	m constraints
Coefficients from objective function	Constraint constants
Constraint constants	Coefficients from objective function

The next theorem, whose proof requires advanced methods, guarantees that a standard minimization problem can be solved by forming a dual standard maximization problem.

Theorem of Duality

The objective function w of a minimization linear programming problem takes on a minimum value if and only if the objective function z of the corresponding dual maximization problem takes on a maximum value. The maximum value of z equals the minimum value of w.

This method is illustrated in the following example.

EXAMPLE 4 Duality

Minimize $w = 3y_1 + 2y_2$

subject to: $y_1 + 3y_2 \geq 6$

$2y_1 + y_2 \geq 3$

with $y_1 \geq 0, \quad y_2 \geq 0.$

SOLUTION Use the given information to write the matrix.

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & 1 & 3 \\ \hline 3 & 2 & 0 \end{array}\right]$$

Transpose to get the following matrix for the dual problem.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 1 & 2 \\ \hline 6 & 3 & 0 \end{array}\right]$$

Write the dual problem from this matrix, as follows:

Maximize $z = 6x_1 + 3x_2$

subject to: $x_1 + 2x_2 \leq 3$

$3x_1 + x_2 \leq 2$

with $x_1 \geq 0, \quad x_2 \geq 0.$

Solve this standard maximization problem using the simplex method. Start by introducing slack variables to give the system

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 3 \\ 3x_1 + x_2 + s_2 &= 2 \\ -6x_1 - 3x_2 + z &= 0 \end{aligned}$$

with $x_1 \geq 0, \quad x_2 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0.$

The first tableau for this system is given below, with the pivot as indicated.

Quotients	x_1	x_2	s_1	s_2	z	
3/1 = 3	1	2	1	0	0	3
2/3	**3**	1	0	1	0	2
	−6	−3	0	0	1	0

The simplex method gives the following as the final tableau.

x_1	x_2	s_1	s_2	z	
0	1	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{7}{5}$
1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{1}{5}$
0	0	$\frac{3}{5}$	$\frac{9}{5}$	1	$\frac{27}{5}$

Since a 1 has been created in the z column, the last row of this final tableau gives the solution to the minimization problem. The minimum value of $w = 3y_1 + 2y_2$, subject to the given constraints, is $27/5$ and occurs when $y_1 = 3/5$ and $y_2 = 9/5$. The minimum value of w, $27/5$, is the same as the maximum value of z. TRY YOUR TURN 2

YOUR TURN 2

Minimize $w = 15y_1 + 12y_2$

subject to: $3y_1 + 5y_2 \geq 20$

$3y_1 + y_2 \geq 18$

with $y_1 \geq 0, y_2 \geq 0$.

Let us summarize the steps in solving a standard minimization linear programming problem by the method of duals.

Solving a Standard Minimum Problem with Duals

1. Find the dual standard maximization problem.
2. Solve the maximization problem using the simplex method.
3. The minimum value of the objective function w is the maximum value of the objective function z.
4. The optimum solution to the minimization problem is given by the entries in the bottom row of the columns corresponding to the slack variables, so long as the entry in the z column is equal to 1.

CAUTION (1) If the final entry in the z column is a value other than 1, divide the bottom row through by that value so that it will become 1. Only then can the solution of the minimization problem be found in the bottom row of the columns corresponding to the slack variables.
(2) Do not simplify an inequality in the dual by dividing out a common factor. For example, if an inequality in the dual is $3x_1 + 3x_2 \leq 6$, do not simplify to $x_1 + x_2 \leq 2$ by dividing out the 3. Doing so will give an incorrect solution to the original problem.

Further Uses of the Dual

The dual is useful not only in solving minimization problems but also in seeing how small changes in one variable will affect the value of the objective function.

EXAMPLE 5 Nutrition

Suppose a dog breeder needs at least 6 units per day of nutrient A and at least 3 units of nutrient B for her golden retrievers, and that the breeder can choose between two different feeds, feed 1 and feed 2. Find the minimum cost for the breeder if each bag of feed 1 costs \$3 and provides 1 unit of nutrient A and 2 units of B, while each bag of feed 2 costs \$2 and provides 3 units of nutrient A and 1 of B.

APPLY IT

SOLUTION If y_1 represents the number of bags of feed 1 and y_2 represents the number of bags of feed 2, the given information leads to the following problem.

$$\begin{aligned} \text{Minimize} \quad & w = 3y_1 + 2y_2 \\ \text{subject to:} \quad & y_1 + 3y_2 \geq 6 \quad \text{Nutrient A} \\ & 2y_1 + y_2 \geq 3 \quad \text{Nutrient B} \\ \text{with} \quad & y_1 \geq 0, \quad y_2 \geq 0. \end{aligned}$$

This standard minimization linear programming problem is the one solved in Example 4 of this section. In that example, the dual was formed and the following tableau was found.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z & \end{matrix} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{7}{5} \\ 1 & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{1}{5} \\ \hline 0 & 0 & \frac{3}{5} & \frac{9}{5} & 1 & \frac{27}{5} \end{array}\right] \end{array}$$

This final tableau shows that the breeder will obtain minimum feed costs by using $3/5$ bag of feed 1 and $9/5$ bags of feed 2 per day, for a daily cost of $27/5 = \$5.40$.

Notice that the solution to the dual (maximization) problem is

$$x_1 = \frac{1}{5} = 0.20 \qquad \text{and} \qquad x_2 = \frac{7}{5} = 1.40.$$

These represent the **shadow costs** of nutrients A and B. The nutrients don't have actual costs because you can't buy a unit of nutrient A or B; all you can buy is a bag of Feed 1 or Feed 2, each of which contains both nutrients A and B. Shadow costs, however, are a convenient way of allocating costs. Suppose that a unit of nutrient A is said to cost \$0.20, its shadow cost, and a unit of nutrient B is said to cost \$1.40. Then the minimum daily cost, which we previously found to be \$5.40 (providing 6 units of A and 3 units of B), can be found by the following procedure.

$$\begin{aligned} (\$0.20 \text{ per unit of A}) \times (6 \text{ units of A}) &= \$1.20 \\ + (\$1.40 \text{ per unit of B}) \times (3 \text{ units of B}) &= \$4.20 \\ \text{Minimum daily cost} &= \$5.40 \end{aligned}$$

Furthermore, the shadow costs allow the breeder to calculate feed costs for small changes in nutrient requirements. For example, an increase of one unit in the requirement for each nutrient would produce a total cost of \$7.00:

$$\begin{array}{rl} \$5.40 & \text{6 units of A, 3 of B} \\ 0.20 & \text{1 extra unit of A} \\ + 1.40 & \text{1 extra unit of B} \\ \hline \$7.00 & \text{Total cost per day} \end{array}$$

Shadow costs only give the exact answer for a limited range. Unfortunately, finding that range is somewhat complicated. In the dog feed example, we can add up to 3 units or delete up to 4 units of A, and shadow costs will give the exact answer. If, however, we add 4 units of A, shadow costs give an answer of \$6.20, while the true cost is \$6.67.

NOTE
Shadow costs become shadow profits in maximization problems. For example, see Exercises 21 and 22.

CAUTION If you wish to use shadow costs, do not simplify an inequality in the original problem by dividing out a common factor. For example, if an inequality in the original problem is $3y_1 + 3y_2 \geq 6$, do not simplify to $y_1 + y_2 \geq 2$ by dividing out the 3. Doing so will give incorrect shadow costs.

TECHNOLOGY NOTE The Solver in Microsoft Excel provides the values of the dual variables. See the *Graphing Calculator and Excel Spreadsheet Manual* available with this book for more details.

4.3 EXERCISES

Find the transpose of each matrix.

1. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 10 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 4 & -2 & 0 & 1 \\ 2 & 0 & 11 & 5 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 5 & -3 & 15 \\ 7 & 14 & 20 & -8 \\ 5 & 0 & -2 & 23 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 11 & 15 \\ 0 & 10 & -6 \\ 4 & 12 & -2 \\ 1 & -1 & 13 \\ 2 & 25 & -1 \end{bmatrix}$

State the dual problem for each linear programming problem.

5. Maximize $z = 4x_1 + 3x_2 + 2x_3$
subject to: $x_1 + x_2 + x_3 \le 5$
$x_1 + x_2 \le 4$
$2x_1 + x_2 + 3x_3 \le 15$
with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

6. Maximize $z = 2x_1 + 7x_2 + 4x_3$
subject to: $4x_1 + 2x_2 + x_3 \le 26$
$x_1 + 7x_2 + 8x_3 \le 33$
with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

7. Minimize $w = 3y_1 + 6y_2 + 4y_3 + y_4$
subject to: $y_1 + y_2 + y_3 + y_4 \ge 150$
$2y_1 + 2y_2 + 3y_3 + 4y_4 \ge 275$
with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0, \quad y_4 \ge 0.$

8. Minimize $w = y_1 + y_2 + 4y_3$
subject to: $y_1 + 2y_2 + 3y_3 \ge 115$
$2y_1 + y_2 + 8y_3 \ge 200$
$y_1 + y_3 \ge 50$
with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

Use the simplex method to solve.

9. Find $y_1 \ge 0$ and $y_2 \ge 0$ such that
$2y_1 + 3y_2 \ge 6$
$2y_1 + y_2 \ge 7$
and $w = 5y_1 + 2y_2$ is minimized.

10. Find $y_1 \ge 0$ and $y_2 \ge 0$ such that
$2y_1 + 3y_2 \ge 15$
$5y_1 + 6y_2 \ge 35$
and $w = 2y_1 + 3y_2$ is minimized.

11. Find $y_1 \ge 0$ and $y_2 \ge 0$ such that
$10y_1 + 5y_2 \ge 100$
$20y_1 + 10y_2 \ge 150$
and $w = 4y_1 + 5y_2$ is minimized.

12. Minimize $w = 29y_1 + 10y_2$
subject to: $3y_1 + 2y_2 \ge 2$
$5y_1 + y_2 \ge 3$
with $y_1 \ge 0, \quad y_2 \ge 0.$

13. Minimize $w = 6y_1 + 10y_2$
subject to: $3y_1 + 5y_2 \ge 15$
$4y_1 + 7y_2 \ge 20$
with $y_1 \ge 0, \quad y_2 \ge 0.$

14. Minimize $w = 3y_1 + 2y_2$
subject to: $y_1 + 2y_2 \ge 10$
$y_1 + y_2 \ge 8$
$2y_1 + y_2 \ge 12$
with $y_1 \ge 0, \quad y_2 \ge 0.$

15. Minimize $w = 2y_1 + y_2 + 3y_3$
subject to: $y_1 + y_2 + y_3 \ge 100$
$2y_1 + y_2 \ge 50$
with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

16. Minimize $w = 4y_1 + 7y_2 + 9y_3$
subject to: $2y_1 + 3y_2 + 4y_3 \ge 45$
$y_1 + 5y_2 + 2y_3 \ge 40$
with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

17. You are given the following linear programming problem (P):

Minimize $z = x_1 + 2x_2$
subject to: $-2x_1 + x_2 \ge 1$
$x_1 - 2x_2 \ge 1$
$x_1 \ge 0, \quad x_2 \ge 0.$

The dual of (P) is (D). Which of the statements below is true? ***Source: Society of Actuaries.***

a. (P) has no feasible solution and the objective function of (D) is unbounded.

b. (D) has no feasible solution and the objective function of (P) is unbounded.

c. The objective functions of both (P) and (D) are unbounded.

d. Both (P) and (D) have optimal solutions.

e. Neither (P) nor (D) has feasible solutions.

18. Suppose the coefficient of 3 in the objective function of Example 4 is changed to -3 and all the constraints are kept the same. Explain why the objective function now has no minimum.

APPLICATIONS

Business and Economics

19. Production Costs A brewery produces regular beer and a lower-carbohydrate "light" beer. Steady customers of the brewery buy 10 units of regular beer and 15 units of light beer monthly. While setting up the brewery to produce the beers, the management decides to produce extra beer, beyond that needed to satisfy customers. The cost per unit of regular beer is $32,000 and the cost per unit of light beer is $50,000. Every unit of regular beer brings in $120,000 in revenue, while every unit of light beer brings in $300,000 in revenue. The brewery wants at least $9,000,000 in revenue. At least 20 additional units of beer can be sold.

a. How much of each type of beer should be made so as to minimize total production costs?

b. Suppose the minimum revenue is increased to $9,500,000. Use shadow costs to calculate the total production costs.

20. Supply Costs The chemistry department at a local college decides to stock at least 900 small test tubes and 600 large test tubes. It wants to buy at least 2700 test tubes to take advantage of a special price. Since the small test tubes are broken twice as often as the large, the department will order at least twice as many small tubes as large.

a. If the small test tubes cost 18 cents each and the large ones, made of a cheaper glass, cost 15 cents each, how many of each size should be ordered to minimize cost?

b. Suppose the minimum number of test tubes is increased to 3000. Use shadow costs to calculate the total cost in this case.

In most examples of this section, the original problem is a minimization problem and the dual is a maximization problem whose solution gives shadow costs. The reverse is true in Exercises 21 and 22. The dual here is a minimization problem whose solution can be interpreted as shadow profits.

21. Agriculture Refer to the original information in Example 2, Section 4.1.

a. Give the dual problem.

b. Use the shadow profits to estimate the farmer's profit if land is cut to 90 acres but capital increases to $21,000.

c. Suppose the farmer has 110 acres but only $19,000. Find the optimum profit and the planting strategy that will produce this profit.

22. Toy Manufacturing A small toy manufacturing firm has 200 squares of felt, 600 oz of stuffing, and 90 ft of trim available to make two types of toys, a small bear and a monkey. The bear requires 1 square of felt and 4 oz of stuffing. The monkey requires 2 squares of felt, 3 oz of stuffing, and 1 ft of trim. The firm makes $1 profit on each bear and $1.50 profit on each monkey.

a. Set up the linear programming problem to maximize profit.

b. Solve the linear programming problem in part a.

c. What is the corresponding dual problem?

d. What is the optimal solution to the dual problem?

e. Use the shadow profits to calculate the profit the firm will make if its supply of felt increases to 210 squares.

f. How much profit will the firm make if its supply of stuffing is cut to 590 oz and its supply of trim is cut to 80 ft?

g. Explain why it makes sense that the shadow profit for trim is 0.

23. Interview Time Joan McKee has a part-time job conducting public opinion interviews. She has found that a political interview takes 45 min and a market interview takes 55 min. She needs to minimize the time she spends doing interviews to allow more time for her full-time job. Unfortunately, to keep her part-time job, she must complete at least 8 interviews each week. Also, she must earn at least $60 per week at this job; she earns $8 for each political interview and $10 for each market interview. Finally, to stay in good standing with her supervisor, she must earn at least 40 bonus points per week; she receives 6 bonus points for each political interview and 5 points for each market interview. How many of each interview should she do each week to minimize the time spent?

24. Animal Food An animal food must provide at least 54 units of vitamins and 60 calories per serving. One gram of soybean meal provides 2.5 units of vitamins and 5 calories. One gram of meat byproducts provides 4.5 units of vitamins and 3 calories. One gram of grain provides 5 units of vitamins and 10 calories. A gram of soybean meal costs 8¢, a gram of meat byproducts 9¢, and a gram of grain 10¢.

a. What mixture of these three ingredients will provide the required vitamins and calories at minimum cost?

b. What is the minimum cost?

c. There is more than one optimal basic solution to this problem. The answer found in part a depends on whether the tie in the minimum ratio rule was broken by pivoting on the second row or third row of the dual. Find the other solution.

25. Feed Costs Refer to Example 5 in this section on minimizing the daily cost of feeds.

a. Find a combination of feeds that will cost $7 and give 7 units of A and 4 units of B.

b. Use the dual variables to predict the daily cost of feed if the requirements change to 5 units of A and 4 units of B. Find a combination of feeds to meet these requirements at the predicted price.

26. Pottery Karla Harby makes three items in her pottery shop: large bowls, small bowls, and pots for plants. A large bowl requires 3 lb of clay and 6 fl oz of glaze. A small bowl requires 2 lb of clay, and 6 fl oz of glaze. A pot requires 4 lb of clay and 2 fl oz of glaze. She must use up 72 lb of old clay and 108 fl oz of old glaze; she can order more if necessary. If Karla can make a large bowl in 5 hours, a small bowl in 6 hours, and a pot in 4 hours, how many of each should she make to minimize her time? What is the minimum time?

Life Sciences

27. Calorie Expenditure Francesca wants to start exercising to burn at least 1500 extra calories per week, but she does not have much spare time for exercise. According to a website, she can burn an average of 3.5 calories per minute walking, 4 calories per minute cycling, and 8 calories per minute swimming. She would like her total time walking and cycling to be at least 3 times as long as she spends swimming. She would also like to walk at least 30 minutes per week. How much time should she spend on each activity not only to meet her goals but also to minimize her total exercise time per week? What is her minimum exercise time per week? ***Source: BrianMac.co.uk.***

28. Health Care Marty McDonald takes vitamin pills. Each day he must have at least 3200 IU of vitamin A, 5 mg of vitamin B_1, and 200 mg of vitamin C. He can choose between pill 1, which costs 10¢ and contains 1600 IU of A, 1 mg of B_1, and 20 mg of C; and pill 2, which costs 20¢ and contains 400 IU of A, 1 mg of B_1, and 70 mg of C. How many of each pill should he buy in order to minimize his cost?

29. Blending Nutrients A biologist must make a nutrient for her algae. The nutrient must contain the three basic elements D, E, and F, and must contain at least 10 kg of D, 12 kg of E, and 20 kg of F. The nutrient is made from three ingredients, I, II, and III. The quantity of D, E, and F in one unit of each of the ingredients is as given in the following chart.

		Ingredient		
		I	*II*	*III*
Kilograms of Elements (per unit of ingredient)	*D*	4	1	10
	E	3	2	1
	F	0	4	5
Cost per unit (in $)		4	7	5

How many units of each ingredient are required to meet the biologist's needs at minimum cost?

YOUR TURN ANSWERS

1. Maximize $z = 24x_1 + 27x_2$

subject to: $3x_1 + 5x_2 \le 25$

$3x_1 + x_2 \le 12$

$4x_1 + 3x_2 \le 27$

with $x_1 \ge 0, \quad x_2 \ge 0.$

2. Minimum is 187/2 when $y_1 = 35/6$ and $y_2 = 1/2$.

4.4 Nonstandard Problems

APPLY IT **How many cars should an auto manufacturer send from each of its two plants to each of two dealerships in order to minimize the cost while meeting each dealership's needs?**

We will learn techniques in this section for answering questions like the one above, which will be answered in Example 2.

So far we have used the simplex method to solve linear programming problems in standard maximum or minimum form only. Now this work is extended to include linear programming problems with mixed $\le$ and $\ge$ constraints.

For example, suppose a new constraint is added to the farmer problem in Example 2 of Section 4.1: To satisfy orders from regular buyers, the farmer must plant a total of at least 60 acres of the three crops. Notice that our solution from Section 4.2 (plant 50 acres of potatoes, no acres of corn, and no acres of cabbage) does not satisfy this constraint, which introduces the new inequality

$$x_1 + x_2 + x_3 \ge 60.$$

As before, this inequality must be rewritten as an equation in which the variables all represent nonnegative numbers. The inequality $x_1 + x_2 + x_3 \ge 60$ means that

$$x_1 + x_2 + x_3 - s_3 = 60$$

for some nonnegative variable s_3. (Remember that s_1 and s_2 are the slack variables in the problem.)

The new variable, s_3, is called a **surplus variable**. The value of this variable represents the excess number of acres (over 60) that may be planted. Since the total number of acres planted is to be no more than 100 but at least 60, the value of s_3 can vary from 0 to 40.

We must now solve the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 + s_1 &= 100 \\ 10x_1 + 4x_2 + 7x_3 + s_2 &= 500 \\ x_1 + x_2 + x_3 - s_3 &= 60 \\ -120x_1 - 40x_2 - 60x_3 + z &= 0, \end{aligned}$$

with x_1, x_2, x_3, s_1, s_2, and s_3 all nonnegative.

Set up the initial simplex tableau.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 0 & 500 \\ 1 & 1 & 1 & 0 & 0 & -1 & 0 & 60 \\ \hline -120 & -40 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

This tableau gives the solution

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad s_1 = 100, \quad s_2 = 500, \quad s_3 = -60.$$

But this is not a feasible solution, since s_3 is negative. This means that the third constraint is not satisfied; we have $x_1 + x_2 + x_3 = 0$, but the sum is supposed to be at least 60. All the variables in any feasible solution must be nonnegative if the solution is to correspond to a point in the feasible region.

When a negative value of a variable appears in the solution, row operations are used to transform the matrix until a solution is found in which all variables are nonnegative. Here the problem is the -1 in a column corresponding to a basic variable. If the number in that row of the right-hand column were 0, we could simply multiply this row by -1 to remove the negative from the column. But we cannot do this with a positive number in the right-hand column. Instead, we find the positive entry that is farthest to the left in the third row (the row containing the -1); namely, the 1 in row 3, column 1. We will pivot using this column. (Actually, any column with a positive entry in row 3 will do; we chose the column farthest to the left arbitrarily.*) Use quotients as before to find the pivot, which is the 10 in row 2, column 1. Then use row operations to get the following tableau.

$$\begin{array}{r} \\ \mathbf{-R_2 + 10R_1 \to R_1} \\ \\ \mathbf{-R_2 + 10R_3 \to R_3} \\ \mathbf{12R_2 + R_4 \to R_4} \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 6 & 3 & 10 & -1 & 0 & 0 & 500 \\ 10 & 4 & 7 & 0 & 1 & 0 & 0 & 500 \\ 0 & 6 & 3 & 0 & -1 & -10 & 0 & 100 \\ \hline 0 & 8 & 24 & 0 & 12 & 0 & 1 & 6000 \end{array}\right] \end{array}$$

Notice from the s_3 column that $-10s_3 = 100$, so s_3 is still negative. Therefore, we apply the procedure again. The 6 in row 3, column 2, is the positive entry farthest to the left in row 3, and by investigating quotients, we see that it is also the pivot. This leads to the following tableau.

$$\begin{array}{r} \\ \mathbf{-R_3 + R_1 \to R_1} \\ \mathbf{-2R_3 + 3R_2 \to R_2} \\ \\ \mathbf{-4R_3 + 3R_4 \to R_4} \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 10 & 0 & 10 & 0 & 400 \\ 30 & 0 & 15 & 0 & 5 & 20 & 0 & 1300 \\ 0 & 6 & 3 & 0 & -1 & -10 & 0 & 100 \\ \hline 0 & 0 & 60 & 0 & 40 & 40 & 3 & 17{,}600 \end{array}\right] \end{array}$$

The value of s_3 is now 0 and the solution is feasible. We now continue with the simplex method until an optimal solution is found. We check for negative indicators, but since

*We use this rule for simplicity. There are, however, more complicated methods for choosing the pivot element that require, on average, fewer pivots to find the solution.

there are none, we have merely to create a 1 in each column corresponding to a basic variable or z.

$$\begin{array}{r} \\ \mathbf{R_1/10 \to R_1} \\ \mathbf{R_2/30 \to R_2} \\ \mathbf{R_3/6 \to R_3} \\ \mathbf{R_4/3 \to R_4} \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 40 \\ 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{130}{3} \\ 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{6} & -\frac{5}{3} & 0 & \frac{50}{3} \\ \hline 0 & 0 & 20 & 0 & \frac{40}{3} & \frac{40}{3} & 1 & \frac{17{,}600}{3} \end{array}\right] \end{array}$$

The solution is

$$x_1 = \frac{130}{3} = 43\frac{1}{3}, \quad x_2 = \frac{50}{3} = 16\frac{2}{3}, \quad x_3 = 0, \quad z = \frac{17{,}600}{3} = 5866.67.$$

For maximum profit with this new constraint, the farmer should plant $43\frac{1}{3}$ acres of potatoes, $16\frac{2}{3}$ acres of corn, and no cabbage. The profit will be \$5866.67, less than the \$6000 profit if the farmer were to plant only 50 acres of potatoes. Because of the additional constraint that at least 60 acres must be planted, the profit is reduced. Notice that $s_1 = 40$. This is the slack variable for the constraint that no more than 100 acres are available. It indicates that 40 of the 100 available acres are still unused.

NOTE
If we ever reach a point where a surplus variable still has a negative solution, but there are no positive elements left in the row, then we have no way to make the surplus variable positive, so the problem has no feasible solution.

The procedure we have followed is a simplified version of the **two-phase method**, which is widely used for solving problems with mixed constraints. To see the complete method, including how to handle some complications that may arise, see *Linear Programming: An Introduction with Applications* by Alan Sultan, Academic Press, 1993.

In the previous section we solved standard minimum problems using duals. If a minimizing problem has mixed $\leq$ and $\geq$ constraints, the dual method cannot be used. We solve such problems with the method presented in this section. To see how, consider the simple fact: When a number t gets smaller, then $-t$ gets larger, and vice versa. For instance, if t goes from 6 to 1 to 0 to -8, then $-t$ goes from -6 to -1 to 0 to 8. Thus, if w is the objective function of a minimizing linear programming problem, the feasible solution that produces the minimum value of w also produces the maximum value of $z = -w$, and vice versa. Therefore, to solve a minimization problem with objective function w, we need only solve the maximization problem with the same constraints and objective function $z = -w$.

In summary, the following steps are involved in solving the nonstandard problems in this section.

Solving a Nonstandard Problem

1. If necessary, convert the problem to a maximization problem.
2. Add slack variables and subtract surplus variables as needed.
3. Write the initial simplex tableau.
4. If any basic variable has a negative value, locate the nonzero number in that variable's column, and note what row it is in.
5. In the row located in Step 4, find the positive entry that is farthest to the left, and note what column it is in.
6. In the column found in Step 5, choose a pivot by investigating quotients.
7. Use row operations to change the other numbers in the pivot column to 0.
8. Continue Steps 4 through 7 until all basic variables are nonnegative. If it ever becomes impossible to continue, then the problem has no feasible solution.
9. Once a feasible solution has been found, continue to use the simplex method until the optimal solution is found.

In the next example, we use this method to solve a minimization problem with mixed constraints.

EXAMPLE 1 Minimization

Minimize $w = 3y_1 + 2y_2$

subject to: $y_1 + 3y_2 \le 6$

$2y_1 + y_2 \ge 3$

with $y_1 \ge 0, \quad y_2 \ge 0.$

SOLUTION Change this to a maximization problem by letting z equal the *negative* of the objective function: $z = -w$. Then find the *maximum* value of

$$z = -w = -3y_1 - 2y_2.$$

The problem can now be stated as follows.

Maximize $z = -3y_1 - 2y_2$

subject to: $y_1 + 3y_2 \le 6$

$2y_1 + y_2 \ge 3$

with $y_1 \ge 0, \quad y_2 \ge 0.$

To begin, we add slack and surplus variables, and rewrite the objective function.

$$\begin{aligned} y_1 + 3y_2 + s_1 &= 6 \\ 2y_1 + y_2 - s_2 &= 3 \\ 3y_1 + 2y_2 + z &= 0 \end{aligned}$$

NOTE

1) Slack variables are used for $\le$ constraints, while surplus variables are used for $\ge$ constraints
2) Remember to convert from z to w as the last step in solving a minimization problem.

Set up the initial simplex tableau.

$$\begin{array}{c} \begin{matrix} y_1 & y_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 6 \\ 2 & 1 & 0 & -1 & 0 & 3 \\ \hline 3 & 2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The solution $y_1 = 0$, $y_2 = 0$, $s_1 = 6$, and $s_2 = -3$, is not feasible. Row operations must be used to get a feasible solution. We start with s_2, which has a -1 in row 2. The positive entry farthest to the left in row 2 is the 2 in column 1. The element in column 1 that gives the smallest quotient is 2, so it becomes the pivot. Pivoting produces the following matrix.

$$\begin{array}{r} \\ -\mathbf{R}_2 + 2\mathbf{R}_1 \to \mathbf{R}_1 \\ \\ -3\mathbf{R}_2 + 2\mathbf{R}_3 \to \mathbf{R}_3 \end{array} \begin{array}{c} \begin{matrix} y_1 & y_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{ccccc|c} 0 & 5 & 2 & 1 & 0 & 9 \\ 2 & 1 & 0 & -1 & 0 & 3 \\ \hline 0 & 1 & 0 & 3 & 2 & -9 \end{array}\right] \end{array}$$

Now $s_2 = 0$, so the solution is feasible. Furthermore, there are no negative indicators, so the solution is optimal. Divide row 1 by 2, row 2 by 2, and row 3 by 2 to find the final solution: $y_1 = 3/2$ and $y_2 = 0$. Since $z = -w = -9/2$, the minimum value is $w = 9/2$.

TRY YOUR TURN 1

YOUR TURN 1

Minimize $w = 6y_1 + 4y_2$

subject to: $3y_1 + 4y_2 \ge 10$

$9y_1 + 7y_2 \le 18$

with $y_1 \ge 0, \quad y_2 \ge 0.$

An important application of linear programming is the problem of minimizing the cost of transporting goods. This type of problem is often referred to as a *transportation problem* or *warehouse problem*. Some problems of this type were included in the exercise sets in previous chapters. The next example is based on Exercise 52 from Section 2.2, in which the transportation costs were set equal to $10,640. We will now use the simplex method to minimize the transportation costs.

EXAMPLE 2 Transportation Problem

An auto manufacturer sends cars from two plants, I and II, to dealerships A and B located in a midwestern city. Plant I has a total of 28 cars to send, and plant II has 8. Dealer A needs 20 cars, and dealer B needs 16. Transportation costs per car based on the distance of each

dealership from each plant are \$220 from I to A, \$300 from I to B, \$400 from II to A, and \$180 from II to B. How many cars should be sent from each plant to each of the two dealerships to minimize transportation costs? Use the simplex method to find the solution.

APPLY IT

SOLUTION To begin, let

y_1 = the number of cars shipped from I to A;

y_2 = the number of cars shipped from I to B;

y_3 = the number of cars shipped from II to A;

and y_4 = the number of cars shipped from II to B.

Plant I has only 28 cars to ship, so

$$y_1 + y_2 \le 28.$$

Similarly, plant II has only 8 cars to ship, so

$$y_3 + y_4 \le 8.$$

Since dealership A needs 20 cars and dealership B needs 16 cars,

$$y_1 + y_3 \ge 20 \quad \text{and} \quad y_2 + y_4 \ge 16.$$

The manufacturer wants to minimize transportation costs, so the objective function is

$$w = 220y_1 + 300y_2 + 400y_3 + 180y_4.$$

Now write the problem as a system of linear equations, adding slack or surplus variables as needed, and let $z = -w$.

$$\begin{aligned}
y_1 + y_2 + s_1 &= 28\\
y_3 + y_4 + s_2 &= 8\\
y_1 + y_3 - s_3 &= 20\\
y_2 + y_4 - s_4 &= 16\\
220y_1 + 300y_2 + 400y_3 + 180y_4 + z &= 0
\end{aligned}$$

Set up the initial simplex tableau.

y_1	y_2	y_3	y_4	s_1	s_2	s_3	s_4	z	
1	1	0	0	1	0	0	0	0	28
0	0	1	1	0	1	0	0	0	8
1	0	1	0	0	0	−1	0	0	20
0	1	0	1	0	0	0	−1	0	16
220	300	400	180	0	0	0	0	1	0

Because $s_3 = -20$, we choose the positive entry farthest to the left in row 3, which is the 1 in column 1. After forming the necessary quotients, we find that the 1 is also the pivot, leading to the following tableau.

	y_1	y_2	y_3	y_4	s_1	s_2	s_3	s_4	z	
$-R_3 + R_1 \to R_1$	0	1	−1	0	1	0	1	0	0	8
	0	0	1	1	0	1	0	0	0	8
	1	0	1	0	0	0	−1	0	0	20
	0	1	0	1	0	0	0	−1	0	16
$-220R_3 + R_5 \to R_5$	0	300	180	180	0	0	220	0	1	−4400

We still have $s_4 = -16$. Verify that the 1 in row 1, column 2, is the next pivot, leading to the following tableau.

	y_1	y_2	y_3	y_4	s_1	s_2	s_3	s_4	z	
	0	1	−1	0	1	0	1	0	0	8
	0	0	1	1	0	1	0	0	0	8
	1	0	1	0	0	0	−1	0	0	20
$-R_1 + R_4 \to R_4$	0	0	1	1	−1	0	−1	−1	0	8
$-300R_1 + R_5 \to R_5$	0	0	480	180	−300	0	−80	0	1	−6800

We still have $s_4 = -8$. Choosing column 3 to pivot, there is a tie between rows 2 and 4. Ordinarily in such cases, we arbitrarily choose the pivot in the row nearest to the top of the matrix. With surplus variables, however, we have the immediate goal of making all basic variables nonnegative. Because choosing row 4 will remove s_4 from the set of basic variables, we will choose as the pivot the 1 in column 3, row 4. The result is the following tableau.

	y_1	y_2	y_3	y_4	s_1	s_2	s_3	s_4	z	
$R_4 + R_1 \to R_1$	0	1	0	1	0	0	0	−1	0	16
$-R_4 + R_2 \to R_2$	0	0	0	0	1	1	1	1	0	0
$-R_4 + R_3 \to R_3$	1	0	0	−1	1	0	0	1	0	12
	0	0	1	1	−1	0	−1	−1	0	8
$-480R_4 + R_5 \to R_5$	0	0	0	−300	180	0	400	480	1	−10640

We now have the feasible solution

$$y_1 = 12, \quad y_2 = 16, \quad y_3 = 8, \quad y_4 = 0, \quad s_1 = 0, \quad s_2 = 0, \quad s_3 = 0, \quad s_4 = 0,$$

with $w = 10{,}640$. But there are still negative indicators in the bottom row, so we can keep going. After two more tableaus, we find that

$$y_1 = 20, \quad y_2 = 8, \quad y_3 = 0, \quad y_4 = 8,$$

with $w = 8240$. Therefore, the manufacturer should send 20 cars from plant I to dealership A and 8 cars to dealership B. From plant II, 8 cars should be sent to dealership B and none to dealership A. The transportation cost will then be \$8240, a savings of \$2400 over the original stated cost of \$10,640. TRY YOUR TURN 2

YOUR TURN 2 Finish the missing steps in Example 2 and show the final tableau.

When one or more of the constraints in a linear programming problem is an equation, rather than an inequality, we add an **artificial variable** to each equation, rather than a slack or surplus variable. The first goal of the simplex method is to eliminate any artificial variables as basic variables, since they must have a value of 0 in the solution. We do this exactly the way we changed the surplus variables from having negative values to being zero. The only difference is that once an artificial variable is made nonbasic, we must never pivot on that column again, because that would change it to a basic variable. We must never carry out any pivot that would result in an artificial variable becoming nonzero.

EXAMPLE 3 Artificial Variables

In the transportation problem discussed in Example 2, it would be more realistic for the dealerships to order exactly 20 and 16 cars, respectively. Solve the problem with these two equality constraints.

SOLUTION Using the same variables, we can state the problem as follows.

$$\begin{aligned} \text{Minimize} \quad & w = 220y_1 + 300y_2 + 400y_3 + 180y_4 \\ \text{subject to:} \quad & y_1 + y_2 \le 28 \\ & y_3 + y_4 \le 8 \\ & y_1 + y_3 = 20 \\ & y_2 + y_4 = 16 \end{aligned}$$

with all variables nonnegative.

The corresponding system of equations requires slack variables s_1 and s_2 and two artificial variables that we shall call a_1 and a_2, to remind us that they require special handling. The system

$$\begin{aligned}
y_1 + y_2 + s_1 &= 28\\
y_3 + y_4 + s_2 &= 8\\
y_1 + y_3 + a_1 &= 20\\
y_2 + y_4 + a_2 &= 16\\
220y_1 + 300y_2 + 400y_3 + 180y_4 + z &= 0
\end{aligned}$$

produces a tableau exactly the same as in Example 2, except that the columns labeled s_3 and s_4 in that example are now labeled a_1 and a_2, and there is initially a 1 rather than a -1 in the columns for a_1 and a_2. The first three pivots are the same as in Example 2, resulting in the following tableau.

$$\left[\begin{array}{cccccccc|c}
y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & a_1 & a_2 & z & \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 16\\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0\\
1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 12\\
0 & 0 & 1 & 1 & -1 & 0 & 1 & 1 & 0 & 8\\
\hline
0 & 0 & 0 & -300 & 180 & 0 & -400 & -480 & 1 & -10640
\end{array}\right]$$

NOTE
Another way to handle this situation is by solving for y_3 and y_4 in terms of y_1 and y_2. Then proceed with the usual method for standard problems.

Ordinarily our next step would be to pivot on the column with the indicator of -480, but that is not allowed because it would make a_2 a basic variable. Similarly, we cannot pivot on the column with the indicator of -400. Instead, we pivot on the column with the indicator of -300. Verify that after this step and one more tableau, we reach the same solution as in Example 2. In other problems, equality constraints can result in a higher cost.

CAUTION If the artificial variables cannot be made equal to zero, the problem has no feasible solution.

Applications requiring the simplex method often have constraints that have a zero on the right-hand side. For example, in Exercise 35 of Section 4.2 a person wants to walk at least twice as many hours as she bikes. This results in one of the constraints $x_1 - 2x_2 \le 0$ or $-x_1 + 2x_2 \ge 0$. For the purposes of using the simplex method to solve problems in the standard maximum form, it is always better to write constraints in the first form, since the first constraint can be readily handled by the basic simplex method by adding a slack variable.

Several linear programming models in actual use are presented on the Web site for this textbook. These models illustrate the usefulness of linear programming. In most real applications, the number of variables is so large that these problems could not be solved without using methods (like the simplex method) that can be adapted to computers.

4.4 EXERCISES

Rewrite each system of inequalities as a system of linear equations, adding slack variables or subtracting surplus variables as necessary.

1. $2x_1 + 3x_2 \le 8$
$x_1 + 4x_2 \ge 7$

2. $3x_1 + 7x_2 \le 9$
$4x_1 + 5x_2 \ge 11$

3. $2x_1 + x_2 + 2x_3 \le 50$
$x_1 + 3x_2 + x_3 \ge 35$
$x_1 + 2x_2 \ge 15$

4. $2x_1 + x_3 \le 40$
$x_1 + x_2 \ge 18$
$x_1 + x_3 \ge 20$

Convert each problem into a maximization problem.

5. Minimize $w = 3y_1 + 4y_2 + 5y_3$
subject to: $y_1 + 2y_2 + 3y_3 \ge 9$
$y_2 + 2y_3 \ge 8$
$2y_1 + y_2 + 2y_3 \ge 6$
with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

6. Minimize $w = 8y_1 + 3y_2 + y_3$
subject to: $7y_1 + 6y_2 + 8y_3 \geq 18$
$4y_1 + 5y_2 + 10y_3 \geq 20$
with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$

7. Minimize $w = y_1 + 2y_2 + y_3 + 5y_4$
subject to: $y_1 + y_2 + y_3 + y_4 \geq 50$
$3y_1 + y_2 + 2y_3 + y_4 \geq 100$
with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0, \quad y_4 \geq 0.$

8. Minimize $w = y_1 + y_2 + 7y_3$
subject to: $5y_1 + 2y_2 + y_3 \geq 125$
$4y_1 + y_2 + 6y_3 \leq 75$
$6y_1 + 8y_2 \geq 84$
with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$

Use the simplex method to solve.

9. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that
$x_1 + 2x_2 \geq 24$
$x_1 + x_2 \leq 40$
and $z = 12x_1 + 10x_2$ is maximized.

10. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that
$2x_1 + x_2 \geq 20$
$2x_1 + 5x_2 \leq 80$
and $z = 6x_1 + 2x_2$ is maximized.

11. Find $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ such that
$x_1 + x_2 + x_3 \leq 150$
$x_1 + x_2 + x_3 \geq 100$
and $z = 2x_1 + 5x_2 + 3x_3$ is maximized.

12. Find $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ such that
$x_1 + x_2 + x_3 \leq 15$
$4x_1 + 4x_2 + 2x_3 \geq 48$
and $z = 2x_1 + x_2 + 3x_3$ is maximized.

13. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that
$x_1 + x_2 \leq 100$
$2x_1 + 3x_2 \leq 75$
$x_1 + 4x_2 \geq 50$
and $z = 5x_1 - 3x_2$ is maximized.

14. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that
$x_1 + 2x_2 \leq 18$
$x_1 + 3x_2 \geq 12$
$2x_1 + 2x_2 \leq 24$
and $z = 5x_1 - 10x_2$ is maximized.

15. Find $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$ such that
$5y_1 + 3y_2 + 2y_3 \leq 150$
$5y_1 + 10y_2 + 3y_3 \geq 90$
and $w = 10y_1 + 12y_2 + 10y_3$ is minimized.

16. Minimize $w = 3y_1 + 2y_2 + 3y_3$
subject to: $2y_1 + 3y_2 + 6y_3 \leq 60$
$y_1 + 4y_2 + 5y_3 \geq 40$
with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$

Solve using artificial variables.

17. Maximize $z = 3x_1 + 2x_2$
subject to: $x_1 + x_2 = 50$
$4x_1 + 2x_2 \geq 120$
$5x_1 + 2x_2 \leq 200$
with $x_1 \geq 0, \quad x_2 \geq 0.$

18. Maximize $z = 5x_1 + 7x_2$
subject to: $x_1 + x_2 = 15$
$2x_1 + 4x_2 \geq 30$
$3x_1 + 5x_2 \geq 10$
with $x_1 \geq 0, \quad x_2 \geq 0.$

19. Minimize $w = 32y_1 + 40y_2 + 48y_3$
subject to: $20y_1 + 10y_2 + 5y_3 = 200$
$25y_1 + 40y_2 + 50y_3 \leq 500$
$18y_1 + 24y_2 + 12y_3 \geq 300$
with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$

20. Minimize $w = 15y_1 + 12y_2 + 18y_3$
subject to: $y_1 + 2y_2 + 3y_3 \leq 12$
$3y_1 + y_2 + 3y_3 \geq 18$
$y_1 + y_2 + y_3 = 10$
with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$

21. Explain how, in any linear programming problem, the value of the objective function can be found without using the number in the lower right-hand corner of the final tableau.

22. Explain why, for a maximization problem, you write the negative of the coefficients of the objective function on the bottom row, while, for a minimization problem, you write the coefficients themselves.

APPLICATIONS

Business and Economics

23. **Transportation** Southwestern Oil supplies two distributors in the Northwest from two outlets, S_1 and S_2. Distributor D_1 needs at least 3000 barrels of oil, and distributor D_2 needs at least 5000 barrels. The two outlets can each furnish up to 5000 barrels of oil. The costs per barrel to ship the oil are given in the table.

		Distributors	
		D_1	D_2
Outlets	S_1	\$30	\$20
	S_2	\$25	\$22

There is also a shipping tax per barrel as given in the table below. Southwestern Oil is determined to spend no more than $40,000 on shipping tax.

	D_1	D_2
S_1	$2	$6
S_2	$5	$4

a. How should the oil be supplied to minimize shipping costs?

b. Find and interpret the values of any nonzero slack or surplus variables.

24. Transportation Change Exercise 23 so that the two outlets each furnish exactly 5000 barrels of oil, with everything else the same. Use artificial variables to solve the problem, following the steps outlined in Example 3.

25. Finance A bank has set aside a maximum of $25 million for commercial and home loans. Every million dollars in commercial loans requires 2 lengthy application forms, while every million dollars in home loans requires 3 lengthy application forms. The bank cannot process more than 72 application forms at this time. The bank's policy is to loan at least four times as much for home loans as for commercial loans. Because of prior commitments, at least $10 million will be used for these two types of loans. The bank earns 10% on commercial loans and 12% on home loans. What amount of money should be allotted for each type of loan to maximize the interest income?

26. Blending Seed Topgrade Turf lawn seed mixture contains three types of seed: bluegrass, rye, and Bermuda. The costs per pound of the three types of seed are 16 cents, 14 cents, and 12 cents, respectively. In each batch there must be at least 25% bluegrass seed, and the amount of Bermuda must be no more than 2/3 the amount of rye. To fill current orders, the company must make at least 6000 lb of the mixture. How much of each kind of seed should be used to minimize cost?

27. Blending Seed Change Exercise 26 so that the company must make exactly 6000 lb of the mixture. Use artificial variables to solve the problem.

28. Investments Lynda Rago has decided to invest a $100,000 inheritance in government securities that earn 7% per year, municipal bonds that earn 6% per year, and mutual funds that earn an average of 10% per year. She will spend at least $40,000 on government securities, and she wants at least half the inheritance to go to bonds and mutual funds. Government securities have an initial fee of 2%, municipal bonds have an initial fee of 1%, and mutual funds have an initial fee of 3%. Lynda has $2400 available to pay initial fees. How much should be invested in each way to maximize the interest yet meet the constraints? What is the maximum interest she can earn?

29. Transportation The manufacturer of a popular personal computer has orders from two dealers. Dealer D_1 wants at least 32 computers, and dealer D_2 wants at least 20 computers. The manufacturer can fill the orders from either of two warehouses, W_1 or W_2. There are 25 computers on hand at W_1, and 30 at W_2. The costs (in dollars) to ship one computer to each dealer from each warehouse are given in the following table.

		Dealer	
		D_1	D_2
Warehouse	W_1	$14	$12
	W_2	$12	$10

a. How should the orders be filled to minimize shipping costs?

b. Find and interpret the values of any nonzero slack or surplus variables.

30. Calorie Expenditure Joe Vetere's exercise regimen includes light calisthenics, swimming, and playing the drums. He has at most 10 hours per week to devote to these activities. He wants the total time he does calisthenics and plays the drums to be at least twice as long as he swims. His neighbors, however, will tolerate no more than 4 hours per week on the drums. According to a website, a 190-pound person like Joe will burn an average of 388 calories per hour doing calisthenics, 518 calories per hour swimming, and 345 calories per hour playing the drums. In Section 4.2, Exercise 36, Joe found that he could maximize calories burned in an exercise routine that did not include playing the drums as part of his exercise plan. ***Source: NutriStrategy.com.***

a. Joe really likes to play the drums and insists that his exercise plan include at least 1 hour of playing the drums per week. With this added constraint, now how many hours per week should Joe spend on each exercise to maximize the number of calories he burns? What is the maximum number of calories he will burn?

b. Without the added constraint from part a, Joe's maximum calorie expenditure was $4313\frac{1}{3}$ calories. Compare this number with the new optimal solution. What conclusions can you draw when additional constraints are placed on a problem?

c. Show how the solution from part a can be found without using the simplex method by considering the constraints and the number of calories for each activity.

31. Blending Chemicals Natural Brand plant food is made from three chemicals, labeled I, II, and III. In each batch of the plant food, the amounts of chemicals II and III must be in the ratio of 4 to 3. The amount of nitrogen must be at least 30 kg. The percent of nitrogen in the three chemicals is 9%, 4%, and 3%, respectively. If the three chemicals cost $1.09, $0.87, and $0.65 per kilogram, respectively, how much of each should be used to minimize the cost of producing at least 750 kg of the plant food?

32. Blending a Soft Drink A popular soft drink called Sugarlo, which is advertised as having a sugar content of no more than 10%, is blended from five ingredients, each of which has some sugar content. Water may also be added to dilute the mixture. The sugar content of the ingredients and their costs per gallon are given below.

	Ingredient					
	1	*2*	*3*	*4*	*5*	*Water*
Sugar Content (%)	0.28	0.19	0.43	0.57	0.22	0
Cost ($/gal)	0.48	0.32	0.53	0.28	0.43	0.04

At least 0.01 of the content of Sugarlo must come from ingredients 3 or 4, 0.01 must come from ingredients 2 or 5, and 0.01 from ingredients 1 or 4. How much of each ingredient should be used in preparing 15,000 gal of Sugarlo to minimize the cost?

33. Blending Gasoline A company is developing a new additive for gasoline. The additive is a mixture of three liquid ingredients, I, II, and III. For proper performance, the total amount of additive must be at least 10 oz per gal of gasoline. However, for safety reasons, the amount of additive should not exceed 15 oz per gal of gasoline. At least 1/4 oz of ingredient I must be used for every ounce of ingredient II, and at least 1 oz of ingredient III must be used for every ounce of ingredient I. If the costs of I, II, and III are \$0.30, \$0.09, and \$0.27 per oz, respectively, find the mixture of the three ingredients that produces the minimum cost of the additive. How much of the additive should be used per gal of gasoline?

YOUR TURN ANSWERS

1. Minimum is 10 when $y_1 = 0$ and $y_2 = 5/2$.

2.
$$\left[\begin{array}{ccccccccc|c} 0 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 8 \\ \hline 0 & 0 & 300 & 0 & 0 & 120 & 220 & 300 & 1 & -8240 \end{array}\right]$$

4 CHAPTER REVIEW

SUMMARY

In this chapter, we introduced the simplex method, which is a procedure for solving any linear programming problem. To apply this method, we first had to write the problem as a standard maximization problem in matrix form. This form tells us an initial basic feasible solution, which the simplex method uses to determine other basic feasible solutions. Each successive iteration of the simplex method gives us a new basic feasible solution, whose objective function value is greater than or equal to the objective function value of the previous basic feasible solution. We then introduced duality, which tells us that every time we solve a linear programming problem, we are actually solving two problems—a maximization problem and a minimization problem. This has far-reaching consequences in the fields of operations research and decision sciences, including the fact that standard minimization problems can be solved by the simplex method. Finally, we extended the simplex method to solve problems that are not standard because they have inequalities going in both directions (and perhaps equalities as well).

Standard Maximum Form A linear programming problem is in standard maximum form if the following conditions are satisfied.

1. The objective function is to be maximized.
2. All variables are nonnegative.
3. All remaining constraints are stated in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \le b \qquad \text{with } b \ge 0.$$

Simplex Method

1. Determine the objective function.
2. Write all the necessary constraints.
3. Convert each constraint into an equation by adding a slack variable in each.
4. Set up the initial simplex tableau.
5. Locate the most negative indicator. If there are two such indicators, choose the one farther to the left.
6. Form the necessary quotients to find the pivot. Disregard any quotients with 0 or a negative number in the denominator. The smallest nonnegative quotient gives the location of the pivot. If all quotients must be disregarded, no maximum solutions exist. If two quotients are both equal and smallest, choose the pivot in the row nearest the top of the matrix.
7. Use row operations to change all other numbers in the pivot column to zero by adding a suitable multiple of the pivot row to a positive multiple of each row.
8. If the indicators are all positive or 0, this is the final tableau. If not, go back to step 5 and repeat the process until a tableau with no negative indicators is obtained.
9. Read the solution from the final tableau.

Standard Minimum Form

A linear programming problem is in standard minimum form if the following conditions are satisfied.

1. The objective function is to be minimized.
2. All variables are nonnegative.
3. All remaining constraints are stated in the form

$$a_1y_1 + a_2y_2 + \cdots + a_ny_n \geq b \qquad \text{with } b \geq 0.$$

Theorem of Duality

The objective function w of a minimization linear programming problem takes on a minimum value if and only if the objective function z of the corresponding dual maximization problem takes on a maximum value. The maximum value of z equals the minimum value of w.

Solving a Standard Minimum Problem with Duals

1. Find the dual standard maximization problem.
2. Solve the maximization problem using the simplex method.
3. The minimum value of the objective function w is the maximum value of the objective function z.
4. The optimum solution is given by the entries in the bottom row of the columns corresponding to the slack variables, as long as the entry in the z column is equal to 1.

Solving a Nonstandard Problem

1. If necessary, convert the problem to a maximization problem.
2. Add slack variables and subtract surplus variables as needed.
3. Write the initial simplex tableau.
4. If any basic variable has a negative value, locate the nonzero number in that variable's column, and note what row it is in.
5. In the row located in step 4, find the positive entry that is farthest to the left, and note what column it is in.
6. In the column found in step 5, choose a pivot by investigating quotients.
7. Use row operations to change the other numbers in the pivot column to 0.
8. Continue steps 4 through 7 until all basic variables are nonnegative. If it ever becomes impossible to continue, then the problem has no feasible solution.
9. Once a feasible solution has been found, continue to use the simplex method until the optimal solution is found.

Artificial Variables

When one or more of the constraints in a linear programming problem is an equation, rather than an inequality, an artificial variable is added to each equation. Artificial variables are handled in the same way as surplus variables, except that once an artificial variable is no longer basic, never pivot on its column. If in the optimal solution an artificial variable has a positive value, then the original problem does not have a solution.

KEY TERMS

simplex method

4.1
standard maximum form
slack variable
simplex tableau
indicators
basic variable
basic feasible solution
pivot

4.2
sensitivity analysis

4.3
standard minimum form
dual
transpose
shadow costs

4.4
surplus variable
two-phase method
artificial variable

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The simplex method can be used to solve all linear programming problems.

2. If the feasible region of a linear programming problem is unbounded, then the objective function value is unbounded.

3. A linear programming problem in standard maximization form always has a feasible solution.

4. A linear programming problem in standard minimization form always has a feasible solution.

5. A linear programming problem in standard maximization form always has a finite optimal solution.

6. The tableau below for a linear program in standard maximization form shows that it has no finite maximum value.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline -1 & 1 & 0 & 1 & 0 & 1 \\ -4 & 0 & 1 & -2 & 0 & 3 \\ \hline -1 & 0 & 0 & 2 & 1 & 4 \end{array}$$

7. One must always use the minimum quotient when choosing a pivot row.

8. If there is a 0 in the right-hand column, we can disregard it when determining the quotients used to choose the pivot row.

9. One must always pick the most negative number in the indicator row when choosing the pivot column.

10. A basic variable can be assigned a value of zero by the simplex method.

11. A slack variable of a linear programming problem in standard maximization form may become negative during the intermediate stages of the simplex method.

12. The dual of the dual of a linear programming problem is the original problem.

13. The simplex method guarantees that each iteration will yield a feasible solution whose objective function value is bigger than the objective function value of all previous solutions.

14. Standard maximization problems can only have slack variables, and standard minimization problems (not solved by the dual method) can only have surplus variables.

PRACTICE AND EXPLORATIONS

15. When is it necessary to use the simplex method rather than the graphical method?

16. What can you conclude if a surplus variable cannot be made nonnegative?

For each problem, (a) add slack variables or subtract surplus variables, and (b) set up the initial simplex tableau.

17. Maximize $z = 2x_1 + 7x_2$

subject to: $4x_1 + 6x_2 \le 60$

$3x_1 + x_2 \le 18$

$2x_1 + 5x_2 \le 20$

$x_1 + x_2 \le 15$

with $x_1 \ge 0, \quad x_2 \ge 0.$

18. Maximize $z = 25x_1 + 30x_2$

subject to: $3x_1 + 5x_2 \le 47$

$x_1 + x_2 \le 25$

$5x_1 + 2x_2 \le 35$

$2x_1 + x_2 \le 30$

with $x_1 \ge 0, \quad x_2 \ge 0.$

19. Maximize $z = 5x_1 + 8x_2 + 6x_3$

subject to: $x_1 + x_2 + x_3 \le 90$

$2x_1 + 5x_2 + x_3 \le 120$

$x_1 + 3x_2 \ge 80$

with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

20. Maximize $z = 4x_1 + 6x_2 + 8x_3$

subject to: $x_1 + x_2 + 2x_3 \ge 200$

$8x_1 + 6x_3 \le 400$

$3x_1 + 5x_2 + x_3 \le 300$

with $x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

Use the simplex method to solve each maximization linear programming problem with the given initial tableau.

21.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 4 & 5 & 2 & 1 & 0 & 0 & 18 \\ 2 & 8 & 6 & 0 & 1 & 0 & 24 \\ \hline -5 & -3 & -6 & 0 & 0 & 1 & 0 \end{array}$$

22.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & 7 & 1 & 0 & 0 & 14 \\ 2 & 3 & 0 & 1 & 0 & 10 \\ \hline -2 & -4 & 0 & 0 & 1 & 0 \end{array}$$

23.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 2 & 2 & 1 & 0 & 0 & 0 & 50 \\ 3 & 1 & 0 & 0 & 1 & 0 & 0 & 20 \\ 1 & 0 & 2 & 0 & 0 & -1 & 0 & 15 \\ \hline -5 & -3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

24.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline 3 & 6 & -1 & 0 & 0 & 0 & 28 \\ 1 & 1 & 0 & 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 0 & 1 & 0 & 16 \\ \hline -1 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Convert each problem into a maximization problem and then solve each problem using both the dual method and the method of Section 4.4.

25. Minimize $w = 10y_1 + 15y_2$

subject to: $y_1 + y_2 \ge 17$

$5y_1 + 8y_2 \ge 42$

with $y_1 \ge 0, \quad y_2 \ge 0.$

26. Minimize $w = 22y_1 + 44y_2 + 33y_3$

subject to: $y_1 + 2y_2 + y_3 \ge 3$

$y_1 + y_3 \ge 3$

$3y_1 + 2y_2 + 2y_3 \ge 8$

with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

27. Minimize $w = 7y_1 + 2y_2 + 3y_3$

subject to: $y_1 + y_2 + 2y_3 \ge 48$

$y_1 + y_2 \ge 12$

$y_3 \ge 10$

$3y_1 + y_3 \ge 30$

with $y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

28. Minimize $w = 3y_1 + 4y_2 + y_3 + 2y_4$

subject to: $4y_1 + 6y_2 + 3y_3 + 8y_4 \geq 19$

$13y_1 + 7y_2 + 2y_3 + 6y_4 \geq 16$

with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0, \quad y_4 \geq 0.$

Use the simplex method to solve each problem. (You may need to use artificial variables.)

29. Maximize $z = 20x_1 + 30x_2$

subject to: $5x_1 + 10x_2 \leq 120$

$10x_1 + 15x_2 \geq 200$

with $x_1 \geq 0, \quad x_2 \geq 0.$

30. Minimize $w = 4y_1 + 2y_2$

subject to: $y_1 + 3y_2 \geq 6$

$2y_1 + 8y_2 \leq 21$

with $y_1 \geq 0, \quad y_2 \geq 0.$

31. Maximize $z = 10x_1 + 12x_2$

subject to: $2x_1 + 2x_2 = 17$

$2x_1 + 5x_2 \geq 22$

$4x_1 + 3x_2 \leq 28$

with $x_1 \geq 0, \quad x_2 \geq 0.$

32. Minimize $w = 24y_1 + 30y_2 + 36y_3$

subject to: $5y_1 + 10y_2 + 15y_3 \geq 1200$

$y_1 + y_2 + y_3 \leq 50$

with $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$

33. What types of problems can be solved using slack, surplus, and artificial variables?

34. What kind of problems can be solved using the method of duals?

35. In solving a linear programming problem, you are given the following initial tableau.

$$\left[\begin{array}{cccccc|c} 4 & 2 & 3 & 1 & 0 & 0 & 9 \\ 5 & 4 & 1 & 0 & 1 & 0 & 10 \\ \hline -6 & -7 & -5 & 0 & 0 & 1 & 0 \end{array}\right]$$

a. What is the problem being solved?

b. If the 1 in row 1, column 4 were a -1 rather than a 1, how would it change your answer to part a?

c. After several steps of the simplex algorithm, the following tableau results.

$$\left[\begin{array}{cccccc|c} 3 & 0 & 5 & 2 & -1 & 0 & 8 \\ 11 & 10 & 0 & -1 & 3 & 0 & 21 \\ \hline 47 & 0 & 0 & 13 & 11 & 10 & 227 \end{array}\right]$$

What is the solution? (List only the values of the original variables and the objective function. Do not include slack or surplus variables.)

d. What is the dual of the problem you found in part a?

e. What is the solution of the dual you found in part d? (Do not perform any steps of the simplex algorithm; just examine the tableau given in part c.)

36. In Chapter 2 we wrote a system of linear equations using matrix notation. We can do the same thing for the system of linear inequalities in this chapter.

a. Find matrices A, B, C, and X such that the maximization problem in Example 1 of Section 4.1 can be written as

Maximize CX

subject to: $AX \leq B$

with $X \geq O.$

(*Hint:* Let B and X be column matrices, and C a row matrix.)

b. Show that the dual of the problem in part a can be written as

Minimize YB

subject to: $YA \geq C$

with $Y \geq O,$

where Y is a row matrix.

c. Show that for any feasible solutions X and Y to the original and dual problems, respectively, $CX \leq YB$. (*Hint:* Multiply both sides of $AX \leq B$ by Y on the left. Then substitute for YA.)

d. For the solution X to the maximization problem and Y to the dual, it can be shown that

$$CX = YB$$

is always true. Verify this for Example 1 of Section 4.1. What is the significance of the value in CX (or YB)?

APPLICATIONS

For Exercises 37–40, (a) select appropriate variables; (b) write the objective functions; (c) write the constraints as inequalities.

Business and Economics

37. Production The Bronze Forge produces and ships three different hand-crafted bronze plates: a dogwood-engraved cake plate, a wheat-engraved bread plate, and a lace-engraved dinner plate. Each cake plate requires \$15 in materials, 5 hours of labor, and \$6 to ship. Each bread plate requires \$10 in materials, 4 hours of labor, and \$5 to ship. Each dinner plate requires \$8 in materials, 4 hours of labor, and \$5 to deliver. The profit on the cake plate is \$15, on the bread plate is \$12, and on the dinner plate is \$5. The company has available up to 2700 hours of labor per week. Each week, they can spend at most \$1500 on materials and \$1200 on delivery. How many of each plate should the company produce to maximize their weekly profit? What is their maximum profit?

38. Investments An investor is considering three types of investments: a high-risk venture into oil leases with a potential return of 15%, a medium-risk investment in stocks with a 9% return, and a relatively safe bond investment with a 5% return. He has \$50,000 to invest. Because of the risk, he will limit his investment in oil leases and stocks to 30% and his investment in oil leases and bonds to 50%. How much should he invest in each to maximize his return, assuming investment returns are as expected?

39. Profit The Aged Wood Winery makes two white wines, Fruity and Crystal, from two kinds of grapes and sugar. One

gallon of Fruity wine requires 2 bushels of Grape A, 2 bushels of Grape B, and 2 lb of sugar, and produces a profit of $12. One gallon of Crystal wine requires 1 bushel of Grape A, 3 bushels of Grape B, and 1 lb of sugar, and produces a profit of $15. The winery has available 110 bushels of grape A, 125 bushels of grape B, and 90 lb of sugar. How much of each wine should be made to maximize profit?

40. Production Costs Cauchy Canners produces canned whole tomatoes and tomato sauce. This season, the company has available 3,000,000 kg of tomatoes for these two products. To meet the demands of regular customers, it must produce at least 80,000 kg of sauce and 800,000 kg of whole tomatoes. The cost per kilogram is $4 to produce canned whole tomatoes and $3.25 to produce tomato sauce. Labor agreements require that at least 110,000 person-hours be used. Each kilogram can of sauce requires 3 minutes for one worker, and each kilogram can of whole tomatoes requires 6 minutes for one worker. How many kilograms of tomatoes should Cauchy use for each product to minimize cost? (For simplicity, assume production of y_1 kg of canned whole tomatoes and y_2 kg of tomato sauce requires $y_1 + y_2$ kg of tomatoes.)

41. Solve Exercise 37.

42. Solve Exercise 38.

43. Solve Exercise 39.

44. Solve Exercise 40.

45. Canning Cauchy Canners produces canned corn, beans, and carrots. Demand for vegetables requires it to produce at least 1000 cases per month. Based on past sales, it should produce at least twice as many cases of corn as of beans, and at least 340 cases of carrots. It costs $10 to produce a case of corn, $15 to produce a case of beans, and $25 to produce a case of carrots.

a. Using the method of surplus variables, find how many cases of each vegetable should be produced to minimize costs. What is the minimum cost?

b. Using the method of duals, find how many cases of each vegetable should be produced to minimize costs. What is the minimum cost?

c. Suppose the minimum number of cases that must be produced each month rises to 1100. Use shadow costs to calculate the total cost in this case.

46. Food Cost A store sells two brands of snacks. A package of Sun Hill costs $3 and contains 10 oz of peanuts, 4 oz of raisins, and 2 oz of rolled oats. A package of Bear Valley costs $2 and contains 2 oz of peanuts, 4 oz of raisins, and 8 oz of rolled oats. Suppose you wish to make a mixture that contains at least 20 oz of peanuts, 24 oz of raisins, and 24 oz of rolled oats.

a. Using the method of surplus variables, find how many packages of each you should buy to minimize the cost. What is the minimum cost?

b. Using the method of duals, find how many packages of each you should buy to minimize the cost. What is the minimum cost?

c. Suppose the minimum amount of peanuts is increased to 28. Use shadow costs to calculate the total cost in this case.

d. Explain why it makes sense that the shadow cost for rolled oats is 0.

Life Sciences

47. Calorie Expenditure Ginger's exercise regimen includes doing tai chi, riding a unicycle, and fencing. She has at most 10 hours per week to devote to these activities. Her fencing partner can work with her at most only 2 hours per week. She wants the total time she does tai chi to be at least twice as long as she unicycles. According to a website, a 130-pound person like Ginger will burn an average of 236 calories per hour doing tai chi, 295 calories per hour riding a unicycle, and 354 calories per hour fencing. ***Source: NutriStrategy.com.***

a. How many hours per week should Ginger spend on each activity to maximize the number of calories she burns? What is the maximum number of calories she will burn?

b. Show how the solution from part a can be found without using the simplex method by considering the constraints and the number of calories for each activity.

EXTENDED APPLICATION

USING INTEGER PROGRAMMING IN THE STOCK-CUTTING PROBLEM

In Chapter 3 Section 3 we noted that some problems require solutions in integers because the resources to be allocated are items that can't be split into pieces, like cargo containers or airplanes. These *integer programming* problems are generally harder than the linear programming problems we have been solving by the simplex method, but often linear programming can be combined with other techniques to solve integer problems. Even if the number of variables and constraints is small, some help from software is usually required. We will introduce integer programming with the basic but important *stock-cutting problem.* (To get a feeling for the issues involved, you may want to try the simple stock-cutting problem given in Exercise 1.)

A paper mill produces rolls of paper that are much wider than most customers require, often as wide as 200 in. The mill then cuts these wide rolls into smaller widths to fill orders for paper rolls to

Source: Northwestern University and the Special Interest Group on Cutting and Packing

be used in printing and packaging and other applications. The stock-cutting problem is the following:

> *Given a list of roll widths and the number of rolls ordered for each width, decide how to cut the raw rolls that come from the paper-making machine into smaller rolls so as to fill all the orders with a minimum amount of waste.*

Another way to state the problem is: What is the minimum number of raw rolls required to fill the orders? This is an integer problem because the customers have ordered whole numbers of rolls, and each roll is cut in a single piece from one of the raw rolls.

As an example, suppose the paper machine produces rolls 100 in. wide. The manufacturer offers rolls in the following six widths: 14 in., 17 in., 31 in., 33 in., 36 in., and 45 in. (We'll call these the standard widths.) The current orders to be filled are as follows:

	Rolls of Paper Ordered					
Width in Inches	14	17	31	33	36	45
Number Ordered	100	123	239	121	444	87

The cutting machine can make four simultaneous cuts, so a raw roll can be cut into as many as five pieces. With luck, all five pieces might be usable for filling orders, but there will usually be unusable waste on the end, and we also might end up with more rolls of some standard width than we need. We'll consider both the end pieces that are too narrow and any unused standard-width rolls as waste, and this is the waste we want to minimize.

The first question is, what are the possible cutting patterns? We're restricted to at most five standard rolls from any given raw roll, and we'll elect to use as much as possible in each raw roll so that the waste remaining at the end will always be less than 14 in. So, for example, 14|36|45| is a possible pattern, but 14|14|14|14|14| is not, because it has too many cuts, and 45|36| is not, because more than 14 in. is left at the end. (Each vertical bar represents a cut; if the piece on the end happens to be a standard width, then we don't need a cut after it, since we've reached the end of the roll.) This is already a tricky problem, and variations of it appear in many industrial applications involving packing objects of different sizes into a fixed space (for example, packing crates into a container for shipment overseas). In the Exercises we'll ask you to write down some more possible patterns, but finding all of them is a job for a computer, and it turns out that there are exactly 33 possible cutting patterns. In Chapter 8 you'll learn some counting techniques that might help you write the program to find all possible patterns.

The next question is, what's the best we can do? We have to use an integral number of 100-in. raw rolls, and we can find the total "roll-inches" ordered by multiplying the width of each standard roll by the number ordered for this width. This computation is a natural one for the matrix notation that you have learned. If W and O are 6×1 column matrices, then the total roll inches used is $W^{\mathrm{T}}O$:

$$W = \begin{bmatrix} 14 \\ 17 \\ 31 \\ 33 \\ 36 \\ 45 \end{bmatrix} \qquad O = \begin{bmatrix} 100 \\ 123 \\ 239 \\ 121 \\ 444 \\ 87 \end{bmatrix} \qquad W^{\mathrm{T}}O = 34{,}792$$

Since each raw roll is 100 in., the best we can do is to use 348 rolls with a total width of 34,800. As a percentage of the raw material, the corresponding waste is

$$\frac{8}{34{,}800} \approx 0.02\%,$$

which represents very low waste. Of course, we'll only reach this target if we can lay out the cutting with perfect efficiency.

As we noted previously, these integer programming problems are difficult, but many mathematical analysis and spreadsheet programs have built-in optimization routines that can handle problems of modest size. We submitted this problem to one such program, giving it the lists of orders and widths and a list of the 33 allowable cutting patterns. Figure 10 shows the seven cutting patterns chosen by the minimizer software, with a graphical representation, and the total number of times each pattern was used.

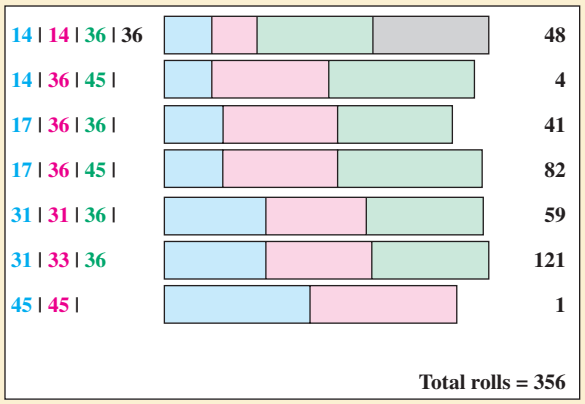


FIGURE 10

With these cutting choices we generate the following numbers of each standard width:

	Solution to Stock-cutting Problem					
Width	14	17	31	33	36	45
Quantity Produced	100	123	239	121	444	88
Quantity Ordered	100	123	239	121	444	87

We figured that the minimum possible number of raw rolls was 348, so we have used only 8 more than the minimum. In the Exercises you'll figure the percentage of waste with this cutting plan.

Manufacturers of glass and sheet metal encounter a two-dimensional version of this problem: They need to cut the rectangular pieces that have been ordered from a larger rectangular piece of glass or metal, laying out the ordered sizes so as to minimize waste.

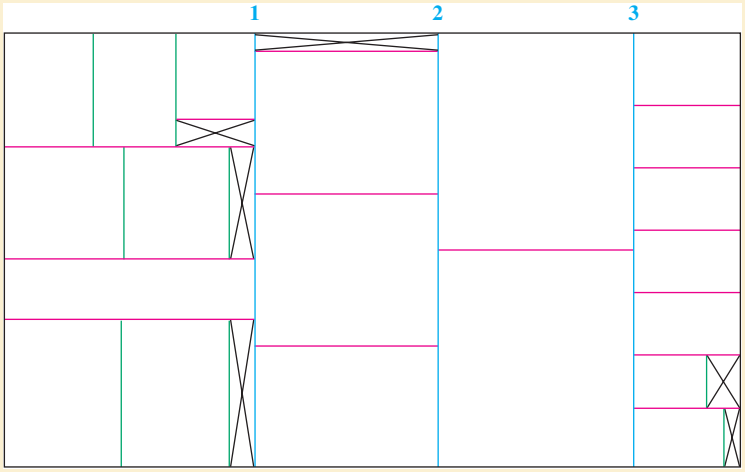


FIGURE 11

Besides the extra dimension, this problem is complicated by another constraint: The typical cutting machine can make only "guillotine cuts" that go completely across the sheet being cut, so a cutting algorithm must usually begin with a few long cuts that cut the original rectangle into strips, followed by crossways cuts that begin to create the order sizes. A typical finished cutting layout might look like Figure 11.

The first cuts would be the three vertical cuts labeled 1, 2, and 3, followed by horizontal cuts in each of the four resulting strips, then vertical cuts in these new rectangles, and so on. Areas of waste are marked with X. An additional complication in designing the layout is that any given stock rectangle can be oriented in two different directions (unless it's square), so the packing problem has many alternative solutions.

In three dimensions, a comparable problem is to fill a shipping container with smaller boxes (rectangular prisms) with the minimum wasted space. These packing problems are complicated geometric versions of a basic problem called the *knapsack problem*:

Given n objects with weights $w_1, w_2, \ldots, w_n$ and cash values $v_1, v_2, \ldots, v_n$ and a knapsack that can hold a weight of at most W, choose the objects that will pack in the greatest value. In the Exercises you can try a small example, but as soon as n gets large, this problem "explodes," that is, the number of possibilities becomes too large for a trial-and-error solution, even with a computer to do the bookkeeping. The development of good algorithms for cutting and packing problems is an active research specialty in the field of optimization.

EXERCISES

1. Suppose you plan to build a raised flower bed using landscape timbers, which come in 8-ft lengths. You want the bed's outer dimensions to be 6 ft by 4 ft, and you will use three layers of timbers. The timbers are 6 in. by 6 in. in cross section, so if you make the bottom and top layers with 6-ft lengths on the sides and 3-ft lengths on the ends, and the middle layer with 5-ft lengths on the sides and 4-ft lengths on the ends, you could build the bed out of the following lengths.

Plan A	
Length	**Number Needed**
3 ft	4
4 ft	2
5 ft	2
6 ft	4

a. What is the smallest number of timbers you can buy to build your bed? How will you lay out the cuts? How much wood will you waste?

b. If you overlap the corners in a different way, you can build the bed with this plan:

Plan B	
Length	**Needed**
3 ft	2
4 ft	4
5 ft	4
6 ft	2

Does plan B allow you to build the bed with fewer 8-ft timbers?

c. What is the smallest length for the uncut timbers that would allow you to build the bed with no waste?

2. For the list of standard paper roll widths given earlier, write down four more possible cutting patterns that use at most four cuts and leave less than 14 in. of waste on the end. See if you can find ones that aren't in the list of patterns returned by the optimizer.

3. Four of the 33 possible patterns use up the raw roll with no waste, that is, the widths add up to exactly 100 in. Find these four patterns.

4. For the computer solution of the cutting problem, figure out the percent of the 356 rolls used that is wasted.

5. In our cutting plan, we elected to use up as much as possible of each 100-in, roll with standard widths. Why might it be a better idea to allow leftover rolls that are *wider* than 14 in.?

6. The following table shows the weights of six objects and their values.

Weight	2	2.5	3	3.5	4	4.5
Value	12	11	7	13	10	11

If your knapsack holds a maximum weight of 9, what is the highest value you can pack in?

7. Suppose that of the original 33 cutting patterns, we allow only the 7 that occur in the final solution. Let y_1 = the number of copies of the first pattern, and similarly define y_2 through y_7. Write the integer programming problem that we are trying to solve.

8. Try solving the problem from Exercise 7 with the Solver in Microsoft Excel. First try it only requiring that the variables be nonnegative, not that they be integers. Then add constraints in the constraint box that the variables be integers.

9. Compare your two answers from Exercise 8 with each other and with the answer given in Figure 10. Discuss the effect on the minimum value of the requirement that the variables take on integer values. Also discuss what these solutions tell you about uniqueness of this integer programming problem.

10. Go to the website WolframAlpha.com and enter: "Minimize $y_1 + y_2$ with $2y_1 + y_2 \geq 100$, $y_1 \geq 0$, $y_2 \geq 0$". Study the solution provided by Wolfram|Alpha. As of this writing, Wolfram|Alpha cannot solve linear programming problems with more than five variables nor does it provide a way to do integer programming. Investigate whether this is still true. If these restrictions no longer exist, try Exercise 8 with Wolfram|Alpha and discuss how it compares with Microsoft Excel.

DIRECTIONS FOR GROUP PROJECT

Suppose you and three of the students from class have met at your house to study and your father questions each of you on what you are learning in college. While this is happening, your mother is busy planning a new raised-bed garden and your sister is attempting to choose which items she will put in a backpack for a field trip. Using the data in Exercises 1 and 6, prepare a presentation for your family on the value of what you're learning in college.

5 Mathematics of Finance

5.1 Simple and Compound Interest

5.2 Future Value of an Annuity

5.3 Present Value of an Annuity; Amortization

Chapter 5 Review

Extended Application: Time, Money, and Polynomials

Buying a car usually requires both some savings for a down payment and a loan for the balance. An exercise in Section 2 calculates the regular deposits that would be needed to save up the full purchase price, and other exercises and examples in this chapter compute the payments required to amortize a loan.

Everybody uses money. Sometimes you work for your money and other times your money works for you. For example, unless you are attending college on a full scholarship, it is very likely that you and your family have either saved money or borrowed money, or both, to pay for your education. When we borrow money, we normally have to pay interest for that privilege. When we save money, for a future purchase or retirement, we are lending money to a financial institution and we expect to earn interest on our investment. We will develop the mathematics in this chapter to understand better the principles of borrowing and saving. These ideas will then be used to compare different financial opportunities and make informed decisions.

5.1 Simple and Compound Interest

APPLY IT **If you can borrow money at 8% interest compounded annually or at 7.9% compounded monthly, which loan would cost less?**

In this section we will learn how to compare different interest rates with different compounding periods. The question above will be answered in Example 8.

Simple Interest

Interest on loans of a year or less is frequently calculated as **simple interest**, a type of interest that is charged (or paid) only on the amount borrowed (or invested) and not on past interest. The amount borrowed is called the **principal**. The **rate** of interest is given as a percentage per year, expressed as a decimal. For example, $6\% = 0.06$ and $11\frac{1}{2}\% = 0.115$. The **time** the money is earning interest is calculated in years. One year's interest is calculated by multiplying the principal times the interest rate, or Pr. If the time that the money earns interest is other than one year, we multiply the interest for one year by the number of years, or Prt.

Simple Interest

$$I = Prt$$

where

P is the principal;

r is the annual interest rate;

t is the time in years.

EXAMPLE 1 **Simple Interest**

To buy furniture for a new apartment, Candace Cooney borrowed $5000 at 8% simple interest for 11 months. How much interest will she pay?

SOLUTION Use the formula $I = Prt$, with $P = 5000$, $r = 0.08$, and $t = 11/12$ (in years). The total interest she will pay is

$$I = 5000(0.08)(11/12) \approx 366.67,$$

or $366.67.

A deposit of P dollars today at a rate of interest r for t years produces interest of $I = Prt$. The interest, added to the original principal P, gives

$$P + Prt = P(1 + rt).$$

This amount is called the *future value* of P dollars at an interest rate r for time t in years. When loans are involved, the future value is often called the *maturity value* of the loan. This idea is summarized as follows.

Future or Maturity Value for Simple Interest

The **future** or **maturity value** A of P dollars at a simple interest rate r for t years is

$$A = P(1 + rt).$$

EXAMPLE 2 Maturity Values

Find the maturity value for each loan at simple interest.

(a) A loan of $2500 to be repaid in 8 months with interest of 4.3%

SOLUTION The loan is for 8 months, or $8/12 = 2/3$ of a year. The maturity value is

$$A = P(1 + rt)$$
$$A = 2500\left[1 + 0.043\left(\frac{2}{3}\right)\right]$$
$$A \approx 2500(1 + 0.028667) = 2571.67,$$

or $2571.67. (The answer is rounded to the nearest cent, as is customary in financial problems.) Of this maturity value,

$$\$2571.67 - \$2500 = \$71.67$$

represents interest.

(b) A loan of $11,280 for 85 days at 7% interest

SOLUTION It is common to assume 360 days in a year when working with simple interest. We shall usually make such an assumption in this book. The maturity value in this example is

$$A = 11{,}280\left[1 + 0.07\left(\frac{85}{360}\right)\right] \approx 11{,}466.43,$$

or $11,466.43.

YOUR TURN 1 Find the maturity value for a $3000 loan at 5.8% interest for 100 days.

TRY YOUR TURN 1

CAUTION When using the formula for future value, as well as all other formulas in this chapter, we often neglect the fact that in real life, money amounts are rounded to the nearest penny. As a consequence, when the amounts are rounded, their values may differ by a few cents from the amounts given by these formulas. For instance, in Example 2(a), the interest in each monthly payment would be $\$2500(0.043/12) \approx \$8.96$, rounded to the nearest penny. After 8 months, the total is $8(\$8.96) = \$71.68$, which is 1¢ more than we computed in the example.

In part (b) of Example 2 we assumed 360 days in a year. Historically, to simplify calculations, it was often assumed that each year had twelve 30-day months, making a year 360 days long. Treasury bills sold by the U.S. government assume a 360-day year in calculating interest. Interest found using a 360-day year is called *ordinary interest* and interest found using a 365-day year is called *exact interest*.

The formula for future value has four variables, P, r, t, and A. We can use the formula to find any of the quantities that these variables represent, as illustrated in the next example.

EXAMPLE 3 Simple Interest

Theresa Cortesini wants to borrow $8000 from Christine O'Brien. She is willing to pay back $8180 in 6 months. What interest rate will she pay?

SOLUTION Use the formula for future value, with $A = 8180$, $P = 8000$, $t = 6/12 = 0.5$, and solve for r.

$$\begin{aligned} A &= P(1 + rt) \\ 8180 &= 8000(1 + 0.5r) \\ 8180 &= 8000 + 4000r && \text{Distributive property} \\ 180 &= 4000r && \text{Subtract 8000.} \\ r &= 0.045 && \text{Divide by 4000.} \end{aligned}$$

Thus, the interest rate is 4.5%.

TRY YOUR TURN 2

YOUR TURN 2 Find the interest rate if $5000 is borrowed, and $5243.75 is paid back 9 months later.

When you deposit money in the bank and earn interest, it is as if the bank borrowed the money from you. Reversing the scenario in Example 3, if you put $8000 in a bank account that pays simple interest at a rate of 4.5% annually, you will have accumulated $8180 after 6 months.

Compound Interest

As mentioned earlier, simple interest is normally used for loans or investments of a year or less. For longer periods compound interest is used. With **compound interest**, interest is charged (or paid) on interest as well as on principal. For example, if $1000 is deposited at 5% interest for 1 year, at the end of the year the interest is $\$1000(0.05)(1) = \$50$. The balance in the account is $\$1000 + \$50 = \$1050$. If this amount is left at 5% interest for another year, the interest is calculated on $1050 instead of the original $1000, so the amount in the account at the end of the second year is $\$1050 + \$1050(0.05)(1) = \$1102.50$. Note that simple interest would produce a total amount of only

$$\$1000[1 + (0.05)(2)] = \$1100.$$

The additional $2.50 is the interest on $50 at 5% for one year.

To find a formula for compound interest, first suppose that P dollars is deposited at a rate of interest r per year. The amount on deposit at the end of the first year is found by the simple interest formula, with $t = 1$.

$$A = P(1 + r \cdot 1) = P(1 + r)$$

If the deposit earns compound interest, the interest earned during the second year is paid on the total amount on deposit at the end of the first year. Using the formula $A = P(1 + rt)$ again, with P replaced by $P(1 + r)$ and $t = 1$, gives the total amount on deposit at the end of the second year.

$$A = [P(1 + r)](1 + r \cdot 1) = P(1 + r)^2$$

In the same way, the total amount on deposit at the end of the third year is

$$P(1 + r)^3.$$

Generalizing, in t years the total amount on deposit is

$$A = P(1 + r)^t,$$

called the **compound amount**.

NOTE Compare this formula for compound interest with the formula for simple interest.

$$\begin{aligned} &\text{Compound interest} && A = P(1 + r)^t \\ &\text{Simple interest} && A = P(1 + rt) \end{aligned}$$

The important distinction between the two formulas is that in the compound interest formula, the number of years, t, is an *exponent*, so that money grows much more rapidly when interest is compounded.

Interest can be compounded more than once per year. Common compounding periods include *semiannually* (two periods per year), *quarterly* (four periods per year), *monthly* (twelve periods per year), or *daily* (usually 365 periods per year). The *interest rate per period*, i, is found by dividing the annual interest rate, r, by the number of compounding periods, m, per year. To find the total number of compounding periods, n, we multiply the number of years, t, by the number of compounding periods per year, m. The following formula can be derived in the same way as the previous formula.

Compound Amount

$$A = P(1 + i)^n$$

where $i = \dfrac{r}{m}$ and $n = mt$,

A is the future (maturity) value;
P is the principal;
r is the annual interest rate;
m is the number of compounding periods per year;
t is the number of years;
n is the number of compounding periods;
i is the interest rate per period.

EXAMPLE 4 Compound Interest

Suppose $1000 is deposited for 6 years in an account paying 4.25% per year compounded annually.

(a) Find the compound amount.

SOLUTION In the formula for the compound amount, $P = 1000$, $i = 0.0425/1$, and $n = 6(1) = 6$. The compound amount is

$$A = P(1 + i)^n$$
$$A = 1000(1.0425)^6.$$

Using a calculator, we get

$$A \approx \$1283.68,$$

the compound amount.

(b) Find the amount of interest earned.

SOLUTION Subtract the initial deposit from the compound amount.

$$\text{Amount of interest} = \$1283.68 - \$1000 = \$283.68$$

EXAMPLE 5 Compound Interest

Find the amount of interest earned by a deposit of $2450 for 6.5 years at 5.25% compounded quarterly.

SOLUTION Interest compounded quarterly is compounded 4 times a year. In 6.5 years, there are $6.5(4) = 26$ periods. Thus, $n = 26$. Interest of 5.25% per year is $5.25\%/4$ per quarter, so $i = 0.0525/4$. Now use the formula for compound amount.

$$A = P(1 + i)^n$$
$$A = 2450(1 + 0.0525/4)^{26} \approx 3438.78$$

Rounded to the nearest cent, the compound amount is $3438.78, so the interest is $3438.78 − $2450 = $988.78.

YOUR TURN 3 Find the amount of interest earned by a deposit of $1600 for 7 years at 4.2% compounded monthly.

TRY YOUR TURN 3

CAUTION As shown in Example 5, compound interest problems involve two rates—the annual rate r and the rate per compounding period i. Be sure you understand the distinction between them. When interest is compounded annually, these rates are the same. In all other cases, $i \neq r$. Similarly, there are two quantities for time: the number of years t and the number of compounding periods n. When interest is compounded annually, these variables have the same value. In all other cases, $n \neq t$.

It is interesting to compare loans at the same rate when simple or compound interest is used. Figure 1 shows the graphs of the simple interest and compound interest formulas with $P = 1000$ at an annual rate of 10% from 0 to 20 years. The future value after 15 years is shown for each graph. After 15 years of compound interest, \$1000 grows to \$4177.25, whereas with simple interest, it amounts to \$2500.00, a difference of \$1677.25.

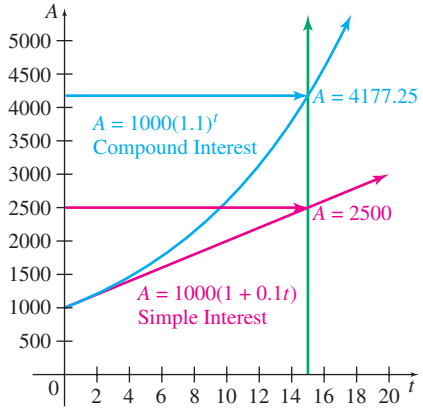


FIGURE 1

TECHNOLOGY NOTE Spreadsheets are ideal for performing financial calculations. Figure 2 shows a Microsoft Excel spreadsheet with the formulas for compound and simple interest used to create columns B and C, respectively, when \$1000 is invested at an annual rate of 10%. Compare row 16 with Figure 1. For more details on the use of spreadsheets in the mathematics of finance, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

	A	B	C
1	period	compound	simple
2	1	1100	1100
3	2	1210	1200
4	3	1331	1300
5	4	1464.1	1400
6	5	1610.51	1500
7	6	1771.561	1600
8	7	1948.7171	1700
9	8	2143.58881	1800
10	9	2357.947691	1900
11	10	2593.74246	2000
12	11	2853.116706	2100
13	12	3138.428377	2200
14	13	3452.27124	2300
15	14	3797.498336	2400
16	15	4177.248169	2500
17	16	4594.972986	2600
18	17	5054.470285	2700
19	18	5559.917313	2800
20	19	6115.909045	2900
21	20	6727.499949	3000

FIGURE 2

We can also solve the compound amount formula for the interest rate, as in the following example.

EXAMPLE 6 Compound Interest Rate

Suppose Carol Merrigan invested $5000 in a savings account that paid quarterly interest. After 6 years the money had accumulated to $6539.96. What was the annual interest rate?

SOLUTION Because $m = 4$ and $t = 6$, the number of compounding periods is $n = 4 \times 6 = 24$. Using this value along with $P = 5000$ and $A = 6539.96$ in the formula for compound amount, we have

$$\begin{aligned} 5000(1 + r/4)^{24} &= 6539.96 \\ (1 + r/4)^{24} &= 1.30799 && \text{Divide both sides by 5000.} \\ 1 + r/4 &= 1.30799^{1/24} \approx 1.01125 && \text{Take both sides to the } 1/24 \text{ power.} \\ r/4 &= 0.01125 && \text{Subtract 1 from both sides.} \\ r &= 0.045 && \text{Multiply both sides by 4.} \end{aligned}$$

The annual interest rate was 4.5%.

TRY YOUR TURN 4

YOUR TURN 4 Find the annual interest rate if $6500 is worth $8665.69 after being invested for 8 years in an account that compounded interest monthly.

Effective Rate

If $1 is deposited at 4% compounded quarterly, a calculator can be used to find that at the end of one year, the compound amount is $1.0406, an increase of 4.06% over the original $1. The actual increase of 4.06% in the money is somewhat higher than the stated increase of 4%. To differentiate between these two numbers, 4% is called the **nominal** or **stated rate** of interest, while 4.06% is called the *effective rate*.* To avoid confusion between stated rates and effective rates, we shall continue to use r for the stated rate and we will use r_E for the effective rate.

EXAMPLE 7 Effective Rate

Find the effective rate corresponding to a stated rate of 6% compounded semiannually.

SOLUTION Here, $i = r/m = 0.06/2 = 0.03$ for $m = 2$ periods. Use a calculator to find that $(1.03)^2 \approx 1.06090$, which shows that $1 will increase to $1.06090, an actual increase of 6.09%. The effective rate is $r_E = 6.09\%$.

Generalizing from this example, the effective rate of interest is given by the following formula.

Effective Rate

The **effective rate** corresponding to a stated rate of interest r compounded m times per year is

$$r_E = \left(1 + \frac{r}{m}\right)^m - 1.$$

EXAMPLE 8 Effective Rate

Joe Vetere needs to borrow money. His neighborhood bank charges 8% interest compounded semiannually. A downtown bank charges 7.9% interest compounded monthly. At which bank will Joe pay the lesser amount of interest?

*When applied to consumer finance, the effective rate is called the annual percentage rate, APR, or annual percentage yield, APY.

APPLY IT **SOLUTION** Compare the effective rates.

Neighborhood bank: $r_E = \left(1 + \frac{0.08}{2}\right)^2 - 1 = 0.0816 = 8.16\%$

Downtown bank: $r_E = \left(1 + \frac{0.079}{12}\right)^{12} - 1 \approx 0.081924 \approx 8.19\%$

The neighborhood bank has the lower effective rate, although it has a higher stated rate.

YOUR TURN 5 Find the effective rate for an account that pays 2.7% compounded monthly.

TRY YOUR TURN 5

Present Value

The formula for compound interest, $A = P(1 + i)^n$, has four variables: A, P, i, and n. Given the values of any three of these variables, the value of the fourth can be found. In particular, if A (the future amount), i, and n are known, then P can be found. Here P is the amount that should be deposited today to produce A dollars in n periods.

EXAMPLE 9 Present Value

Rachel Reeve must pay a lump sum of $6000 in 5 years. What amount deposited today at 6.2% compounded annually will amount to $6000 in 5 years?

SOLUTION Here $A = 6000$, $i = 0.062$, $n = 5$, and P is unknown. Substituting these values into the formula for the compound amount gives

$$6000 = P(1.062)^5$$

$$P = \frac{6000}{(1.062)^5} \approx 4441.49,$$

or $4441.49. If Rachel leaves $4441.49 for 5 years in an account paying 6.2% compounded annually, she will have $6000 when she needs it. To check your work, use the compound interest formula with $P = \$4441.49$, $i = 0.062$, and $n = 5$. You should get $A = \$6000.00$.

As Example 9 shows, $6000 in 5 years is approximately the same as $4441.49 today (if money can be deposited at 6.2% compounded annually). An amount that can be deposited today to yield a given sum in the future is called the *present value* of the future sum. Generalizing from Example 9, by solving $A = P(1 + i)^n$ for P, we get the following formula for present value.

Present Value for Compound Interest

The **present value** of A dollars compounded at an interest rate i per period for n periods is

$$P = \frac{A}{(1 + i)^n} \quad \text{or} \quad P = A(1 + i)^{-n}.$$

EXAMPLE 10 Present Value

Find the present value of $16,000 in 9 years if money can be deposited at 6% compounded semiannually.

SOLUTION In 9 years there are $2 \cdot 9 = 18$ semiannual periods. A rate of 6% per year is 3% in each semiannual period. Apply the formula with $A = 16{,}000$, $i = 0.03$, and $n = 18$.

$$P = \frac{A}{(1 + i)^n} = \frac{16{,}000}{(1.03)^{18}} \approx 9398.31$$

A deposit of $9398.31 today, at 6% compounded semiannually, will produce a total of $16,000 in 9 years.

YOUR TURN 6 Find the present value of $10,000 in 7 years if money can be deposited at 4.25% compounded quarterly.

TRY YOUR TURN 6

We can solve the compound amount formula for n also, as the following example shows.

EXAMPLE 11 Compounding Time

Suppose the $2450 from Example 5 is deposited at 5.25% compounded quarterly until it reaches at least $10,000. How much time is required?

Method 1 Graphing Calculator

SOLUTION Graph the functions $y = 2450(1 + 0.0525/4)^x$ and $y = 10{,}000$ in the same window, and then find the point of intersection. As Figure 3 shows, the functions intersect at $x = 107.8634$. Note, however, that interest is only added to the account every quarter, so we must wait 108 quarters, or $108/4 = 27$ years, for the money to be worth at least $10,000.

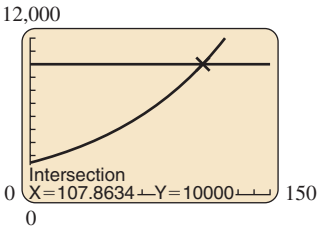


FIGURE 3

Method 2 Using Logarithms (Optional)

The goal is to solve the equation

$$2450(1 + 0.0525/4)^n = 10{,}000.$$

Divide both sides by 2450, and simplify the expression in parentheses to get

$$1.013125^n = \frac{10{,}000}{2450}.$$

Now take the logarithm (either base 10 or base e) of both sides to get

$$\begin{aligned} \log(1.013125^n) &= \log(10{,}000/2450) \\ n\log(1.013125) &= \log(10{,}000/2450) && \text{Use logarithm property } \log x^r = r\log x. \\ n &= \frac{\log(10{,}000/2450)}{\log(1.013125)} && \text{Divide both sides by } \log(1.013125). \\ &\approx 107.86. \end{aligned}$$

As in Method 1, this means that we must wait 108 quarters, or $108/4 = 27$ years, for the money to be worth at least $10,000.

FOR REVIEW

For a review of logarithmic functions, please refer to Appendix B if you are using *Finite Mathematics*, or to Section 10.5 if you are using *Finite Mathematics and Calculus with Applications*. The only property of logarithms that is needed to find the compounding time is $\log x^r = r\log x$. Logarithms may be used in base 10, using the LOG button on a calculator, or in base e, using the LN button.

EXAMPLE 12 Price Doubling

Suppose the general level of inflation in the economy averages 8% per year. Find the number of years it would take for the overall level of prices to double.

SOLUTION To find the number of years it will take for $1 worth of goods or services to cost $2, find n in the equation

$$2 = 1(1 + 0.08)^n,$$

where $A = 2$, $P = 1$, and $i = 0.08$. This equation simplifies to

$$2 = (1.08)^n.$$

Solving this equation using either a graphing calculator or logarithms, as in Example 11, shows that $n = 9.00647$. Thus, the overall level of prices will double in about 9 years.

TRY YOUR TURN 7

YOUR TURN 7 Find the time needed for $3800 deposited at 3.5% compounded semiannually to be worth at least $7000.

You can quickly estimate how long it takes a sum of money to double, when compounded annually, by using either the **rule of 70** or the **rule of 72**. The rule of 70 (used for

small rates of growth) says that for $0.001 \le r < 0.05$, the value of $70/100r$ gives a good approximation of the doubling time. The rule of 72 (used for larger rates of growth) says that for $0.05 \le r \le 0.12$, the value of $72/100r$ approximates the doubling time well. In Example 12, the inflation rate is 8%, so the doubling time is approximately $72/8 = 9$ years.*

Continuous Compounding

Suppose that a bank, in order to attract more business, offers to not just compound interest every quarter, or every month, or every day, or even every hour, but constantly? This type of compound interest, in which the number of times a year that the interest is compounded becomes infinite, is known as **continuous compounding.** To see how it works, look back at Example 5, where we found that $2450, when deposited for 6.5 years at 5.25% compounded quarterly, resulted in a compound amount of $3438.78. We can find the compound amount if we compound more often by putting different values of n in the formula $A = 2450(1 + 0.0525/n)^{6.5n}$, as shown in the following table.

Compounding n Times Annually		
n	Type of Compounding	Compound Amount
4	quarterly	$3438.78
12	monthly	$3443.86
360	daily	$3446.34
8640	every hour	$3446.42

Notice that as n becomes larger, the compound amount also becomes larger, but by a smaller and smaller amount. In this example, increasing the number of compounding periods a year from 360 to 8640 only earns 8¢ more. It is shown in calculus that as n becomes infinitely large, $P(1 + r/n)^{nt}$ gets closer and closer to Pe^{rt}, where e is a very important irrational number whose approximate value is 2.718281828. To calculate interest with continuous compounding, use the e^x button on your calculator. You will learn more about the number e if you study calculus, where e plays as important a role as π does in geometry.

Continuous Compounding

If a deposit of P dollars is invested at a rate of interest r compounded continuously for t years, the compound amount is

$$A = Pe^{rt} \text{ dollars.}$$

EXAMPLE 13 Continuous Compounding

Suppose that $2450 is deposited at 5.25% compounded continuously.

(a) Find the compound amount and the interest earned after 6.5 years.

SOLUTION Using the formula for continuous compounding with $P = 2450$, $r = 0.0525$, and $t = 6.5$, the compound amount is

$$A = 2450e^{0.0525(6.5)} \approx 3446.43.$$

The compound amount is $3446.43, which is just a penny more than if it had been compounded hourly, or 9¢ more than daily compounding. Because it makes so little difference, continuous compounding has dropped in popularity in recent years. The interest in this case is $3446.43 − 2450 = $996.43, or $7.65 more than if it were compounded quarterly, as in Example 5.

*To see where the rule of 70 and the rule of 72 come from, see the section on Taylor Series in *Calculus with Applications* by Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey, Pearson, 2012.

(b) Find the effective rate.

SOLUTION As in Example 7, the effective rate is just the amount of interest that $1 would earn in one year, or

$$e^{0.0525} - 1 \approx 0.0539,$$

or 5.39%. In general, the effective rate for interest compounded continuously at a rate r is just $e^r - 1$.

(c) Find the time required for the original $2450 to grow to $10,000.

SOLUTION Similar to the process in Example 11, we must solve the equation

$$10{,}000 = 2450e^{0.0525t}.$$

Divide both sides by 2450, and solve the resulting equation as in Example 11, either by taking logarithms of both sides or by using a graphing calculator to find the intersection of the graphs of $y = 2450e^{0.0525t}$ and $y = 10{,}000$. If you use logarithms, you can take advantage of the fact that $\ln(e^x) = x$, where $\ln x$ represents the logarithm in base e. In either case, the answer is 26.79 years.

Notice that unlike in Example 11, you don't need to wait until the next compounding period to reach this amount, because interest is being added to the account continuously.

TRY YOUR TURN 8

YOUR TURN 8 Find the interest earned on $5000 deposited at 3.8% compounded continuously for 9 years.

At this point, it seems helpful to summarize the notation and the most important formulas for simple and compound interest. We use the following variables.

P = principal or present value
A = future or maturity value
r = annual (stated or nominal) interest rate
t = number of years
m = number of compounding periods per year
i = interest rate per period $\quad i = r/m$
n = total number of compounding periods $\quad n = tm$
r_E = effective rate

Simple Interest	Compound Interest	Continuous Compounding
$A = P(1 + rt)$	$A = P(1 + i)^n$	$A = Pe^{rt}$
$P = \dfrac{A}{1 + rt}$	$P = \dfrac{A}{(1 + i)^n} = A(1 + i)^{-n}$	$P = Ae^{-rt}$
	$r_E = \left(1 + \dfrac{r}{m}\right)^m - 1$	$r_E = e^r - 1$

5.1 EXERCISES

1. What factors determine the amount of interest earned on a fixed principal?
2. In your own words, describe the *maturity value* of a loan.
3. What is meant by the *present value* of money?
4. We calculated the loan in Example 2(b) assuming 360 days in a year. Find the maturity value using 365 days in a year. Which is more advantageous to the borrower?

Find the simple interest.

5. $25,000 at 3% for 9 months
6. $4289 at 4.5% for 35 weeks
7. $1974 at 6.3% for 25 weeks
8. $6125 at 1.25% for 6 months

Find the simple interest. Assume a 360-day year.

9. $8192.17 at 3.1% for 72 days

10. $7236.15 at 4.25% for 30 days

Find the maturity value and the amount of simple interest earned.

11. $3125 at 2.85% for 7 months

12. $12,000 at 5.3% for 11 months

13. If $1500 earned simple interest of $56.25 in 6 months, what was the simple interest rate?

14. If $23,500 earned simple interest of $1057.50 in 9 months, what was the simple interest rate?

15. Explain the difference between simple interest and compound interest.

16. What is the difference between r and i?

17. What is the difference between t and n?

18. In Figure 1, one line is straight and the other is curved. Explain why this is, and which represents each type of interest.

Find the compound amount for each deposit and the amount of interest earned.

19. $1000 at 6% compounded annually for 8 years

20. $1000 at 4.5% compounded annually for 6 years

21. $470 at 5.4% compounded semiannually for 12 years

22. $15,000 at 6% compounded monthly for 10 years

23. $8500 at 8% compounded quarterly for 5 years

24. $9100 at 6.4% compounded quarterly for 9 years

Find the interest rate for each deposit and compound amount.

25. $8000 accumulating to $11,672.12, compounded quarterly for 8 years

26. $12,500 accumulating to $20,077.43, compounded quarterly for 9 years

27. $4500 accumulating to $5994.79, compounded monthly for 5 years

28. $6725 accumulating to $10,353.47, compounded monthly for 7 years

Find the effective rate corresponding to each nominal rate.

29. 4% compounded quarterly

30. 6% compounded quarterly

31. 7.25% compounded semiannually

32. 6.25% compounded semiannually

Find the present value (the amount that should be invested now to accumulate the following amount) if the money is compounded as indicated.

33. $12,820.77 at 4.8% compounded annually for 6 years

34. $36,527.13 at 5.3% compounded annually for 10 years

35. $2000 at 6% compounded semiannually for 8 years

36. $2000 at 7% compounded semiannually for 8 years

37. $8800 at 5% compounded quarterly for 5 years

38. $7500 at 5.5% compounded quarterly for 9 years

39. How do the nominal or stated interest rate and the effective interest rate differ?

40. If interest is compounded more than once per year, which rate is higher, the stated rate or the effective rate?

Using either logarithms or a graphing calculator, find the time required for each initial amount to be at least equal to the final amount.

41. $5000, deposited at 4% compounded quarterly, to reach at least $9000

42. $8000, deposited at 3% compounded quarterly, to reach at least $23,000

43. $4500, deposited at 3.6% compounded monthly, to reach at least $11,000

44. $6800, deposited at 5.4% compounded monthly, to reach at least $15,000

Find the doubling time for each of the following levels of inflation using (a) logarithms or a graphing calculator, and (b) the rule of 70 or 72, whichever is appropriate.

45. 3.3%

46. 6.25%

For each of the following amounts at the given interest rate compounded continuously, find (a) the future value after 9 years, (b) the effective rate, and (c) the time to reach $10,000.

47. $5500 at 3.1%

48. $4700 at 4.65%

APPLICATIONS

Business and Economics

49. Loan Repayment Tanya Kerchner borrowed $7200 from her father to buy a used car. She repaid him after 9 months, at an annual interest rate of 6.2%. Find the total amount she repaid. How much of this amount is interest?

50. Delinquent Taxes An accountant for a corporation forgot to pay the firm's income tax of $321,812.85 on time. The government charged a penalty based on an annual interest rate of 13.4% for the 29 days the money was late. Find the total amount (tax and penalty) that was paid. (Use a 365-day year.)

51. Savings A $1500 certificate of deposit held for 75 days was worth $1521.25. To the nearest tenth of a percent, what interest rate was earned? Assume a 360-day year.

52. Bond Interest A bond with a face value of $10,000 in 10 years can be purchased now for $5988.02. What is the simple interest rate?

53. Stock Growth A stock that sold for $22 at the beginning of the year was selling for $24 at the end of the year. If the stock paid a dividend of $0.50 per share, what is the simple interest rate on an investment in this stock? (*Hint:* Consider the interest to be the increase in value plus the dividend.)

54. Wealth A 1997 article in *The New York Times* discussed how long it would take for Bill Gates, the world's second richest person at the time (behind the Sultan of Brunei), to become the world's first trillionaire. His birthday is October 28, 1955, and on

July 16, 1997, he was worth $42 billion. (*Note:* A trillion dollars is 1000 billion dollars.) ***Source: The New York Times.***

a. Assume that Bill Gates's fortune grows at an annual rate of 58%, the historical growth rate through 1997 of Microsoft stock, which made up most of his wealth in 1997. Find the age at which he becomes a trillionaire. (*Hint:* Use the formula for interest compounded annually, $A = P(1 + i)^n$, with $P = 42$. Graph the future value as a function of n on a graphing calculator, and find where the graph crosses the line $y = 1000$.)

b. Repeat part a using 10.5% growth, the average return on all stocks since 1926. ***Source: CNN.***

c. What rate of growth would be necessary for Bill Gates to become a trillionaire by the time he is eligible for Social Security on January 1, 2022, after he has turned 66?

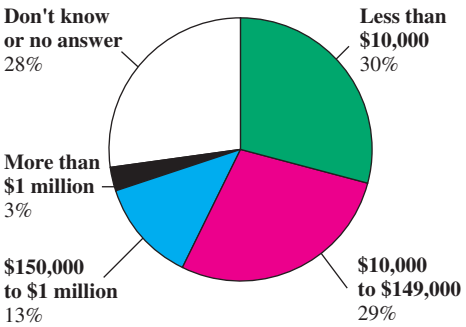
d. *Forbes* magazine's listings of billionaires for 2006 and 2010 have given Bill Gates's worth as roughly $50.0 billion and $53.0 billion, respectively. What was the rate of growth of his wealth between 2006 and 2010? ***Source: Forbes.***

55. Student Loan Upon graduation from college, Kelly was able to defer payment on his $40,000 subsidized Stafford student loan for 6 months. Since the interest will no longer be paid on his behalf, it will be added to the principal until payments begin. If the interest is 6.54% compounded monthly, what will the principal amount be when he must begin repaying his loan? ***Source: SallieMae.***

56. Comparing Investments Two partners agree to invest equal amounts in their business. One will contribute $10,000 immediately. The other plans to contribute an equivalent amount in 3 years, when she expects to acquire a large sum of money. How much should she contribute at that time to match her partner's investment now, assuming an interest rate of 6% compounded semiannually?

57. Comparing Investments As the prize in a contest, you are offered $1000 now or $1210 in 5 years. If money can be invested at 6% compounded annually, which is larger?

58. Retirement Savings The pie graph below shows the percent of baby boomers aged 46–49 who said they had investments with a total value as shown in each category. ***Source: The New York Times.***



Figures add to more than 100% because of rounding.

Note that 30% have saved less than $10,000. Assume the money is invested at an average rate of 8% compounded quarterly. What will the top numbers in each category amount to in 20 years, when this age group will be ready for retirement?

Negative Interest **Under certain conditions, Swiss banks pay negative interest: they charge you. (You didn't think all that secrecy was free?) Suppose a bank "pays" −2.4% interest compounded annually. Find the compound amount for a deposit of $150,000 after each period.**

59. 4 years **60.** 8 years

61. Investment In the New Testament, Jesus commends a widow who contributed 2 mites to the temple treasury (Mark 12: 42–44). A mite was worth roughly 1/8 of a cent. Suppose the temple invested those 2 mites at 4% interest compounded quarterly. How much would the money be worth 2000 years later?

62. Investments Eric Cobbe borrowed $5200 from his friend Frank Cronin to pay for remodeling work on his house. He repaid the loan 10 months later with simple interest at 7%. Frank then invested the proceeds in a 5-year certificate of deposit paying 6.3% compounded quarterly. How much will he have at the end of 5 years? (*Hint:* You need to use both simple and compound interest.)

63. Investments Suppose $10,000 is invested at an annual rate of 5% for 10 years. Find the future value if interest is compounded as follows.

a. Annually **b.** Quarterly **c.** Monthly

d. Daily (365 days) **e.** Continuously

64. Investments In Exercise 63, notice that as the money is compounded more often, the compound amount becomes larger and larger. Is it possible to compound often enough so that the compound amount is $17,000 after 10 years? Explain.

The following exercise is from an actuarial examination. *Source: The Society of Actuaries.*

65. Savings On January 1, 2000, Jack deposited $1000 into bank X to earn interest at a rate of j per annum compounded semiannually. On January 1, 2005, he transferred his account to bank Y to earn interest at the rate of k per annum compounded quarterly. On January 1, 2008, the balance of bank Y is $1990.76. If Jack could have earned interest at the rate of k per annum compounded quarterly from January 1, 2000, through January 1, 2008, his balance would have been $2203.76. Calculate the ratio k/j.

66. Interest Rate In 1995, O. G. McClain of Houston, Texas, mailed a $100 check to a descendant of Texas independence hero Sam Houston to repay a $100 debt of McClain's great-great-grandfather, who died in 1835, to Sam Houston. A bank estimated the interest on the loan to be $420 million for the 160 years it was due. Find the interest rate the bank was using, assuming interest is compounded annually. ***Source: The New York Times.***

67. Comparing CD Rates Marine Bank offered the following CD (Certificates of Deposit) rates. The rates are annual percentage yields, or effective rates, which are higher than the corresponding nominal rates. Assume quarterly compounding. Solve for r to approximate the corresponding nominal rates to the nearest hundredth. ***Source: Marine Bank.***

Term	6 mo	Special! 9 mo	1 yr	2 yr	3 yr
APY%	2.50	5.10	4.25	4.50	5.25

68. Effective Rate A Web site for E*TRADE Financial claims that they have "one of the highest yields in the nation" on a 6-month CD. The stated yield was 5.46%; the actual rate was not stated. Assuming monthly compounding, find the actual rate. ***Source: E*TRADE.***

69. Effective Rate On August 18, 2006, Centennial Bank of Fountain Valley, California, paid 5.5% interest, compounded monthly, on a 1-year CD, while First Source Bank of South Bend, Indiana, paid 5.63% compounded annually. What are the effective rates for the two CDs, and which bank pays a higher effective rate? ***Source: Bankrate.com.***

70. Savings A department has ordered 8 new Dell computers at a cost of $2309 each. The order will not be delivered for 6 months. What amount could the department deposit in a special 6-month CD paying 4.79% compounded monthly to have enough to pay for the machines at time of delivery?

71. Buying a House Steve May wants to have $30,000 available in 5 years for a down payment on a house. He has inherited $25,000. How much of the inheritance should he invest now to accumulate $30,000, if he can get an interest rate of 5.5% compounded quarterly?

72. Rule of 70 On the day of their first grandchild's birth, a new set of grandparents invested $10,000 in a trust fund earning 4.5% compounded monthly.

a. Use the rule of 70 to estimate how old the grandchild will be when the trust fund is worth $20,000.

b. Use your answer to part a to determine the actual amount that will be in the trust fund at that time. How close was your estimate in part a?

Doubling Time **Use the ideas from Example 12 to find the time it would take for the general level of prices in the economy to double at each average annual inflation rate.**

73. 4% **74.** 5%

75. Doubling Time The consumption of electricity has increased historically at 6% per year. If it continues to increase at this rate indefinitely, find the number of years before the electric utilities will need to double their generating capacity.

76. Doubling Time Suppose a conservation campaign coupled with higher rates causes the demand for electricity to increase at only 2% per year, as it has recently. Find the number of years before the utilities will need to double generating capacity.

77. Mitt Romney According to *The New York Times*, "During the fourteen years [Mitt Romney] ran it, Bain Capital's investments reportedly earned an annual rate of return of over 100 percent, potentially turning an initial investment of $1 million into more than $14 million by the time he left in 1998." ***Source: The New York Times.***

a. What rate of return, compounded annually, would turn $1 million into $14 million by 1998?

b. The actual rate of return of Bain Capital during the 14 years that Romney ran it was 113%. ***Source: The American.*** How much would $1 million, compounded annually at this rate, be worth after 14 years?

YOUR TURN ANSWERS

1. $3048.33 **2.** 6.5% **3.** $545.75
4. 3.6% **5.** 2.73% **6.** $7438.39
7. 18 years **8.** $2038.80

5.2 Future Value of an Annuity

APPLY IT **If you deposit $1500 each year for 6 years in an account paying 8% interest compounded annually, how much will be in your account at the end of this period?**

In this section and the next, we develop future value and present value formulas for such periodic payments. To develop these formulas, we must first discuss sequences.

Geometric Sequences

If a and r are nonzero real numbers, the infinite list of numbers $a, ar, ar^2, ar^3, ar^4, \ldots, ar^n, \ldots$ is called a **geometric sequence**. For example, if $a = 3$ and $r = -2$, we have the sequence

$$3, 3(-2), 3(-2)^2, 3(-2)^3, 3(-2)^4, \ldots,$$

or

$$3, -6, 12, -24, 48, \ldots.$$

In the sequence $a, ar, ar^2, ar^3, ar^4, \ldots$, the number a is called the **first term** of the sequence, ar is the **second term**, ar^2 is the **third term**, and so on. Thus, for any $n \geq 1$,

ar^{n-1} is the nth term of the sequence.

Each term in the sequence is r times the preceding term. The number r is called the **common ratio** of the sequence.

CAUTION Do not confuse r, the ratio of two successive terms in a geometric series, with r, the annual interest rate. Different letters might have been helpful, but the usage in both cases is almost universal.

EXAMPLE 1 Geometric Sequence

Find the seventh term of the geometric sequence $5, 20, 80, 320, \ldots$.

SOLUTION The first term in the sequence is 5, so $a = 5$. The common ratio, found by dividing the second term by the first, is $r = 20/5 = 4$. We want the seventh term, so $n = 7$. Use ar^{n-1}, with $a = 5$, $r = 4$, and $n = 7$.

$$ar^{n-1} = (5)(4)^{7-1} = 5(4)^6 = 20{,}480$$

Next, we need to find the sum S_n of the first n terms of a geometric sequence, where

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}. \tag{1}$$

If $r = 1$, then

$$S_n = \underbrace{a + a + a + a + \cdots + a}_{n \text{ terms}} = na.$$

If $r \neq 1$, multiply both sides of equation (1) by r to get

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n. \tag{2}$$

Now subtract corresponding sides of equation (1) from equation (2).

$$\begin{aligned} rS_n &= \quad ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\ -S_n &= -(a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}) \\ rS_n - S_n &= -a + ar^n \\ S_n(r-1) &= a(r^n - 1) \qquad \text{Factor.} \\ S_n &= \frac{a(r^n-1)}{r-1} \qquad \text{Divide both sides by } r-1. \end{aligned}$$

This result is summarized below.

Sum of Terms

If a geometric sequence has first term a and common ratio r, then the sum S_n of the first n terms is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1.$$

EXAMPLE 2 Sum of a Geometric Sequence

Find the sum of the first six terms of the geometric sequence $3, 12, 48, \ldots$.

SOLUTION Here $a = 3$, $r = 4$, and $n = 6$. Find S_6 by the formula above.

$$\begin{aligned} S_6 &= \frac{3(4^6 - 1)}{4 - 1} \qquad n = 6, a = 3, r = 4. \\ &= \frac{3(4096 - 1)}{3} \\ &= 4095 \end{aligned}$$

YOUR TURN 1 Find the sum of the first 9 terms of the geometric series $4, 12, 36, \ldots$.

TRY YOUR TURN 1

Ordinary Annuities

A sequence of equal payments made at equal periods of time is called an **annuity**. If the payments are made at the end of the time period, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity**. The time between payments is the **payment period**, and the time from the beginning of the first payment period to the end of the last period is called the **term** of the annuity. The **future value of the annuity**, the final sum on deposit, is defined as the sum of the compound amounts of all the payments, compounded to the end of the term.

Two common uses of annuities are to accumulate funds for some goal or to withdraw funds from an account. For example, an annuity may be used to save money for a large purchase, such as an automobile, a college education, or a down payment on a home. An annuity also may be used to provide monthly payments for retirement. We explore these options in this and the next section.

APPLY IT

For example, suppose $1500 is deposited at the end of each year for the next 6 years in an account paying 8% per year compounded annually. Figure 4 shows this annuity. To find the future value of the annuity, look separately at each of the $1500 payments. The first of these payments will produce a compound amount of

$$1500(1 + 0.08)^5 = 1500(1.08)^5.$$

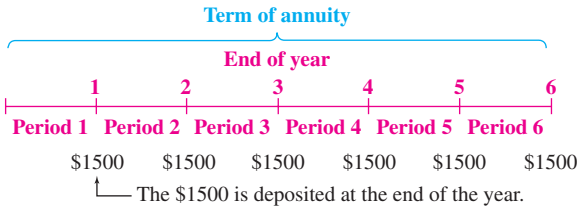


FIGURE 4

Use 5 as the exponent instead of 6, since the money is deposited at the *end* of the first year and earns interest for only 5 years. The second payment of $1500 will produce a compound amount of $1500(1.08)^4$. As shown in Figure 5, the future value of the annuity is

$$1500(1.08)^5 + 1500(1.08)^4 + 1500(1.08)^3 + 1500(1.08)^2 + 1500(1.08)^1 + 1500.$$

(The last payment earns no interest at all.)

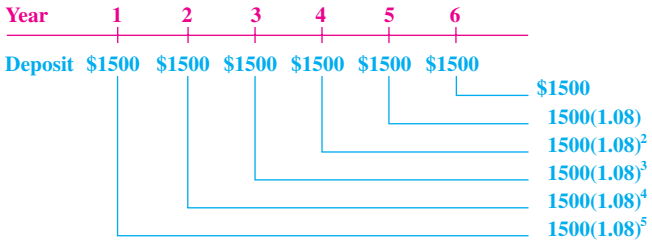


FIGURE 5

Reading this sum in reverse order, we see that it is the sum of the first six terms of a geometric sequence, with $a = 1500$, $r = 1.08$, and $n = 6$. Thus, the sum equals

$$\frac{a(r^n - 1)}{r - 1} = \frac{1500[(1.08)^6 - 1]}{1.08 - 1} \approx \$11{,}003.89.$$

To generalize this result, suppose that payments of R dollars each are deposited into an account at the end of each period for n periods, at a rate of interest i per period. The first payment of R dollars will produce a compound amount of $R(1 + i)^{n-1}$ dollars, the second

payment will produce $R(1 + i)^{n-2}$ dollars, and so on; the final payment earns no interest and contributes just R dollars to the total. If S represents the future value (or sum) of the annuity, then (as shown in Figure 6),

$$S = R(1 + i)^{n-1} + R(1 + i)^{n-2} + R(1 + i)^{n-3} + \cdots + R(1 + i) + R,$$

or, written in reverse order,

$$S = R + R(1 + i)^1 + R(1 + i)^2 + \cdots + R(1 + i)^{n-1}.$$

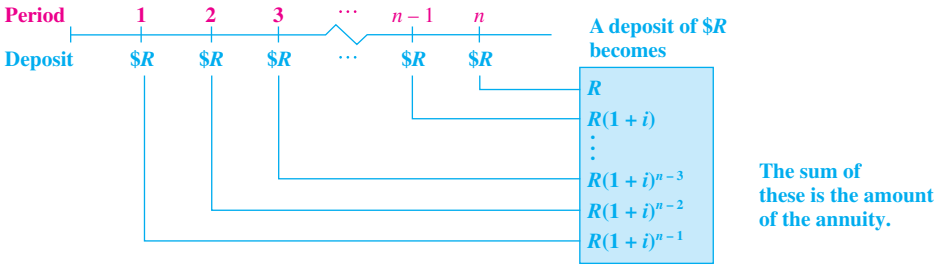


FIGURE 6

This result is the sum of the first n terms of the geometric sequence having first term R and common ratio $1 + i$. Using the formula for the sum of the first n terms of a geometric sequence,

$$S = \frac{R[(1 + i)^n - 1]}{(1 + i) - 1} = \frac{R[(1 + i)^n - 1]}{i} = R\left[\frac{(1 + i)^n - 1}{i}\right].$$

The quantity in brackets is commonly written $s_{\overline{n}|i}$ (read "s-angle-n at i"), so that

$$S = R \cdot s_{\overline{n}|i}.$$

Values of $s_{\overline{n}|i}$ can be found with a calculator.

A formula for the future value of an annuity S of n payments of R dollars each at the end of each consecutive interest period, with interest compounded at a rate i per period, follows.* Recall that this type of annuity, with payments at the *end* of each time period, is called an ordinary annuity.

Future Value of an Ordinary Annuity

$$S = R\left[\frac{(1 + i)^n - 1}{i}\right] \quad \text{or} \quad S = Rs_{\overline{n}|i}$$

where

S is the future value;

R is the periodic payment;

i is the interest rate per period;

n is the number of periods.

TECHNOLOGY NOTE

A calculator will be very helpful in computations with annuities. The TI-84 Plus graphing calculator has a special FINANCE menu that is designed to give any desired result after entering the basic information. If your calculator does not have this feature, many calculators can easily be programmed to evaluate the formulas introduced in this section and the next. We include these programs in the *Graphing Calculator and Excel Spreadsheet Manual* available for this text.

*We use S for the future value here, instead of A as in the compound interest formula, to help avoid confusing the two formulas.

EXAMPLE 3 Ordinary Annuity

Bethany Ward is an athlete who believes that her playing career will last 7 years. To prepare for her future, she deposits $22,000 at the end of each year for 7 years in an account paying 6% compounded annually. How much will she have on deposit after 7 years?

SOLUTION Her payments form an ordinary annuity, with $R = 22{,}000$, $n = 7$, and $i = 0.06$. Using the formula for future value of an annuity,

$$S = 22{,}000\left[\frac{(1.06)^7 - 1}{0.06}\right] \approx 184{,}664.43,$$

or $184,664.43. Note that she made 7 payments of $22,000, or $154,000. The interest that she earned is $184,664.43 − $154,000 = $30,664.43. TRY YOUR TURN 2

YOUR TURN 2 Find the accumulated amount after 11 years if $250 is deposited every month in an account paying 3.3% interest compounded monthly.

Sinking Funds

A fund set up to receive periodic payments as in Example 3 is called a **sinking fund**. The periodic payments, together with the interest earned by the payments, are designed to produce a given sum at some time in the future. For example, a sinking fund might be set up to receive money that will be needed to pay off the principal on a loan at some future time. If the payments are all the same amount and are made at the end of a regular time period, they form an ordinary annuity.

EXAMPLE 4 Sinking Fund

Experts say that the baby boom generation (Americans born between 1946 and 1960) cannot count on a company pension or Social Security to provide a comfortable retirement, as their parents did. It is recommended that they start to save early and regularly. Nancy Hart, a baby boomer, has decided to deposit $200 each month for 20 years in an account that pays interest of 7.2% compounded monthly.

(a) How much will be in the account at the end of 20 years?

SOLUTION This savings plan is an annuity with $R = 200$, $i = 0.072/12$, and $n = 12(20)$. The future value is

$$S = 200\left[\frac{(1 + (0.072/12))^{12(20)} - 1}{0.072/12}\right] \approx 106{,}752.47,$$

or $106,752.47. Figure 7 shows a calculator graph of the function

$$S = 200\left[\frac{(1 + (x/12))^{12(20)} - 1}{x/12}\right]$$

where r, the annual interest rate, is designated x. The value of the function at $x = 0.072$, shown at the bottom of the window, agrees with our result above.

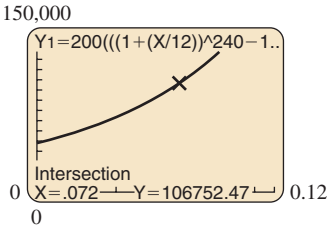


FIGURE 7

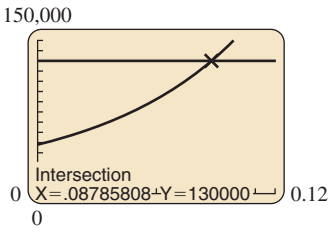


FIGURE 8

(b) Nancy believes she needs to accumulate $130,000 in the 20-year period to have enough for retirement. What interest rate would provide that amount?

SOLUTION

Method 1 Graphing Calculator

One way to answer this question is to solve the equation for S in terms of x with $S = 130{,}000$. This is a difficult equation to solve. Although trial and error could be used, it would be easier to use the graphing calculator graph in Figure 7. Adding the line $y = 130{,}000$ to the graph and then using the capability of the calculator to find the intersection point with the curve shows the annual interest rate must be at least 8.79% to the nearest hundredth. See Figure 8.

Method 2 TVM Solver

Using the TVM Solver under the FINANCE menu on the TI-84 Plus calculator, enter 240 for N (the number of periods), 0 for PV (present value), −200 for PMT (negative because the money is being paid out), 130000 for FV (future value), and 12 for P/Y (payments per year). Put the cursor next to I% (payment) and press SOLVE. The result, shown in Figure 9, indicates that an interest rate of 8.79% is needed.

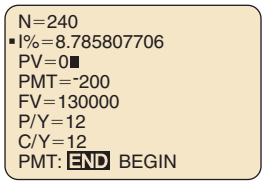


FIGURE 9

In Example 4 we used sinking fund calculations to determine the amount of money that accumulates over time through monthly payments and interest. We can also use this formula to determine the amount of money necessary to periodically invest at a given interest rate to reach a particular goal. Start with the annuity formula

$$S = R\left[\frac{(1+i)^n - 1}{i}\right],$$

and multiply both sides by $i/[(1+i)^n - 1]$ to derive the following formula.

Sinking Fund Payment

$$R = \frac{Si}{(1+i)^n - 1} \quad \text{or} \quad R = \frac{S}{s_{\overline{n}|i}}$$

where

R is the periodic payment;

S is the future value;

i is the interest rate per period;

n is the number of periods.

EXAMPLE 5 Sinking Fund Payment

Suppose Nancy, in Example 4, cannot get the higher interest rate to produce $130,000 in 20 years. To meet that goal, she must increase her monthly payment. What payment should she make each month?

SOLUTION Nancy's goal is to accumulate $130,000 in 20 years at 7.2% compounded monthly. Therefore, the future value is $S = 130{,}000$, the monthly interest rate is $i = 0.072/12$, and the number of periods is $n = 12(20)$. Use the sinking fund payment formula to find the payment R.

$$R = \frac{(130{,}000)(0.072/12)}{(1 + (0.072/12))^{12(20)} - 1} \approx 243.5540887$$

Nancy will need payments of $243.56 each month for 20 years to accumulate at least $130,000. Notice that $243.55 is not quite enough, so round up here. Figure 10(a) shows the point of intersection of the graphs of

$$Y_1 = X\left[\frac{(1 + 0.072/12)^{12(20)} - 1}{0.072/12}\right]$$

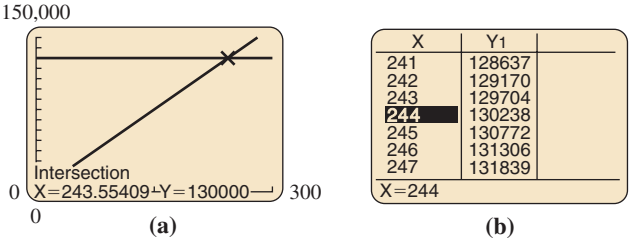


FIGURE 10

YOUR TURN 3 Find the quarterly payment needed to produce $13,500 in 14 years at 3.75% interest compounded quarterly.

and $Y_2 = 130{,}000$. The result agrees with the answer we found analytically. The table shown in Figure 10(b) confirms that the payment should be between \$243 and \$244.

TRY YOUR TURN 3

TECHNOLOGY NOTE

We can also use a graphing calculator or spreadsheet to make a table of the amount in a sinking fund. In the formula for future value of an annuity, simply let n be a variable with values from 1 to the total number of payments. Figure 11(a) shows the beginning of such a table generated on a TI-84 Plus for Example 5. Figure 11(b) shows the beginning of the same table using Microsoft Excel.

X	Y1
1	243.55
2	488.56
3	735.04
4	983
5	1232.5
6	1483.4
7	1735.8

X=1

(a)

n	Amount in Fund
1	243.55
2	488.56
3	735.04
4	983.00
5	1232.45
6	1483.40
7	1735.85
8	1989.81
9	2245.30
10	2502.32
11	2760.89
12	3021.00

(b)

FIGURE 11

Annuities Due

The formula developed above is for *ordinary annuities*—those with payments made at the *end* of each time period. These results can be modified slightly to apply to **annuities due**—annuities in which payments are made at the *beginning* of each time period. To find the future value of an annuity due, treat each payment as if it were made at the *end* of the *preceding* period. That is, find $s_{\overline{n}|i}$ for *one additional period*; to compensate for this, subtract the amount of one payment.

Thus, the **future value of an annuity due** of n payments of R dollars each at the beginning of consecutive interest periods, with interest compounded at the rate of i per period, is

$$S = R\left[\frac{(1+i)^{n+1}-1}{i}\right] - R \quad \text{or} \quad S = Rs_{\overline{n+1}|i} - R.$$

TECHNOLOGY NOTE

The finance feature of the TI-84 Plus can be used to find the future value of an annuity due as well as an ordinary annuity. If this feature is not built in, you may wish to program your calculator to evaluate this formula, too.

EXAMPLE 6 Future Value of an Annuity Due

Find the future value of an annuity due if payments of \$500 are made at the beginning of each quarter for 7 years, in an account paying 6% compounded quarterly.

SOLUTION In 7 years, there are $n = 28$ quarterly periods. Add one period to get $n + 1 = 29$, and use the formula with $i = 0.06/4 = 0.015$.

$$S = 500\left[\frac{(1.015)^{29}-1}{0.015}\right] - 500 \approx 17{,}499.35$$

The account will have a total of \$17,499.35 after 7 years.

TRY YOUR TURN 4

YOUR TURN 4 Find the future value of an annuity due with \$325 made at the beginning of each month for 5 years in an account paying 3.3% compounded monthly.

5.2 EXERCISES

Find the fifth term of each geometric sequence.

1. $a = 3; \quad r = 2$

2. $a = 7; \quad r = 5$

3. $a = -8; \quad r = 3$

4. $a = -6; \quad r = 2$

5. $a = 1; \quad r = -3$

6. $a = 12; \quad r = -2$

7. $a = 256; \quad r = \frac{1}{4}$

8. $a = 729; \quad r = \frac{1}{3}$

Find the sum of the first four terms for each geometric sequence.

9. $a = 1; \quad r = 2$

10. $a = 4; \quad r = 4$

11. $a = 5; \quad r = \frac{1}{5}$

12. $a = 6; \quad r = \frac{1}{2}$

13. $a = 128; \quad r = -\frac{3}{2}$

14. $a = 64; \quad r = -\frac{3}{4}$

15. Explain how a geometric sequence is related to an ordinary annuity.

16. Explain the difference between an ordinary annuity and an annuity due.

Find the future value of each ordinary annuity. Interest is compounded annually.

17. $R = 100; \quad i = 0.06; \quad n = 4$

18. $R = 1000; \quad i = 0.06; \quad n = 5$

19. $R = 25{,}000; \quad i = 0.045; \quad n = 36$

20. $R = 29{,}500; \quad i = 0.058; \quad n = 15$

Find the future value of each ordinary annuity, if payments are made and interest is compounded as given. Then determine how much of this value is from contributions and how much is from interest.

21. $R = 9200$; 10% interest compounded semiannually for 7 years

22. $R = 1250$; 5% interest compounded semiannually for 18 years

23. $R = 800$; 6.51% interest compounded semiannually for 12 years

24. $R = 4600$; 8.73% interest compounded quarterly for 9 years

25. $R = 12{,}000$; 4.8% interest compounded quarterly for 16 years

26. $R = 42{,}000$; 10.05% interest compounded semiannually for 12 years

27. What is meant by a sinking fund? Give an example of a sinking fund.

28. List some reasons for establishing a sinking fund.

Determine the interest rate needed to accumulate the following amounts in a sinking fund, with monthly payments as given.

29. Accumulate $56,000, monthly payments of $300 over 12 years

30. Accumulate $120,000, monthly payments of $500 over 15 years

Find the periodic payment that will amount to each given sum under the given conditions.

31. $S = \$10{,}000$; interest is 5% compounded annually; payments are made at the end of each year for 12 years.

32. $S = \$150{,}000$; interest is 6% compounded semiannually; payments are made at the end of each semiannual period for 11 years.

Find the amount of each payment to be made into a sinking fund so that enough will be present to accumulate the following amounts. Payments are made at the end of each period.

33. $8500; money earns 8% compounded annually; there are 7 annual payments.

34. $2750; money earns 5% compounded annually; there are 5 annual payments.

35. $75,000; money earns 6% compounded semiannually for $4\frac{1}{2}$ years.

36. $25,000; money earns 5.7% compounded quarterly for $3\frac{1}{2}$ years.

37. $65,000; money earns 7.5% compounded quarterly for $2\frac{1}{2}$ years.

38. $9000; money earns 4.8% compounded monthly for $2\frac{1}{2}$ years.

Find the future value of each annuity due. Assume that interest is compounded annually.

39. $R = 600; \quad i = 0.06; \quad n = 8$

40. $R = 1700; \quad i = 0.04; \quad n = 15$

41. $R = 16{,}000; \quad i = 0.05; \quad n = 7$

42. $R = 4000; \quad i = 0.06; \quad n = 11$

Find the future value of each annuity due. Then determine how much of this value is from contributions and how much is from interest.

43. Payments of $1000 made at the beginning of each semiannual period for 9 years at 8.15% compounded semiannually

44. $750 deposited at the beginning of each month for 15 years at 5.9% compounded monthly

45. $250 deposited at the beginning of each quarter for 12 years at 4.2% compounded quarterly

46. $1500 deposited at the beginning of each semiannual period for 11 years at 5.6% compounded semiannually

APPLICATIONS

Business and Economics

47. Comparing Accounts Laurie Campbell deposits $12,000 at the end of each year for 9 years in an account paying 8% interest compounded annually.

a. Find the final amount she will have on deposit.

b. Laurie's brother-in-law works in a bank that pays 6% compounded annually. If she deposits money in this bank instead of the one above, how much will she have in her account?

c. How much would Laurie lose over 9 years by using her brother-in-law's bank?

48. Savings Matthew Pastier is saving for a Plasma HDTV. At the end of each month he puts $100 in a savings account that pays 2.25% interest compounded monthly. How much is in the account after 2 years? How much did Matthew deposit? How much interest did he earn?

49. Savings A typical pack-a-day smoker spends about $136.50 per month on cigarettes. Suppose the smoker invests that amount each month in a savings account at 4.8% interest compounded monthly. What would the account be worth after 40 years? ***Source: MSN.***

50. Retirement Planning A 45-year-old man puts $2500 in a retirement account at the end of each quarter until he reaches the age of 60, then makes no further deposits. If the account pays 6% interest compounded quarterly, how much will be in the account when the man retires at age 65?

51. Retirement Planning At the end of each quarter, a 50-year-old woman puts $3000 in a retirement account that pays 5% interest compounded quarterly. When she reaches 60, she withdraws the entire amount and places it in a mutual fund that pays 6.9% interest compounded monthly. From then on she deposits $300 in the mutual fund at the end of each month. How much is in the account when she reaches age 65?

Individual Retirement Accounts **Suppose a 40-year-old person deposits $4000 per year in an Individual Retirement Account until age 65. Find the total in the account with the following assumptions of interest rates. (Assume quarterly compounding, with payments of $1000 made at the end of each quarter period.) Find the total amount of interest earned.**

52. 6% **53.** 8% **54.** 4% **55.** 10%

56. Savings Greg Tobin needs $10,000 in 8 years.

a. What amount should he deposit at the end of each quarter at 8% compounded quarterly so that he will have his $10,000?

b. Find Greg's quarterly deposit if the money is deposited at 6% compounded quarterly.

57. Buying Equipment Harv, the owner of Harv's Meats, knows that he must buy a new deboner machine in 4 years. The machine costs $12,000. In order to accumulate enough money to pay for the machine, Harv decides to deposit a sum of money at the end of each 6 months in an account paying 6% compounded semiannually. How much should each payment be?

58. Buying a Car Amanda Perdaris wants to have a $20,000 down payment when she buys a new car in 6 years. How much money must she deposit at the end of each quarter in an account paying 3.2% compounded quarterly so that she will have the down payment she desires?

59. Savings Stacy Schrank is paid on the first day of the month and $80 is automatically deducted from her pay and deposited in a savings account. If the account pays 2.5% interest compounded monthly, how much will be in the account after 3 years and 9 months?

60. Savings A father opened a savings account for his daughter on the day she was born, depositing $1000. Each year on her birthday he deposits another $1000, making the last deposit on her 21st birthday. If the account pays 5.25% interest compounded annually, how much is in the account at the end of the day on his daughter's 21st birthday? How much interest has been earned?

61. Savings Beth Dahlke deposits $2435 at the beginning of each semiannual period for 8 years in an account paying 6% compounded semiannually. She then leaves that money alone, with no further deposits, for an additional 5 years. Find the final amount on deposit after the entire 13-year period.

62. Savings David Kurzawa deposits $10,000 at the beginning of each year for 12 years in an account paying 5% compounded annually. He then puts the total amount on deposit in another account paying 6% compounded semiannually for another 9 years. Find the final amount on deposit after the entire 21-year period.

In Exercises 63 and 64, use a graphing calculator to find the value of *i* that produces the given value of *S*. (See Example 4(b).)

63. Retirement To save for retirement, Karla Harby put $300 each month into an ordinary annuity for 20 years. Interest was compounded monthly. At the end of the 20 years, the annuity was worth $147,126. What annual interest rate did she receive?

64. Rate of Return Caroline DiTullio made payments of $250 per month at the end of each month to purchase a piece of property. At the end of 30 years, she completely owned the property, which she sold for $330,000. What annual interest rate would she need to earn on an annuity for a comparable rate of return?

65. Lottery In a 1992 Virginia lottery, the jackpot was $27 million. An Australian investment firm tried to buy all possible combinations of numbers, which would have cost $7 million. In fact, the firm ran out of time and was unable to buy all combinations but ended up with the only winning ticket anyway. The firm received the jackpot in 20 equal annual payments of $1.35 million. Assume these payments meet the conditions of an ordinary annuity. ***Source: The Washington Post.***

a. Suppose the firm can invest money at 8% interest compounded annually. How many years would it take until the investors would be further ahead than if they had simply invested the $7 million at the same rate? (*Hint:* Experiment with different values of n, the number of years, or use a graphing calculator to plot the value of both investments as a function of the number of years.)

b. How many years would it take in part a at an interest rate of 12%?

66. Buying Real Estate Vicki Kakounis sells some land in Nevada. She will be paid a lump sum of $60,000 in 7 years. Until then, the buyer pays 8% simple interest quarterly.

a. Find the amount of each quarterly interest payment on the $60,000.

b. The buyer sets up a sinking fund so that enough money will be present to pay off the $60,000. The buyer will make semiannual payments into the sinking fund; the account pays 6% compounded semiannually. Find the amount of each payment into the fund.

67. Buying Rare Stamps Phil Weaver bought a rare stamp for his collection. He agreed to pay a lump sum of $4000 after 5 years. Until then, he pays 6% simple interest semiannually on the $4000.

a. Find the amount of each semiannual interest payment.

b. Paul sets up a sinking fund so that enough money will be present to pay off the $4000. He will make annual payments into the fund. The account pays 8% compounded annually. Find the amount of each payment.

68. Down Payment A conventional loan, such as for a car or a house, is similar to an annuity but usually includes a down payment. Show that if a down payment of D dollars is made at the beginning of the loan period, the future value of all the payments, including the down payment, is

$$S = D(1 + i)^n + R\left[\frac{(1 + i)^n - 1}{i}\right].$$

YOUR TURN ANSWERS

1. 39,364 **2.** $39,719.98
3. $184.41 **4.** $21,227.66

5.3 Present Value of an Annuity; Amortization

APPLY IT

What monthly payment will pay off a $17,000 car loan in 36 monthly payments at 6% annual interest?

The answer to this question is given in Example 2 in this section. We shall see that it involves finding the present value of an annuity.

Suppose that at the end of each year, for the next 10 years, $500 is deposited in a savings account paying 7% interest compounded annually. This is an example of an ordinary annuity. The **present value of an annuity** is the amount that would have to be deposited in one lump sum today (at the same compound interest rate) in order to produce exactly the same balance at the end of 10 years. We can find a formula for the present value of an annuity as follows.

Suppose deposits of R dollars are made at the end of each period for n periods at interest rate i per period. Then the amount in the account after n periods is the future value of this annuity:

$$S = R \cdot s_{\overline{n}|i} = R\left[\frac{(1 + i)^n - 1}{i}\right].$$

On the other hand, if P dollars are deposited today at the same compound interest rate i, then at the end of n periods, the amount in the account is $P(1 + i)^n$. If P is the present value of the annuity, this amount must be the same as the amount S in the formula above; that is,

$$P(1 + i)^n = R\left[\frac{(1 + i)^n - 1}{i}\right].$$

FOR REVIEW

Recall from Section R.6 that for any nonzero number a, $a^0 = 1$. Also, by the product rule for exponents, $a^x \cdot a^y = a^{x+y}$. In particular, if a is any nonzero number, $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$.

To solve this equation for P, multiply both sides by $(1 + i)^{-n}$.

$$P = R(1 + i)^{-n}\left[\frac{(1 + i)^n - 1}{i}\right]$$

Use the distributive property; also recall that $(1 + i)^{-n}(1 + i)^n = 1$.

$$P = R\left[\frac{(1 + i)^{-n}(1 + i)^n - (1 + i)^{-n}}{i}\right] = R\left[\frac{1 - (1 + i)^{-n}}{i}\right]$$

The amount P is the *present value of the annuity*. The quantity in brackets is abbreviated as $a_{\overline{n}|i}$, so

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i}.$$

(The symbol $a_{\overline{n}|i}$ is read "a-angle-n at i." Compare this quantity with $s_{\overline{n}|i}$ in the previous section.) The formula for the present value of an annuity is summarized on the next page.

Present Value of an Ordinary Annuity

The present value P of an annuity of n payments of R dollars each at the end of consecutive interest periods with interest compounded at a rate of interest i per period is

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right] \quad \text{or} \quad P = Ra_{\overline{n}|i}.$$

CAUTION Don't confuse the formula for the present value of an annuity with the one for the future value of an annuity. Notice the difference: the numerator of the fraction in the present value formula is $1-(1+i)^{-n}$, but in the future value formula, it is $(1+i)^n - 1$.

TECHNOLOGY NOTE The financial feature of the TI-84 Plus calculator can be used to find the present value of an annuity by choosing that option from the menu and entering the required information. If your calculator does not have this built-in feature, it will be useful to store a program to calculate present value of an annuity in your calculator. A program is given in the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

EXAMPLE 1 Present Value of an Annuity

John Cross and Wendy Mears are both graduates of the Brisbane Institute of Technology (BIT). They both agree to contribute to the endowment fund of BIT. John says that he will give $500 at the end of each year for 9 years. Wendy prefers to give a lump sum today. What lump sum can she give that will equal the present value of John's annual gifts, if the endowment fund earns 7.5% compounded annually?

SOLUTION Here, $R = 500$, $n = 9$, and $i = 0.075$, and we have

$$P = R \cdot a_{\overline{9}|0.075} = 500\left[\frac{1-(1.075)^{-9}}{0.075}\right] \approx 3189.44.$$

Therefore, Wendy must donate a lump sum of $3189.44 today. TRY YOUR TURN 1

YOUR TURN 1 Find the present value of an annuity of $120 at the end of each month put into an account yielding 4.8% compounded monthly for 5 years.

One of the most important uses of annuities is in determining the equal monthly payments needed to pay off a loan, as illustrated in the next example.

EXAMPLE 2 Car Payments

A car costs $19,000. After a down payment of $2000, the balance will be paid off in 36 equal monthly payments with interest of 6% per year on the unpaid balance. Find the amount of each payment.

APPLY IT

SOLUTION A single lump sum payment of $17,000 today would pay off the loan. So, $17,000 is the present value of an annuity of 36 monthly payments with interest of $6\%/12 = 0.5\%$ per month. Thus, $P = 17{,}000$, $n = 36$, $i = 0.005$, and we must find the monthly payment R in the formula

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$17{,}000 = R\left[\frac{1-(1.005)^{-36}}{0.005}\right]$$

$$R \approx 517.17.$$

A monthly payment of $517.17 will be needed. TRY YOUR TURN 2

YOUR TURN 2 Find the car payment in Example 2 if there are 48 equal monthly payments and the interest rate is 5.4%.

Each payment in Example 2 includes interest on the unpaid balance, with the remainder going to reduce the loan. For example, the first payment of \$517.17 includes interest of $0.005(\$17{,}000) = \$85$ and is divided as follows.

monthly payment	interest due	to reduce the balance
\$517.17	− \$85	= \$432.17

At the end of this section, amortization schedules show that this procedure does reduce the loan to \$0 after all payments are made (the final payment may be slightly different).

Amortization

A loan is **amortized** if both the principal and interest are paid by a sequence of equal periodic payments. In Example 2, a loan of \$17,000 at 6% interest compounded monthly could be amortized by paying \$517.17 per month for 36 months.

The periodic payment needed to amortize a loan may be found, as in Example 2, by solving the present value equation for R.

Amortization Payments

A loan of P dollars at interest rate i per period may be amortized in n equal periodic payments of R dollars made at the end of each period, where

$$R = \frac{P}{\left[\dfrac{1-(1+i)^{-n}}{i}\right]} = \frac{Pi}{1-(1+i)^{-n}} \quad \text{or} \quad R = \frac{P}{a_{\overline{n}|i}}.$$

EXAMPLE 3 Home Mortgage

The Perez family buys a house for \$275,000, with a down payment of \$55,000. They take out a 30-year mortgage for \$220,000 at an annual interest rate of 6%.

(a) Find the amount of the monthly payment needed to amortize this loan.

SOLUTION Here $P = 220{,}000$ and the monthly interest rate is $i = 0.06/12 = 0.005$.* The number of monthly payments is $12(30) = 360$. Therefore,

$$R = \frac{220{,}000}{a_{\overline{360}|0.005}} = \frac{220{,}000}{\left[\dfrac{1-(1.005)^{-360}}{0.005}\right]} = 1319.01.$$

Monthly payments of \$1319.01 are required to amortize the loan.

(b) Find the total amount of interest paid when the loan is amortized over 30 years.

SOLUTION The Perez family makes 360 payments of \$1319.01 each, for a total of \$474,843.60. Since the amount of the loan was \$220,000, the total interest paid is

$$\$474{,}843.60 - \$220{,}000 = \$254{,}843.60.$$

This large amount of interest is typical of what happens with a long mortgage. A 15-year mortgage would have higher payments but would involve significantly less interest.

(c) Find the part of the first payment that is interest and the part that is applied to reducing the debt.

SOLUTION During the first month, the entire \$220,000 is owed. Interest on this amount for 1 month is found by the formula for simple interest, with r = annual interest rate and t = time in years.

$$I = Prt = 220{,}000(0.06)\frac{1}{12} = \$1100$$

*Mortgage rates are quoted in terms of annual interest, but it is always understood that the monthly rate is 1/12 of the annual rate and that interest is compounded monthly.

YOUR TURN 3 Find the monthly payment and total amount of interest paid in Example 3 if the mortgage is for 15 years and the interest rate is 7%.

At the end of the month, a payment of \$1319.01 is made; since \$1100 of this is interest, a total of

$$\$1319.01 - \$1100 = \$219.01$$

is applied to the reduction of the original debt. **TRY YOUR TURN 3**

It can be shown that the unpaid balance after x payments is given by the function

$$y = R\left[\frac{1 - (1 + i)^{-(n-x)}}{i}\right],$$

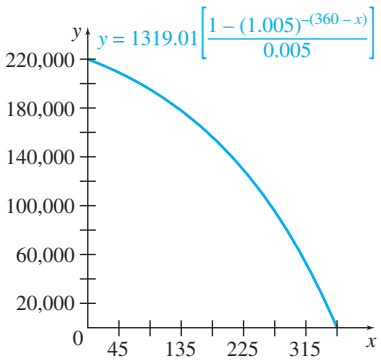


FIGURE 12

although this formula will only give an approximation if R is rounded to the nearest penny. For example, the unrounded value of R in Example 3 is 1319.011155. When this value is put into the above formula, the unpaid balance is found to be

$$y = 1319.011155\left[\frac{1 - (1.005)^{-359}}{0.005}\right] = 219{,}780.99,$$

while rounding R to 1319.01 in the above formula gives an approximate balance of \$219,780.80. A graph of this function is shown in Figure 12.

We can find the unpaid balance after any number of payments, x, by finding the y-value that corresponds to x. For example, the remaining balance after 5 years or 60 payments is shown at the bottom of the graphing calculator screen in Figure 13(a). You may be surprised that the remaining balance on a \$220,000 loan is as large as \$204,719.41. This is because most of the early payments on a loan go toward interest, as we saw in Example 3(c).

By adding the graph of $y = (1/2)220{,}000 = 110{,}000$ to the figure, we can find when half the loan has been repaid. From Figure 13(b) we see that 252 payments are required. Note that only 108 payments remain at that point, which again emphasizes the fact that the earlier payments do little to reduce the loan.

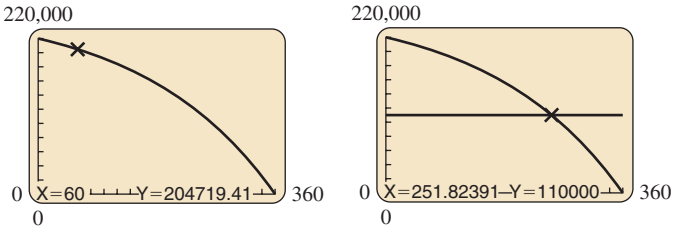


FIGURE 13

Amortization Schedules

In the preceding example, 360 payments are made to amortize a \$220,000 loan. The loan balance after the first payment is reduced by only \$219.01, which is much less than $(1/360)(220{,}000) \approx \$611.11$. Therefore, even though equal *payments* are made to amortize a loan, the loan *balance* does not decrease in equal steps. This fact is very important if a loan is paid off early.

EXAMPLE 4 Early Payment

Ami Aigen borrows \$1000 for 1 year at 12% annual interest compounded monthly. Verify that her monthly loan payment is \$88.8488, which is rounded to \$88.85. After making three payments, she decides to pay off the remaining balance all at once. How much must she pay?

SOLUTION Since nine payments remain to be paid, they can be thought of as an annuity consisting of nine payments of \$88.85 at 1% interest per period. The present value of this annuity is

$$88.8488\left[\frac{1 - (1.01)^{-9}}{0.01}\right] \approx 761.08.$$

So Ami's remaining balance, computed by this method, is \$761.08.

An alternative method of figuring the balance is to consider the payments already made as an annuity of three payments. At the beginning, the present value of this annuity was

$$88.85\left[\frac{1-(1.01)^{-3}}{0.01}\right] \approx 261.31.$$

So she still owes the difference $1000 − $261.31 = $738.69. Furthermore, she owes the interest on this amount for 3 months, for a total of

$$(738.69)(1.01)^3 \approx \$761.07.$$

This balance due differs from the one obtained by the first method by 1 cent because the monthly payment and the other calculations were rounded to the nearest penny. If we had used the more accurate value of $R = 88.8488$ and not rounded any intermediate answers, both methods would have given the same value of $761.08. TRY YOUR TURN 4

YOUR TURN 4 Find the remaining balance in Example 4 if the balance was to be paid off after four months. Use the unrounded value for R of $88.8488.

Although most people would not quibble about a difference of 1 cent in the balance due in Example 4, the difference in other cases (larger amounts or longer terms) might be more than that. A bank or business must keep its books accurately to the nearest penny, so it must determine the balance due in such cases unambiguously and exactly. This is done by means of an **amortization schedule**, which lists how much of each payment is interest and how much goes to reduce the balance, as well as how much is owed after *each* payment.

EXAMPLE 5 Amortization Table

Determine the exact amount Ami Aigen in Example 4 owes after three monthly payments.

SOLUTION An amortization table for the loan is shown below. It is obtained as follows. The annual interest rate is 12% compounded monthly, so the interest rate per month is $12\%/12 = 1\% = 0.01$. When the first payment is made, 1 month's interest—namely $0.01(1000) = \$10$—is owed. Subtracting this from the $88.85 payment leaves $78.85 to be applied to repayment. Hence, the principal at the end of the first payment period is $1000 − $78.85 = $921.15, as shown in the "payment 1" line of the chart.

When payment 2 is made, 1 month's interest on $921.15 is owed, namely $0.01(921.15) = \$9.21$. Subtracting this from the $88.85 payment leaves $79.64 to reduce the principal. Hence, the principal at the end of payment 2 is $921.15 − $79.64 = $841.51. The interest portion of payment 3 is based on this amount, and the remaining lines of the table are found in a similar fashion.

The schedule shows that after three payments, she still owes $761.08, an amount that agrees with the first method in Example 4.

Amortization Table

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	$1000.00
1	$88.85	$10.00	$78.85	$921.15
2	$88.85	$9.21	$79.64	$841.51
3	$88.85	$8.42	$80.43	$761.08
4	$88.85	$7.61	$81.24	$679.84
5	$88.85	$6.80	$82.05	$597.79
6	$88.85	$5.98	$82.87	$514.92
7	$88.85	$5.15	$83.70	$431.22
8	$88.85	$4.31	$84.54	$346.68
9	$88.85	$3.47	$85.38	$261.30
10	$88.85	$2.61	$86.24	$175.06
11	$88.85	$1.75	$87.10	$87.96
12	$88.84	$0.88	$87.96	$0.00

The amortization schedule in Example 5 is typical. In particular, note that all payments are the same except the last one. It is often necessary to adjust the amount of the final payment to account for rounding off earlier and to ensure that the final balance is exactly 0.

An amortization schedule also shows how the periodic payments are applied to interest and principal. The amount going to interest decreases with each payment, while the amount going to reduce the principal increases with each payment.

TECHNOLOGY NOTE

A graphing calculator program to produce an amortization schedule is available in the *Graphing Calculator and Excel Spreadsheet Manual* available with this book. The TI-84 Plus includes a built-in program to find the amortization payment. Spreadsheets are another useful tool for creating amortization tables. Microsoft Excel has a built-in feature for calculating monthly payments. Figure 14 shows an Excel amortization table for Example 5. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual*, also available with this book.

	A	B	C	D	E	F
1	Pmt#	Payment	Interest	Principal	End Prncpl	
2	0				1000	
3	1	88.85	10.00	78.85	921.15	
4	2	88.85	9.21	79.64	841.51	
5	3	88.85	8.42	80.43	761.08	
6	4	88.85	7.61	81.24	679.84	
7	5	88.85	6.80	82.05	597.79	
8	6	88.85	5.98	82.87	514.92	
9	7	88.85	5.15	83.70	431.22	
10	8	88.85	4.31	84.54	346.68	
11	9	88.85	3.47	85.38	261.30	
12	10	88.85	2.61	86.24	175.06	
13	11	88.85	1.75	87.10	87.96	
14	12	88.85	0.88	87.97	-0.01	

FIGURE 14

EXAMPLE 6 Paying Off a Loan Early

Suppose that in Example 2, the car owner decides that she can afford to make payments of $700 rather than $517.17. How much earlier would she pay off the loan? How much interest would she save?

SOLUTION Putting $R = 700$, $P = 17{,}000$, and $i = 0.005$ into the formula for the present value of an annuity gives

$$17{,}000 = 700\left[\frac{1 - 1.005^{-n}}{0.005}\right].$$

Multiply both sides by 0.005 and divide by 700 to get

$$\frac{85}{700} = 1 - 1.005^{-n},$$

or

$$1.005^{-n} = 1 - \frac{85}{700}.$$

Solve this using either logarithms or a graphing calculator, as in Example 11 in Section 5.1, to get $n = 25.956$. This means that 25 payments of $700, plus a final, smaller payment, would be sufficient to pay off the loan. Create an amortization table to verify that the final payment would be $669.47 (the sum of the principal after the penultimate payment plus the interest on that principal for the final month). The loan would then be paid off after 26 months, or 10 months early.

The original loan required 36 payments of $517.17, or $36(517.17) = \$18{,}618.12$, although the amount is actually $18,618.24 because the final payment was $517.29, as an amortization table would show. With larger payments, the car owner paid $25(700) + 669.47 = \$18{,}169.47$. Therefore, the car owner saved $18,618.24 − $18,169.47 = $448.77 in interest by making larger payments each month.

5.3 EXERCISES

1. Explain the difference between the present value of an annuity and the future value of an annuity. For a given annuity, which is larger? Why?
2. What does it mean to amortize a loan?

Find the present value of each ordinary annuity.

3. Payments of $890 each year for 16 years at 6% compounded annually
4. Payments of $1400 each year for 8 years at 6% compounded annually
5. Payments of $10,000 semiannually for 15 years at 5% compounded semiannually
6. Payments of $50,000 quarterly for 10 years at 4% compounded quarterly
7. Payments of $15,806 quarterly for 3 years at 6.8% compounded quarterly
8. Payments of $18,579 every 6 months for 8 years at 5.4% compounded semiannually

Find the lump sum deposited today that will yield the same total amount as payments of $10,000 at the end of each year for 15 years at each of the given interest rates.

9. 4% compounded annually
10. 6% compounded annually

Find (a) the payment necessary to amortize each loan; (b) the total payments and the total amount of interest paid based on the calculated monthly payments, and (c) the total payments and total amount of interest paid based upon an amortization table.

11. $2500; 6% compounded quarterly; 6 quarterly payments
12. $41,000; 8% compounded semiannually; 10 semiannual payments
13. $90,000; 6% compounded annually; 12 annual payments
14. $140,000; 8% compounded quarterly; 15 quarterly payments
15. $7400; 6.2% compounded semiannually; 18 semiannual payments
16. $5500; 10% compounded monthly; 24 monthly payments

Suppose that in the loans described in Exercises 13–16, the borrower paid off the loan after the time indicated below. Calculate the amount needed to pay off the loan, using either of the two methods described in Example 4.

17. After 3 years in Exercise 13
18. After 5 quarters in Exercise 14
19. After 3 years in Exercise 15
20. After 7 months in Exercise 16

Use the amortization table in Example 5 to answer the questions in Exercises 21–24.

21. How much of the fourth payment is interest?
22. How much of the eleventh payment is used to reduce the debt?
23. How much interest is paid in the first 4 months of the loan?
24. How much interest is paid in the last 4 months of the loan?
25. What sum deposited today at 5% compounded annually for 8 years will provide the same amount as $1000 deposited at the end of each year for 8 years at 6% compounded annually?
26. What lump sum deposited today at 8% compounded quarterly for 10 years will yield the same final amount as deposits of $4000 at the end of each 6-month period for 10 years at 6% compounded semiannually?

Find the monthly house payments necessary to amortize each loan. Then calculate the total payments and the total amount of interest paid.

27. $199,000 at 7.01% for 25 years
28. $175,000 at 6.24% for 30 years
29. $253,000 at 6.45% for 30 years
30. $310,000 at 5.96% for 25 years

Suppose that in the loans described in Exercises 13–16, the borrower made a larger payment, as indicated below. Calculate (a) the time needed to pay off the loan, (b) the total amount of the payments, and (c) the amount of interest saved, compared with part c of Exercises 13–16.

31. $16,000 in Exercise 13
32. $18,000 in Exercise 14
33. $850 in Exercise 15
34. $400 in Exercise 16

APPLICATIONS

Business and Economics

35. **House Payments** Calculate the monthly payment and total amount of interest paid in Example 3 with a 15-year loan, and then compare with the results of Example 3.
36. **Installment Buying** Stereo Shack sells a stereo system for $600 down and monthly payments of $30 for the next 3 years. If the interest rate is 1.25% per month on the unpaid balance, find

    a. the cost of the stereo system.

    b. the total amount of interest paid.
37. **Car Payments** Hong Le buys a car costing $14,000. He agrees to make payments at the end of each monthly period for 4 years. He pays 7% interest, compounded monthly.

    a. What is the amount of each payment?

    b. Find the total amount of interest Le will pay.
38. **Credit Card Debt** Tom Shaffer charged $8430 on his credit card to relocate for his first job. When he realized that the interest rate for the unpaid balance was 27% compounded monthly, he decided not to charge any more on that account. He wants to have this account paid off by the end of 3 years,

so he arranges to have automatic payments sent at the end of each month.

a. What monthly payment must he make to have the account paid off by the end of 3 years?

b. How much total interest will he have paid?

39. New Car In Spring 2010, some dealers offered a cash-back allowance of $2250 or 0.9% financing for 36 months on an Acura TL. ***Source: cars.com.***

a. Determine the payments on an Acura TL if a buyer chooses the 0.9% financing option and needs to finance $30,000 for 36 months, compounded monthly. Find the total amount the buyer will pay for this option.

b. Determine the payments on an Acura TL if a buyer chooses the cash-back option and now needs to finance only $27,750. At the time, it was possible to get a new car loan at 6.33% for 48 months, compounded monthly. Find the total amount the buyer will pay for this option.

c. Discuss which deal is best and why.

40. New Car In Spring 2010, some dealers offered a cash-back allowance of $1500 or 1.9% financing for 36 months on a Volkswagen Tiguan. ***Source: cars.com.***

a. Determine the payments on a Volkswagen Tiguan if a buyer chooses the 1.9% financing option and needs to finance $25,000 for 36 months, compounded monthly. Find the total amount the buyer will pay for this option.

b. Determine the payments on a Volkswagen Tiguan if a buyer chooses the cash-back option and now needs to finance only $23,500. At the time, it was possible to get a new car loan at 6.33% for 48 months, compounded monthly. Find the total amount the buyer will pay for this option.

c. Discuss which deal is best and why.

41. Lottery Winnings In most states, the winnings of million-dollar lottery jackpots are divided into equal payments given annually for 20 years. (In Colorado, the results are distributed over 25 years.) This means that the present value of the jackpot is worth less than the stated prize, with the actual value determined by the interest rate at which the money could be invested. ***Source: The New York Times Magazine.***

a. Find the present value of a $1 million lottery jackpot distributed in equal annual payments over 20 years, using an interest rate of 5%.

b. Find the present value of a $1 million lottery jackpot distributed in equal annual payments over 20 years, using an interest rate of 9%.

c. Calculate the answer for part a using the 25-year distribution time in Colorado.

d. Calculate the answer for part b using the 25-year distribution time in Colorado.

Student Loans **Student borrowers now have more options to choose from when selecting repayment plans. The standard plan repays the loan in up to 10 years with equal monthly payments. The extended plan allows up to 25 years to repay the loan. *Source: U.S. Department of Education.* A student borrows $55,000 at 6.80% compounded monthly.**

42. Find the monthly payment and total interest paid under the standard plan over 10 years.

43. Find the monthly payment and total interest paid under the extended plan over 25 years.

Installment Buying **In Exercises 44–46, prepare an amortization schedule showing the first four payments for each loan.**

44. An insurance firm pays $4000 for a new printer for its computer. It amortizes the loan for the printer in 4 annual payments at 8% compounded annually.

45. Large semitrailer trucks cost $110,000 each. Ace Trucking buys such a truck and agrees to pay for it by a loan that will be amortized with 9 semiannual payments at 8% compounded semiannually.

46. One retailer charges $1048 for a laptop computer. A firm of tax accountants buys 8 of these laptops. They make a down payment of $1200 and agree to amortize the balance with monthly payments at 6% compounded monthly for 4 years.

47. Investment In 1995, Oseola McCarty donated $150,000 to the University of Southern Mississippi to establish a scholarship fund. What is unusual about her is that the entire amount came from what she was able to save each month from her work as a washer woman, a job she began in 1916 at the age of 8, when she dropped out of school. ***Sources: The New York Times.***

a. How much would Ms. McCarty have to put into her savings account at the end of every 3 months to accumulate $150,000 over 79 years? Assume she received an interest rate of 5.25% compounded quarterly.

b. Answer part a using a 2% and a 7% interest rate.

48. Loan Payments When Nancy Hart opened her law office, she bought $14,000 worth of law books and $7200 worth of office furniture. She paid $1200 down and agreed to amortize the balance with semiannual payments for 5 years, at 8% compounded semiannually.

a. Find the amount of each payment.

b. Refer to the text and Figure 13. When her loan had been reduced below $5000, Nancy received a large tax refund and decided to pay off the loan. How many payments were left at this time?

49. House Payments Ian Desrosiers buys a house for $285,000. He pays $60,000 down and takes out a mortgage at 6.5% on the balance. Find his monthly payment and the total amount of interest he will pay if the length of the mortgage is

a. 15 years;

b. 20 years;

c. 25 years.

d. Refer to the text and Figure 13. When will half the 20-year loan in part b be paid off?

50. House Payments The Chavara family buys a house for $225,000. They pay $50,000 down and take out a 30-year mortgage on the balance. Find their monthly payment and the total amount of interest they will pay if the interest rate is

a. 6%;

b. 6.5%;

c. 7%.

d. Refer to the text and Figure 13. When will half the 7% loan in part c be paid off?

51. Refinancing a Mortgage Fifteen years ago, the Budai family bought a home and financed $150,000 with a 30-year mortgage at 8.2%.

a. Find their monthly payment, the total amount of their payments, and the total amount of interest they will pay over the life of this loan.

b. The Budais made payments for 15 years. Estimate the unpaid balance using the formula

$$y = R\left[\frac{1 - (1 + i)^{-(n-x)}}{i}\right],$$

and then calculate the total of their remaining payments.

c. Suppose interest rates have dropped since the Budai family took out their original loan. One local bank now offers a 30-year mortgage at 6.5%. The bank fees for refinancing are $3400. If the Budais pay this fee up front and refinance the balance of their loan, find their monthly payment. Including the refinancing fee, what is the total amount of their payments? Discuss whether or not the family should refinance with this option.

d. A different bank offers the same 6.5% rate but on a 15-year mortgage. Their fee for financing is $4500. If the Budais pay this fee up front and refinance the balance of their loan, find their monthly payment. Including the refinancing fee, what is the total amount of their payments? Discuss whether or not the family should refinance with this option.

52. Inheritance Deborah Harden has inherited $25,000 from her grandfather's estate. She deposits the money in an account offering 6% interest compounded annually. She wants to make equal annual withdrawals from the account so that the money (principal and interest) lasts exactly 8 years.

a. Find the amount of each withdrawal.

b. Find the amount of each withdrawal if the money must last 12 years.

53. Charitable Trust The trustees of a college have accepted a gift of $150,000. The donor has directed the trustees to deposit the money in an account paying 6% per year, compounded semiannually. The trustees may make equal withdrawals at the end of each 6-month period; the money must last 5 years.

a. Find the amount of each withdrawal.

b. Find the amount of each withdrawal if the money must last 6 years.

Amortization **Prepare an amortization schedule for each loan.**

54. A loan of $37,948 with interest at 6.5% compounded annually, to be paid with equal annual payments over 10 years.

55. A loan of $4836 at 7.25% interest compounded semi-annually, to be repaid in 5 years in equal semiannual payments.

56. Perpetuity A *perpetuity* is an annuity in which the payments go on forever. We can derive a formula for the present value of a perpetuity by taking the formula for the present value of an annuity and looking at what happens when n gets larger and larger. Explain why the present value of a perpetuity is given by

$$P = \frac{R}{i}.$$

57. Perpetuity Using the result of Exercise 56, find the present value of perpetuities for each of the following.

a. Payments of $1000 a year with 4% interest compounded annually

b. Payments of $600 every 3 months with 6% interest compounded quarterly

YOUR TURN ANSWERS

1. $6389.86 **2.** $394.59

3. $1977.42, $135,935.60 **4.** $679.84

5 CHAPTER REVIEW

SUMMARY

In this chapter we introduced the mathematics of finance. We first extended simple interest calculations to compound interest, which is interest earned on interest previously earned. We then developed the mathematics associated with the following financial concepts.

- In an annuity, money continues to be deposited at regular intervals, and compound interest is earned on that money as well.
- In an ordinary annuity, payments are made at the end of each time period, and the compounding period is the same as the time between payments, which simplifies the calculations.
- An annuity due is slightly different, in that the payments are made at the beginning of each time period.
- A sinking fund is like an ordinary annuity; a fund is set up to receive periodic payments. The payments plus the compound interest will produce a desired sum by a certain date.
- The present value of an annuity is the amount that would have to be deposited today to produce the same amount as the annuity at the end of a specified time.
- An amortization table shows how a loan is paid back after a specified time. It shows the payments broken down into interest and principal.

We have presented a lot of new formulas in this chapter. By answering the following questions, you can decide which formula to use for a particular problem.

1. Is simple or compound interest involved?

   Simple interest is normally used for investments or loans of a year or less; compound interest is normally used in all other cases.

2. If simple interest is being used, what is being sought: interest amount, future value, present value, or interest rate?

3. If compound interest is being used, does it involve a lump sum (single payment) or an annuity (sequence of payments)?

   a. For a lump sum, what is being sought: present value, future value, number of periods at interest, or effective rate?

   b. For an annuity,

   i. Is it an ordinary annuity (payment at the end of each period) or an annuity due (payment at the beginning of each period)?

   ii. What is being sought: present value, future value, or payment amount?

Once you have answered these questions, choose the appropriate formula and work the problem. As a final step, consider whether the answer you get makes sense. For instance, present value should always be less than future value. The amount of interest or the payments in an annuity should be fairly small compared to the total future value.

List of Variables

r is the annual interest rate.

i is the interest rate per period.

t is the number of years.

n is the number of periods.

m is the number of periods per year.

P is the principal or present value.

A is the future value of a lump sum.

S is the future value of an annuity.

R is the periodic payment in an annuity.

$$i = \frac{r}{m} \qquad n = tm$$

	Simple Interest	Compound Interest	Continuous Compounding
Interest	$I = Prt$	$I = A - P$	$I = A - P$
Future Value	$A = P(1 + rt)$	$A = P(1 + i)^n$	$A = Pe^{rt}$
Present Value	$P = \dfrac{A}{1 + rt}$	$P = \dfrac{A}{(1 + i)^n} = A(1 + i)^{-n}$	$P = Ae^{-rt}$
Effective Rate		$r_E = \left(1 + \dfrac{r}{m}\right)^m - 1$	$r_E = e^{r-1}$

Ordinary Annuity Future Value $S = R\left[\dfrac{(1 + i)^n - 1}{i}\right] = R \cdot s_{\overline{n}|i}$

Present Value $P = R\left[\dfrac{1 - (1 + i)^{-n}}{i}\right] = R \cdot a_{\overline{n}|i}$

Annuity Due Future Value $S = R\left[\dfrac{(1 + i)^{n+1} - 1}{i}\right] - R$

Sinking Fund Payment $R = \dfrac{Si}{(1 + i)^n - 1} = \dfrac{S}{s_{\overline{n}|i}}$

Amortization Payments $R = \dfrac{Pi}{1 - (1 + i)^{-n}} = \dfrac{P}{a_{\overline{n}|i}}$

KEY TERMS

5.1
simple interest
principal
rate
time
future value
maturity value
compound interest
compound amount
nominal (stated) rate
effective rate
present value
rule of 70
rule of 72
continuous compounding

5.2
geometric sequence
terms
common ratio
annuity
ordinary annuity
payment period
future value of an annuity
term of an annuity
future value of an ordinary annuity
sinking fund
annuity due
future value of an annuity due

5.3
present value of an annuity
amortize a loan
amortization schedule

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. For a particular interest rate, compound interest is always better than simple interest.

2. The sequence 1, 2, 4, 6, 8, . . . is a geometric sequence.

3. If a geometric sequence has first term 3 and common ratio 2, then the sum of the first 5 terms is $S_5 = 93$.

4. The value of a sinking fund should decrease over time.

5. For payments made on a mortgage, the (noninterest) portion of the payment applied on the principal increases over time.

6. On a 30-year conventional home mortgage, at recent interest rates, it is common to pay more money on the interest on the loan than the actual loan itself.

7. One can use the amortization payments formula to calculate the monthly payment of a car loan.

8. The effective rate formula can be used to calculate the present value of a loan.

9. The following calculation gives the monthly payment on a $25,000 loan, compounded monthly at a rate of 5% for a period of six years:

$$25{,}000\left[\frac{(1 + 0.05/12)^{72} - 1}{0.05/12}\right].$$

10. The following calculation gives the present value of an annuity of $5,000 payments at the end of each year for 10 years. The fund earns 4.5% compounded annually.

$$5000\left[\frac{1 - (1.045)^{-10}}{0.045}\right]$$

PRACTICE AND EXPLORATION

Find the simple interest for each loan.

11. $15,903 at 6% for 8 months

12. $4902 at 5.4% for 11 months

13. $42,368 at 5.22% for 7 months

14. $3478 at 6.8% for 88 days (assume a 360-day year)

15. For a given amount of money at a given interest rate for a given time period, does simple interest or compound interest produce more interest?

Find the compound amount in each loan.

16. $2800 at 7% compounded annually for 10 years

17. $19,456.11 at 8% compounded semiannually for 7 years

18. $312.45 at 5.6% compounded semiannually for 16 years

19. $57,809.34 at 6% compounded quarterly for 5 years

Find the amount of interest earned by each deposit.

20. $3954 at 8% compounded annually for 10 years

21. $12,699.36 at 5% compounded semiannually for 7 years

22. $12,903.45 at 6.4% compounded quarterly for 29 quarters

23. $34,677.23 at 4.8% compounded monthly for 32 months

24. What is meant by the present value of an amount A?

Find the present value of each amount.

25. $42,000 in 7 years, 6% compounded monthly

26. $17,650 in 4 years, 4% compounded quarterly

27. $1347.89 in 3.5 years, 6.77% compounded semiannually

28. $2388.90 in 44 months, 5.93% compounded monthly

29. Write the first five terms of the geometric sequence with $a = 2$ and $r = 3$.

30. Write the first four terms of the geometric sequence with $a = 4$ and $r = 1/2$.

31. Find the sixth term of the geometric sequence with $a = -3$ and $r = 2$.

32. Find the fifth term of the geometric sequence with $a = -2$ and $r = -2$.

33. Find the sum of the first four terms of the geometric sequence with $a = -3$ and $r = 3$.

34. Find the sum of the first five terms of the geometric sequence with $a = 8000$ and $r = -1/2$.

35. Find $s_{\overline{30}|0.02}$.

36. Find $s_{\overline{20}|0.06}$.

37. What is meant by the future value of an annuity?

Find the future value of each annuity and the amount of interest earned.

38. $500 deposited at the end of each 6-month period for 10 years; money earns 6% compounded semiannually.

39. $1288 deposited at the end of each year for 14 years; money earns 4% compounded annually.

40. $4000 deposited at the end of each quarter for 7 years; money earns 5% compounded quarterly.

41. $233 deposited at the end of each month for 4 years; money earns 4.8% compounded monthly.

42. $672 deposited at the beginning of each quarter for 7 years; money earns 4.4% compounded quarterly.

43. $11,900 deposited at the beginning of each month for 13 months; money earns 6% compounded monthly.

44. What is the purpose of a sinking fund?

Find the amount of each payment that must be made into a sinking fund to accumulate each amount.

45. $6500; money earns 5% compounded annually for 6 years.

46. $57,000; money earns 4% compounded semiannually for $8\frac{1}{2}$ years.

47. $233,188; money earns 5.2% compounded quarterly for $7\frac{3}{4}$ years.

48. $1,056,788; money earns 7.2% compounded monthly for $4\frac{1}{2}$ years.

Find the present value of each ordinary annuity.

49. Deposits of $850 annually for 4 years at 6% compounded annually

50. Deposits of $1500 quarterly for 7 years at 5% compounded quarterly

51. Payments of $4210 semiannually for 8 years at 4.2% compounded semiannually

52. Payments of $877.34 monthly for 17 months at 6.4% compounded monthly

53. Give two examples of the types of loans that are commonly amortized.

Find the amount of the payment necessary to amortize each loan. Calculate the total interest paid.

54. $80,000; 5% compounded annually; 9 annual payments

55. $3200; 8% compounded quarterly; 12 quarterly payments

56. $32,000; 6.4% compounded quarterly; 17 quarterly payments

57. $51,607; 8% compounded monthly; 32 monthly payments

Find the monthly house payments for each mortgage. Calculate the total payments and interest.

58. $256,890 at 5.96% for 25 years

59. $177,110 at 6.68% for 30 years

A portion of an amortization table is given below for a $127,000 loan at 8.5% interest compounded monthly for 25 years.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
1	$1022.64	$899.58	$123.06	$126,876.94
2	$1022.64	$898.71	$123.93	$126,753.01
3	$1022.64	$897.83	$124.81	$126,628.20
4	$1022.64	$896.95	$125.69	$126,502.51
5	$1022.64	$896.06	$126.58	$126,375.93
6	$1022.64	$895.16	$127.48	$126,248.45
7	$1022.64	$894.26	$128.38	$126,120.07
8	$1022.64	$893.35	$129.29	$125,990.78
9	$1022.64	$892.43	$130.21	$125,860.57
10	$1022.64	$891.51	$131.13	$125,729.44
11	$1022.64	$890.58	$132.06	$125,597.38
12	$1022.64	$889.65	$132.99	$125,464.39

Use the table to answer the following questions.

60. How much of the fifth payment is interest?

61. How much of the twelfth payment is used to reduce the debt?

62. How much interest is paid in the first 3 months of the loan?

63. How much has the debt been reduced at the end of the first year?

APPLICATIONS

Business and Economics

64. Personal Finance Jane Fleming owes $5800 to her mother. She has agreed to repay the money in 10 months at an interest rate of 5.3%. How much will she owe in 10 months? How much interest will she pay?

65. Business Financing Julie Ward needs to borrow $9820 to buy new equipment for her business. The bank charges her 6.7% simple interest for a 7-month loan. How much interest will she be charged? What amount must she pay in 7 months?

66. Business Financing An accountant loans $28,000 at simple interest to her business. The loan is at 6.5% and earns $1365 interest. Find the time of the loan in months.

67. Business Investment A developer deposits $84,720 for 7 months and earns $4055.46 in simple interest. Find the interest rate.

68. Personal Finance In 3 years Beth Rechsteiner must pay a pledge of $7500 to her college's building fund. What lump sum can she deposit today, at 5% compounded semiannually, so that she will have enough to pay the pledge?

69. Personal Finance Tom, a graduate student, is considering investing $500 now, when he is 23, or waiting until he is 40 to invest $500. How much more money will he have at the age of 65 if he invests now, given that he can earn 5% interest compounded quarterly?

70. Pensions Pension experts recommend that you start drawing at least 40% of your full pension as early as possible. Suppose you have built up a pension of $12,000-annual payments by working 10 years for a company. When you leave to accept a better job, the company gives you the option of collecting half of the full pension when you reach age 55 or the full pension at age 65. Assume an interest rate of 8% compounded annually. By age 75, how much will each plan produce? Which plan would produce the larger amount? ***Source: Smart Money.***

71. Business Investment A firm of attorneys deposits $5000 of profit-sharing money at the end of each semiannual period for $7\frac{1}{2}$ years. Find the final amount in the account if the deposits earn 10% compounded semiannually. Find the amount of interest earned.

72. Business Financing A small resort must add a swimming pool to compete with a new resort built nearby. The pool will cost $28,000. The resort borrows the money and agrees to repay it with equal payments at the end of each quarter for $6\frac{1}{2}$ years at an interest rate of 8% compounded quarterly. Find the amount of each payment.

73. Business Financing The owner of Eastside Hallmark borrows $48,000 to expand the business. The money will be repaid in equal payments at the end of each year for 7 years. Interest is 6.5%. Find the amount of each payment and the total amount of interest paid.

74. Personal Finance To buy a new computer, Mark Nguyen borrows $3250 from a friend at 4.2% interest compounded annually for 4 years. Find the compound amount he must pay back at the end of the 4 years.

75. Effective Rate On May 21, 2010, Ascencia (a division of PBI Bank) paid 1.49% interest, compounded monthly, on a 1-year CD, while giantbank.com paid 1.45% compounded daily. What are the effective rates for the two CDs, and which bank paid a higher effective rate? ***Source: Bankrate.com.***

76. Home Financing When the Lee family bought their home, they borrowed $315,700 at 7.5% compounded monthly for 25 years. If they make all 300 payments, repaying the loan on schedule, how much interest will they pay? (Assume the last payment is the same as the previous ones.)

77. New Car In Spring 2010, some dealers offered the following options on a Chevrolet HHR: a cash-back allowance of $4000, 0% financing for 60 months, or 3.9% financing for 72 months. ***Source: cars.com.***

a. Determine the payments on a Chevrolet HHR if a buyer chooses the 0% financing option and needs to finance $16,000 for 60 months. Find the total amount of the payments.

b. Repeat part a for the 3.9% financing option for 72 months.

c. Determine the payments on a Chevrolet HHR if a buyer chooses the cash-back option and now needs to finance only $12,000. At the time, it was possible to get a new car loan at 6.33% for 48 months, compounded monthly. Find the total amount of the payments.

d. Discuss which deal is best and why.

78. New Car In Spring 2010, some dealers offered a cash-back allowance of $5000 or 0% financing for 72 months on a Chevrolet Silverado. ***Source: cars.com.***

a. Determine the payments on a Chevrolet Silverado if a buyer chooses the 0% financing option and needs to finance $30,000 for 72 months. Find the total amount of the payments.

b. Determine the payments on a Chevrolet Silverado if a buyer chooses the cash-back option and now needs to finance only $25,000. At the time, it was possible to get a new car loan at 6.33% for 48 months, compounded monthly. Find the total amount of the payments.

c. Discuss which deal is best and why.

d. Find the interest rate at the bank that would make the total amount of payments for the two options equal.

79. Buying and Selling a House The Bahary family bought a house for $191,000. They paid $40,000 down and took out a 30-year mortgage for the balance at 6.5%.

a. Find their monthly payment.

b. How much of the first payment is interest?

After 180 payments, the family sells its house for $238,000. They must pay closing costs of $3700 plus 2.5% of the sale price.

c. Estimate the current mortgage balance at the time of the sale using one of the methods from Example 4 in Section 3.

d. Find the total closing costs.

e. Find the amount of money they receive from the sale after paying off the mortgage.

The following exercise is from an actuarial examination. *Source: The Society of Actuaries.*

80. Death Benefit The proceeds of a $10,000 death benefit are left on deposit with an insurance company for 7 years at an annual effective interest rate of 5%. The balance at the end of 7 years is paid to the beneficiary in 120 equal monthly payments of X, with the first payment made immediately. During the payout period, interest is credited at an annual effective interest rate of 3%. Calculate X. Choose one of the following.

a. 117 **b.** 118 **c.** 129 **d.** 135 **e.** 158

81. Investment *The New York Times* posed a scenario with two individuals, Sue and Joe, who each have $1200 a month to spend on housing and investing. Each takes out a mortgage for $140,000. Sue gets a 30-year mortgage at a rate of 6.625%. Joe gets a 15-year mortgage at a rate of 6.25%. Whatever money is left after the mortgage payment is invested in a mutual fund with a return of 10% annually. ***Source: The New york Times.***

a. What annual interest rate, when compounded monthly, gives an effective annual rate of 10%?

b. What is Sue's monthly payment?

c. If Sue invests the remainder of her $1200 each month, after the payment in part b, in a mutual fund with the interest rate in part a, how much money will she have in the fund at the end of 30 years?

d. What is Joe's monthly payment?

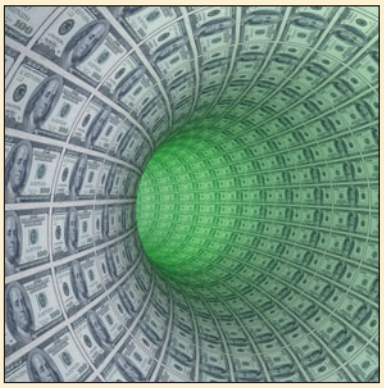
e. You found in part d that Joe has nothing left to invest until his mortgage is paid off. If he then invests the entire $1200 monthly in a mutual fund with the interest rate in part a, how much money will he have at the end of 30 years (that is, after 15 years of paying the mortgage and 15 years of investing)?

f. Who is ahead at the end of the 30 years and by how much?

g. Discuss to what extent the difference found in part f is due to the different interest rates or to the different amounts of time.

EXTENDED APPLICATION

TIME, MONEY, AND POLYNOMIALS



A *time line* is often helpful for evaluating complex investments. For example, suppose you buy a $1000 CD at time t_0. After one year $2500 is added to the CD at t_1. By time t_2, after another year, your money has grown to $3851 with interest. What rate of interest, called *yield to maturity* (YTM), did your money earn? A time line for this situation is shown in Figure 15.

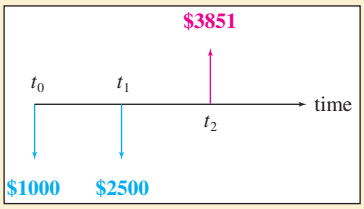


FIGURE 15

Assuming interest is compounded annually at a rate i, and using the compound interest formula, gives the following description of the YTM.

$$1000(1 + i)^2 + 2500(1 + i) = 3851$$

Source: COMAP.

To determine the yield to maturity, we must solve this equation for i. Since the quantity $1 + i$ is repeated, let $x = 1 + i$ and first solve the second-degree (quadratic) polynomial equation for x.

$$1000x^2 + 2500x - 3851 = 0$$

We can use the quadratic formula with $a = 1000$, $b = 2500$, and $c = -3851$.

$$x = \frac{-2500 \pm \sqrt{2500^2 - 4(1000)(-3851)}}{2(1000)}$$

We get $x = 1.0767$ and $x = -3.5767$. Since $x = 1 + i$, the two values for i are $0.0767 = 7.67\%$ and $-4.5767 = -457.67\%$. We reject the negative value because the final accumulation is greater than the sum of the deposits. In some applications, however, negative rates may be meaningful. By checking in the first equation, we see that the yield to maturity for the CD is 7.67%.

Now let us consider a more complex but realistic problem. Suppose Austin Caperton has contributed for 4 years to a retirement fund. He contributed $6000 at the beginning of the first year. At the beginning of the next 3 years, he contributed $5840, $4000, and $5200, respectively. At the end of the fourth year, he had $29,912.38 in his fund. The interest rate earned by the fund varied between 21% and -3%, so Caperton would like to know the YTM $= i$ for his hard-earned retirement dollars. From a time line (see Figure 16), we set up the following equation in $1 + i$ for Caperton's savings program.

$$6000(1 + i)^4 + 5840(1 + i)^3 + 4000(1 + i)^2 + 5200(1 + i) = 29{,}912.38$$

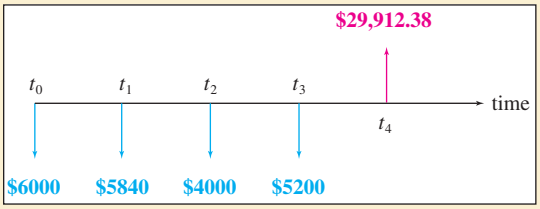


FIGURE 16

Let $x = 1 + i$. We need to solve the fourth-degree polynomial equation

$$f(x) = 6000x^4 + 5840x^3 + 4000x^2 + 5200x - 29{,}912.38 = 0.$$

There is no simple way to solve a fourth-degree polynomial equation, so we will use a graphing calculator.

We expect that $0 < i < 1$, so that $1 < x < 2$. Let us calculate $f(1)$ and $f(2)$. If there is a change of sign, we will know that there is a solution to $f(x) = 0$ between 1 and 2. We find that

$$f(1) = -8872.38 \quad \text{and} \quad f(2) = 139{,}207.62.$$

Using a graphing calculator, we find that there is one positive solution to this equation, $x = 1.14$, so $i = \text{YTM} = 0.14 = 14\%$.

EXERCISES

1. Lorri Morgan received \$50 on her 16th birthday, and \$70 on her 17th birthday, both of which she immediately invested in the bank, with interest compounded annually. On her 18th birthday, she had \$127.40 in her account. Draw a time line, set up a polynomial equation, and calculate the YTM.
2. At the beginning of the year, Blake Allvine invested \$10,000 at 5% for the first year. At the beginning of the second year, he added \$12,000 to the account. The total account earned 4.5% for the second year.
   - **a.** Draw a time line for this investment.
   - **b.** How much was in the fund at the end of the second year?
   - **c.** Set up and solve a polynomial equation and determine the YTM. What do you notice about the YTM?
3. On January 2 each year for 3 years, Michael Bailey deposited bonuses of \$1025, \$2200, and \$1850, respectively, in an account. He received no bonus the following year, so he made no deposit. At the end of the fourth year, there was \$5864.17 in the account.
   - **a.** Draw a time line for these investments.
   - **b.** Write a polynomial equation in x $(x = 1 + i)$ and use a graphing calculator to find the YTM for these investments.
   - **c.** Go to the website WolframAlpha.com, and ask it to solve the polynomial from part b. Compare this method of solving the equation with using a graphing calculator.
4. Don Beville invested yearly in a fund for his children's college education. At the beginning of the first year, he invested \$1000; at the beginning of the second year, \$2000; at the third through the sixth, \$2500 each year, and at the beginning of the seventh, he invested \$5000. At the beginning of the eighth year, there was \$21,259 in the fund.
   - **a.** Draw a time line for this investment program.
   - **b.** Write a seventh-degree polynomial equation in $1 + i$ that gives the YTM for this investment program.
   - **c.** Use a graphing calculator to show that the YTM is less than 5.07% and greater than 5.05%.
   - **d.** Use a graphing calculator to calculate the solution for $1 + i$ and find the YTM.
   - **e.** Go to the website WolframAlpha.com, and ask it to solve the polynomial from part b. Compare this method of solving the equation with using a graphing calculator.
5. People often lose money on investments. Melissa Fischer invested \$50 at the beginning of each of 2 years in a mutual fund, and at the end of 2 years her investment was worth \$90.
   - **a.** Draw a time line and set up a polynomial equation in $1 + i$. Solve for i.
   - **b.** Examine each negative solution (rate of return on the investment) to see if it has a reasonable interpretation in the context of the problem. To do this, use the compound interest formula on each value of i to trace each \$50 payment to maturity.

DIRECTIONS FOR GROUP PROJECT

Assume that you are in charge of a group of financial analysts and that you have been asked by the broker at your firm to develop a time line for each of the people listed in the exercises above. Prepare a report for each client that presents the YTM for each investment strategy. Make sure that you describe the methods used to determine the YTM in a manner that the average client should understand.

6 Logic

6.1 Statements
6.2 Truth Tables and Equivalent Statements
6.3 The Conditional and Circuits
6.4 More on the Conditional
6.5 Analyzing Arguments and Proofs
6.6 Analyzing Arguments with Quantifiers
Chapter 6 Review
Extended Application: Logic Puzzles

The rules of a game often include complex conditional statements, such as "if you roll doubles, you can roll again, but if you roll doubles twice in a row, you lose a turn." As exercises in this chapter illustrate, logical analysis of complex statements helps us clarify not only the rules of games but any precise use of language, from legal codes to medical diagnoses.

In 1943, Thomas Watson, head of IBM, made the now-infamous prediction, "I think there is a world market for maybe five computers." In 1977, Ken Olson, founder of Digital Equipment Corp., prophesied, "There is no reason anyone would want a computer in their home." ***Source: Microsoft Corp.*** Perhaps such predictions were so wrong because of the difficulty, in the early days of computing, of foreseeing later uses of the computer, such as for communication, shopping, entertainment, or information retrieval. It's rather amazing that such modern conveniences are made possible by a machine made of components based on mathematical logic. Even when you are using your computer for some seemingly nonmathematical activity, components, known as gates or switches, that have been developed from concepts that you will learn about in this chapter, are making it all possible.

6.1 Statements

APPLY IT **How can a complex statement be analyzed to determine whether it is a truthful statement?**

You will be asked to answer this question in Exercise 89.

Symbolic logic uses formal mathematics with symbols to represent statements and arguments in everyday language. Logic can help us determine whether a complex statement is true, as well as determine whether a conclusion necessarily follows from a set of assumptions.

Many kinds of sentences occur in ordinary language, including factual statements, opinions, commands, and questions. Symbolic logic discusses statements and opinions that may be true or false, but not commands or questions.

Statements

A **statement** is a declarative sentence that is either true or false, but not both simultaneously. For example, both of the following are statements:

Mount McKinley is the tallest mountain in North America.

$$15 - 9 = 10.$$

The first sentence is true, while the second is false. None of the following sentences are considered statements in logic, however, because they cannot be identified as being either true or false:

Tie your shoes.

Are we having fun yet?

This sentence is false.

The first sentence is a command or suggestion, while the second is a question. The third sentence is a paradox: If the sentence is true, then the sentence itself tells us that it must be false. But if what it says is false, then the sentence must be true. We avoid such paradoxes by disallowing statements that refer to themselves.

When one or more simple statements are combined with **logical connectives** such as *and*, *or*, *not*, and *if . . . then*, the result is called a **compound statement**, while the simple statements that make up the compound statement are called **component statements**.

EXAMPLE 1 Compound Statements

Decide whether each statement is compound.

(a) George Washington was the first U.S. president, and John Adams was his vice president.

SOLUTION This statement is compound using the connective *and.* The component statements are "George Washington was the first U.S. president" and "John Adams was his vice president."

(b) If what you've told me is true, then we are in great peril.

SOLUTION This statement is also compound. The component statements "what you've told me is true" and "we are in great peril" are linked with the connective *if . . . then.*

(c) We drove across New Mexico toward the town with the curious name Truth or Consequences.

SOLUTION This statement is not compound. Even though *or* is a connective, here it is part of the name of the city. It is not connecting two statements.

(d) The money is not there.

SOLUTION Most logicians consider this statement to be compound, even though it has just one component statement, and we will do so in this book. The connective *not* is applied to the component statement "The money is there." TRY YOUR TURN 1

YOUR TURN 1 Is the following statement compound? "I bought Ben and Jerry's ice cream."

We will study the *and*, *or*, and *not* connectives in detail in this section, and return to the *if . . . then* connective in the third section of this chapter.

Negation The **negation** of the statement "I play the guitar" is "I do not play the guitar." There are equivalent ways to say the same thing, such as "It is not true that I play the guitar." The negation of a true statement is false, and the negation of a false statement is true.

EXAMPLE 2 Negation

Give the negation of each statement.

(a) California is the most populous state in the country.

SOLUTION Form the negation using the word *not*: "California is not the most populous state in the country."

(b) It is not raining today.

SOLUTION If a statement already has the word *not*, we can remove it to form the negation: "It is raining today." TRY YOUR TURN 2

YOUR TURN 2 Write the negation of the following statement. "Wal-Mart is not the largest corporation in the USA."

EXAMPLE 3 Negation

Write the negation of each inequality.

(a) $x > 11$

SOLUTION The negation of "x is greater than 11" is "x is *not* greater than 11," or $x \le 11$.

(b) $4x + 9y \le 36$

SOLUTION The negation is $4x + 9y > 36$. TRY YOUR TURN 3

YOUR TURN 3 Write the negation of the following inequality. $4x + 2y < 5$

Symbols To simplify work with logic, symbols are used. Statements are represented with letters, such as p, q, or r, while several symbols for connectives are shown in the following table. The table also gives the type of compound statement having the given connective.

Logic Symbols		
Connective	**Symbol**	**Type of Statement**
and	$\wedge$	Conjunction
or	$\vee$	Disjunction
not	$\sim$	Negation

The symbol $\sim$ represents the connective *not*. If p represents the statement "Barack Obama was president in 2010" then $\sim p$ represents "Barack Obama was not president in 2010." The statement $\sim p$ could also be translated as "It is not true that Barack Obama was president in 2010." There is usually more than one way to express a negation, and so your answer may not always agree exactly with ours. We recommend avoiding convoluted wording.

In applications, choosing meaningful letters will help you remember what the letter represents. While p may be perfectly good for representing a generic statement, a statement such as "Django is a good dog" might be better represented by the letter d.

EXAMPLE 4 Symbolic Statements

Let h represent "My backpack is heavy," and r represent "It's going to rain." Write each symbolic statement in words.

(a) $h \wedge r$

SOLUTION From the table, $\wedge$ represents *and*, so the statement represents

My backpack is heavy and it's going to rain.

(b) $\sim h \vee r$

SOLUTION The *not* applies only to the first symbol, not the entire expression:

My backpack is not heavy or it's going to rain.

(c) $\sim (h \vee r)$

SOLUTION Because of the parentheses, the *not* applies to the entire expression:

It is not the case that either my backpack is heavy or it's going to rain.

(d) $\sim (h \wedge r)$

SOLUTION Again, the *not* applies to the entire expression:

It is not the case that both my backpack is heavy and it's going to rain.

TRY YOUR TURN 4

YOUR TURN 4 Write the symbolic statement in words. $h \wedge \sim r$.

The statement in Example 4(c) is usually translated, "Neither h nor r," as in "Neither is my backpack heavy nor is it going to rain."

The negation of the negation of a statement is simply the statement itself. For example, the negation of the statement in Example 4(d) is $h \wedge r$, or "My backpack is heavy and it's going to rain." Symbolically, $\sim(\sim p)$ is equivalent to p.

We can represent the fact that the negation of a true statement is false and the negation of a false statement is true using a table known as a **truth table**, which shows all possible combinations of truth values for the component statements, as well as the corresponding truth value for the compound statement under consideration. Here is the truth table for negation.

Truth Table for the Negation *not p*

p	$\sim p$
T	F
F	T

Conjunction

The word logicians use for "p *and* q," denoted $p \wedge q$, is **conjunction**. In everyday language, *and* conveys the idea that both component statements are true. For example, the statement

My birthday is in April and yours is in May

would be true only if my birthday is indeed in April and yours is in May. If either part were false, the statement would be considered false. We represent this definition of $p \wedge q$ symbolically using the following truth table for conjunction. Notice that there are four possible combinations of truth values for p and q.

Truth Table for the Conjunction *p and q*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Although the order of the rows in a truth table is arbitrary, you should follow the order that we use, which is not only standard but also organized and easy to remember. A different method can make it hard to compare answers, and with a disorganized method one could easily miss a case or even have duplicate cases. Notice that in the columns labeled with a single variable (p or q), the rightmost column (the q column) alternates T, F, T, F. The next column to the left (the p column) has two T's followed by two F's. When we introduce truth tables with three variables, the leftmost column will have four T's followed by four F's.

EXAMPLE 5 Truth Value

Let p represent "$5 > 3$" and let q represent "$6 < 0$." Find the truth value of $p \wedge q$.

SOLUTION

Here p is true and q is false. Looking in the second row of the conjunction truth table shows that $p \wedge q$ is false. TRY YOUR TURN 5

YOUR TURN 5 Let p represent "$7 < 2$" and q represent "$4 > 3$." Find the truth value of $\sim p \wedge q$.

TECHNOLOGY NOTE

Some calculators have logic functions that allow the user to test the truth or falsity of statements involving $=$, $\neq$, $>$, $\geq$, $<$, and $\leq$. On the TI-84 Plus calculator, functions from the TEST menu, illustrated in Figure 1, return a 1 when a statement is true and a 0 when a statement is false. Figure 2 shows the input of $4 > 9$ and the corresponding output of 0, indicating that the statement is false.

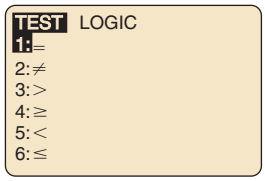


FIGURE 1

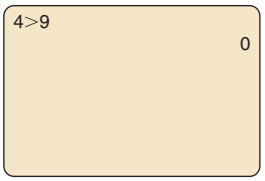


FIGURE 2

Example 5 can also be completed with the help of a TI-84 Plus calculator. Using the LOGIC menu, illustrated in Figure 3, we input $5 > 3$ and $6 < 0$ into the calculator. The output is zero, shown in Figure 4, which means the statement is false.

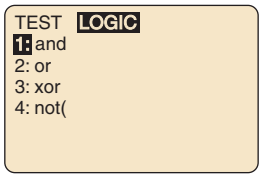


FIGURE 3

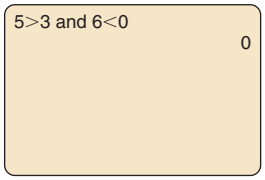


FIGURE 4

There's another word that has the same logical meaning as the word *and*, namely *but*. For example, the statement

I was not ready yesterday, but I am ready today

conveys the same logical meaning as "I was not ready yesterday, and I am ready today," with the additional idea of contrast between my status on the two days. Contrast may be relevant in normal conversation, but it makes no difference in logic.

Disjunction

In ordinary language, the word *or* can be ambiguous. For example, the statement

Those with a passport or driver's license will be admitted

means that anyone will be admitted who has a passport or a driver's license or both. On the other hand, the statement

You can have a piece of cake or a piece of fruit

probably means you cannot have both.

In logic, the word *or* has the first meaning, known as *inclusive disjunction* or just **disjunction**. It is written with the symbol $\vee$, so that $p \vee q$ means "p or q or both." It is only false when both component statements are false. The truth table for disjunction is given below.

Truth Table for the Disjunction *p* or *q*

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

NOTE
In English, the *or* is often interpreted as exclusive disjunction, not the way it is used in logic. For more on the exclusive disjunction, see Exercises 35–38 in the next section.

Compound Statements

The next two examples show how we can find the truth value of compound statements.

EXAMPLE 6 Truth Value of a Compound Statement

Suppose p is false, q is true, and r is false. What is the truth value of the compound statement $\sim p \wedge (q \vee \sim r)$?

SOLUTION Here parentheses are used to group q and $\sim r$ together. Work first inside the parentheses. Since r is false, $\sim r$ will be true. Since $\sim r$ is true and q is true and an *or* statement is true when either component is true, $q \vee \sim r$ must be true. An *and* statement is only

true when both components are true. Since $\sim p$ is true and $q \vee \sim r$ is true, the statement $\sim p \wedge (q \vee \sim r)$ is true.

The preceding paragraph may be interpreted using a short-cut symbolic method, letting T represent a true statement and F represent a false statement:

$$\begin{array}{ll} \sim p \wedge (q \vee \sim r) & \\ \sim \mathrm{F} \wedge (\mathrm{T} \vee \sim \mathrm{F}) & p \text{ is false, } q \text{ is true, } r \text{ is false.} \\ \mathrm{T} \wedge (\mathrm{T} \vee \mathrm{T}) & \sim \mathrm{F} \text{ gives T.} \\ \mathrm{T} \wedge \mathrm{T} & \mathrm{T} \vee \mathrm{T} \text{ gives T.} \\ \mathrm{T}. & \mathrm{T} \wedge \mathrm{T} \text{ gives T.} \end{array}$$

The T in the final row indicates that the compound statement is true. TRY YOUR TURN 6

YOUR TURN 6 If p is false, q is true, and r is false, find the truth value of the statement: $(\sim p \wedge q) \vee r$.

EXAMPLE 7 Mathematical Statements

Let p represent the statement "$3 > 2$," q represent "$5 < 4$," and r represent "$3 < 8$." Decide whether the following statements are *true* or *false*.

(a) $\sim p \wedge \sim q$

SOLUTION Since p is true, $\sim p$ is false. By the *and* truth table, if one part of an "and" statement is false, the entire statement is false. This makes $\sim p \wedge \sim q$ false.

(b) $\sim(p \wedge q)$

SOLUTION First, work within the parentheses. Since p is true and q is false, $p \wedge q$ is false by the *and* truth table. Next, apply the negation. The negation of a false statement is true, making $\sim(p \wedge q)$ a true statement.

(c) $(\sim p \wedge r) \vee (\sim q \wedge \sim p)$

SOLUTION Here p is true, q is false, and r is true. This makes $\sim p$ false and $\sim q$ true. By the *and* truth table, the statement $\sim p \wedge r$ is false, and the statement $\sim q \wedge \sim p$ is also false. Finally,

$$\begin{array}{ccccc} (\sim p \wedge r) & \vee & (\sim q \wedge \sim p) \\ \downarrow & & \downarrow \\ \mathrm{F} & \vee & \mathrm{F}, \end{array}$$

which is false by the *or* truth table. (For an alternate solution, see Example 3(b) in the next section.) TRY YOUR TURN 7

YOUR TURN 7 Let p represent "$7 < 2$," q represent "$4 > 3$," and r represent "$2 > 8$." Find the truth value of $p \vee (\sim q \wedge r)$.

NOTE The expression $\sim p \wedge \sim q$ in part (a) is often expressed in English as "Neither p nor q." We saw at the end of Example 4 that the expression $\sim(p \vee q)$ can also be expressed as "Neither p nor q." As we shall see in the next section, the expressions $\sim p \wedge \sim q$ and $\sim(p \vee q)$ are equivalent.

TECHNOLOGY NOTE

Figure 5 shows the results of using a T1-84 Plus calculator to solve parts (a) and (b) of Example 7.

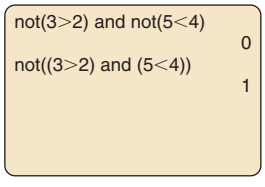


FIGURE 5

6.1 EXERCISES

Decide whether each of the following is a statement. If it is a statement, decide whether or not it is compound.

1. Montevideo is the capital of Uruguay.
2. John Jay was the first chief justice of the United States.
3. Don't feed the animals.
4. Do unto others as you would have them do unto you.
5. $2 + 2 = 5$ and $3 + 3 = 7$
6. $x < 7$ or $x > 14$
7. Got milk?
8. Is that all there is?
9. I am not a crook.
10. China does not have a population of more than 1 billion.
11. She enjoyed the comedy team of Penn and Teller.
12. The New Hampshire motto is "Live free or die."
13. If I get an A, I will celebrate.
14. If it's past 8:00, then we are late.

Write a negation for each statement.

15. My favorite flavor is chocolate.
16. This is not the time to complain.

Give a negation of each inequality.

17. $y > 12$
18. $x < -6$
19. $q \geq 5$
20. $r \leq 19$
21. Try to negate the sentence "The exact number of words in this sentence is 10" and see what happens. Explain the problem that arises.
22. Explain why the negation of "$r > 4$" is not "$r < 4$."

Let b represent the statement "I'm getting better" and d represent the statement "My parrot is dead." Translate each symbolic statement into words.

23. $\sim b$
24. $\sim d$
25. $\sim b \vee d$
26. $b \wedge \sim d$
27. $\sim(b \wedge \sim d)$
28. $\sim(b \vee d)$

Use the concepts introduced in this section to answer Exercises 29–34.

29. If q is false, what must be the truth value of $(p \wedge \sim q) \wedge q$?
30. If q is true, what must be the truth value of $q \vee (q \wedge \sim p)$?
31. If $p \wedge q$ is true, then q must be _____.
32. If $p \vee q$ is false, and p is false, then q must be _____.
33. If $\sim(p \vee q)$ is true, what must be the truth values of each of the component statements?
34. If $\sim(p \wedge q)$ is false, what must be the truth values of the component statements?

Let p represent a false statement and let q represent a true statement. Find the truth value of each compound statement.

35. $\sim p$
36. $\sim q$
37. $p \vee q$
38. $p \wedge q$
39. $p \vee \sim q$
40. $\sim p \wedge q$
41. $\sim p \vee \sim q$
42. $p \wedge \sim q$
43. $\sim(p \wedge \sim q)$
44. $\sim(\sim p \vee \sim q)$
45. $\sim[\sim p \wedge (\sim q \vee p)]$
46. $\sim[(\sim p \wedge \sim q) \vee \sim q]$
47. Is the statement $3 \geq 1$ a conjunction or a disjunction? Why?
48. Why is the statement $6 \geq 2$ true? Why is $6 \geq 6$ true?

Let p represent a true statement, and q and r represent false statements. Find the truth value of each compound statement.

49. $(p \wedge r) \vee \sim q$
50. $(q \vee \sim r) \wedge p$
51. $p \wedge (q \vee r)$
52. $(\sim p \wedge q) \vee \sim r$
53. $\sim(p \wedge q) \wedge (r \vee \sim q)$
54. $(\sim p \wedge \sim q) \vee (\sim r \wedge q)$
55. $\sim[(\sim p \wedge q) \vee r]$
56. $\sim[r \vee (\sim q \wedge \sim p)]$

Let p represent the statement "$2 > 7$," let q represent the statement "$8 \leq 6$," and let r represent the statement "$19 \leq 19$". Find the truth value of each compound statement.

57. $p \wedge r$
58. $p \vee \sim q$
59. $\sim q \vee \sim r$
60. $\sim p \wedge \sim r$
61. $(p \wedge q) \vee r$
62. $\sim p \vee (\sim r \vee \sim q)$
63. $(\sim r \wedge q) \vee \sim p$
64. $\sim(p \vee \sim q) \vee \sim r$

APPLICATIONS

Business and Economics

Income Tax **The following excerpts appear in a guide for preparing income tax reports. *Source: Your Income Tax 2010.***

a. *Which filing status should you use?*

b. *Scholarships and fellowships of a degree candidate are tax free to the extent that the grants pay for tuition and course-related fees, books, supplies, and equipment that are required for courses.*

c. *You do not want to withhold too little from your pay and you do not want to withhold too much.*

d. *An individual does not have to be your biological child to be a "qualifying" child.*

65. Which of these excerpts are statements?
66. Which of these excerpts are compound statements?

67. Write the negation of excerpt d.

68. Determine p and q to symbolically represent excerpt c.

Technology **For Exercises 69–74, let *a* represent the statement "Apple Computer, Inc. developed the iPad," and *j* represent the statement, "Steve Jobs is the CEO of Apple Computer, Inc." Convert each compound statement into symbols.**

69. Apple Computer, Inc. developed the iPad, and Steve Jobs is the CEO of Apple Computer, Inc.

70. Apple Computer, Inc. did not develop the iPad, and Steve Jobs is not the CEO of Apple Computer, Inc.

71. Apple Computer, Inc. did not develop the iPad, or Steve Jobs is the CEO of Apple Computer, Inc.

72. Apple Computer, Inc. developed the iPad, or Steve Jobs is the CEO of Apple Computer, Inc.

73. Suppose the statements that Apple Computer, Inc. developed the iPad and that Steve Jobs is CEO of Apple Computer, Inc. are both true. Which of Exercises 69–72 are true statements?

74. Suppose the statements that Apple Computer, Inc. developed the iPad and that Steve Jobs is CEO of Apple Computer, Inc. are both false. Which of Exercises 69–72 are true statements?

Life Sciences

Medicine **The following excerpts appear in a home medical reference book. *Source: American College of Physicians Complete Home Medical Guide.***

a. *Can you climb one or two flights of stairs without shortness of breath or heaviness or fatigue in your legs?*

b. *Regularly doing exercises that concentrate on strengthening particular muscle groups and improving overall flexibility can help prevent back pain and keep you mobile.*

c. *If you answered yes to all of the questions above, you are reasonably fit.*

d. *These chemical compounds act as natural antidepressants, and they can help you feel more relaxed.*

e. *You may find that exercise helps you cope with stress.*

75. Which of these excerpts are statements?

76. Which of these excerpts are compound statements?

77. Write the negation of excerpt e.

Social Sciences

Law **The following excerpts appear in a guide to common laws. *Source: Law for Dummies.***

a. *If you're involved in an accident, regardless of who you think is at fault, stop your vehicle and inspect the damage.*

b. *Don't be pressured into signing a contract.*

c. *You can file a civil lawsuit yourself, or your attorney can do it for you.*

d. *You can't marry unless you're at least 18 years old or unless you have the permission of your parents or guardian.*

e. *The Bill of Rights defines the fundamental rights of all Americans.*

78. Which of these excerpts are statements?

79. Which of these excepts are compound statements?

80. Write the negation of excerpt e.

81. **Philosophy** Read each of the following quotes from ancient philosophers. Provide an argument why these quotes may or may not be called statements. ***Source: Masterworks of Philosophy.***

a. "A friend is a friend of someone."—Socrates

b. "Every art, and every science reduced to a teachable form, and in like manner every action and moral choice, aims, it is thought, at some good: for which reason a common and by no means a bad description of what the Chief Good is, 'that which all things aim at.' "—Aristotle

c. "Furthermore, Friendship helps the young to keep from error: the old, in respect of attention and such deficiencies in action as their weakness makes them liable to; and those who are in their prime, in respect of noble deeds, because they are thus more able to devise plans and carry them out."—Aristotle

82. **Bible** Read each of the following quotes from the biblical book Proverbs. Provide an argument why these quotes may or may not be called statements. ***Source: TNIV Bible.***

a. "A gentle answer turns away wrath."—Proverbs 15:1

b. "The hot-tempered stir up dissension, but those who are patient calm a quarrel."—Proverbs 15:18

c. "When justice is done, it brings joy to the righteous but terror to evildoers."—Proverbs 21:15

d. "Do not exploit the poor because they are poor and do not crush the needy in court."—Proverbs 22:22

e. "Apply your heart to instruction and your ears to words of knowledge."—Proverbs 23:12

General Interest

Football **For Exercises 83–88, let *n* represent the statement "New Orleans won the Super Bowl" and *m* represent the statement "Peyton Manning is the best quarterback." Convert each compound statement into symbols.**

83. New Orleans won the Super Bowl but Peyton Manning is not the best quarterback.

84. New Orleans did not win the Super Bowl or Peyton Manning is not the best quarterback.

85. New Orleans did not win the Super Bowl or Peyton Manning is the best quarterback.

86. New Orleans did not win the Super Bowl but Peyton Manning is the best quarterback.

87. Neither did New Orleans win the Super Bowl nor is Peyton Manning the best quarterback.

88. Either New Orleans won the Super Bowl or Peyton Manning is the best quarterback, and it is not the case that both New Orleans won the Super Bowl and Peyton Manning is the best quarterback.

89. **APPLY IT** Suppose the statements that New Orleans won the Super Bowl and that Peyton Manning is the best quarterback are both true. Which of Exercises 83–88 are true statements?

90. Suppose the statements that New Orleans won the Super Bowl and that Peyton Manning is the best quarterback are both false. Which of Exercises 83–88 are true statements?

YOUR TURN ANSWERS

1. No **2.** Wal-Mart is the largest corporation in the USA. **3.** $4x + 2y \geq 5$ **4.** My backpack is heavy, and it's not going to rain. **5.** True **6.** True **7.** False

6.2 Truth Tables and Equivalent Statements

APPLY IT **When using a search engine on the Internet, you are asked to supply key words. How does the search engine connect these key words?**
You will be asked to explore this question in Exercise 51.

In the previous section, we created truth tables for some simple logical expressions. We will now create truth tables for more complex statements, and determine for what values of the individual component statements the complex statement is true. We will continue to use the following standard format for listing the possible truth values in compound statements involving two statements.

p	q	Compound Statement
T	T	
T	F	
F	T	
F	F	

EXAMPLE 1 Truth Tables

(a) Construct a truth table for $(\sim p \wedge q) \vee \sim q$.

SOLUTION Begin by listing all possible combinations of truth values for p and q. Then find the truth values of $\sim p \wedge q$. Start by listing the truth values of $\sim p$, which are the opposite of those of p.

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

Use only the "$\sim p$" column and the "q" column, along with the *and* truth table, to find the truth values of $\sim p \wedge q$. List them in a separate column.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Next include a column for $\sim q$.

p	q	$\sim p$	$\sim p \wedge q$	$\sim q$
T	T	F	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Finally, make a column for the entire compound statement. To find the truth values, use *or* to combine $\sim p \wedge q$ with $\sim q$.

p	q	$\sim p$	$\sim p \wedge q$	$\sim q$	$(\sim p \wedge q) \vee \sim q$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

YOUR TURN 1 Construct a truth table for $p \wedge (\sim p \vee q)$. If p and q are both true, find the truth value of $p \wedge (\sim p \vee q)$.

(b) Suppose both p and q are true. Find the truth value of $(\sim p \wedge q) \vee \sim q$.

SOLUTION Look in the first row of the final truth table above, where both p and q have truth value T. Read across the row to find that the compound statement is false.

TRY YOUR TURN 1

EXAMPLE 2 Truth Table

Construct a truth table for the following statement:

I'm bringing the food, or Heather Peck's bringing the food and I'm not.

SOLUTION If we let i represent "I'm bringing the food" and h represent "Heather Peck is bringing the food," the statement can be represented symbolically as $i \vee (h \wedge \sim i)$. Proceed as shown in the following truth table.

i	h	$\sim i$	$h \wedge \sim i$	$i \vee (h \wedge \sim i)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

YOUR TURN 2 Construct a truth statement for the following statement: "I do not order pizza, or you do not make dinner and I order pizza."

Notice from the truth table above that the only circumstances under which the original statement is false is when both statements "I'm bringing the food" and "Heather Peck is bringing the food" are false.

TRY YOUR TURN 2

If a compound statement involves three component statements p, q, and r, we will use the following format in setting up the truth table.

p	q	r	Compound Statement
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

As we mentioned in the previous section, the rightmost column (the r column) alternates T, F, T, F. The next column to the left (the q column) has two T's followed by two F's, and then repeats this pattern. The leftmost column has four T's followed by four F's.

EXAMPLE 3 Truth Tables

(a) Construct a truth table for $(\sim p \wedge r) \vee (\sim q \wedge \sim p)$.

SOLUTION This statement has three component statements, p, q, and r. The truth table thus requires eight rows to list all possible combinations of truth values of p, q, and r. The final truth table is formed in much the same way as in the previous examples.

p	q	r	$\sim p$	$\sim p \wedge r$	$\sim q$	$\sim q \wedge \sim p$	$(\sim p \wedge r) \vee (\sim q \wedge \sim p)$
T	T	T	F	F	F	F	F
T	T	F	F	F	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	T	F	F
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	F	T	T	T

(b) Suppose p is true, q is false, and r is true. Find the truth value of $(\sim p \wedge r) \vee (\sim q \wedge \sim p)$.

SOLUTION By the third row of the truth table in part (a), the compound statement is false. (This is an alternate method for working part (c) of Example 7 of the previous section.)

Notice that the truth table in Example 3(a) has three component statements and eight rows. The truth tables for the conjunction and disjunction have two component statements, and each table has four rows. The truth table for the negation has one component and two rows. These are summarized in the next table. Can we use this information to determine the number of rows in a truth table with n components?

Number of Rows in a Truth Table

Number of Statements	Number of Rows
1	$2 = 2^1$
2	$4 = 2^2$
3	$8 = 2^3$

One strategy for solving this type of problem is noticing a pattern and using *inductive reasoning*, or reasoning that uses particular facts to find a general rule. If n is a counting number, and a logical statement is composed of n component statements, we can use inductive reasoning to conjecture that its truth table will have 2^n rows. This can be proved using ideas in Chapter 8.

Intuitively, it's not hard to see why adding a statement doubles the number of rows. For example, if we wanted to construct a truth table with the four statements p, q, r, and s, we could start with the truth table for p, q, and r, which has eight rows. We must let s have the value of T for each of these rows, and we must also let s have the value of F for each of these rows, giving a total of 16 rows.

Number of Rows in a Truth Table

A logical statement having n component statements will have 2^n rows in its truth table.

Alternative Method for Constructing Truth Tables

After making a reasonable number of truth tables, some people prefer the shortcut method shown in Example 4, which repeats Examples 1 and 3.

EXAMPLE 4 Truth Tables

Construct the truth table for each statement.

(a) $(\sim p \wedge q) \vee \sim q$

SOLUTION Start by inserting truth values for $\sim p$ and for q.

p	q	$(\sim p \wedge q) \vee \sim q$
T	T	F T
T	F	F F
F	T	T T
F	F	T F

Next, use the *and* truth table to obtain the truth values of $\sim p \wedge q$.

p	q	$(\sim p \wedge q) \vee \sim q$
T	T	F F T
T	F	F F F
F	T	T T T
F	F	T F F

Now disregard the two preliminary columns of truth values for $\sim p$ and for q, and insert truth values for $\sim q$.

p	q	$(\sim p \wedge q) \vee \sim q$
T	T	F F
T	F	F T
F	T	T F
F	F	F T

Finally, use the *or* truth table.

p	q	$(\sim p \wedge q) \vee \sim q$
T	T	F F F
T	F	F T T
F	T	T T F
F	F	F T T

These steps can be summarized as follows.

p	q	$(\sim p \wedge q) \vee \sim q$
T	T	F F T F F
T	F	F F F T T
F	T	T T T T F
F	F	T F F T T
		① ② ① ④ ③

The circled numbers indicate the order in which the various columns of the truth table were found.

(b) $(\sim p \wedge r) \vee (\sim q \wedge \sim p)$

SOLUTION Work as follows.

p	q	r	$(\sim p \wedge r) \vee (\sim q \wedge \sim p)$
T	T	T	F F T F F F F
T	T	F	F F F F F F F
T	F	T	F F T F T F F
T	F	F	F F F F T F F
F	T	T	T T T T F F T
F	T	F	T F F F F F T
F	F	T	T T T T T T T
F	F	F	T F F T T T T
			① ② ① ⑤ ③ ④ ③

Equivalent Statements

One application of truth tables is illustrated by showing that two statements are equivalent; by definition, two statements are **equivalent** if they have the same truth value in *every* possible situation. The columns of each truth table that were the last to be completed will be exactly the same for equivalent statements.

EXAMPLE 5 Equivalent Statements

Are the statements

$$\sim p \wedge \sim q \qquad \text{and} \qquad \sim(p \vee q)$$

equivalent?

SOLUTION To find out, make a truth table for each statement, with the following results.

p	q	$\sim p \wedge \sim q$
T	T	F
T	F	F
F	T	F
F	F	T

p	q	$\sim(p \vee q)$
T	T	F
T	F	F
F	T	F
F	F	T

Since the truth values are the same in all cases, as shown in the columns in color, the statements $\sim p \wedge \sim q$ and $\sim(p \vee q)$ are equivalent. Equivalence is written with a three-bar symbol, $\equiv$. Using this symbol, $\sim p \wedge \sim q \equiv \sim(p \vee q)$.

In the same way, the statements $\sim p \vee \sim q$ and $\sim(p \wedge q)$ are equivalent. We call these equivalences **De Morgan's Laws**.*

De Morgan's Laws

For any statements p and q,

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

$$\sim(p \wedge q) \equiv \sim p \vee \sim q.$$

DeMorgan's Laws can be used to find the negations of certain compound statements.

*Augustus De Morgan (1806–1871) was born in India, where his father was serving as an English army officer. After studying mathematics at Cambridge, he prepared for a career in law because his performance on the challenging Tripos Exam in mathematics was not very good. Nevertheless, De Morgan was offered a position as chair of the department of mathematics at London University. He wrote and taught mathematics with great clarity and dedication, and he was also an excellent flutist.

EXAMPLE 6 Negation

Find the negation of each statement, applying De Morgan's Laws to simplify.

(a) It was a dark and stormy night.

SOLUTION If we first rephrase the statement as "The night was dark and the night was stormy," we can let d represent "the night was dark" and s represent "the night was stormy." The original compound statement can then be written $d \wedge s$. The negation of $d \wedge s$ is $\sim(d \wedge s)$. Using the second of De Morgan's Laws, $\sim(d \wedge s) \equiv \sim d \vee \sim s$. In words this reads:

The night was not dark or the night was not stormy.

In retrospect, this should be obvious. If it's not true that the night was dark and stormy, then either the night wasn't dark or it wasn't stormy.

(b) Either John will play the guitar or George will not sing.

SOLUTION In this book, we interpret "either . . . or" as disjunction; the word "either" just highlights that a disjunction is about to occur. Using the first of De Morgan's Laws, $\sim(j \vee \sim g) \equiv \sim j \wedge \sim(\sim g)$. Using the observation from the last section that the negation of a negation of a statement is simply the statement, we can simplify the last statement to $\sim j \wedge g$. In words this reads:

John will not play the guitar and George will sing.

(c) $(\sim p \wedge q) \vee \sim r$

SOLUTION After negating, we apply the first of De Morgan's Laws, changing $\vee$ to $\wedge$:

$$\sim[(\sim p \wedge q) \vee \sim r] \equiv \sim(\sim p \wedge q) \wedge \sim(\sim r).$$

The last term simplifies to r. Applying the second of De Morgan's Laws to the first term yields

$$\sim(\sim p \wedge q) \wedge r \equiv (p \vee \sim q) \wedge r,$$

where we have replaced $\sim(\sim p)$ with p. (From now on, we will make this simplification without mentioning it.)

YOUR TURN 3 Find the negation of the following statement: "You do not make dinner or I order pizza."

TRY YOUR TURN 3

6.2 EXERCISES

Give the number of rows in the truth table for each compound statement.

1. $p \vee \sim r$
2. $p \wedge (r \wedge \sim s)$
3. $(\sim p \wedge q) \vee (\sim r \vee \sim s) \wedge r$
4. $[(p \vee q) \wedge (r \wedge s)] \wedge (t \vee \sim p)$
5. $[(\sim p \wedge \sim q) \wedge (\sim r \wedge s \wedge \sim t)] \wedge (\sim u \vee \sim v)$
6. $[(\sim p \wedge \sim q) \vee (\sim r \vee \sim s)] \vee [(\sim m \wedge \sim n) \wedge (u \wedge \sim v)]$
7. If the truth table for a certain compound statement has 64 rows, how many distinct component statements does it have?
8. Is it possible for the truth table of a compound statement to have exactly 48 rows? Why or why not?

Construct a truth table for each compound statement.

9. $\sim p \wedge q$
10. $\sim p \vee \sim q$
11. $\sim(p \wedge q)$
12. $p \vee \sim q$
13. $(q \vee \sim p) \vee \sim q$
14. $(p \wedge \sim q) \wedge p$
15. $\sim q \wedge (\sim p \vee q)$
16. $\sim p \vee (\sim q \wedge \sim p)$
17. $(p \vee \sim q) \wedge (p \wedge q)$
18. $(\sim p \wedge \sim q) \vee (\sim p \vee q)$
19. $(\sim p \wedge q) \wedge r$
20. $r \vee (p \wedge \sim q)$
21. $(\sim p \wedge \sim q) \vee (\sim r \vee \sim p)$
22. $(\sim r \vee \sim p) \wedge (\sim p \vee \sim q)$
23. $\sim(\sim p \wedge \sim q) \vee (\sim r \vee \sim s)$
24. $(\sim r \vee s) \wedge (\sim p \wedge q)$

Write the negation of each statement, applying De Morgan's Laws to simplify.

25. It's vacation, and I am having fun.

26. Rachel Reeve was elected president and Joanne Dill was elected treasurer.

27. Either the door was unlocked or the thief broke a window.

28. Sue brings the wrong book or she forgets the notes.

29. I'm ready to go, but Naomi Bahary isn't.

30. You can lead a horse to water, but you cannot make him drink.

31. $12 > 4$ or $8 = 9$

32. $2 + 3 = 5$ and $12 + 13 = 15$

33. Larry or Moe is out sick today.

34. You and I have gone through a lot.

35. Complete the truth table for *exclusive disjunction.* The symbol $\veebar$ represents "one or the other is true, but not both."

p	q	$p \veebar q$
T	T	
T	F	
F	T	
F	F	

Exclusive disjunction

Decide whether the following compound statements are true or false. Remember from Exercise 35 that $\veebar$ is the exclusive disjunction; that is, assume "either p or q is true, but not both."

36. $(3 + 1 = 4) \veebar (2 + 5 = 7)$

37. $(3 + 1 = 4) \veebar (2 + 5 = 9)$

38. $(3 + 1 = 7) \veebar (2 + 5 = 7)$

39. Let p represent $2\sqrt{6} - 4\sqrt{5} > -1$, q represent $\dfrac{14 - 7\sqrt{8}}{2.5 - \sqrt{5}} > -22$, and s represent $\dfrac{7 - \dfrac{5}{\sqrt{3}}}{\sqrt{8} - 2} < \dfrac{\sqrt{3}}{\sqrt{2}}$.
Use the LOGIC menu on a graphing calculator to find the truth value of each statement.

a. $p \wedge q$ **b.** $\sim p \wedge q$

c. $\sim(p \vee q)$ **d.** $(s \wedge \sim p) \vee (\sim s \wedge q)$

APPLICATIONS

Business and Economics

40. Income Tax The following statement appears in a guide for preparing income tax reports. Use one of De Morgan's Laws to write the negation of this statement. ***Source: Your Income Tax 2010.***

> Tips of less than $20 per month are taxable but are not subject to withholding.

41. Warranty The following statement appears in an iPhone warranty guide. Use one of De Morgan's Laws to negate this statement. ***Source: AppleCare Protection Plan.***

> Service will be performed at the location, or the store may send the Covered Equipment to an Apple repair service location to be repaired.

42. eBay The eBay Buyer Protection plan guarantees that a buyer will receive an item and that it will be as described, or eBay will cover the purchase price plus shipping. Let r represent "A buyer will receive an item," d represent "It will be as described," and e represent "eBay will cover the purchase price plus shipping." Write the guarantee symbolically, and then construct a truth table for the statement, putting the variables in the order r, d, and e. Under what conditions would the guarantee be false? ***Source: eBay®.***

43. Guarantees The guarantee on a brand of vacuum cleaner reads: "You will be completely satisfied or we will refund your money without asking any questions." Let s represent "You will be completely satisfied," r represent "We will refund your money," and q represent "We will ask you questions." Write the guarantee symbolically and then construct a truth table for the statement, putting the variables in the order s, r, q. Under what conditions would the guarantee be false?

Life Sciences

44. Medicine The following statements appear in a home medical reference book. Define p and q so that the statements can be written symbolically. Then negate each statement. ***Source: American College of Physicians Complete Home Medical Guide.***

a. Tissue samples may be taken from almost anywhere in the body, and the procedure used depends on the site.

b. The procedure can be carried out quickly in the doctor's office and is not painful.

c. Fluid samples may be examined for infection, or the cells in the fluid may be separated and examined to detect other abnormalities.

Social Sciences

45. Law Attorneys sometimes use the phrase "and/or." This phrase corresponds to which usage of the word *or*: inclusive or exclusive?

46. Law The following statement appears in a guide to common laws. Define p and q so that the statement can be written symbolically. Then negate the statement. ***Source: Law for Dummies.***

> You can file a complaint yourself as you would if you were using small claims court, or your attorney can file one for you.

47. Presidential Quote Use both of De Morgan's Laws to rewrite the negation of the following quote made by President John F. Kennedy at Vanderbilt University on March 18, 1963:

> Liberty without learning is always in peril, and learning without liberty is always in vain. ***Source: Masters of Chiasmus.***

48. Politician Senator Pompous B. Blowhard made the following campaign promise: "I will cut taxes and eliminate the deficit, or I will not run for reelection." Let c represent "I will cut taxes," let e represent "I will eliminate the deficit," and let r represent "I will run for reelection." Write the promise symbolically and then construct a truth table for the statement, putting the variables in the order c, e, r. Under what conditions would the promise be false?

General Interest

49. Yahtzee® The following statement appears in the instructions for the Milton Bradley game Yahtzee®. ***Source: Milton Bradley Company.*** Negate the statement.

You could reroll the die again for your Large Straight or set aside the 2 Twos and roll for your Twos or for 3 of a Kind.

50. Logic Puzzles Raymond Smullyan is one of today's foremost writers of logic puzzles. Smullyan proposed a question, based on the classic Frank Stockton short story, in which a prisoner must make a choice between two doors: Behind one is a beautiful lady, and behind the other is a hungry tiger. What if each door has a sign, and the prisoner knows that only one sign is true? The sign on Door 1 reads: In this room there is a lady and in the other room there is a tiger. The sign on Door 2 reads: In one of these rooms there is a lady and in one of these rooms there is a tiger. With this information, determine what is behind each door. ***Source: The Lady or the Tiger? And Other Logic Puzzles.***

51. APPLY IT Describe how a search engine on the Internet uses key words and logical connectives to locate information.

YOUR TURN ANSWERS

1.

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

true

2.

p	d	$\sim d$	$\sim d \wedge p$	$\sim p$	$\sim p \vee (\sim d \wedge p)$
T	T	F	F	F	F
T	F	T	T	F	T
F	T	F	F	T	T
F	F	T	F	T	T

3. You make dinner and I do not order pizza.

6.3 The Conditional and Circuits

APPLY IT **How can logic be used in the design of electrical circuits?**
This question will be answered in this section.

Conditionals

A **conditional** statement is a compound statement that uses the connective *if . . . then*, or anything equivalent. For example:

If it rains, *then* I carry my umbrella.

If the president comes, *then* security will be tight.

If the check doesn't arrive today, I will call to find out why.

In the last statement, the word *then* was implied but not explicitly stated; the sentence is equivalent to the statement "If the check doesn't arrive today, *then* I will call to find out why." In each of these statements, the component after the word *if* gives a condition under which the last component is true. The last component is the statement coming after the word *then* (or, in the third statement, the implied *then*). There may be other conditions under which the last component is true. In the second statement, for example, it might be true that even if the president doesn't come, security will be tight because the vice president is coming.

The conditional is written with an arrow, so that "if p, then q" is symbolized as

$$p \rightarrow q.$$

We read $p \rightarrow q$ as "p implies q" or "if p, then q." In the conditional $p \rightarrow q$, the statement p is the **antecedent**, while q is the **consequent**.

There are many equivalent forms of the conditional. For example, the statement

Winners never quit

can be rephrased as

If you are a winner, then you never quit.

There are other forms of the conditional that we will study in the next section.

Just as we defined conjunction and disjunction using a truth table, we will now do the same for the conditional. To see how such a table should be set up, we will analyze the following statement that might be made by a cereal company:

If you eat Wheat Crunchies, then you'll be full of energy.

Let e represent "You eat Wheat Crunchies" and f represent "You'll be full of energy." As before, there are four possible combinations of truth values for the two component statements.

Wheat Crunchies Analysis			
Possibility	**Eat Wheat Crunchies?**	**Full of Energy?**	
1	Yes	Yes	e is T, f is T
2	Yes	No	e is T, f is F
3	No	Yes	e is F, f is T
4	No	No	e is F, f is F

Let's consider each of these possibilities.

1. If you eat Wheat Crunchies and indeed find that you are full of energy, then you must conclude that the company's claim is true, so place T in the first row of the truth table. This does not necessarily mean that eating Wheat Crunchies caused you to be full of energy; perhaps you are full of energy for some other reason unrelated to what you ate for breakfast.
2. If you eat Wheat Crunchies and are not full of energy, then the company's claim is false, so place F in the second row of the truth table.
3. If you don't eat Wheat Crunchies and yet find that you are full of energy, this doesn't invalidate the company's claim. They only promised results if you ate Wheat Crunchies; they made no claims about what would happen if you don't eat Wheat Crunchies. We will, therefore, place T in the third row of the truth table.
4. If you don't eat Wheat Crunchies and are not full of energy, you can't very well blame the company. Because they can still claim that their promise is true, we will place T in the fourth row.

The discussion above leads to the following truth table.

Truth Table for the Conditional *if p, then q*

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth table for the conditional leads to some counterintuitive conclusions, because *if . . . then* sometimes has other connotations in English, as the following examples illustrate.

EXAMPLE 1 Truth Value

Suppose you get a 61 on the test and you pass the course. Find the truth value of the following statement:

If you get a 70 or higher on the test, then you pass the course.

SOLUTION The first component of this conditional is false, while the second is true. According to the third line of the truth table for the conditional, this statement is true.

This result may surprise you. Even though the statement only says what happens if you get a 70 or higher on the test, perhaps you also infer from the statement that if you don't get 70 or higher, you won't pass the course. This is a common interpretation of *if . . . then*, but it is not consistent with our truth table above. This interpretation is called the *biconditional* and will be discussed in the next section.

EXAMPLE 2 Truth Value

Find the truth value of each of the following statements.

(a) If the earth is shaped like a cube, then elephants are smaller than mice.

SOLUTION Both components of this conditional are false. According to the fourth line of the truth table for the conditional, this statement is true.

YOUR TURN 1 Find the truth value of the following statement. "If Little Rock is the capital of Arkansas, then New York City is the capital of New York."

(b) If the earth is shaped like a cube, then George Washington was the first president of the United States.

SOLUTION The first component of this conditional is false, while the second is true. According to the third line of the truth table for the conditional, this statement is true, even though it may seem like an odd statement. TRY YOUR TURN 1

The following observations come from the truth table for $p \to q$.

Special Characteristics of Conditional Statements

1. $p \to q$ is false only when the antecedent (p) is *true* and the consequent (q) is *false.*
2. If the antecedent (p) is *false,* then $p \to q$ is automatically *true.*
3. If the consequent (q) is *true,* then $p \to q$ is automatically *true.*

EXAMPLE 3 Conditional Statement

Write *true* or *false* for each statement. Here T represents a true statement, and F represents a false statement.

(a) $\text{T} \to (6 = 3)$

SOLUTION Since the antecedent is true, while the consequent, $6 = 3$, is false, the given statement is false by the first point mentioned above.

(b) $(5 < 2) \to \text{F}$

SOLUTION The antecedent is false, so the given statement is true by the second observation.

YOUR TURN 2 Determine if the following statement is *true* or *false:* $(4 > 5) \to \text{F}$.

(c) $(3 \neq 2 + 1) \to \text{T}$

SOLUTION The consequent is true, making the statement true by the third characteristic of conditional statements. TRY YOUR TURN 2

EXAMPLE 4 Truth Value

Given that p, q, and r are all false, find the truth value of the statement

$$(p \to \sim q) \to (\sim r \to q).$$

SOLUTION Using the shortcut method explained in Example 6 of Section 6.1, we can replace p, q, and r with F (since each is false) and proceed as before, using the negation and conditional truth tables as necessary.

$$\begin{aligned}
&(p \to \sim q) \to (\sim r \to q) \\
&(\text{F} \to \sim\text{F}) \to (\sim\text{F} \to \text{F}) && p, q, r \text{ are false.} \\
&(\text{F} \to \text{T}) \to (\text{T} \to \text{F}) && \text{Use the negation truth table.} \\
&\text{T} \to \text{F} && \text{Use the conditional truth table.} \\
&\text{F}
\end{aligned}$$

YOUR TURN 3 If p, q, and r are all false, find the truth value of the statement $\sim q \rightarrow (p \rightarrow r)$.

The statement $(p \rightarrow \sim q) \rightarrow (\sim r \rightarrow q)$ is false when p, q, and r are all false.

TRY YOUR TURN 3

Truth tables for compound statements involving conditionals are found using the techniques described in the previous section. The next example shows how this is done.

EXAMPLE 5 Truth Table

Construct a truth table for each statement.

(a) $(\sim p \rightarrow \sim q) \wedge \sim(p \vee \sim q)$

SOLUTION First insert the truth values of $\sim p$ and of $\sim q$. Then find the truth values of $\sim p \rightarrow \sim q$.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Next use p and $\sim q$ to find the truth values of $p \vee \sim q$ and then $\sim(p \vee \sim q)$.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$
T	T	F	F	T	T	F
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	F

Finally, add the column for the statement $(\sim p \rightarrow \sim q) \wedge \sim(p \vee \sim q)$.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$	$(\sim p \rightarrow \sim q) \wedge \sim(p \vee \sim q)$
T	T	F	F	T	T	F	F
T	F	F	T	T	T	F	F
F	T	T	F	F	F	T	F
F	F	T	T	T	T	F	F

(b) $(p \rightarrow q) \rightarrow (\sim p \vee q)$

SOLUTION Go through steps similar to the ones above.

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \rightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As the truth table in Example 5(a) shows, the statement $(\sim p \rightarrow \sim q) \wedge \sim(p \vee \sim q)$ is always false, regardless of the truth values of the components. Such a statement is called a **contradiction**. Most contradictions consist of two statements that cannot simultaneously be true that are united with a conjunction. The simplest such statement is $p \wedge \sim p$.

Notice from the truth table in Example 5(b) that the statement $(p \to q) \to (\sim p \vee q)$ is always true, regardless of the truth values of the components. Such a statement is called a **tautology**. Other examples of tautologies (as can be checked by forming truth tables) include $p \vee \sim p$, $p \to p$, $(\sim p \vee \sim q) \to \sim(q \wedge p)$, and so on. An important point here is that there are compound statements (such as $p \vee \sim p$) that are true (or false) independent of the truth values of the component parts; the truth depends on the logical structure alone. By the way, the truth tables in Example 5 also could have been found by the alternative method shown in the previous section.

Notice from the third and fifth columns of the truth table in Example 5(b) that $p \to q$ and $\sim p \vee q$ are equivalent.

Writing a Conditional as an *or* Statement

$p \to q$ is equivalent to $\sim p \vee q$.

EXAMPLE 6 Equivalent Statements

Write the following statement without using *if . . . then.*

If this is your first time, we're glad you're here.

SOLUTION Letting f represent "This is your first time" and g represent "We're glad you're here," the conditional may be restated as $\sim f \vee g$, or in words:

This is not your first time or we're glad you're here. TRY YOUR TURN 4

YOUR TURN 4 Write the following statement without using *if . . . then*: "If you do the homework, then you will pass the quiz."

Since

$$p \to q \equiv \sim p \vee q,$$

we can take the negation of both sides of this equivalence to get

$$\sim(p \to q) \equiv \sim(\sim p \vee q).$$

By applying De Morgan's Law to the right side, we have

$$\begin{aligned} \sim(p \to q) &\equiv \sim(\sim p) \wedge \sim q \\ &\equiv p \wedge \sim q. \end{aligned}$$

This gives us an equivalent form for the negation of the conditional.

Negation of $p \to q$

The negation of $p \to q$ is $p \wedge \sim q$.

EXAMPLE 7 Negation

Write the negation of each statement.

(a) If you go to the left, I'll go to the right.

SOLUTION Let y represent "You go to the left" and i represent "I'll go to the right." Then the original statement is represented $y \to i$. As we showed earlier, the negation of this is $y \wedge \sim i$, which can be translated into words as

You go to the left and I won't go to the right.

(b) It must be alive if it is breathing.

SOLUTION First, we must restate the given statement in *if . . . then* form:

If it is breathing, then it must be alive.

Based on our earlier discussion, the negation is

It is breathing and it is not alive. TRY YOUR TURN 5

YOUR TURN 5 Write the negation of the following statement: "If you are on time, then we will be on time."

A common error occurs when students try to write the negation of a conditional statement as another conditional statement. As seen in Example 7, the negation of a conditional statement is written as a conjunction.

APPLY IT

Circuits

One of the first nonmathematical applications of symbolic logic was seen in the master's thesis of Claude Shannon in 1937. Shannon showed how the logic developed almost a century earlier by British mathematician George Boole could be used as an aid in designing electrical **circuits**. His work was immediately taken up by the designers of computers. These computers, then in the developmental stage, could be simplified and built for less money using the ideas of Shannon.

To see how Shannon's ideas work, look at the electrical switch shown in Figure 6. We assume that current will flow through this switch when it is closed and not when it is open.

Open circuit

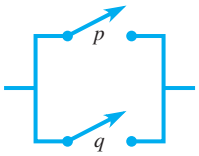
FIGURE 6

Figure 7 shows two switches connected in **series**; in such a circuit, current will flow only when both switches are closed. Note how closely a series circuit corresponds to the conjunction $p \wedge q$. We know that $p \wedge q$ is true only when both p and q are true.

Series circuit

FIGURE 7

A circuit corresponding to the disjunction $p \vee q$ can be found by drawing a **parallel** circuit, as in Figure 8. Here, current flows if either *p or q* is closed or if both *p and q* are closed.



Parallel circuit

FIGURE 8

The circuit in Figure 9 corresponds to the statement $(p \vee q) \wedge \sim q$, which is a compound statement involving both a conjunction and a disjunction.

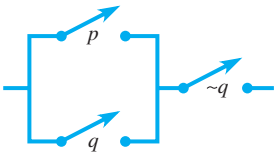


FIGURE 9

The way that logic is used to simplify an electrical circuit depends on the idea of equivalent statements, from Section 6.2. Recall that two statements are equivalent if they have exactly the same truth table final column. The symbol $\equiv$ is used to indicate that the two statements are equivalent. Some of the equivalent statements that we shall need are shown in the following box.

Equivalent Statements

1. a.	$p \vee q \equiv q \vee p$	Commutative Laws
b.	$p \wedge q \equiv q \wedge p$	
2. a.	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	Associative Laws
b.	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	
3. a.	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive Laws
b.	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
4. a.	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's Laws
b.	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	
5. a.	$p \vee p \equiv p$	Idempotent Laws
b.	$p \wedge p \equiv p$	
6. a.	$(p \wedge q) \vee p \equiv p$	Absorption Laws
b.	$(p \vee q) \wedge p \equiv p$	
7.	$\sim(\sim p) \equiv p$	Double Negative
8.	$p \to q \equiv \sim p \vee q$	Conditional as an "or"
9.	$p \to q \equiv \sim q \to \sim p$	Contrapositive

If T represents any true statement and F represents any false statement, then

10. a.	$p \vee \mathrm{T} \equiv \mathrm{T}$	Identity Laws
b.	$p \wedge \mathrm{T} \equiv p$	
c.	$p \vee \mathrm{F} \equiv p$	
d.	$p \wedge \mathrm{F} \equiv \mathrm{F}$	
11. a.	$p \vee \sim p \equiv \mathrm{T}$	Negation Laws
b.	$p \wedge \sim p \equiv \mathrm{F}$	

This list may seem formidable, but if we break it down, it turns out to be not so bad.

First, notice that the first six equivalences come in pairs, in which $\vee$ is replaced with $\wedge$ and vice versa. This illustrates the **Principle of Duality**, which states that if a logical equivalence contains no logical operators other than $\vee$, $\wedge$, and $\sim$ (that is, it does not contain $\rightarrow$), then we may replace $\vee$ with $\wedge$ and vice versa, and replace T with F and vice versa, and the new equivalence is still true. This means that we need to keep track of just one from each pair of equivalences, although we have put both in the table for completeness.

Next, notice that equivalences 1 and 2 are just the familiar Commutative and Associative laws for addition or multiplication, which means you can rearrange the order and rearrange the parentheses in an expression involving only $\wedge$ or only $\vee$. You are asked to prove these using a truth table in Exercises 61–68.

Equivalence 3 is just the familiar Distributive Law of multiplication over addition. In normal multiplication, however, you *cannot* distribute $2 + (3 \times 5)$ to get $(2 + 3) \times (2 + 5)$. In other words, you cannot distribute addition over multiplication. But you can distribute $\wedge$ over $\vee$ and vice versa. You are asked to prove these using a truth table in Exercises 65 and 66.

De Morgan's Laws, given as equivalence 4, were discussed in the previous section.

Equivalence 5, the Idempotent Laws, may seem trivially true, so you may wonder why we bother stating them. The reason is that they are useful in simplifying circuits, as we shall see in Example 8. Equivalence 7, the Double Negative, is similar in this regard.

Equivalence 6, the Absorption Laws, may be less obvious, but they, too, are useful in simplifying circuits, as we shall see in Example 9. Their proofs are in Exercises 67 and 68.

Equivalence 8, the Conditional as an "or" statement, was discussed earlier in this section. Equivalence 9, the Contrapositive, is very important and will be discussed in the next section.

Equivalences 10 and 11, the Identity and Negation laws, should be fairly obvious. We only list them here because, just like equivalences 5 and 7, they can help us simplify circuits. Notice that the Principle of Duality applies to these equivalences.

Circuits can be used as models of compound statements, with a closed switch corresponding to T, while an open switch corresponds to F. The method for simplifying circuits is explained in the following example.

EXAMPLE 8 Circuit

Simplify the circuit in Figure 10.

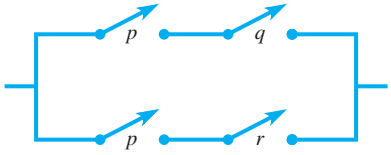


FIGURE 10

SOLUTION At the top of Figure 10, p and q are connected in series, and at the bottom, p and r are connected in series. These are interpreted as the compound statements $p \wedge q$ and $p \wedge r$, respectively. These two conjunctions are connected in parallel, as indicated by the figure treated as a whole. Therefore, we write the disjunction of the two conjunctions:

$$(p \wedge q) \vee (p \wedge r).$$

(Think of the two switches labeled "p" as being controlled by the same handle.) By the Distributive Law (equivalence statement 3),

$$(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r),$$

which has the circuit of Figure 11. This new circuit is logically equivalent to the one in Figure 10 and yet contains only three switches instead of four—which might well lead to a large savings in manufacturing costs.

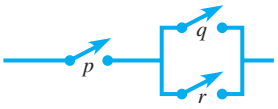


FIGURE 11

EXAMPLE 9 Circuit

Simplify the circuit in Figure 12.

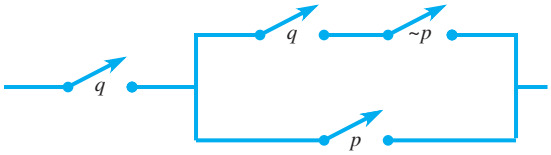


FIGURE 12

SOLUTION The diagram shows q in series with a circuit containing two parallel components, one containing q and $\sim p$, and the other containing p. Thus the circuit can be represented by the logical statement

$$q \wedge [(q \wedge \sim p) \vee p].$$

Notice that for the expression in brackets, we can use the Commutative Law to put the p first, and then we can use the Distributive Law, which leads to a series of simplifications.

$$\begin{aligned}
q \wedge [(q \wedge \sim p) \vee p] &\equiv q \wedge [p \vee (q \wedge \sim p)] && \text{Commutative Law}\\
&\equiv q \wedge [(p \vee q) \wedge (p \vee \sim p)] && \text{Distributive Law}\\
&\equiv q \wedge [(p \vee q) \wedge \mathrm{T}] && \text{Negation Law}\\
&\equiv q \wedge (p \vee q) && \text{Identity Law}\\
&\equiv (p \vee q) \wedge q && \text{Commutative Law}\\
&\equiv (q \vee p) \wedge q && \text{Commutative Law}\\
&\equiv q && \text{Absorption Law}
\end{aligned}$$

The circuit simplified to a single switch! We hope this convinces you of the power of logic. Notice that we used the Commutative Law three times simply to get terms into a form in which another equivalence applies. This could be avoided if we just added more equivalences to our previous list, such as the Distributive Laws in the reverse order. This would, however, make our list even longer, so we will not do that, even though it means that we need an occasional extra step in our proofs.

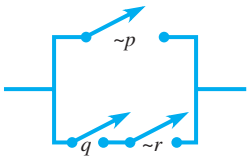


FIGURE 13

EXAMPLE 10 Circuit

Draw a circuit for $p \rightarrow (q \wedge \sim r)$.

SOLUTION By the equivalent statement Conditional as an "or," $p \rightarrow q$ is equivalent to $\sim p \vee q$. This equivalence gives $p \rightarrow (q \wedge \sim r) \equiv \sim p \vee (q \wedge \sim r)$, which has the circuit diagram in Figure 13.

6.3 EXERCISES

In Exercises 1–6, decide whether each statement is true or false, and explain why.

1. If the antecedent of a conditional statement is false, the conditional statement is true.
2. If the consequent of a conditional statement is true, the conditional statement is true.
3. If q is true, then $(p \wedge q) \rightarrow q$ is true.
4. If p is true, then $\sim p \rightarrow (q \vee r)$ is true.
5. Given that $\sim p$ is true and q is false, the conditional $p \rightarrow q$ is true.
6. Given that $\sim p$ is false and q is false, the conditional $p \rightarrow q$ is true.

7. In a few sentences, explain how we determine the truth value of a conditional statement.

8. Explain why the statement "If $3 = 5$, then $4 = 6$" is true.

Tell whether each conditional is true or false. Here T represents a true statement and F represents a false statement.

9. $\text{F} \to (4 \neq 7)$

10. $(6 \geq 6) \to \text{F}$

11. $(4 = 11 - 7) \to (8 > 0)$

12. $(4^2 \neq 16) \to (4 - 4 = 8)$

Let d represent "She dances tonight," let s represent "He sings loudly," and let e represent "I'm leaving early." Express each compound statement in words.

13. $d \to (e \wedge s)$

14. $(d \wedge s) \to e$

15. $\sim s \to (d \vee \sim e)$

16. $(\sim d \vee \sim e) \to \sim s$

Let d represent "My dog ate my homework," let f represent "I receive a failing grade," and let g represent "I'll run for governor." Express each compound statement in symbols.

17. My dog ate my homework, or if I receive a failing grade, then I'll run for governor.

18. I'll run for governor, and if I receive a failing grade, then my dog did not eat my homework.

19. I'll run for governor if I don't receive a failing grade.

20. I won't receive a failing grade if my dog didn't eat my homework.

Find the truth value of each statement. Assume that p and r are false, and q is true.

21. $\sim r \to p$

22. $\sim q \to r$

23. $\sim p \to (q \wedge r)$

24. $(\sim r \vee p) \to p$

25. $\sim q \to (p \wedge r)$

26. $(\sim p \wedge \sim q) \to (p \wedge \sim r)$

27. $(p \to \sim q) \to (\sim p \wedge \sim r)$

28. $(p \to \sim q) \wedge (p \to r)$

29. Explain why, if we know that p is true, we also know that

$$[r \vee (p \vee s)] \to (p \vee q)$$

is true, even if we are not given the truth values of q, r, and s.

30. Construct a true statement involving a conditional, a conjunction, a disjunction, and a negation (not necessarily in that order), that consists of component statements p, q, and r, with all of these component statements false.

31. Using the table of equivalent statements rather than a truth table, explain why the statement $(\sim p \to \sim q) \wedge \sim(p \vee \sim q)$ in Example 5(a) must be a contradiction.

32. What is the minimum number of times that F must appear in the final column of a truth table for us to be assured that the statement is not a tautology?

Construct a truth table for each statement. Identify any tautologies or contradictions.

33. $\sim q \to p$

34. $p \to \sim q$

35. $(p \vee \sim p) \to (p \wedge \sim p)$

36. $(p \wedge \sim q) \wedge (p \to q)$

37. $(p \vee q) \to (q \vee p)$

38. $(\sim p \to \sim q) \to (p \wedge q)$

39. $r \to (p \wedge \sim q)$

40. $[(r \vee p) \wedge \sim q] \to p$

41. $(\sim r \to s) \vee (p \to \sim q)$

42. $(\sim p \wedge \sim q) \to (\sim r \to \sim s)$

Write each statement as an equivalent statement that does not use the *if . . . then* connective. Remember that $p \to q$ is equivalent to $\sim p \vee q$.

43. If your eyes are bad, your whole body will be full of darkness.

44. If you meet me halfway, this will work.

45. I'd buy that car if I had the money.

46. I would watch out if I were you.

Write the negation of each statement. Remember that the negation of $p \to q$ is $p \wedge \sim q$.

47. If you ask me, I will do it.

48. If you are not part of the solution, you are part of the problem.

49. If you don't love me, I won't be happy.

50. If he's my brother, then he's not heavy.

Use truth tables to decide which of the pairs of statements are equivalent.

51. $p \to q$; $\sim p \vee q$

52. $\sim(p \to q)$; $p \wedge \sim q$

53. $p \to q$; $q \to p$

54. $q \to p$; $\sim p \to \sim q$

55. $p \to \sim q$; $\sim p \vee \sim q$

56. $p \to q$; $\sim q \to \sim p$

57. $p \wedge \sim q$; $\sim q \to \sim p$

58. $\sim p \wedge q$; $\sim p \to q$

In some approaches to logic, the only connectives are $\sim$ and $\to$, and the other logical connectives are defined in terms of these.* Verify this by using a truth table to demonstrate the following equivalences.

59. $p \wedge q \equiv \sim(p \to \sim q)$

60. $p \vee q \equiv \sim p \to q$

In Exercises 61–68, construct a truth table to prove each law.

61. $p \vee q \equiv q \vee p$, the Commutative Law for $\vee$

62. $p \wedge q \equiv q \wedge p$, the Commutative Law for $\wedge$

63. $p \vee (q \vee r) \equiv (p \vee q) \vee r$, the Associative Law for $\vee$

64. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$, the Associative Law for $\wedge$

65. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, the Distributive Law for $\vee$ over $\wedge$

66. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, the Distributive Law for $\wedge$ over $\vee$

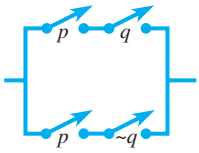
*For example, see Stefan Bilaniuk, *A Problem Course in Mathematical Logic*, http://euclid.trentu.ca/math/sb/pcml/welcome.html.

67. $(p \wedge q) \vee p \equiv p$, the first Absorption Law

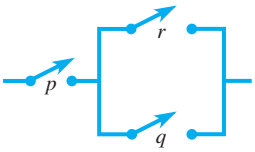
68. $(p \vee q) \wedge p \equiv p$, the second Absorption Law

Write a logical statement representing each circuit. Simplify each circuit when possible.

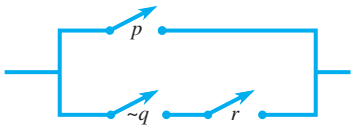
69.



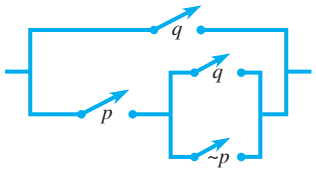
70.



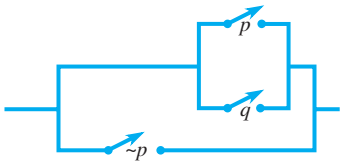
71.



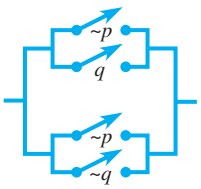
72.



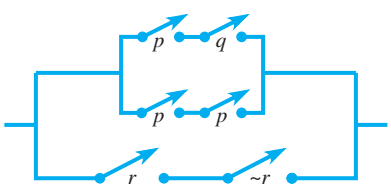
73.



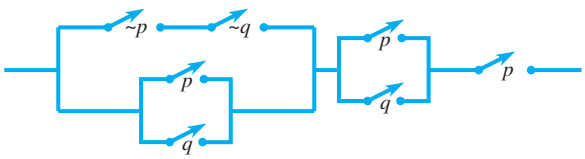
74.



75.



76.



Draw circuits representing the following statements as they are given. Simplify if possible.

77. $p \wedge (q \vee \sim p)$

78. $(\sim p \wedge \sim q) \wedge \sim r$

79. $(p \vee q) \wedge (\sim p \wedge \sim q)$

80. $(\sim q \wedge \sim p) \vee (\sim p \vee q)$

81. $[(p \vee q) \wedge r] \wedge \sim p$

82. $[(\sim p \wedge \sim r) \vee \sim q] \wedge (\sim p \wedge r)$

83. $\sim q \rightarrow (\sim p \rightarrow q)$

84. $\sim p \rightarrow (\sim p \vee \sim q)$

85. $[(p \wedge q) \vee p] \wedge [(p \vee q) \wedge q]$

86. $[(p \wedge q) \vee (p \wedge q)] \vee (p \wedge r)$

87. Explain why the circuit

will always have exactly one open switch. What does this circuit simplify to?

88. Refer to Figures 10 and 11 in Example 8. Suppose the cost of the use of one switch for an hour is 3¢. By using the circuit in Figure 11 rather than the circuit in Figure 10, what is the savings for a year of 365 days, assuming that the circuit is in continuous use?

APPLICATIONS

Business and Economics

89. Income Tax The following statements appear in a guide for preparing income tax reports. Rewrite each statement with an equivalent statement using *or.* ***Source: Your Income Tax 2010.***

a. If you are married at the end of the year, you may file a joint return with your spouse.

b. A bequest received by an executor from an estate is tax free if it is not compensation for services.

c. If a course improves your current job skills but leads to qualification for a new profession, the course is not deductable.

90. Warranty A protection plan for the iPhone states: "Service will be performed at the location, or the store may send the Covered Equipment to an Apple repair service location to be repaired." ***Source: AppleCare Protection Plan.***

a. Rewrite the statement with an equivalent statement using the *if . . . then* connective.

b. Write the negation of the statement.

91. Stocks An investor announces, "If the value of my portfolio exceeds $100,000 or the price of my stock in Ford Motor Company falls below $50 per share, then I will sell all my shares of Ford stock and I will give the proceeds to the United Way." Let v represent "The value of my portfolio exceeds $100,000," p represent "The price of my stock in Ford Motor Company falls below $50 per share," s represent "I will sell all

my shares of Ford stock," and g represent "I will give the proceeds to the United Way."

a. Convert the statement into symbols.

b. Find the truth value of the statement if my portfolio is worth $80,000, Ford Motor stock is at $56 per share, I sell my Ford Motor stock, and I keep the proceeds.

c. Explain under what circumstances the statement will be true or false.

d. Find the negation of the statement.

Life Sciences

92. **Medicine** The following statement appears in a home medical reference book. Rewrite this statement replacing the *if . . . then* with an *or* statement. Then negate that statement. ***Source: American College of Physicians Complete Home Medical Guide.***

If you are exercising outside, it is important to protect your skin and eyes from the sun.

Social Sciences

93. The following statements appear in a guide to common law. Write an equivalent statement using the *if . . . then* connective. ***Source: Law for Dummies.***

a. You can file a civil lawsuit yourself, or your attorney can do it for you.

b. Your driver's license may come with restrictions, or restrictions may sometimes be added on later.

c. You can't marry unless you're at least 18 years old, or unless you have the permission of your parents or guardian.

YOUR TURN ANSWERS

1. False **2.** True **3.** True

4. You do not do the homework, or you will pass the quiz.

5. You are on time and we will not be on time.

6.4 More on the Conditional

APPLY IT **Is it possible to rewrite statements in a medical guide in one or more ways?**
This question will be answered in Exercise 44.

The conditional can be written in several ways. For example, the word *then* might not be explicitly stated, so the statement "If I'm here, then you're safe" might become "If I'm here, you're safe." Also, the consequent (the *then* component) might come before the antecedent (the *if* component), as in "You're safe if I'm here." In this section we will examine other translations of the conditional.

Alternative Forms of the Conditional

Another way to express the conditional is with the word *sufficient*. For example, the statement

If it rains in the valley, then snow is falling on the mountain can be written

Rain in the valley is sufficient for snow to fall on the mountain.

This statement claims that one condition guaranteeing snow to fall on top of the mountain is rain in the valley. The statement doesn't claim that this is the only condition under which snow falls on the mountain; perhaps snow falls on the mountain on days when it's perfectly clear in the valley. On the other hand, assuming the statement to be true, you won't see rain in the valley unless snow is falling on the mountain. In other words,

Snow falling on the mountain is necessary for rain in the valley.

To summarize this discussion, the statement "if p, then q" is sometimes stated in the form "p is sufficient for q" and sometimes in the form "q is necessary for p." To keep track of these two forms, notice that the antecedent is the sufficient part, while the consequent is the necessary part.

There are other common translations of $p \rightarrow q$. We have collected the most common ones in the box on the next page.

Common Translations of $p \rightarrow q$

The conditional $p \rightarrow q$ can be translated in any of the following ways.

If p, then q.	p is sufficient for q.
If p, q.	q is necessary for p.
p implies q.	All p's are q's.
p only if q.	q if p.
q when p.	

The translation of $p \rightarrow q$ into these various word forms does not in any way depend on the truth or falsity of $p \rightarrow q$.

EXAMPLE 1 Equivalent Statements

Write the following statement in eight different equivalent ways, using the common translations of $p \rightarrow q$ in the box above:

If you answer this survey, then you will be entered in the drawing.

SOLUTION

If you answer this survey, you will be entered in the drawing.

Answering this survey implies that you will be entered in the drawing.

You answer this survey only if you will be entered in the drawing.

You will be entered in the drawing when you answer this survey.

Answering this survey is sufficient for you to be entered in the drawing.

Being entered in the drawing is necessary for you to answer this survey.

All who answer this survey will be entered in the drawing.

You will be entered in the drawing if you answer this survey.

EXAMPLE 2 Equivalent Statements

Write each statement in the form "if p, then q."

(a) Possession of a valid identification card is necessary for admission.

SOLUTION If you are admitted, then you possess a valid identification card.

(b) You should use this door only if there is an emergency.

SOLUTION If you should use this door, then there is an emergency.

(c) All who are weary can come and rest.

SOLUTION If you are weary, then you can come and rest. TRY YOUR TURN 1

YOUR TURN 1 Write each statement in the form "if p, then q."

(a) Being happy is sufficient for you to clap your hands.

(b) All who seek shall find.

CAUTION Notice that "p only if q" is not the same as "p if q." The first means $p \rightarrow q$, while the second means $q \rightarrow p$.

EXAMPLE 3 Symbolic Statements

Let r represent "A triangle is a right triangle" and s represent "The sum of the squares of the two sides equals the square of the hypotenuse." Write each statement in symbols.

(a) A triangle is a right triangle if the sum of the squares of the two sides equals the square of the hypotenuse.

SOLUTION

$$s \rightarrow r$$

(b) A triangle is a right triangle only if the sum of the squares of the two sides equals the square of the hypotenuse.

SOLUTION

$$r \rightarrow s$$

Converse, Inverse, and Contrapositive

Example 3(b) gives a statement of the Pythagorean theorem. Example 3(a) gives what is called the **converse** of the Pythagorean theorem. The converse of a statement $p \rightarrow q$ is the statement $q \rightarrow p$. For example, the converse of the statement

If today is Monday, then we have to put out the garbage

is the statement

If we have to put out the garbage, then today is Monday.

Although both the Pythagorean theorem and its converse are true, the converse of a true statement is not necessarily true, as we shall see in Example 4.

A second statement related to the conditional is the **inverse**, in which both the antecedent and consequent are negated. The inverse of the first statement above is

If today is not Monday, then we do not have to put out the garbage.

The inverse of a statement $p \rightarrow q$ is the statement $\sim p \rightarrow \sim q$. The inverse, like the converse, is not necessarily true, even if the original statement is true.

The third and final statement related to the conditional is the **contrapositive**, in which the antecedent and consequent are negated and interchanged. The contrapositive of the first statement above is

If we do not have to put out the garbage, then today is not Monday.

The contrapositive of a statement $p \rightarrow q$ is the statement $\sim q \rightarrow \sim p$. We will show in a moment that the contrapositive is logically equivalent to the original statement, so that if one is true, the other is also true. First, we summarize the three statements related to the original conditional statement below.

Related Conditional Statements

Original Statement	$p \rightarrow q$	(If p, then q.)
Converse	$q \rightarrow p$	(If q, then p.)
Inverse	$\sim p \rightarrow \sim q$	(If not p, then not q.)
Contrapositive	$\sim q \rightarrow \sim p$	(If not q, then not p.)

In the following truth table, we include the conditional $p \rightarrow q$ as well as the three related forms. Notice that the original conditional and the contrapositive are equivalent; that is, whenever one is true, so is the other. Notice also that the converse and the inverse are equivalent to each other. Finally, notice that the original conditional is not equivalent to the converse or the inverse; for example, in line 2 of the table, where p is true and q is false, the original conditional and the contrapositive are both false, but the inverse and converse are true.

		Original	Converse	Inverse	Contrapositive
		Equivalent			Equivalent
			Equivalent		
p	q	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

This discussion is summarized in the following sentence.

Equivalences
The original statement and the contrapositive are equivalent, and the converse and the inverse are equivalent.

EXAMPLE 4 Related Conditional Statements

Consider the following statement:

If Cauchy is a cat, then Cauchy is a mammal.

Write each of the following.

(a) The converse

SOLUTION Let c represent "Cauchy is a cat" and m represent "Cauchy is a mammal." Then the original statement is $c \to m$, and the converse is $m \to c$, or

If Cauchy is a mammal, then Cauchy is a cat.

Notice that in this case the original statement is true, while the converse is false, because Cauchy might be a mammal that is not a cat, such as a horse.

(b) The inverse

SOLUTION The inverse of $c \to m$ is $\sim c \to \sim m$, or

If Cauchy is not a cat, then Cauchy is not a mammal.

The inverse, like the converse, is false in this case.

(c) The contrapositive

SOLUTION The contrapositive of $c \to m$ is $\sim m \to \sim c$, or

If Cauchy is not a mammal, then Cauchy is not a cat.

In this case the contrapositive is true, as it must be if the original statement is true.

YOUR TURN 2 Write the converse, inverse, and contrapositive of the following statement: "If I get another ticket, then I lose my license." Which is equivalent to the original statement?

Biconditionals

As we saw earlier, both the Pythagorean theorem and its converse are true. That is, letting r represent "A triangle is a right triangle" and s represent "The sum of the squares of the two sides equals the square of the hypotenuse," both $s \to r$ and $r \to s$ are true. In words, we say s *if and only if* r (often abbreviated s iff r). (Recall that *only if* is the opposite of *if . . . then*.) This is called the **biconditional** and is written $s \leftrightarrow r$. You can think of the biconditional as being defined by the equivalence

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

or by the following truth table.

Truth Table for the Biconditional *p if and only if q*

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE 5 **Biconditional Statements**

Tell whether each statement is true or false.

(a) George Washington is the first president of the United States if and only if John Adams is the second president of the United States.

SOLUTION Since both component statements are true, the first row of the truth table tells us that this biconditional is true.

(b) Alaska is one of the original 13 states if and only if kangaroos can fly.

SOLUTION Since both component statements are false, the last row of the truth table tells us that this biconditional is true.

(c) $2 + 2 = 4$ if and only if $7 > 10$.

SOLUTION The first component statement is true, but the second is false. The second row of the truth table tells us that this biconditional is false. TRY YOUR TURN 3

YOUR TURN 3 Determine if the following statement is *true* or *false*. "New York City is the capital of the United States if and only if Paris is the capital of France."

Notice in the truth table and the previous example that when p and q have the same truth value, $p \leftrightarrow q$ is true; when p and q have different truth values, $p \leftrightarrow q$ is false.

In this and the previous two sections, truth tables have been derived for several important types of compound statements. The summary that follows describes how these truth tables may be remembered.

Summary of Basic Truth Tables

1. $\sim p$, the **negation** of p, has truth value opposite of p.
2. $p \wedge q$, the **conjunction**, is true only when both p and q are true.
3. $p \vee q$, the **disjunction**, is false only when both p and q are false.
4. $p \rightarrow q$, the **conditional**, is false only when p is true and q is false.
5. $p \leftrightarrow q$, the **biconditional**, is true only when p and q have the same truth value.

6.4 EXERCISES

For each given statement, write (a) the converse, (b) the inverse, and (c) the contrapositive in *if . . . then* form. In some of the exercises, it may be helpful to restate the statement in *if . . . then* form.

1. If the exit is ahead, then I don't see it.
2. If I finish reading this novel, then I'll write a review.
3. If I knew you were coming, I'd have cleaned the house.
4. If I'm the bottom, you're the top.
5. Mathematicians wear pocket protectors.
6. Beggars can't be choosers.
7. $p \rightarrow \sim q$
8. $\sim q \rightarrow \sim p$
9. $p \rightarrow (q \vee r)$ (*Hint:* Use one of De Morgan's Laws as necessary.)
10. $(r \vee \sim q) \rightarrow p$ (*Hint:* Use one of De Morgan's Laws as necessary.)

11. Discuss the equivalences that exist among the direct conditional statement, the converse, the inverse, and the contrapositive.

12. State the contrapositive of "If the square of a natural number is even, then the natural number is even." The two statements must have the same truth value. Use several examples and inductive reasoning to decide whether both are true or both are false.

Write each statement in the form "if *p* . . . then *q*."

13. Your signature implies that you accept the conditions.

14. His tardiness implies that he doesn't care.

15. You can take this course pass/fail only if you have prior permission.

16. You can purchase this stock only if you have $1000.

17. You can skate on the pond when the temperature is below 10°.

18. The party will be stopped when more than 200 people attend.

19. Eating 10 hot dogs is sufficient to make someone sick.

20. Two hours in the desert sun is sufficient to give the typical person a sunburn.

21. A valid passport is necessary for travel to France.

22. Support from the party bosses is necessary to get the nomination.

23. For a number to have a real square root, it is necessary that it be nonnegative.

24. For a number to have a real square root, it is sufficient that it be nonnegative.

25. All brides are beautiful.

26. All passengers for Hempstead must change trains at Jamaica station.

27. A number is divisible by 3 if the sum of its digits is divisible by 3.

28. A number is even if its last digit is even.

29. One of the following statements is not equivalent to all the others. Which one is it?

a. *r* only if *s*.

b. *r* implies *s*.

c. If *r*, then *s*.

d. *r* is necessary for *s*.

30. Use the statement "Being 65 years old is sufficient for being eligible for Medicare" to explain why "*p* is sufficient for *q*" is equivalent to "if *p*, then *q*."

31. Use the statement "Being over 21 is necessary for entering this club" to explain why "*p* is necessary for *q*" is equivalent to "if *q*, then *p*."

32. Explain why the statement "Elephants can fly if and only if Africa is the smallest continent" is true.

Identify each statement as *true* or *false*.

33. $5 = 9 - 4$ if and only if $8 + 2 = 10$.

34. $3 + 1 \neq 6$ if and only if $8 \neq 8$.

35. $8 + 7 \neq 15$ if and only if $3 \times 5 \neq 9$.

36. $6 \times 2 = 14$ if and only if $9 + 7 \neq 16$.

37. China is in Asia if and only if Mexico is in Europe.

38. The moon is made of green cheese if and only if Hawaii is one of the United States.

Construct a truth table for each statement.

39. $(\sim p \wedge q) \leftrightarrow (p \rightarrow q)$

40. $(p \leftrightarrow \sim q) \leftrightarrow (\sim p \vee q)$

APPLICATIONS

Business and Economics

41. Income Tax The following excerpts appear in a guide for preparing income tax reports. Write each of the statements in *if . . . then* form. ***Source: Your Income Tax 2010.***

a. All employer contributions must be reported on Form 8889.

b. Certain tax benefits may be claimed by married persons only if they file jointly.

c. A child is not a qualifying child if he or she provides over half of his or her own support.

42. Income Tax The following excerpt appears in a guide for preparing income tax reports. Write the statement in *if . . . then* form. Find the contrapositive of this statement. ***Source: Your Income Tax 2010.***

A refund of your state or local income tax is not taxable if you did not previously claim the tax as an itemized deduction in a prior year.

43. Credit Cards The following statement appeared in a card member agreement for a Chase Visa Card. Write the converse, inverse, and contrapositive. Which statements are equivalent? ***Source: JP Morgan Chase & Co.***

If your account is in default, we may close your account without notice.

Life Sciences

44. APPLY IT The following statement is from a home medical guide. Write the statement in four different ways, using the common translations of $p \rightarrow q$. (See Example 1.) ***Source: American College of Physicians Complete Home Medical Guide.***

When you sleep well, you wake up feeling refreshed and alert.

45. Polar Bears The following statement is with regard to polar bear cubs. ***Source: National Geographic.***

If there are triplets, the most persistent stands to gain an extra meal and it may eat at the expense of another.

a. Use symbols to write this statement.

b. Write the contrapositive of this statement.

Social Sciences

46. Law The following statements appear in a guide to common laws. Write the statement in *if . . . then* form. ***Source: Law for Dummies.***

a. When you buy a car, you must register it right away.

b. All states have financial responsibility laws.

c. Your insurer pays the legitimate claims of the injured party and defends you in court if you're sued.

47. Law The following statements appear in a guide to common laws. Write the converse, inverse, and contrapositive of the statements. Which statements are equivalent? ***Source: Law for Dummies.***

a. If you are married, then you can't get married again.

b. If you pay for your purchase with a credit card, you are protected by the Fair Credit Billing Act.

c. If you hit a parked car, you're expected to make a reasonable effort to locate the owner.

48. Philosophy Aristotle once said, "If liberty and equality, as is thought by some, are chiefly to be found in democracy, they will be best attained when all persons alike share in the government to the utmost." Write the contrapositive of this statement. ***Source: Bartlett's Familiar Quotations.***

49. Political Development It has been argued that political development in Western Europe will increase if and only if social assimilation is increasing. ***Source: International Organizational.***

a. Express this statement symbolically, and construct a truth table for it.

b. The author of the article quoted above says that it is true that political development in Western Europe is increasing, but it is false that social assimilation is increasing. What can then be said about the original statement?

50. Libya Referring to Libya's offer to pay $2.7 billion in compensation for the families of those killed in the 1988 crash of Pan Am flight 103, a White House official said, "This is a necessary step, but it is not sufficient [for the United States to drop sanctions against Libya]." Letting d represent "the United States drops sanctions against Libya" and l represent "Libya offers to pay compensation," write the White House official's statement as a statement in symbolic logic. ***Source: The New York Times.***

51. Education It has been argued that "a high level of education . . . comes close to being a necessary [condition for democracy]." For the purpose of this exercise, consider "comes close to being" as meaning "is," and assume the statement refers to a country. Write the statement in *if . . . then* form, and then write the converse, inverse, and contrapositive of the statement. Which one of these is equivalent to the original statement? ***Source: The American Political Science Review.***

52. Political Alliances According to political scientist Howard Rosenthal, "the presence of [a Modéré] incumbent can be regarded as a necessary condition for a R.P.F. alliance." Write the statement in *if . . . then* form, and then write the converse, inverse, and contrapositive of the statement. Which one of these is equivalent to the original statement? ***Source: The American Political Science Review.****

*The R.P.F. and Modéré are the names of political parties in France.

53. Test of Reasoning A test devised by psychologist Peter Wason is designed to test how people reason. ***Source: Science News.*** As an example of this test, volunteers are given the rule, "If a card has a D on one side, then it must have a 3 on the other side." Volunteers view four cards displaying D, F, 3, and 7, respectively. They are told that each card has a letter on one side and a number on the other. Which cards do they need to turn over to determine if the rule has been violated? In Wason's experiments, fewer than one-fourth of the participants gave the correct answer.

54. Test of Reasoning In another example of a Wason test (see previous exercise), volunteers are given the rule, "If an employee works on the weekend, then that person gets a day off during the week." Volunteers are given four cards displaying "worked on the weekend," "did not work on the weekend," "did get a day off," and "did not get a day off." Volunteers were told that one side of the card tells whether an employee worked on the weekend, and the other side tells whether an employee got a day off. Which cards must be turned over to determine if the rule has been violated? In a set of experiments, volunteers told to take the perspective of the employees tended to give the correct answer, while volunteers told to take the perspective of employers tended to turn over the second and third card.

General Interest

55. Sayings Rewrite each of the following statements as a conditional in *if . . . then* form. Then write two statements that are equivalent to the *if . . . then* statements. (*Hint:* Write the contrapositive of the statement and rewrite the conditional statement using *or.*)

a. Nothing ventured, nothing gained.

b. The best things in life are free.

c. Every cloud has a silver lining.

56. Sayings Think of some wise sayings that have been around for a long time, and state them in *if . . . then* form.

57. Games Statements similar to the ones below appear in the instructions for various Milton Bradley games. Rewrite each statement in *if . . . then* form and then write an equivalent statement using *or*. ***Source: Milton Bradley Company.***

a. You can score in this box only if the dice show any sequence of four numbers.

b. When two or more words are formed in the same play, each is scored.

c. All words labeled as a part of speech are permitted.

YOUR TURN ANSWERS

1. a. If you are happy, then you clap your hands.
b. If you seek, then you shall find.

2. *Converse:* If I lose my license, then I get another ticket. *Inverse:* If I didn't get another ticket, then I didn't lose my license. *Contrapositive:* If I didn't lose my license, then I didn't get another ticket. The contrapositive is equivalent.

3. False.

6.5 Analyzing Arguments and Proofs

APPLY IT **If I could be in two places at the same time, then I'd come to your game. I did not come to your game, so can I conclude that I can't be in two places at the same time?**

This question will be analyzed using truth tables in Example 2 of this section.

In this section, we will analyze and construct logical arguments, or proofs, that can be used to determine whether a given set of statements produces a sensible conclusion. A logical argument is made up of **premises** and **conclusions**. The premises are statements that we accept for the sake of the argument. It is not the purpose of logic to determine whether or not the premises are actually true. In logic, we suppose the premises to be true and then ask what statements follow using the laws of logic. The statements that follow are the conclusions. The argument is considered *valid* if the conclusions must be true when the premises are true.

Valid and Invalid Arguments

An argument is **valid** if the fact that all the premises are true forces the conclusion to be true. An argument that is not valid is **invalid**, or a **fallacy**.

It is very important to note that *valid* and *true* are not the same—an argument can be valid even though the conclusion is false. (See the discussion after Example 3.)

We will begin by using truth tables to determine whether certain types of arguments are valid or invalid. We will then use the results from these examples to demonstrate a more powerful method of proving that an argument is valid or invalid. As our first example, consider the following argument.

If it's after midnight, then I must go to sleep.
It's after midnight.

I must go to sleep.

Here we use the common method of placing one premise over another, with the conclusion below a line. Alternatively, we could indicate that the last line is a conclusion using "therefore," as in "Therefore, I must go to sleep."

To test the validity of this argument, we begin by identifying the component statements found in the argument. We will use the generic variables p and q here, rather than meaningful variable names, so the argument will have a generic form.

p represents "It's after midnight";
q represents "I must go to sleep."

Now we write the two premises and the conclusion in symbols:

Premise 1: $p \to q$
Premise 2: p

Conclusion: q .

To decide if this argument is valid, we must determine whether the conjunction of both premises implies the conclusion for all possible cases of truth values for p and q. Therefore, write the conjunction of the premises as the antecedent of a conditional statement, and the conclusion as the consequent.

$$\underset{\text{Premise}}{[(p \to q)} \quad \underset{\text{and}}{\wedge} \quad \underset{\text{premise}}{p]} \quad \underset{\text{implies}}{\to} \quad \underset{\text{conclusion.}}{q}$$

Finally, construct the truth table for the conditional statement, as shown on the next page.

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the final column, shown in color, indicates that the conditional statement that represents the argument is true for all possible truth values of p and q, the statement is a tautology. Thus, the argument is valid.

The pattern of the argument in the preceding example,

$$\begin{array}{l} p \to q \\ \underline{p\quad\quad} \\ q\quad\quad , \end{array}$$

is a common one and is called **Modus Ponens**, or the *law of detachment.*

In summary, to test the validity of an argument using a truth table, go through the steps in the box that follows.

Testing the Validity of an Argument with a Truth Table

1. Assign a letter to represent each component statement in the argument.
2. Express each premise and the conclusion symbolically.
3. Form the symbolic statement of the entire argument by writing the *conjunction* of *all* the premises as the antecedent of a conditional statement and the conclusion of the argument as the consequent.
4. Complete the truth table for the conditional statement formed in step 3. If it is a tautology, then the argument is valid; otherwise, it is invalid.

EXAMPLE 1 Determining Validity

Determine whether the argument is *valid* or *invalid.*

If I win the lottery, then I'll buy a new house.
I bought a new house.

I won the lottery.

SOLUTION Let p represent "I win the lottery" and let q represent "I'll buy a new house." Using these symbols, the argument can be written in the form

$$\begin{array}{l} p \to q \\ \underline{q\quad\quad} \\ p\quad\quad . \end{array}$$

To test for validity, construct a truth table for the statement

$$[(p \to q) \wedge q] \to p.$$

p	q	$p \to q$	$(p \to q) \wedge q$	$[(p \to q) \wedge q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The third row of the final column of the truth table shows F, and this is enough to conclude that the argument is invalid. Even if the premises are true, the conclusion that I won the lottery is not necessarily true. Perhaps I bought a new house with money I inherited.

If a conditional and its converse were logically equivalent, then an argument of the type found in Example 1 would be valid. Since a conditional and its converse are *not* equivalent, the argument is an example of what is sometimes called the **Fallacy of the Converse**.

EXAMPLE 2 Determining Validity

Determine whether the argument is *valid* or *invalid.*

If I could be in two places at the same time, I'd come to your game.
I did not come to your game.

I cannot be in two places at the same time.

APPLY IT

SOLUTION If p represents "I could be in two places at the same time" and q represents "I'd come to your game," the argument becomes

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \sim p \quad . \end{array}$$

The symbolic statement of the entire argument is

$$[(p \to q) \land \sim q] \to \sim p.$$

The truth table for this argument, shown below, indicates a tautology, and the argument is valid.

p	q	$p \to q$	$\sim q$	$(p \to q) \land \sim q$	$\sim p$	$[(p \to q) \land \sim q] \to \sim p$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

The pattern of reasoning of this example is called **Modus Tollens**, or the *law of contraposition.*

With reasoning similar to that used to name the fallacy of the converse, the fallacy

$$\begin{array}{l} p \to q \\ \underline{\sim p} \\ \sim q \end{array}$$

is often called the **Fallacy of the Inverse**. An example of such a fallacy is "If it rains, I get wet. It doesn't rain. Therefore, I don't get wet."

EXAMPLE 3 Determining Validity

Determine whether the argument is *valid* or *invalid.*

I'll win this race or I'll eat my hat.
I didn't win this race.

I'll eat my hat.

SOLUTION Let p represent "I'll win this race" and let q represent "I'll eat my hat." Using these symbols, the argument can be written in the form

$$\begin{array}{l} p \lor q \\ \underline{\sim p} \\ q \quad . \end{array}$$

Set up a truth table for

$$[(p \lor q) \land \sim p] \to q.$$

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$[(p \vee q) \wedge \sim p] \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

The statement is a tautology and the argument is valid. Any argument of this form is valid by the law of **Disjunctive Syllogism**.

Suppose you notice a few months later that I didn't win the race or eat my hat. You object that the conclusion of the argument in Example 3 is false. Did that make the argument invalid? No. The problem is that I lied about the first premise, namely, that I would win the race or eat my hat. A valid argument only guarantees that *if* all the premises are true, then the conclusion must also be true. Whether the premises of an argument are actually true is a separate issue. I might have a perfectly valid argument, yet you disagree with my conclusion because you disagree with one or more of my premises.

EXAMPLE 4 Determining Validity

Determine whether the following argument is *valid* or *invalid*.

If you make your debt payments late, you will damage your credit record.
If you damage your credit record, you will have difficulty getting a loan.

If you make your debt payments late, you will have difficulty getting a loan.

SOLUTION Let p represent "you make your debt payments late," let q represent "you will damage your credit record," and let r represent "you will have difficulty getting a loan." The argument takes on the general form

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ p \rightarrow r. \end{array}$$

Make a truth table for the following statement:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r).$$

It will require eight rows.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

This argument is valid since the final statement is a tautology. The pattern of argument shown in this example is called **Reasoning by Transitivity**, or the *law of hypothetical syllogism.*

A summary of the valid and invalid forms of argument presented so far is given on the following page.

Valid Argument Forms

Modus Ponens	Modus Tollens	Disjunctive Syllogism	Reasoning by Transitivity
$p \to q$	$p \to q$	$p \vee q$	$p \to q$
p	$\sim q$	$\sim p$	$q \to r$
q	$\sim p$	q	$p \to r$

Invalid Argument Forms (Fallacies)

Fallacy of the Converse	Fallacy of the Inverse
$p \to q$	$p \to q$
q	$\sim p$
p	$\sim q$

When an argument contains three or more premises, it will be necessary to determine the truth values of the conjunction of all of them. Remember that if *at least one* premise in a conjunction of several premises is false, then the entire conjunction is false. This will be used in the next example.

EXAMPLE 5 Determining Validity

Determine whether the following argument is *valid* or *invalid.*

If Ed reaches the semifinals, then Liz will be happy.
If Liz is not happy, then Roxanne will bake her a cake.
Roxanne did not bake Liz a cake.
Therefore, Ed reaches the semifinals.

SOLUTION Let e represent "Ed reaches the semifinals," l represent "Liz is happy," and r represent "Roxanne bakes Liz a cake." The symbolic form of the argument is as follows.

1. $e \to l$ Premise
2. $\sim l \to r$ Premise
3. $\sim r$ Premise
e Conclusion

Our strategy will be to apply the valid argument forms either to the premises or to conclusions that have already been reached. Notice, for example, that we can apply Modus Tollens to lines 2 and 3 to reach $\sim(\sim l) \equiv l$. We write this as follows, giving the statements and the argument form used.

4. l 2, 3, Modus Tollens

From statements 1 and 4, can we conclude e? Only if we use the Fallacy of the Converse! Thus this argument appears to be invalid.

Someone might object that the argument is actually valid by a clever proof that we haven't discovered. We could refute this objection by creating a truth table, as we did in Examples 1 through 4. A less tedious approach is to find a truth value for all the components e, l, and r that make the premises true and the conclusion false. If we can do that, we know for certain that the conclusion does not follow from the premises. Write the premises and the conclusion with the desired truth value next to each.

YOUR TURN 1 Determine whether the following argument is *valid* or *invalid*. You watch television tonight or you write your paper tonight. If you write your paper tonight, then you will get a good grade. You get a good grade. Therefore, you did not watch television tonight.

1.	$e \to l$	T
2.	$\sim l \to r$	T
3.	$\sim r$	T
	e	F

In statement 3, to make $\sim r$ true, we know that r must be false. From the conclusion, e must be false. Statement 1 is then automatically true. To make statement 2 true, simply make l true, so $\sim l$ is false. In summary, the following assignment of truth values make all the premises true, yet the conclusion is false.

e = "Ed reaches the semifinals" = F
l = "Liz is happy" = T
r = "Roxanne bakes Liz a cake" = F

TRY YOUR TURN 1

Let's summarize what we have seen.

- To show that an argument is valid, prove it using the four valid argument forms discussed in this section. We can also use any of the laws given in the Equivalent Statement box in Section 6.3.
- To show that an argument is invalid, give an assignment of truth values that makes the premises true and the conclusion false.

It's not always clear which of these two strategies you should try first, but if one doesn't work, try the other. An argument may be valid, but it may not be provable using only the rules learned so far. In the next section we will study additional rules for proving arguments. When all else fails, you can always create a truth table, but if the argument has many variables, this could be very tedious.

EXAMPLE 6 Determining Validity

Lewis Carroll* gave humorous logic puzzles in his book *Symbolic Logic*. In each puzzle, he presented several premises, and the reader was to find valid conclusions. Here is one of his puzzles. What is the valid conclusion? ***Source: The Complete Works of Lewis Carroll.***

Babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

SOLUTION First, write each premise in *if . . . then* form.

If you are a baby, then you are illogical.
If you can manage a crocodile, then you are not despised.
If you are illogical, then you are despised.

Let b represent "you are a baby," i represent "you are illogical," m represent "you can manage a crocodile," and d represent "you are despised." The statements can then be written symbolically as

1.	$b \to i$	Premise
2.	$m \to \sim d$	Premise
3.	$i \to d$	Premise

Notice that we can combine the first and third statements using reasoning by transitivity.

4.	$b \to d$	1, 3, Transitivity

*Lewis Carroll is the pseudonym for Charles Dodgson (1832–1898), mathematician and author of *Alice in Wonderland*.

How can we combine this with statement 2, which has not yet been used? Both statements 2 and 4 have d at the end, but in statement 2 the d is negated. Use the contrapositive to get statement 2 in a more useful form.

5. $d \rightarrow \sim m$ 2, Contrapositive

Now we can use transitivity.

6. $b \rightarrow \sim m$ 4, 5, Transitivity

In words, the conclusion is "If you are a baby, then you cannot manage a crocodile," or, as Lewis Carroll put it, "Babies cannot manage crocodiles."

Often there are different ways to do a proof. In the previous example, you might first apply contrapositive to statement 2, and then combine the result with statement 3 using transitivity, and finally combine that result with statement 1 using transitivity. In the exercises in this textbook, your proofs might look different from those in the back of the book but still be correct.

EXAMPLE 7 Determining Validity

Determine whether the following argument is *valid* or *invalid.*

If tomorrow is Saturday and sunny, then it is a beach day.
Tomorrow is Saturday.
Tomorrow is not a beach day.
Therefore, tomorrow must not be sunny.

SOLUTION Let t represent "tomorrow is Saturday," s represent "tomorrow is sunny," and b represent "tomorrow is a beach day." We give a proof that this argument is valid.

1. $(t \wedge s) \rightarrow b$ Premise
2. t Premise
3. $\sim b$ Premise
4. $\sim(t \wedge s)$ 1, 3, Modus Tollens
5. $\sim t \vee \sim s$ 4, De Morgan's Law
6. $\sim s$ 2, 5, Disjunctive Syllogism

How did we figure out this proof? We started by looking for a rule of logic that combines two of the three premises. Notice that statement 3 has the negation of the right side of statement 1; this is a clue that Modus Tollens might be helpful. Once we get statement 4, using one of De Morgan's Laws is an obvious choice, as it should be whenever we see a negation over $\wedge$ or $\vee$. Statement 2 still hasn't been used at this point, but notice that it is the negation of one part of the $\vee$ in statement 5, leading us to try Disjunctive Syllogism.

TRY YOUR TURN 2

YOUR TURN 2 Determine whether the following argument is *valid* or *invalid.* You will not put money in the parking meter or you will not buy a cup of coffee. If you do not put money in the parking meter, you will get a ticket. You did not get a ticket. Therefore, you did not buy a cup of coffee.

EXAMPLE 8 Determining Validity

Determine whether the following argument is *valid* or *invalid.*

If your teeth are white, you use Extreme Bright toothpaste.
If your teeth are white, you will be more attractive.
Therefore, if you use Extreme Bright toothpaste, you will be more attractive.

SOLUTION Let w represent "your teeth are white," t represent "you use Extreme Bright toothpaste," and a represent "you will be more attractive."

In symbolic form, this argument can be written as follows.

$$\begin{array}{l} w \to t \\ \underline{w \to a} \\ t \to a \end{array}$$

This looks like a misguided attempt at using reasoning by transitivity and so appears invalid. To show this, we need to find a way to make the premises true and the conclusion false. The only way the conclusion, $t \to a$, can be false is if t is true and a is false. Both premises will automatically be true if we make w false. Thus the argument is invalid. The following assignment of truth values make all the premises true and the conclusion false.

w = "your teeth are white" = F
t = "you use Extreme Bright toothpaste" = T
a = "you will be more attractive" = F

6.5 EXERCISES

Each of the following arguments is either valid by one of the forms of valid arguments discussed in this section or is a fallacy by one of the forms of invalid arguments discussed. (See the summary boxes.) Decide whether the argument is *valid* or *invalid*, and give the form that applies.

1. If she weighs the same as a duck, she's made of wood.
If she's made of wood, she's a witch.

If she weighs the same as a duck, she's a witch.

2. If passing is out of the question, there's no point in going to class.
If there's no point in going to class, then attending college makes no sense.

If passing is out of the question, then attending college makes no sense.

3. If I had the money, I'd go on vacation.
I have the money.

I go on vacation.

4. If I were a rabbit, I'd hop away.
I am a rabbit.

I hop away.

5. If you want to make trouble, the door is that way.
The door is that way.

You want to make trouble.

6. If you finish the test, you can leave early.
You can leave early.

You finish the test.

7. If Andrew Crowley plays, the opponent gets shut out.
The opponent does not get shut out.

Andrew Crowley does not play.

8. If you want to follow along, we're on p. 315.
We're not on p. 315.

You don't want to follow along.

9. "If we evolved a race of Isaac Newtons, that would not be progress." (quote from Aldous Huxley)
We have not evolved a race of Isaac Newtons.

That is progress.

10. "If I have seen farther than others, it is because I stood on the shoulders of giants." (quote from Sir Isaac Newton)
I have not seen farther than others.

I have not stood on the shoulders of giants.

11. Something is rotten in the state of Denmark, or my name isn't Hamlet.
My name is Hamlet.

Something is rotten in the state of Denmark.

12. "We shall conquer together or we shall die together." (quote from Winston Churchill)
We shall not die together.

We shall conquer together.

Determine whether each argument is valid or invalid. If it is valid, give a proof. If it is invalid, give an assignment of truth values to the variables that makes the premises true and the conclusion false.

13. $p \vee q$
$\underline{p}$
$\sim q$

14. $p \vee \sim q$
$\underline{p}$
$\sim q$

15. $p \to q$
$\underline{q \to p}$
$p \wedge q$

16. $\sim p \to q$
$\underline{p}$
$\sim q$

17. $\sim p \to \sim q$
$\underline{q}$
p

18. $p \to \sim q$
$\underline{q}$
$\sim p$

19. $p \to q$
$\sim q$
$\sim p \to r$
r

20. $p \vee q$
$\sim p$
$r \to \sim q$
$\sim r$

21. $p \to q$
$q \to r$
$\sim r$
$\sim p$

22. $p \to q$
$r \to \sim q$
$p \to \sim r$

23. $p \to q$
$q \to \sim r$
p
$r \vee s$
s

24. $p \to q$
$\sim p \to r$
$s \to \sim q$
$\sim r \to \sim s$

Use a truth table, similar to those in Examples 1–4, to prove each rule of logic. The rules in Exercises 25–27 are known as *simplification*, *amplification*, and *conjunction*, respectively.

25. $p \wedge q$
p

26. p
$p \vee q$

27. p
q
$p \wedge q$

28. Lori Hales made the following observation: "If I want to determine whether an argument leading to the statement

$$[(p \to q) \wedge \sim q] \to \sim p$$

is valid, I only need to consider the lines of the truth table that lead to T for the column headed $(p \to q) \wedge \sim q$." Lori was very perceptive. Can you explain why her observation was correct?

APPLICATIONS

For Exercises 29–37, determine whether each of the following arguments is *valid* or *invalid*. If it is valid, give a proof. If it is invalid, give an assignment of truth values to the variables that makes the premises true and the conclusion false.

Business and Economics

29. Investment If Alex invests in AT&T, Sophia will invest in Sprint Nextel. Victor will invest in Verizon or Alex invests in AT&T. Victor will not invest in Verizon. Therefore, Sophia will invest in Sprint Nextel.

30. Credit If you make your debt payments late, you will damage your credit record. If you damage your credit record, you will have trouble getting a loan. You do not have trouble getting a loan. Therefore, you do not make your debt payments late.

31. Stock Market It is a bearish market. If prices are rising, then it is not a bearish market. If prices are not rising, then the investor will sell stocks. Therefore, the investor will not sell stocks.

Life Sciences

32. Classification If the animal is a reptile, then it belongs to the chordate phylum. The animal either belongs to the echinoderm phylum or the chordate phylum. It does not belong to the echinoderm phylum. Therefore, it is not a reptile.

33. Classification The animal is a spider or it is an insect. If it is a spider, then it has eight legs and it has two main body parts. It does not have eight legs or it does not have two main body parts. Therefore, it is an insect.

Social Sciences

34. Politics If Boehme runs for senator, then Hoffman will run for governor. If Tobin runs for congressman, then Hoffman will not run for governor. Therefore, if Tobin runs for congressman, then Boehme will not run for senator.

General Interest

35. Marriage If I am married to you, then we are one. If we are one, then you are not really a part of me. We are one or you are really a part of me. Therefore, if I am married to you, then you are really a part of me.

36. Love If I were your husband and you were my wife, then I'd never stop loving you. I've stopped loving you. Therefore, I am not your husband or you are not my wife.

37. Baseball The Yankees will be in the World Series or the Phillies won't be there. If the Phillies are not in the World Series, the National League cannot win. In fact, the National League wins. Therefore, the Yankees are in the World Series.

38. Time Suppose that you ask someone for the time and you get the following response:

> "If I tell you the time, then we'll start chatting. If we start chatting, then you'll want to meet me at a truck stop. If we meet at a truck stop, then we'll discuss my family. If we discuss my family, then you'll find out that my daughter is available for marriage. If you find out that she is available for marriage, then you'll want to marry her. If you want to marry her, then my life will be miserable since I don't want my daughter married to some fool who can't afford a $20 watch."

Use Reasoning by Transitivity to draw a valid conclusion.

Lewis Carroll Exercises 39–44 are from problems in Lewis Carroll's book *Symbolic Logic*. Write each premise in symbols, and then give a conclusion that uses all the premises and yields a valid argument. *Source: The Complete Works of Lewis Carroll.*

39. Let d be "it is a duck," p be "it is my poultry," o be "one is an officer," and w be "one waltzes."

a. No ducks waltz.

b. No officers ever decline to waltz.

c. All my poultry are ducks.

d. Give a conclusion that yields a valid argument.

40. Let l be "one is able to do logic," j be "one is fit to serve on a jury," s be "one is sane," and y be "he is your son."

a. Everyone who is sane can do logic.

b. No lunatics are fit to serve on a jury.

c. None of your sons can do logic.

d. Give a conclusion that yields a valid argument.

41. Let h be "one is honest," p be "one is a pawnbroker," b be "one is a promise breaker," t be "one is trustworthy," c be "one is very communicative," and w be "one is a wine drinker."

a. Promise breakers are untrustworthy.

b. Wine drinkers are very communicative.

c. A man who keeps his promise is honest.

d. No teetotalers are pawnbrokers. (*Hint:* Assume "teetotaler" is the opposite of "wine drinker.")

e. One can always trust a very communicative person.

f. Give a conclusion that yields a valid argument.

42. Let g be "it is a guinea pig," i be "it is hopelessly ignorant of music," s be "it keeps silent while the *Moonlight Sonata* is being played," and a be "it appreciates Beethoven."

a. Nobody who really appreciates Beethoven fails to keep silent while the *Moonlight Sonata* is being played.

b. Guinea pigs are hopelessly ignorant of music.

c. No one who is hopelessly ignorant of music ever keeps silent while the *Moonlight Sonata* is being played.

d. Give a conclusion that yields a valid argument.

43. Let s be "it begins with 'Dear Sir'," c be "it is crossed," d be "it is dated," f be "it is filed," i be "it is in black ink," t be "it is in the third person," r be "I can read it," p be "it is on blue paper," o be "it is on one sheet," and b be "it is written by Brown."

a. All the dated letters in this room are written on blue paper.

b. None of them are in black ink, except those that are written in the third person.

c. I have not filed any of them that I can read.

d. None of them that are written on one sheet are undated.

e. All of them that are not crossed are in black ink.

f. All of them written by Brown begin with "Dear Sir."

g. All of them written on blue paper are filed.

h. None of them written on more than one sheet are crossed.

i. None of them that begin with "Dear Sir" are written in the third person.

j. Give a conclusion that yields a valid argument.

44. Let p be "he is going to a party," b be "he brushes his hair," s be "he has self-command," l be "he looks fascinating," o be "he is an opium eater," t be "he is tidy," and w be "he wears white kid gloves."

a. No one who is going to a party ever fails to brush his hair.

b. No one looks fascinating if he is untidy.

c. Opium eaters have no self-command.

d. Everyone who has brushed his hair looks fascinating.

e. No one wears white kid gloves unless he is going to a party. (*Hint:* "not a unless b" $\equiv a \rightarrow b$.)

f. A man is always untidy if he has no self-command.

g. Give a conclusion that yields a valid argument.

YOUR TURN ANSWERS

1. Invalid;

t = "you watch television tonight" = T,

p = "you write your paper tonight" = F,

g = "you get a good grade" = T.

2. Valid

1.	$\sim m \vee \sim c$	Premise
2.	$\sim m \rightarrow t$	Premise
3.	$\sim t$	Premise
4.	m	2,3, Modus Tollens
5.	$\sim c$	1,4 Disjunctive Syllogism

6.6 Analyzing Arguments with Quantifiers

APPLY IT

If some U.S. presidents won the popular vote and George W. Bush is the U.S. president, did he win the popular vote?

This question will be analyzed in Example 6 of this section.

There is a subtle but important point that we have ignored in some of the logic puzzles of the previous section. Consider the statement "Babies are illogical" in Example 6 in that section. We reinterpreted this statement as "If you are a baby, then you are illogical." Letting b represent "you are a baby" and i represent "you are illogical," we wrote this statement as $b \rightarrow i$. This was perfectly adequate for solving that puzzle. But now suppose we want to add two more premises: "Madeleine is a baby" and "Noa is a baby." It would seem that two valid conclusions are: "Madeleine is illogical" and "Noa is illogical." But how do we write these statements symbolically? We can't very well have b represent both "Madeleine is a baby" and "Noa is a baby." We might try letting these two statements be represented by m and n, but then it's not clear how we can combine m or n with $b \rightarrow i$.

Logicians solve this problem with the use of **quantifiers**. The words *all*, *each*, *every*, and *no(ne)* are called **universal quantifiers**, while words and phrases such as *some, there exists*, and *(for) at least one* are called **existential quantifiers**. Quantifiers are used extensively in mathematics to indicate *how many* cases of a particular situation exist.

In the previous example, we could let $b(x)$ represent "x is a baby," where x could represent any baby. Then $b(m)$ could represent "Madeleine is a baby" and $b(n)$ could represent "Noa is a baby." We want to say that $b(x) \rightarrow i(x)$ *for all* x. Logicians use the symbol $\forall$ to represent "for all." We could then represent the statement "Babies are illogical" as

$$\forall x\,[b(x) \rightarrow i(x)].$$

NOTE
The important point to note here is that when a statement contains the word "all," a universal quantifier ($\forall$) is usually called for, and when it contains the word "some," an existential quantifier ($\exists$) is usually called for.

Suppose, instead, that we don't want to claim that all babies are illogical, but that some babies are illogical, or, equivalently, that there exists someone who is a baby and who is illogical. Logicians use the symbol $\exists$ to represent "there exists." We could then write the statement "Some babies are illogical" as

$$\exists x\,[b(x) \wedge i(x)].$$

Negation of Quantifiers

We must be careful in forming the negation of a statement involving quantifiers. Suppose we wish to say that the statement "Babies are illogical" is false. This is *not* the same as saying "Babies are logical." Maybe some babies are illogical and some are logical. Let $b(x)$ represent "x is a baby" and $i(x)$ represent "x is illogical." To deny the claim that for all x, the statement $b(x) \rightarrow i(x)$ is true, is equivalent to saying that for some x, the statement $b(x) \rightarrow i(x)$ is false. Thus,

$$\sim\{\forall x\,[b(x) \rightarrow i(x)]\} \equiv \exists x\,\{\sim[b(x) \rightarrow i(x)]\}.$$

Recall from Section 6.3 that the right side of this last statement is equivalent to $\exists x\,\{[b(x) \wedge \sim i(x)]\}$. Therefore,

$$\sim\{\forall x\,[b(x) \rightarrow i(x)]\} \equiv \exists x\,\{[b(x) \wedge \sim i(x)]\}.$$

This last statement makes intuitive sense; it says that if it's false that all babies are illogical, then there exists someone who is a baby and who is not illogical. In other words, some babies are logical.

In a similar way, suppose we wish to say that the statement "Some babies are illogical" is false. To deny the existence of an x such that $b(x) \wedge i(x)$ is the same as saying that for all x, the statement $b(x) \wedge i(x)$ is false. Thus,

$$\sim\{\exists x\,[b(x) \wedge i(x)]\} \equiv \forall x\,\{\sim[b(x) \wedge i(x)]\}.$$

Recall from Section 6.3 that $\sim(p \rightarrow q) \equiv p \wedge \sim q$, so negating both sides of this equivalence and replacing q with $\sim q$ implies $\sim(p \wedge q) \equiv p \rightarrow \sim q$. We can use this fact to rewrite the previous equivalence as

$$\sim\{\exists x\,[b(x) \wedge i(x)]\} \equiv \forall x\,\{[b(x) \rightarrow \sim i(x)]\}.$$

This equivalence makes intuitive sense. It says that if it's false that some babies are illogical, then all babies are not illogical, which can be phrased more clearly as "All babies are logical."

Let's summarize the above discussion.

Negations of Quantified Statements

Statement	Symbolic	Negation	Symbolic for Negation
$s(x)$ is true for all x.	$\forall x\,[s(x)]$	$s(x)$ is false for some x.	$\exists x\,[\sim s(x)]$
$s(x)$ is true for some x.	$\exists x\,[s(x)]$	$s(x)$ is false for all x.	$\forall x\,[\sim s(x)]$

Essentially, negation changes a $\forall$ into a $\exists$ and vice versa.

EXAMPLE 1 Negation

Write each statement symbolically. Then write the negation symbolically, and translate the negation back into words.

(a) Some Texans eat quiche.

SOLUTION Let $t(x)$ represent "x is a Texan" and $q(x)$ represent "x eats quiche." The statement can be written as

$$\exists x\,[t(x) \wedge q(x)].$$

Its negation is

$$\forall x\,\{\sim[t(x) \wedge q(x)]\},$$

which is equivalent to

$$\forall x\,[t(x) \rightarrow \sim q(x)].$$

In words, "All Texans don't eat quiche," which can be expressed more clearly as "No Texan eats quiche."

(b) Some Texans do not eat quiche.

SOLUTION Using the notation from part (a), the statement can be written as

$$\exists x\,[t(x) \wedge \sim q(x)].$$

Its negation is

$$\forall x\,\{\sim[t(x) \wedge \sim q(x)]\},$$

which is equivalent to

$$\forall x\,[t(x) \rightarrow q(x)].$$

In words, "All Texans eat quiche."

(c) No Texans eat quiche.

SOLUTION This is the same as saying "All Texans do not eat quiche," which can be written as

$$\forall x\,[t(x) \rightarrow \sim q(x)].$$

Its negation is

$$\exists x\{\sim[t(x) \rightarrow \sim q(x)]\},$$

which is equivalent to

$$\exists x\,[t(x) \wedge q(x)].$$

In words, "Some Texans eat quiche." Notice that the statements in parts (a) and (c) are negations of each other. TRY YOUR TURN 1

YOUR TURN 1 Write each statement symbolically. Then write the negation symbolically, and translate the negation back into words.

a. All college students study.
b. Some professors are not organized.

Just as we had rules for analyzing arguments without quantifiers in the last section, there are also rules for analyzing arguments with quantifiers. Instead of presenting such rules, we will instead demonstrate a visual technique based on **Euler diagrams**, illustrated in the following examples. Euler (pronounced "oiler") diagrams are named after the great Swiss mathematician Leonhard Euler (1707–1783).

EXAMPLE 2 Determining Validity

Represent the following argument symbolically. Is the argument valid?

All elephants have wrinkles.
Babar is an elephant.

Babar has wrinkles.

SOLUTION If we let $e(x)$ represent "x is an elephant," $w(x)$ represent "x has wrinkles," and b represent "Babar," we could represent the argument symbolically as follows.

$$\begin{array}{l} \forall x\,[e(x) \rightarrow w(x)] \\ \underline{e(b)} \\ w(b) \end{array}$$

Notice that this argument resembles Modus Ponens from the previous section, but with a quantifier, so we suspect that the argument is valid. To verify our suspicion, we draw regions to represent the first premise. One is the region for "elephants." Since all elephants have wrinkles, the region for "elephants" goes inside the region for "have wrinkles," as in Figure 14(a).

The second premise, "Babar is an elephant," suggests that "Babar" should go inside the region representing "elephants." Let b represent "Babar." Figure 14(b) shows that "Babar" is also inside the region for "have wrinkles." Therefore, if both premises are true, the conclusion that "Babar has wrinkles" must also be true. Euler diagrams illustrate that this argument is valid. TRY YOUR TURN 2

YOUR TURN 2 Represent the following argument symbolically. Is the argument valid?
All insects are arthropods.
A bee is an insect.
A bee is an arthropod.

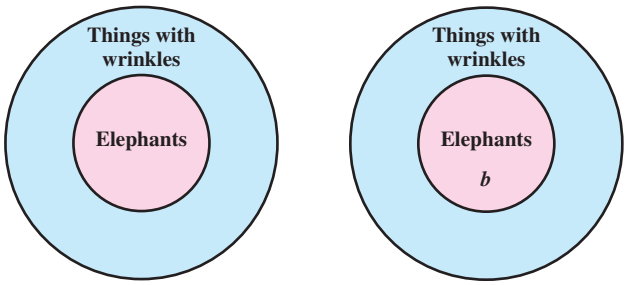


FIGURE 14

EXAMPLE 3 Determining Validity

Represent the following argument symbolically. Is the argument valid?

All frogs are slimy.
Kermit is not slimy.
Kermit is not a frog.

SOLUTION If we let $f(x)$ represent "x is a frog," $s(x)$ represent "x is slimy," and k represent "Kermit," we could represent the argument symbolically as follows.

$$\begin{array}{l} \forall x\,[f(x) \rightarrow s(x)] \\ \underline{\sim s(k)} \\ \sim f(k) \end{array}$$

This argument resembles Modus Tollens with a quantifier, so it is probably valid. We will use an Euler diagram to verify this.

In Figure 15(a) on the next page, the region for "frogs" is drawn entirely inside the region for "slimy." Since "Kermit is *not* slimy," place a k for "Kermit" *outside* the region for "slimy." (See Figure 15(b).) Placing the k outside the region for "slimy" automatically places it outside the region for "frogs." Thus, if the first two premises are true, the conclusion that Kermit is not a frog must also be true, so the argument is valid. TRY YOUR TURN 3

YOUR TURN 3 Represent the following argument symbolically. Is the argument valid?
All birds have wings.
Rover does not have wings.
Rover is not a bird.

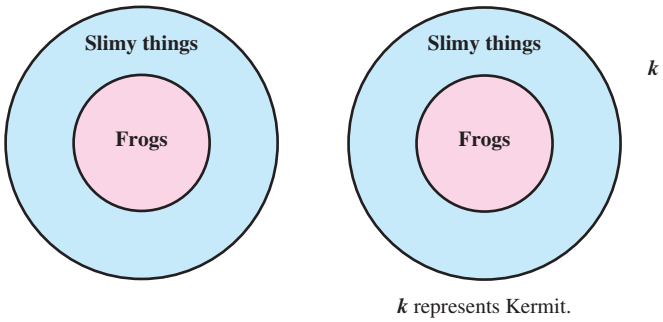


FIGURE 15

EXAMPLE 4 Determining Validity

Represent the following argument symbolically. Is the argument valid?

All well-run businesses generate profits.
Monsters, Inc., generates profits.
Monsters, Inc., is a well-run business.

SOLUTION If we let $w(x)$ represent "x is a well-run business," $p(x)$ represent "x generates profits," and m represent "Monsters, Inc.," we could represent the argument symbolically as follows.

$$\forall x\,[w(x) \rightarrow p(x)]$$
$$p(m)$$
$$\overline{w(m)}$$

This argument resembles the Fallacy of the Converse with a quantifier. We will use an Euler diagram to verify that the argument is invalid.

The region for "well-run business" goes entirely inside the region for "generates a profit." (See Figure 16.) It is not clear where to put the m for "Monsters, Inc." It must go inside the region for "generates profits," but it could go inside or outside the region "well-run business." Even if the premises are true, the conclusion may or may not be true, so the argument is invalid. TRY YOUR TURN 4

YOUR TURN 4 Represent the following argument symbolically. Is the argument valid?
Every man has his price.
Sam has his price.
Sam is a man.

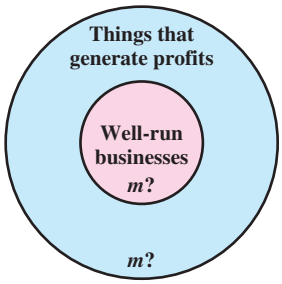


FIGURE 16

As we mentioned in the previous section, the validity of an argument is not the same as the truth of its conclusion. The argument in Example 4 was invalid, but the conclusion "Monsters, Inc." may or may not be true. We cannot make a valid conclusion.

EXAMPLE 5 Determining Validity

Represent the following argument symbolically. Is the argument valid?

All squirrels eat nuts.
All those who eat nuts are healthy.
All who are healthy avoid cigarettes.
All squirrels avoid cigarettes.

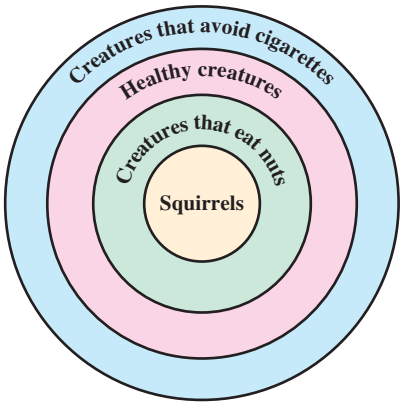


FIGURE 17

SOLUTION If we let $s(x)$ represent "x is a squirrel," $e(x)$ represent "x eats nuts," $h(x)$ represent "x is healthy," and $a(x)$ represent "x avoids cigarettes," we could represent the argument symbolically as follows.

$$\begin{array}{l} \forall x\,[s(x) \rightarrow e(x)] \\ \forall x\,[e(x) \rightarrow h(x)] \\ \underline{\forall x\,[h(x) \rightarrow a(x)]} \\ \forall x\,[s(x) \rightarrow a(x)] \end{array}$$

This argument should remind you of Reasoning by Transitivity. We will use the Euler diagram in Figure 17 to verify that the argument is valid. If each premise is true, then the conclusion must be true because the region for "squirrels" lies completely within the region for "avoid cigarettes." Thus, the argument is valid.

Our last example in this section illustrates an argument with the word "some," which means that an existential quantifier is needed.

EXAMPLE 6 Determining Validity

Represent the following argument symbolically. Is the argument valid?

Some U.S. presidents have won the popular vote.
George W. Bush is president of the United States.
George W. Bush won the popular vote.

APPLY IT

SOLUTION If we let $p(x)$ represent "x is a U.S. president," $v(x)$ represent "x won the popular vote," and w represent "George W. Bush," we could represent the argument symbolically as follows.

$$\begin{array}{l} \exists x\,[p(x) \wedge v(x)] \\ \underline{p(w)} \\ v(w) \end{array}$$

This argument doesn't resemble any of those in the previous section. An Euler diagram might help us see whether this argument is valid or not. The first premise is sketched in Figure 18(a). We have indicated that some U.S. presidents have won the popular vote by putting an x in the region that belongs to both the set of presidential candidates who have won the popular vote and the set of U.S. presidents. There are two possibilities for w, as shown in Figure 18(b).

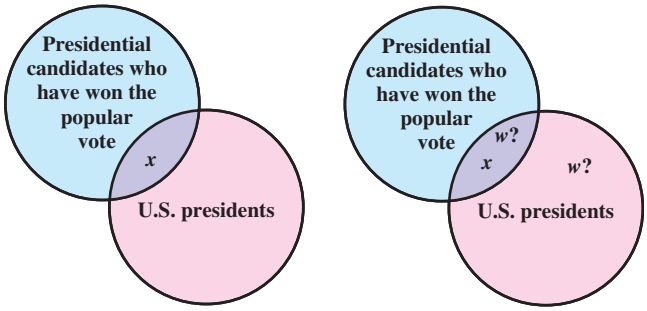


FIGURE 18

One possibility is that Bush won the popular vote; the other is that Bush did not win the popular vote. Since the truth of the premises does not force the conclusion to be true, the argument is invalid.

TRY YOUR TURN 5

YOUR TURN 5 Represent the following argument symbolically. Is the argument valid?
Some vegetarians eat eggs.
Sarah is a vegetarian.
Sarah eats eggs.

This argument is not valid regardless of whether George W. Bush won the popular vote. In fact, in 2000 George W. Bush was elected president even though he received 50,456,002 votes, compared with 50,999,897 for Al Gore. Bush did win the popular vote in 2004. ***Source: Federal Election Commission.***

6.6 EXERCISES

For Exercises 1–6, (a) write the statement symbolically, (b) write the negative of the statement in part (a), and (c) translate your answer from part (b) into words.

1. Some books are bestsellers.

2. Every dog has his day.

3. No CEO sleeps well at night.

4. There's no place like Alaska.

5. All the leaves are brown.

6. Some days are better than other days.

In Exercises 7–20, (a) represent the argument symbolically, and (b) use an Euler diagram to determine if the argument is valid.

7. Graduates want to find good jobs.
Theresa Cortesini is a graduate.
Theresa Cortesini wants to find a good job.

8. All sophomores have earned at least 60 credits.
Lucy Banister is a sophomore.
Lucy Banister has earned at least 60 credits.

9. All professors are covered with chalk dust.
Otis Taylor is covered with chalk dust.
Otis Taylor is a professor.

10. All dinosaurs are extinct.
The dodo is extinct.
The dodo is a dinosaur.

11. All accountants use spreadsheets.
Nancy Hart does not use spreadsheets.
Nancy Hart is not an accountant.

12. All fish have gills.
Whales do not have gills.
Whales are not fish.

13. Some people who are turned down for a mortgage have a second income.
All people who are turned down for a loan need a mortgage broker.
Some people with a second income need a mortgage broker.

14. Some residents of Minnesota don't like snow.
All skiers like snow.
Some residents of Minnesota are not skiers.

15. Some who wander are lost.
Marty McDonald wanders.
Marty McDonald is lost.

16. Some old houses have root cellars.
My house has a root cellar.
My house is old.

17. Some psychologists are university professors.
Some psychologists have a private practice.
Some university professors have a private practice.

18. Someone who is responsible must pay for this.
Neil Hunnewell is responsible.
Neil Hunnewell must pay for this.

19. Everybody is either a saint or a sinner.
Some people are not saints.
Some people are sinners.

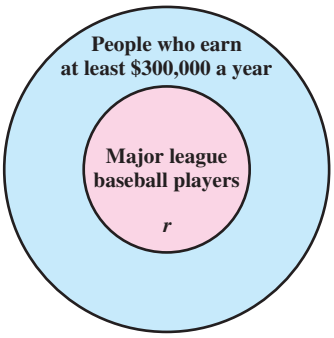
20. If you're here for the first time, we want to make you feel welcome.
If you're here after being gone for a while, we want to make you feel welcome.
Some people are here for the first time or are here after being gone for a while.
There are some people here whom we want to make feel welcome.

21. Refer to Example 4. If the second premise and the conclusion were interchanged, would the argument then be valid?

22. Refer to Example 5. Give a different conclusion than the one given there, so that the argument is still valid.

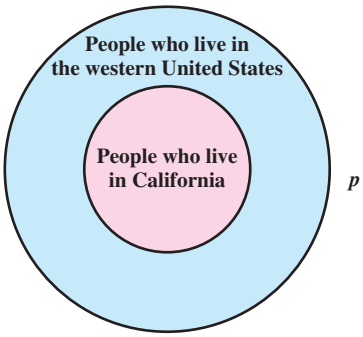
Construct a valid argument based on the Euler diagram shown.

23.



r represents Ryan Howard.

24.



p represents Phyllis Crittenden.

As mentioned in the text, an argument can have a true conclusion yet be invalid. In these exercises, each argument has a true conclusion. Identify each argument as valid or invalid.

25. All houses have roofs.
All roofs have nails.
All houses have nails.

26. All platypuses have bills.
All ducks have bills.
A platypus is not a duck.

27. All mammals have fur.
All tigers have fur.
All tigers are mammals.

28. All mammals have fur.
All tigers are mammals.
All tigers have fur.

29. California is adjacent to Arizona.
Arizona is adjacent to Nevada.
California is adjacent to Nevada.

30. Seattle is northwest of Boise.
Seattle is northwest of Salt Lake City.
Boise is northwest of Salt Lake City.

31. A rectangle has four sides.
A square has four sides.
A square is a rectangle.

32. No integer is irrational.
The number π is irrational.
The number π is not an integer.

33. Explain the difference between the following statements:

All students did not pass the test.
Not all students passed the test.

34. Write the following statement using *every:* There is no one here who has not done that at one time or another.

APPLICATIONS

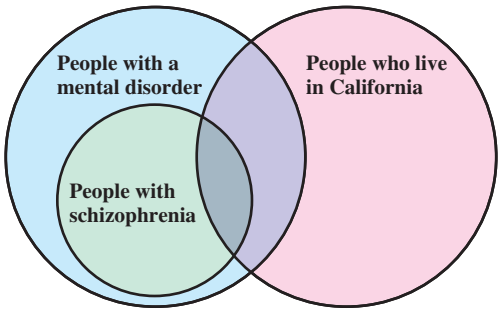
Business and Economics

35. Advertising Incorrect use of quantifiers often is heard in everyday language. Suppose you hear that a local electronics chain is having a 30% off sale, and the radio advertisement states, "All items are not available in all stores." Do you think that, literally translated, the ad really means what it says? What do you think really is meant? Explain your answer.

36. Portfolios Repeat Exercise 35 for the following: "All people don't have the time to devote to maintaining their financial portfolios properly."

Life Sciences

37. Schizophrenia The following diagram shows the relationship between people with schizophrenia, people with a mental disorder, and people who live in California.



Which of the following conclusions are valid?

a. All those with schizophrenia have a mental disorder.

b. All those with schizophrenia live in California.

c. Some of those with a mental disorder live in California.

d. Some of those who live in California do not have schizophrenia.

e. All those with a mental disorder have schizophrenia.

Social Sciences

Constitution **Each of the following exercises gives a passage from the U.S. Constitution, followed by another statement. (a) Translate the passage from the Constitution into a statement in symbolic logic. (b) Find a valid conclusion using as premises the passage from the Constitution and the statement that follows it. (c) Illustrate the argument with an Euler diagram.**

38. All legislative powers herein granted shall be vested in a Congress of the United States . . . (Article 1, Section 1)

The power to collect taxes is a legislative power herein granted.

39. No person shall be a Representative who shall not have attained to the age of twenty-five years, and been seven years a citizen of the United States, and who shall not, when elected, be an inhabitant of that State in which he shall be chosen. (Article 1, Section 2)

John Boehner is a Representative.

40. No bill of attainder or ex post facto law shall be passed. (Article 1, Section 9)

The law forbidding members of the Communist Party to serve as an officer or as an employee of a labor union was a bill of attainder. ***Source: United States V. Brown.***

41. No State shall enter into any treaty, alliance, or confederation. (Article 1, Section 10)

Texas is a state.

General Interest

42. Bible Write the negation of each of the following quotes from the Bible. ***Source: NIV Bible.***

a. "Everyone who hears about this will laugh with me."—Genesis 21:6

b. "Someone came to destroy your lord the king."—1 Samuel 26:15

c. "There is no one who does good."—Psalm 53:3

d. "Everyone is the friend of one who gives gifts."—Proverbs 19:6

e. "Everyone who quotes proverbs will quote this proverb about you: 'Like mother, like daughter.'"—Ezekiel 16:44

Animals **In Exercises 43–48, the premises marked *A*, *B*, and *C* are followed by several possible conclusions. Take each conclusion in turn, and check whether the resulting argument is *valid* or *invalid*.**

A. *All kittens are cute animals.*

B. *All cute animals are admired by animal lovers.*

C. *Some dangerous animals are admired by animal lovers.*

43. Some kittens are dangerous animals.

44. Some cute animals are dangerous.

45. Some dangerous animals are cute.

46. Kittens are not dangerous animals.

47. All kittens are admired by animal lovers.

48. Some things admired by animal lovers are dangerous animals.

YOUR TURN ANSWERS

1. a. $\forall x\,[c(x) \to s(x)]$; negation: $\exists x\,[c(x) \wedge \sim s(x)]$, "Some college students do not study."

b. $\exists x\,[p(x) \wedge \sim o(x)]$; negation: $\forall x\,[p(x) \to o(x)]$, "All professors are organized."

2. $\forall x\,[i(x) \to a(x)]$

$$\frac{i(b)}{a(b)}$$

Valid.

3. $\forall x\,[b(x) \to w(x)]$

$$\frac{\sim w(r)}{\sim b(r)}$$

Valid.

4. $\forall x\,[m(x) \to p(x)]$

$$\frac{p(s)}{m(s)}$$

Invalid.

5. $\exists x\,[v(x) \wedge e(x)]$

$$\frac{v(s)}{e(s)}$$

Invalid.

6 CHAPTER REVIEW

SUMMARY

In this chapter we introduced symbolic logic, which uses letters to represent statements, and symbols for words such as

- *and* (conjunction, denoted by $\wedge$),
- *or* (disjunction, denoted by $\vee$),
- *not* (negation, denoted by $\sim$),
- *if . . . then* (conditional, denoted by $\to$), and
- *if and only if* (biconditional, denoted by $\leftrightarrow$).

Statements are declarative sentences that are either true or false, but not both simultaneously. Using logical connectives, two or more statements can be combined to form a compound statement. Truth values of various compound statements were explored using truth tables. We saw that two logical statements are equivalent (denoted as $\equiv$) if they have the same truth value. We used symbolic logic to design circuits, and then used logical equivalences to simplify the circuits. We saw that the contrapositive, a statement related to the conditional, is equivalent to the original conditional statement, but that two other related statements, the inverse and the converse, are not equivalent to the original conditional statement. We next explored how to prove valid arguments and give counterexamples to invalid arguments. Finally, we discussed the quantifiers *for all* (denoted by $\forall$) and *there exists* (denoted by $\exists$), and we used Euler diagrams to determine the validity of an argument involving quantifiers.

Truth Tables for Logical Operators

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Writing a Conditional as an *or* Statement

$p \to q \equiv \sim p \vee q$

Negation of a Conditional Statement

$\sim(p \to q) \equiv p \wedge \sim q$

Equivalent Statements

1a.	$p \vee q \equiv q \vee p$	Commutative Laws
b.	$p \wedge q \equiv q \wedge p$	
2a.	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	Associative Laws
b.	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	
3a.	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive Laws
b.	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
4a.	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's Laws
b.	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	
5a.	$p \vee p \equiv p$	Idempotent Laws
b.	$p \wedge p \equiv p$	
6a.	$(p \wedge q) \vee p \equiv p$	Absorption Laws
b.	$(p \vee q) \wedge p \equiv p$	
7.	$\sim(\sim p) \equiv p$	Double Negative
8.	$p \to q \equiv \sim p \vee q$	Conditional as an "or"
9.	$p \to q \equiv \sim q \to \sim p$	Contrapositive

If T represents any true statement and F represents any false statement, then

10a.	$p \vee \mathrm{T} \equiv \mathrm{T}$	Identity Laws
b.	$p \wedge \mathrm{T} \equiv p$	
c.	$p \vee \mathrm{F} \equiv p$	
d.	$p \wedge \mathrm{F} \equiv \mathrm{F}$	
11a.	$p \vee \sim p \equiv \mathrm{T}$	Negation Laws
b.	$p \wedge \sim p \equiv \mathrm{F}$	

Common Translations of $p \to q$

If p, then q.	p is sufficient for q.
If p, q.	q is necessary for p.
p implies q.	All p's are q's.
p only if q.	q if p.
q when p.	

Related Conditional Statements

The contrapositive is equivalent to the original statement. However, the converse and inverse are not equivalent to the original statement, although they are equivalent to each other.

Original Statement	$p \to q$
Converse	$q \to p$
Inverse	$\sim p \to \sim q$
Contrapositive	$\sim q \to \sim p$

Valid Argument Forms

Modus Ponens	Modus Tollens	Disjunctive Syllogism	Reasoning by Transitivity
$p \to q$	$p \to q$	$p \vee q$	$p \to q$
p	$\sim q$	$\sim p$	$q \to r$
q	$\sim p$	q	$p \to r$

Invalid Argument Forms (Fallacies)

Fallacy of the Converse	Fallacy of the Inverse
$p \to q$	$p \to q$
q	$\sim p$
p	$\sim q$

KEY TERMS

6.1
statement
logical connective
compound statement
component statement
negation
truth table
conjunction
disjunction

6.2
equivalent
De Morgan's Laws

6.3
conditional
antecedent
consequent
contradiction
tautology
circuit
series
parallel
Principle of Duality

6.4
converse
inverse
contrapositive
biconditional

6.5
premise
conclusion
valid
invalid
fallacy
Modus Ponens
Fallacy of the Converse
Modus Tollens
Fallacy of the Inverse
Disjunctive Syllogism
Reasoning by Transitivity

6.6
quantifier
universal quantifier
existential quantifier
Euler diagram

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. A compound statement is a negation, a conjunction, a disjunction, a conditional, or a biconditional.

2. A truth table with 5 variables has 10 rows.

3. A truth table can have an odd number of rows.

4. Using one of De Morgan's Laws, the negation of a disjunction may be written as a conditional.

5. The negation of a conditional statement is a disjunction.

6. Elements in a circuit that are in parallel are connected in a logic statement with an *or*.

7. A tautology might be false.

8. A statement might be true even though its inverse is false.

9. The conclusion of a valid argument must be true.

10. The conclusion of a fallacy must be false.

11. Euler diagrams can be used to determine whether an argument with quantifiers is valid or invalid.

12. The negation of a statement with the universal quantifier involves the existential quantifier.

PRACTICE AND EXPLORATIONS

Write the negation of each statement.

13. If she doesn't pay me, I won't have enough cash.

14. We played the Titans and the Titans won.

Let l represent "He loses the election" and let w represent "He wins the hearts of the voters." Write each statement in symbols.

15. He loses the election, but he wins the hearts of the voters.

16. If he wins the hearts of the voters, then he doesn't lose the election.

17. He loses the election only if he doesn't win the hearts of the voters.

18. He loses the election if and only if he doesn't win the hearts of the voters.

Using the same statements as for Exercises 15–18, write each mathematical statement in words.

19. $\sim l \wedge w$

20. $\sim(l \vee \sim w)$

Assume that p is true and that q and r are false. Find the truth value of each statement.

21. $\sim q \wedge \sim r$

22. $r \vee (p \wedge \sim q)$

23. $r \rightarrow (s \vee r)$ (The truth value of the statement s is unknown.)

24. $p \leftrightarrow (p \rightarrow q)$

25. Explain in your own words why, if p is a statement, the biconditional $p \leftrightarrow \sim p$ must be false.

26. State the necessary conditions for

a. a conditional statement to be false;

b. a conjunction to be true;

c. a disjunction to be false.

Construct a truth table for each statement. Is the statement a tautology?

27. $p \wedge (\sim p \vee q)$

28. $\sim(p \wedge q) \rightarrow (\sim p \vee \sim q)$

Write each conditional statement in *if . . . then* form.

29. All mathematicians are loveable.

30. You can have dessert only if you eat your vegetables.

31. Having at least as many equations as unknowns is necessary for a system to have a unique solution.

32. Having a feasible region is sufficient for a linear programming problem to have a minimum.

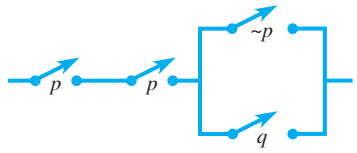
For each statement, write (a) the converse, (b) the inverse, and (c) the contrapositive.

33. If the proposed regulations have been approved, then we need to change the way we do business.

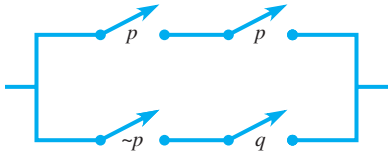
34. $(p \vee q) \rightarrow \sim r$ (Use one of De Morgan's Laws to simplify.)

Write a logical statement representing each circuit. Simplify each circuit when possible.

35.



36.



Draw the circuit representing each statement as it is given. Simplify if possible.

37. $(p \wedge q) \vee (p \wedge p)$

38. $p \wedge (p \vee q)$

For Exercises 39 and 40, consider the exclusive disjunction introduced in Exercises 35–38 of Section 6.2. In exclusive disjunction, denoted by $p \veebar q$, either p or q is true, but not both.

39. Use a truth table to show that
$p \veebar q \equiv (p \vee q) \wedge \sim(p \wedge q)$.

40. Use a truth table to show that
$p \veebar q \equiv \sim[(p \vee q) \rightarrow (p \wedge q)]$.

41. Consider the statement "If this year is 2010, then $1 + 1 = 3$."

a. Is the above statement true?

b. Was the above statement true in 2010?

c. Discuss how a statement such as the one above changes its truth value over time and how this agrees or disagrees with your intuitive notion of whether a conditional statement is true or false.

42. a. Consider the statement "If Shakespeare didn't write *Hamlet*, then someone else did." Explain why this statement is true.

b. Consider the statement "If Shakespeare hadn't written *Hamlet*, then someone else would have." Explain why this statement should be false in any reasonable logic.

c. If we let s represent "Shakespeare wrote *Hamlet*" and e represent "someone else wrote *Hamlet*," then the logic of this chapter leads us to represent either the statement of part (a) or part (b) as $\sim s \rightarrow e$. Discuss the issue of how this statement could be true in part (a) but false in part (b). ***Source: Formal Logic: Its Scope and Limit.***

Each of the following arguments is either valid by one of the forms of valid arguments discussed in this chapter or is a fallacy by one of the forms of invalid arguments discussed. Decide whether the argument is valid or invalid, and give the form that applies.

43. If you're late one more time, you'll be docked.
You're late one more time.

You'll be docked.

44. If the company makes a profit, its stock goes up.
The company doesn't make a profit.

Its stock doesn't go up.

45. The instructor is late or my watch is wrong.
My watch is not wrong.

The instructor is late.

46. If the parent is loving, then the child will be happy.
If the child is happy, then the teacher can teach.

If the parent is loving, then the teacher can teach.

47. If you play that song one more time, I'm going nuts.
I'm going nuts.

You play that song one more time.

48. If it's after five, the store is closed.
The store is not closed.

It's not after five.

Determine whether each argument is *valid* or *invalid*. If it is valid, give a proof. If it is invalid, give an assignment of truth values to the variables that makes the premises true and the conclusion false.

49. If we hire a new person, then we'll spend more on training. If we rewrite the manual, then we won't spend more on training. We rewrite the manual. Therefore, we don't hire a new person.

50. It is not true that Smith or Jones received enough votes to qualify. But if Smith got enough votes to qualify, then the election was rigged. Therefore, the election was not rigged.

51. $\sim p \rightarrow \sim q$
$q \rightarrow p$

$p \vee q$

52. $p \rightarrow q$
$r \rightarrow \sim q$

$p \rightarrow \sim r$

In Exercises 53 and 54, (a) write the statement symbolically, (b) write the negation of the statement in part (a), and (c) translate your answer from part (b) into words.

53. All dogs have a license.

54. Some cars have manual transmissions.

In Exercises 55 and 56, (a) represent the argument symbolically, and (b) use an Euler diagram to determine whether the argument is valid.

55. All members of that fraternity do well academically.
Jordan Enzor is a member of that fraternity.

Jordan Enzor does well academically.

56. Some members of that fraternity don't do well academically.
Bruce Collin does well academically.

Bruce Collin is not a member of that fraternity.

57. Construct a truth table for $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$. Are these two statements equivalent?

58. a. Convert the statement $p \rightarrow (q \rightarrow r)$ into an equivalent statement with an "or," but without the conditional.

b. Convert the statement $(p \rightarrow q) \rightarrow r$ into an equivalent statement with an "or," but without the conditional.

59. a. Construct a truth table for the statement $(p \wedge \sim p) \rightarrow q$.

b. Based on the truth table for part a, explain why any conclusion may be reached from a contradictory premise.

APPLICATIONS

Business and Economics

Income Tax **The following excerpts appear in a guide for preparing income tax reports. *Source: Your Income Tax 2010.***

a. *The tax law provides several tax benefits for people attending school.*

b. *If you use the Tax Table, you do not have to compute your tax mathematically.*

c. *You may not deduct the expenses of seeking your first job.*

d. *Make sure the names used when you and your spouse file your joint return match the names you have provided to the Social Security Administration.*

60. Which of these excerpts are statements?

61. Which of these excerpts are compound statements?

62. Write the negation of statement c.

63. Write statement b as an equivalent statement using the connective *or*.

64. Write the contrapositive of statement b.

Life Sciences

65. Medicine The following statements appear in a home medical reference book. Write each conditional statement in *if . . . then* form. ***Source: American College of Physicians Complete Home Medical Guide.***

a. When you exercise regularly, your heart becomes stronger and more efficient.

b. All teenagers need to be aware of the risks of drinking and driving.

c. You may need extra immunizations if you are visiting a country that has a high incidence of infectious diseases.

d. Food is essential (necessary) for good health.

Social Sciences

66. Law The following statements appear in a guide to common laws. Write the negation of each statement. ***Source: Law For Dummies.***

a. If you can't afford to hire an attorney, the judge will arrange for you to have legal representation.

b. Your attorney may discover it in the course of working on your case, or the other side may unearth it.

c. You don't have a Constitutional right to make a phone call from jail, but you may be allowed to make one call.

67. Democratization According to one sociologist, "If, for example, democratization always occurs in the company of widening splits within ruling oligarchies (that is, such splits are active candidates for necessary conditions of democratization), a valid causal story will most likely connect democratization with such splits." Let w represent "widening splits occur within ruling oligarchies," d represent "democratization occurs," and v represent "a valid causal story connects democratization with splits." Represent the sociologist's statement as a statement in logic. ***Source: Sociological Theory.***

68. Philosophy In an article in *Skeptical Inquirer*, Ralph Estling said, "Positivists such as [Stephen] Hawking tell us that reality cannot be dealt with . . . [and that] we can only deal with what we can measure in some way. . . . This seems sensible . . . until we watch as our physicists slowly slide down the slippery slopes . . . [and] tell us that only the measurable is real." ***Source: Skeptical Inquirer.***

a. Draw an Euler diagram with three circles showing real things, things we can measure, and things we can deal with. Your diagram should simultaneously illustrate the following statements:

i. Real things are not things we can deal with.

ii. Only things we can measure are things we can deal with.

iii. Only things we can measure are real things.

b. Using the Euler diagram from part a, explain why things that we can measure and deal with are not real, assuming the previous statements are true.

c. Philosopher Timothy Chambers suggests adding the plausible premise, "All things we can measure are things we can deal with." Explain why this premise, added to the previous three, implies that nothing is real. ***Source: The Mathematics Teacher.***

General Interest

Scrabble **The following excerpts can be found on the box to the game, Scrabble. *Source: Milton Bradley Company.***

a. *Turn all the letter tiles facedown at the side of the board or pour them into the bag or other container, and shuffle.*

b. *Any word may be challenged before the next player starts a turn.*

c. *If the word challenged is acceptable, the challenger loses his or her next turn.*

d. *When the game ends, each player's score is reduced by the sum of his or her unplayed letters.*

69. Which of these excerpts are statements?

70. Which of these excerpts are compound statements?

Lewis Carroll **The following exercises are from problems by Lewis Carroll. Write each premise in symbols, and then give a conclusion that uses all the premises and yields a valid argument. *Source: The Complete Works of Lewis Carroll.***

71. Let s be "the puppy lies still," g be "the puppy is grateful to be lent a skipping rope," l be "the puppy is lame," and w be "the puppy cares to do worsted work."

a. Puppies that will not lie still are always grateful for the loan of a skipping rope.

b. A lame puppy would not say "thank you" if you offered to lend it a skipping rope.

c. None but lame puppies ever care to do worsted work.

d. Give a conclusion that yields a valid argument.

72. Let o be "the bird is an ostrich," h be "the bird is at least 9 feet high," a be "the bird is in this aviary," m be "the bird belongs to me," and p be "the bird lives on mince pies."

a. No birds, except ostriches, are 9 feet high. (*Hint:* Interpret as: If a bird is at least 9 feet high, it is an ostrich.)

b. There are no birds in this aviary that belong to any one but me.

c. No ostrich lives on mince pies.

d. I have no birds less than 9 feet high.

e. Give a conclusion that yields a valid argument.

73. Let f be "the kitten loves fish," t be "the kitten is teachable," a be "the kitten has a tail," g be "the kitten will play with a gorilla," w be "the kitten has whiskers," and e be "the kitten has green eyes."

a. No kitten that loves fish is unteachable.

b. No kitten without a tail will play with a gorilla.

c. Kittens with whiskers always love fish.

d. No teachable kitten has green eyes.

e. No kittens have tails unless they have whiskers. (*Hint:* "$(\sim a)$ unless b" $\equiv a \rightarrow b$)

f. Give a conclusion that yields a valid argument.

74. Let u be "the writer understands human nature," c be "the writer is clever," p be "the writer is a true poet," r be "the writer can stir the hearts of men," s be "the writer is Shakespeare," and h be "the writer wrote *Hamlet.*"

a. All writers who understand human nature are clever.

b. No one is a true poet unless he can stir the hearts of men. (*Hint:* "$(\sim a)$ unless b" $\equiv a \rightarrow b$)

c. Shakespeare wrote *Hamlet.*

d. No writer who does not understand human nature can stir the hearts of men.

e. None but a true poet could have written *Hamlet.*

f. Give a conclusion that yields a valid argument.

EXTENDED APPLICATION

LOGIC PUZZLES

Some people find that logic puzzles, which appear in periodicals such as *World-Class Logic Problems* (Penny Press) and *Logic Puzzles* (Dell), provide hours of enjoyment. They are based on deductive reasoning, and players answer questions based on clues given. The following explanation on solving such problems appeared in an issue of *World-Class Logic Problems*.

HOW TO SOLVE LOGIC PROBLEMS

Solving logic problems is entertaining and challenging. All the information you need to solve a logic problem is given in the introduction and clues, and in illustrations, when provided. If you've never solved a logic problem before, our sample should help you get started. Fill in the Sample Solving Chart in Figure 19 as you follow our explanation. We use a "•" to signify "Yes" and an "×" to signify "No."

Five couples were married last week, each on a different weekday. From the information provided, determine the woman (one is Cathy) and man (one is Paul) who make up each couple, as well as the day on which each couple was married.

1. Anne was married on Monday, but not to Wally.
2. Stan's wedding was on Wednesday. Rob was married on Friday, but not to Ida.
3. Vern (who married Fran) was married the day after Eve.

Anne was married Monday (1), so put a "•" at the intersection of Anne and Monday. Put "×" in all the other days in Anne's row and all the other names in the Monday column. (Whenever you establish a relationship, as we did here, be sure to place "×" at the intersections of all relationships that become impossible as a result.)

SAMPLE SOLVING CHART:

	PAUL	ROB	STAN	VERN	WALLY	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
ANNE										
CATHY										
EVE										
FRAN										
IDA										
MONDAY										
TUESDAY										
WEDNESDAY										
THURSDAY										
FRIDAY										

FIGURE 19

Anne wasn't married to Wally (1), so put an "×" at the intersection of Anne and Wally. Stan's wedding was Wednesday (2), so put a "•" at the intersection of Stan and Wednesday (don't forget the "×"s). Stan didn't marry Anne, who was married Monday, so put an "×" at the intersection of Anne and Stan. Rob was married Friday, but not to Ida (2), so put a "•" at the intersection of Rob and Friday, and "×" at the intersections of Rob and Ida and Ida and Friday. Rob also didn't marry Anne, who was married Monday, so put an "×" at the intersection of Anne and Rob. Now your chart should look like Figure 20 on the next page.

Vern married Fran (3), so put a "•" at the intersection of Vern and Fran. This leaves Anne's only possible husband as Paul, so put a "•" at the intersection of Anne and Paul and Paul and Monday. Vern and Fran's wedding was the day after Eve's (3), which wasn't

	PAUL	ROB	STAN	VERN	WALLY	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
ANNE		×	×		×	•	×	×	×	×
CATHY						×				
EVE						×				
FRAN						×				
IDA		×				×				×
MONDAY		×	×							
TUESDAY		×	×							
WEDNESDAY	×	×	•	×	×					
THURSDAY		×	×							
FRIDAY	×	•	×	×	×					

FIGURE 20

Monday [Anne], so Vern's wasn't Tuesday. It must have been Thursday [see chart], so Eve's was Wednesday (3). Put "•" at the intersections of Vern and Thursday, Fran and Thursday, and Eve and Wednesday. Now your chart should look like Figure 21.

	PAUL	ROB	STAN	VERN	WALLY	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
ANNE	•	×	×	×	×	•	×	×	×	×
CATHY	×			×		×		×	×	
EVE	×			×		×	×	•	×	×
FRAN	×	×	×	•	×	×	×	×	•	×
IDA	×	×		×		×		×	×	×
MONDAY	•	×	×	×	×					
TUESDAY	×	×	×	×						
WEDNESDAY	×	×	•	×	×					
THURSDAY	×	×	×	•	×					
FRIDAY	×	•	×	×	×					

FIGURE 21

The chart shows that Cathy was married Friday, Ida was married Tuesday, and Wally was married Tuesday. Ida married Wally, and Cathy's wedding was Friday, so she married Rob. After this information is filled in, Eve could only have married Stan. You've completed the puzzle, and your chart should now look like Figure 22

	PAUL	ROB	STAN	VERN	WALLY	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
ANNE	•	×	×	×	×	•	×	×	×	×
CATHY	×	•	×	×	×	×	×	×	×	•
EVE	×	×	•	×	×	×	×	•	×	×
FRAN	×	×	×	•	×	×	×	×	•	×
IDA	×	×	×	×	•	×	•	×	×	×
MONDAY	•	×	×	×	×					
TUESDAY	×	×	×	×	•					
WEDNESDAY	×	×	•	×	×					
THURSDAY	×	×	×	•	×					
FRIDAY	×	•	×	×	×					

FIGURE 22

In summary: Anne and Paul, Monday; Cathy and Rob, Friday; Eve and Stan, Wednesday; Fran and Vern, Thursday; Ida and Wally, Tuesday.

In some problems, it may be necessary to make a logical guess based on facts you've established. When you do, always look for clues or other facts that disprove it. If you find that your guess is incorrect, eliminate it as a possibility.

EXERCISES

1. ***Water, Water, Everywhere*** After an invigorating workout, five fitness-conscious friends know that nothing is more refreshing than a tall, cool glass of mineral water! Each person (including Annie) has a different, favorite form of daily exercise (one likes to rollerblade), and each drinks a different brand of mineral water (one is Crystal Spring). From the information provided, determine the type of exercise and brand of water each person prefers. ***Source: World-Class Logic Problems Special.***

   a. The one who bicycles in pursuit of fitness drinks Bevé.

   b. Tim enjoys aerobicizing every morning before work. Ben is neither the one who drinks Sparkling Creek nor the one who imbibes Bevé.

   c. Page (who is neither the one who jogs nor the one who walks to keep in shape) drinks Purity. Meg drinks Mountain Clear, but not after jogging.

		EXERCISE					WATER				
		AEROBICIZING	BICYCLING	JOGGING	ROLLERBLADING	WALKING	BEVÉ	CRYSTAL SPRING	MOUNTAIN CLEAR	PURITY	SPARKLING CREEK
PERSON	ANNIE										
	BEN										
	MEG										
	PAGE										
	TIM										
WATER	BEVÉ										
	CRYSTAL SPRING										
	MOUNTAIN CLEAR										
	PURITY										
	SPARKLING CREEK										

2. ***Let's Get Physical*** The Anytown Community Center, in conjunction with the Board of Education's adult-outreach program, has scheduled a week-long series of lectures this fall on topics in physics. The goals are to increase awareness of the physical sciences and to attract renowned scientists (including Dr. Denton) to the community. Each of the five lectures will be held on a different weekday, and each will feature a different physicist lecturing on a different topic (one is magnetism). So far, the community has shown great interest in their upcoming physical training! From the information provided, determine the physicist who will speak on each weekday and the topic of his or her lecture. ***Source: World-Class Logic Problems Special.***

   a. Dr. Hoo, who is from Yale, will not be lecturing on Thursday. Dr. Zhivago's lecture will be exactly three days after the lecture on chaos theory.

   b. If Dr. Jay is lecturing on Thursday, then the person giving the kinetic-energy lecture will appear on Tuesday; otherwise, Dr. Jay will speak on Tuesday, and kinetic energy will be the topic of Monday's lecture.

c. Dr. Know (who is not giving the lecture on quantum mechanics) is not the Harvard physicist who will speak on Monday. The photonics lecture will not be given on either Wednesday or Thursday.

		PHYSICIST					TOPIC				
		DR. DENTON	DR. HOO	DR. JAY	DR. KNOW	DR. ZHIVAGO	CHAOS THEORY	KINETIC ENERGY	MAGNETISM	PHOTONICS	QUANTUM MECHANICS
WEEKDAY	MONDAY										
	TUESDAY										
	WEDNESDAY										
	THURSDAY										
	FRIDAY										
TOPIC	CHAOS THEORY										
	KINETIC ENERGY										
	MAGNETISM										
	PHOTONICS										
	QUANTUM MECHANICS										

3. ***What's in Store?*** I had a day off from work yesterday, so I figured it was the perfect time to do some shopping. I hit the road shortly after breakfast and visited five stores (one was Bullseye). At each shop, I had intended to buy a different one of five items (a pair of andirons, a Crock-pot, pruning shears, a pair of sneakers, or a toaster oven). Unfortunately, no store had the item I was looking for in stock. The trips weren't a total loss, however, as I purchased a different item (a CD, a fondue pot, a garden gnome, spark plugs, or a winter coat) that had caught my eye in each store. Despite my failure to acquire any of the things I had sought, there were a couple of positive outcomes. I now have some nifty new things that I know I'll enjoy, and I have a shopping list written and ready to go for my next day off! From the information provided, determine the order in which I visited the five stores, as well as the item I sought and the item I bought at each store. ***Source: World-Class Logic Problems Special.***

a. I went to the store where I bought a CD (which isn't where I sought pruning shears) immediately after I visited PJ Nickle but immediately before I went to the shop where I intended to buy a Crock-pot.

b. I didn't purchase the fondue pot at Costington's. I went into one store intending to buy a Crock-pot, but came out with a garden gnome instead. I didn't go to PJ Nickle for a pair of andirons.

c. The store at which I sought a toaster oven (which wasn't the third one I visited) isn't the place where I eventually bought a winter coat. Neither Lacy's nor S-Mart was the fourth shop I visited.

d. I went to Costington's immediately after I visited the shop where I sought pruning shears (which wasn't S-Mart) but immediately before I went to the store where I purchased a set of spark plugs.

4. ***High Five*** Otis Lifter is the elevator operator at Schwarzenbach Tower, downtown Brownsville's tallest building. Since he works in such a towering edifice, Otis gets a chance to chat with his passengers on the way to their destinations. Five people who always have a friendly word for Otis work on the Schwarzenbach's top floors. Each person works on a different floor, which is home to a different company (one is the Watershed Co.). Each company is in a different business (one is a real-estate agency). Otis is content with his job, but he'll be the first to tell you that, like any profession, it has its ups and downs! From the information provided, can you determine the floor (41st through 45th) to which Otis took each person, as well as the name of his or her company and the type of business it conducts? ***Source: World-Class Logic Problems Special.***

a. Edwina's company is exactly 1 floor above Nelson & Leopold but exactly 1 floor below the accounting firm. Brierwood Ltd. is on the 44th floor.

b. Zed's company is exactly 1 floor above Ogden's. Keith's business is on the 45th floor. Trish works at the public-relations firm (which is exactly 2 floors above Glyptic).

c. Glyptic and the Thebes Group are the literary agency and the Web-design firm, in some order. The Web-design firm is not on the 41st floor.

		STORE					ITEM SOUGHT					ITEM BOUGHT				
		BULLSEYE	COSTINGTON'S	LACY'S	PJ NICKLE	S-MART	ANDIRONS	CROCK-POT	PRUNING SHEARS	SNEAKERS	TOASTER OVEN	CD	FONDUE POT	GARDEN GNOME	SPARK PLUGS	WINTER COAT
ORDER	FIRST															
	SECOND															
	THIRD															
	FOURTH															
	FIFTH															
ITEM BOUGHT	CD															
	FONDUE POT															
	GARDEN GNOME															
	SPARK PLUGS															
	WINTER COAT															
ITEM SOUGHT	ANDIRONS															
	CROCK-POT															
	PRUNING SHEARS															
	SNEAKERS															
	TOASTER OVEN															

		FLOOR					COMPANY					BUSINESS				
		41st	42nd	43rd	44th	45th	BRIERWOOD LTD.	GLYPTIC	NELSON & LEOPOLD	THEBES GROUP	WATERSHED CO.	ACCOUNTING	LITERARY AGENCY	PUBLIC RELATIONS	REAL ESTATE	WEB DESIGN
PERSON	EDWINA															
	KEITH															
	OGDEN															
	TRISH															
	ZED															
BUSINESS	ACCOUNTING															
	LITERARY AGENCY															
	PUBLIC RELATIONS															
	REAL ESTATE															
	WEB DESIGN															
COMPANY	BRIERWOOD LTD.															
	GLYPTIC															
	NELSON & LEOPOLD															
	THEBES GROUP															
	WATERSHED CO.															

5. ***First Ratings*** At long last, Macrocosm Industries has released Q Sphere, its new video-game console. To fully demonstrate the Q Sphere's capabilities, each of Macrocosm's five Q Sphere games (including Idle Hands) is a different genre. Anxious to be the first publication to spotlight this new gaming system, *All Game* magazine featured reviews of the Q Sphere games in its latest issue. Each game was played extensively by a different *All Game* staff reviewer (including Chadwick) and given a different rating (from lowest to highest, "don't bother," "just okay," "pretty cool," "almost perfect," or "totally awesome"). In the end, though, true video-game aficionados will want to try all of the Q Sphere games for themselves, despite the ratings! From the information provided, determine the genre of the game reviewed by each *All Game* staff member (identified by first and last names—one surname is Ploof), as well as the rating given to each game. ***Source: World-Class Logic Problems Special.***

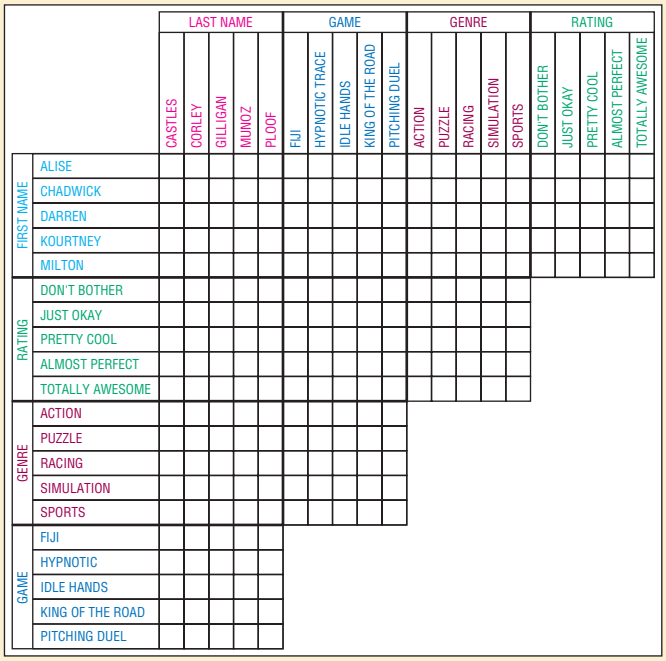
a. At least one game was given a lower rating than the sports game (which is called Pitching Duel). The person surnamed Corley reviewed the action game.

b. The puzzle game's rating was "just okay," which was higher than the rating Darren Castles gave.

c. King of the Road is the racing game. The person surnamed Munoz (who isn't Milton) gave the "totally awesome" rating.

d. The person surnamed Gilligan reviewed the simulation game, which isn't Hypnotic Trace (which was rated "almost perfect").

e. Alise gave one game a "pretty cool" rating. Kourtney spent many hours playing Fiji in order to write her review.



DIRECTIONS FOR GROUP PROJECT

Construct your own logic puzzle.

7 Sets and Probability

7.1 Sets
7.2 Applications of Venn Diagrams
7.3 Introduction to Probability
7.4 Basic Concepts of Probability
7.5 Conditional Probability; Independent Events
7.6 Bayes' Theorem
Chapter 7 Review
Extended Application: Medical Diagnosis

The study of probability begins with counting. An exercise in Section 2 of this chapter counts trucks carrying different combinations of early, late, and extra late peaches from the orchard to canning facilities. You'll see trees in another context in Section 5, where we use branching tree diagrams to calculate conditional probabilities.

Our lives are bombarded by seemingly chance events — the chance of rain, the risk of an accident, the possibility the stock market increases, the likelihood of winning the lottery — all whose particular outcome may appear to be quite random. The field of probability attempts to quantify the likelihood of chance events and helps us to prepare for this uncertainty. In short, probability helps us to better understand the world in which we live. In this chapter and the next, we introduce the basic ideas of probability theory and give a sampling of some of its uses. Since the language of sets and set operations is used in the study of probability, we begin there.

7.1 Sets

APPLY IT **In how many ways can two candidates win the 50 states plus the District of Columbia in a U.S. presidential election?**

Using knowledge of sets, we will answer this question in Exercise 69.

Think of a **set** as a well-defined collection of objects in which it is possible to determine if a given object is included in the collection. A set of coins might include one of each type of coin now put out by the U.S. government. Another set might be made up of all the students in your English class. By contrast, a collection of young adults does not constitute a set unless the designation "young adult" is clearly defined. For example, this set might be defined as those aged 18 to 29.

In mathematics, sets are often made up of numbers. The set consisting of the numbers 3, 4, and 5 is written

$$\{3, 4, 5\},$$

with set braces, { }, enclosing the numbers belonging to the set. The numbers 3, 4, and 5 are called the **elements** or **members** of this set. To show that 4 is an element of the set $\{3, 4, 5\}$, we use the symbol $\in$ and write

$$4 \in \{3, 4, 5\},$$

read "4 is an element of the set containing 3, 4, and 5." Also, $5 \in \{3, 4, 5\}$.

To show that 8 is *not* an element of this set, place a slash through the symbol:

$$8 \notin \{3, 4, 5\}.$$

Sets often are named with capital letters, so that if

$$B = \{5, 6, 7\},$$

then, for example, $6 \in B$ and $10 \notin B$.

It is possible to have a set with no elements. Some examples are the set of counting numbers less than one, the set of foreign-born presidents of the United States, and the set of men more than 10 feet tall. A set with no elements is called the **empty set** (or **null set**) and is written $\emptyset$.

CAUTION Be careful to distinguish between the symbols 0, $\emptyset$, $\{0\}$, and $\{\emptyset\}$. The symbol 0 represents a *number*; $\emptyset$ represents a *set* with 0 elements; $\{0\}$ represents a set with one element, 0; and $\{\emptyset\}$ represents a set with one element, $\emptyset$.

We use the symbol $n(A)$ to indicate the *number* of elements in a finite set A. For example, if $A = \{a, b, c, d, e\}$, then $n(A) = 5$. Using this notation, we can write the information in the previous Caution as $n(\emptyset) = 0$ and $n(\{0\}) = n(\{\emptyset\}) = 1$.

Two sets are *equal* if they contain the same elements. The sets $\{5, 6, 7\}$, $\{7, 6, 5\}$, and $\{6, 5, 7\}$ all contain exactly the same elements and are equal. In symbols,

$$\{5, 6, 7\} = \{7, 6, 5\} = \{6, 5, 7\}.$$

This means that the ordering of the elements in a set is unimportant. Note that each element of the set is only listed once. Sets that do not contain exactly the same elements are *not equal*. For example, the sets $\{5, 6, 7\}$ and $\{7, 8, 9\}$ do not contain exactly the same elements and, thus, are not equal. To indicate that these sets are not equal, we write

$$\{5, 6, 7\} \neq \{7, 8, 9\}.$$

Sometimes we are interested in a common property of the elements in a set, rather than a list of the elements. This common property can be expressed by using **set-builder notation**, for example,

$$\{x \mid x \text{ has property } P\}$$

(read "the set of all elements x such that x has property P") represents the set of all elements x having some stated property P.

EXAMPLE 1 Sets

Write the elements belonging to each set.

(a) $\{x \mid x \text{ is a natural number less than } 5\}$

SOLUTION The natural numbers less than 5 make up the set $\{1, 2, 3, 4\}$.

(b) $\{x \mid x \text{ is a state that borders Florida}\}$

SOLUTION The states that border Florida make up the set {Alabama, Georgia}.

TRY YOUR TURN 1

YOUR TURN 1 Write the elements belonging to the set. $\{x \mid x$ is a state whose name begins with the letter O$\}$.

The **universal set** for a particular discussion is a set that includes all the objects being discussed. In elementary school arithmetic, for instance, the set of whole numbers might be the universal set, while in a college algebra class the universal set might be the set of real numbers. The universal set will be specified when necessary, or it will be clearly understandable from the context of the problem.

Subsets

Sometimes every element of one set also belongs to another set. For example, if

$$A = \{3, 4, 5, 6\}$$

and

$$B = \{2, 3, 4, 5, 6, 7, 8\},$$

then every element of A is also an element of B. This is an example of the following definition.

Subset

Set A is a **subset** of set B (written $A \subseteq B$) if every element of A is also an element of B.
Set A is a *proper subset* (written $A \subset B$) if $A \subseteq B$ and $A \neq B$.

To indicate that A is *not* a subset of B, we write $A \not\subseteq B$.

EXAMPLE 2 Sets

Decide whether the following statements are *true* or *false*.

(a) $\{3, 4, 5, 6\} = \{4, 6, 3, 5\}$

SOLUTION Both sets contain exactly the same elements, so the sets are equal and the given statement is true. (The fact that the elements are listed in a different order does not matter.)

YOUR TURN 2 Decide if the following statement is *true* or *false*.
$\{2, 4, 6\} \subseteq \{6, 2, 4\}$

(b) $\{5, 6, 9, 12\} \subseteq \{5, 6, 7, 8, 9, 10, 11\}$

SOLUTION The first set is not a subset of the second because it contains an element, 12, that does not belong to the second set. Therefore, the statement is false.

TRY YOUR TURN 2

The empty set, $\emptyset$, by default, is a subset of every set A since it is impossible to find an element of the empty set (it has no elements) that is not an element of the set A. Similarly, A is a subset of itself, since every element of A is also an element of the set A.

Subset Properties

For any set A,

$$\emptyset \subseteq A \quad \text{and} \quad A \subseteq A.$$

EXAMPLE 3 Subsets

List all possible subsets for each set.

(a) $\{7, 8\}$

SOLUTION There are 4 subsets of $\{7, 8\}$:

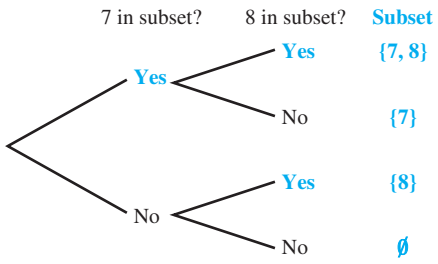
$$\emptyset, \quad \{7\}, \quad \{8\}, \quad \text{and} \quad \{7, 8\}.$$

(b) $\{a, b, c\}$

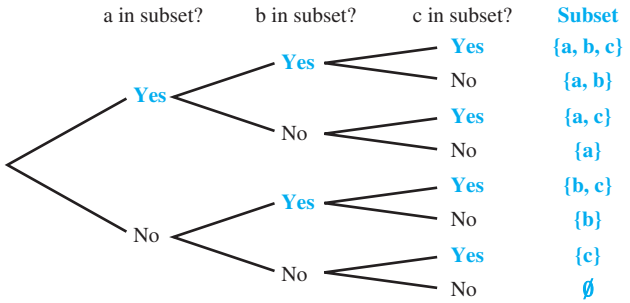
SOLUTION There are 8 subsets of $\{a, b, c\}$:

$$\emptyset, \quad \{a\}, \quad \{b\}, \quad \{c\}, \quad \{a, b\}, \quad \{a, c\}, \quad \{b, c\}, \quad \text{and} \quad \{a, b, c\}.$$

A good way to find the subsets of $\{7, 8\}$ and the subsets of $\{a, b, c\}$ in Example 3 is to use a **tree diagram**—a systematic way of listing all the subsets of a given set. Figure 1 shows tree diagrams for finding the subsets of $\{7, 8\}$ and $\{a, b, c\}$.



(a) Find the subsets of $\{7, 8\}$.



(b) Find the subsets of $\{a, b, c\}$.

FIGURE 1

As Figure 1 shows, there are two possibilities for each element (either it's in the subset or it's not), so a set with 2 elements has $2 \cdot 2 = 2^2 = 4$ subsets, and a set with 3 elements has $2^3 = 8$ subsets. This idea can be extended to a set with any finite number of elements, which leads to the following conclusion.

Number of Subsets

A set of k distinct elements has 2^k subsets.

In other words, if $n(A) = k$, then $n(\text{the set of all subsets of } A) = 2^k$.

EXAMPLE 4 Subsets

Find the number of subsets for each set.

(a) $\{3, 4, 5, 6, 7\}$

SOLUTION This set has 5 elements; thus, it has 2^5 or 32 subsets.

(b) $\{x \mid x \text{ is a day of the week}\}$

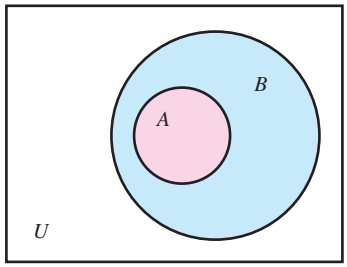
SOLUTION This set has 7 elements and therefore has $2^7 = 128$ subsets.

(c) $\emptyset$

SOLUTION Since the empty set has 0 elements, it has $2^0 = 1$ subset—itself.

TRY YOUR TURN 3

YOUR TURN 3 Find the number of subsets for the set $\{x \mid x \text{ is a season of the year}\}$.



$A \subseteq B$

FIGURE 2

Figure 2 shows a set A that is a subset of set B. The rectangle and everything inside it represents the universal set, U. Such diagrams, called **Venn diagrams**—after the English logician John Venn (1834–1923), who invented them in 1876—are used to help illustrate relationships among sets. Venn diagrams are very similar to Euler diagrams, described in Section 6.6. Euler diagrams are used in logic to denote variables having a certain property or not, while Venn diagrams are used in the context of sets to denote something being an element of a set or not.

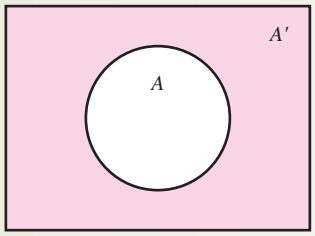
Set Operations

It is possible to form new sets by combining or manipulating one or more existing sets. Given a set A and a universal set U, the set of all elements of U that do *not* belong to A is called the *complement* of set A. For example, if set A is the set of all the female students in a class, and U is the set of all students in the class, then the complement of A would be the set of all male students in the class. The complement of set A is written A', read "A-prime."

Complement of a Set

Let A be any set, with U representing the universal set. Then the **complement** of A, colored pink in the figure, is

$$A' = \{x \mid x \notin A \text{ and } x \in U\}.$$



(Recall that the rectangle represents the universal set U.)

EXAMPLE 5 Set Operations

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $A = \{1, 2, 4, 5, 7\}$, and $B = \{2, 4, 5, 7, 9, 11\}$. Find each set.

(a) A'

SOLUTION Set A' contains the elements of U that are not in A.

$$A' = \{3, 6, 8, 9, 10, 11\}$$

(b) $B' = \{1, 3, 6, 8, 10\}$

(c) $\emptyset' = U$ and $U' = \emptyset$

(d) $(A')' = A$

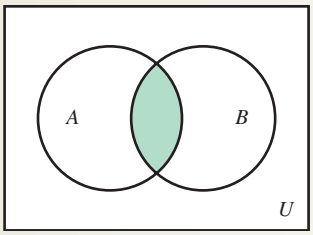
Given two sets A and B, the set of all elements belonging to *both* set A and set B is called the *intersection* of the two sets, written $A \cap B$. For example, the elements that belong to both set $A = \{1, 2, 4, 5, 7\}$ and set $B = \{2, 4, 5, 7, 9, 11\}$ are 2, 4, 5, and 7, so that

$$A \cap B = \{1, 2, 4, 5, 7\} \cap \{2, 4, 5, 7, 9, 11\} = \{2, 4, 5, 7\}.$$

Intersection of Two Sets

The **intersection** of sets A and B, shown in green in the figure, is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$



EXAMPLE 6 Set Operations

Let $A = \{3, 6, 9\}$, $B = \{2, 4, 6, 8\}$, and the universal set $U = \{0, 1, 2, \ldots, 10\}$. Find each set.

(a) $A \cap B$

SOLUTION

$$A \cap B = \{3, 6, 9\} \cap \{2, 4, 6, 8\} = \{6\}$$

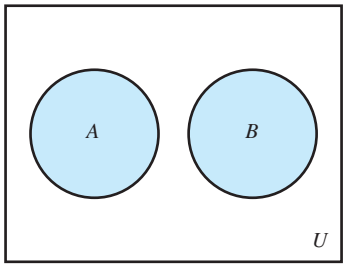
(b) $A \cap B'$

SOLUTION

$$A \cap B' = \{3, 6, 9\} \cap \{0, 1, 3, 5, 7, 9, 10\} = \{3, 9\}$$

YOUR TURN 4 For the sets in Example 6, find $A' \cap B$.

TRY YOUR TURN 4



A and B are disjoint sets.

FIGURE 3

Two sets that have no elements in common are called *disjoint sets*. For example, there are no elements common to both $\{50, 51, 54\}$ and $\{52, 53, 55, 56\}$, so these two sets are disjoint, and

$$\{50, 51, 54\} \cap \{52, 53, 55, 56\} = \emptyset.$$

This result can be generalized as follows.

Disjoint Sets

For any sets A and B, if A and B are **disjoint sets**, then $A \cap B = \emptyset$.

Figure 3 shows a pair of disjoint sets.

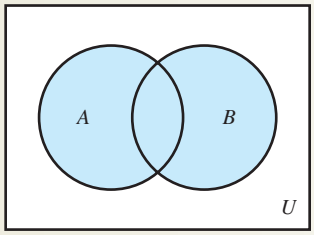
The set of all elements belonging to set A, to set B, or to both sets is called the *union* of the two sets, written $A \cup B$. For example,

$$\{1, 3, 5\} \cup \{3, 5, 7, 9\} = \{1, 3, 5, 7, 9\}.$$

Union of Two Sets

The **union** of sets A and B, shown in blue in the figure, is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}.$$



EXAMPLE 7 Union of Sets

Let $A = \{1, 3, 5, 7, 9, 11\}$, $B = \{3, 6, 9, 12\}$, $C = \{1, 2, 3, 4, 5\}$, and the universal set $U = \{0, 1, 2, \ldots, 12\}$. Find each set.

(a) $A \cup B$

SOLUTION Begin by listing the elements of the first set, $\{1, 3, 5, 7, 9, 11\}$. Then include any elements from the second set *that are not already listed*. Doing this gives

$$\begin{aligned} A \cup B &= \{1, 3, 5, 7, 9, 11\} \cup \{3, 6, 9, 12\} = \{1, 3, 5, 7, 9, 11, 6, 12\} \\ &= \{1, 3, 5, 6, 7, 9, 11, 12\}. \end{aligned}$$

(b) $(A \cup B) \cap C'$

SOLUTION Begin with the expression in parentheses, which we calculated in part (a), and then intersect this with C'.

$$\begin{aligned} (A \cup B) \cap C' &= \{1, 3, 5, 6, 7, 9, 11, 12\} \cap \{0, 6, 7, 8, 9, 10, 11, 12\} \\ &= \{6, 7, 9, 11, 12\} \end{aligned}$$

TRY YOUR TURN 5

YOUR TURN 5 For the sets in Example 7, find $A \cup (B \cap C')$.

NOTE

1. As Example 7 shows, when forming sets, do not list the same element more than once. In our final answer, we listed the elements in numerical order to make it easier to see what elements are in the set, but the set is the same, regardless of the order of the elements.
2. As shown in the definitions, an element is in the *intersection* of sets A and B if it is in A *and* B. On the other hand, an element is in the *union* of sets A and B if it is in A *or* B (or both).
3. In mathematics, "A *or* B" implies A or B, or both. Since "or" includes the possibility of both, we will usually omit the words "or both".

EXAMPLE 8 Stocks

The following table gives the 52-week low and high prices, the closing price, and the change from the previous day for six stocks in the Standard & Poor's 100 on February 19, 2010. ***Source: New York Stock Exchange.***

	New York Stock Exchange			
Stock	Low	High	Close	Change
AT&T	21.44	28.73	25.10	−0.55
Coca-Cola	37.44	59.45	55.72	−0.34
FedEx	34.02	92.59	81.76	+2.07
McDonald's	50.44	65.75	64.74	+0.40
PepsiCo	45.39	64.48	62.66	+0.21
Walt Disney	15.14	32.75	31.23	+0.68

Let the universal set U consist of the six stocks listed in the table. Let A contain all stocks with a high price greater than \$50, B all stocks with a closing price between \$60 and \$70, C all stocks with a positive price change, and D all stocks with a low price less than \$40. Find the following.

a. A'

SOLUTION Set A' contains all the listed stocks that are not in set A, or those with a high price less than or equal to \$50, so

$$A' = \{\text{AT\&T, Disney}\}.$$

b. $A \cap C$

SOLUTION The intersection of A and C will contain those stocks that are in both sets A and C, or those with a high price greater than \$50 *and* a positive price change.

$$A \cap C = \{\text{FedEx, McDonald's, PepsiCo}\}$$

c. $D \cup B$

SOLUTION The union of D and B contains all stocks with a low price less than \$40 *or* a closing price between \$60 and \$70.

$$D \cup B = \{\text{AT\&T, Coca-Cola, FedEx, McDonald's, PepsiCo, Disney}\} = U$$

EXAMPLE 9 Employment

A department store classifies credit applicants by gender, marital status, and employment status. Let the universal set be the set of all applicants, M be the set of male applicants, S be the set of single applicants, and E be the set of employed applicants. Describe each set in words.

a. $M \cap E$

SOLUTION The set $M \cap E$ includes all applicants who are both male *and* employed; that is, employed male applicants.

b. $M' \cup S$

SOLUTION This set includes all applicants who are female (not male) *or* single. *All* female applicants and *all* single applicants are in this set.

c. $M' \cap S'$

SOLUTION These applicants are female *and* married (not single); thus, $M' \cap S'$ is the set of all married female applicants.

d. $M \cup E'$

SOLUTION $M \cup E'$ is the set of applicants that are male *or* unemployed. The set includes *all* male applicants and *all* unemployed applicants.

7.1 EXERCISES

In Exercises 1–10, write true or false for each statement.

1. $3 \in \{2, 5, 7, 9, 10\}$

2. $6 \in \{-2, 6, 9, 5\}$

3. $9 \notin \{2, 1, 5, 8\}$

4. $3 \notin \{7, 6, 5, 4\}$

5. $\{2, 5, 8, 9\} = \{2, 5, 9, 8\}$

6. $\{3, 7, 12, 14\} = \{3, 7, 12, 14, 0\}$

7. $\{\text{all whole numbers greater than 7 and less than 10}\} = \{8, 9\}$

8. $\{x \mid x \text{ is an odd integer; } 6 \le x \le 18\} = \{7, 9, 11, 15, 17\}$

9. $0 \in \emptyset$

10. $\emptyset \in \{\emptyset\}$

Let $A = \{2, 4, 6, 10, 12\}$, $B = \{2, 4, 8, 10\}$, $C = \{4, 8, 12\}$, $D = \{2, 10\}$, $E = \{6\}$, and $U = \{2, 4, 6, 8, 10, 12, 14\}$. Insert $\subseteq$ or $\not\subseteq$ to make the statement true.

11. A ___ U

12. E ___ A

13. A ___ E

14. B ___ C

15. $\emptyset$ ___ A

16. $\{0, 2\}$ ___ D

17. D ___ B

18. A ___ C

19. Repeat Exercises 11–18 except insert $\subset$ or $\not\subset$ to make the statement true.

20. What is set-builder notation? Give an example.

Insert a number in each blank to make the statement true, using the sets for Exercises 11–18.

21. There are exactly ___ subsets of A.

22. There are exactly ___ subsets of B.

23. There are exactly ___ subsets of C.

24. There are exactly ___ subsets of D.

Insert $\cap$ or $\cup$ to make each statement true.

25. $\{5, 7, 9, 19\}$ ___ $\{7, 9, 11, 15\} = \{7, 9\}$

26. $\{8, 11, 15\}$ ___ $\{8, 11, 19, 20\} = \{8, 11\}$

27. $\{2, 1, 7\} __ \{1, 5, 9\} = \{1, 2, 5, 7, 9\}$

28. $\{6, 12, 14, 16\} __ \{6, 14, 19\} = \{6, 12, 14, 16, 19\}$

29. $\{3, 5, 9, 10\} __ \emptyset = \emptyset$

30. $\{3, 5, 9, 10\} __ \emptyset = \{3, 5, 9, 10\}$

31. $\{1, 2, 4\} __ \{1, 2, 4\} = \{1, 2, 4\}$

32. $\{0, 10\} __ \{10, 0\} = \{0, 10\}$

33. Describe the intersection and union of sets. How do they differ?

34. Is it possible for two nonempty sets to have the same intersection and union? If so, give an example.

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $X = \{2, 4, 6, 8\}$, $Y = \{2, 3, 4, 5, 6\}$, and $Z = \{1, 2, 3, 8, 9\}$. List the members of each set, using set braces.

35. $X \cap Y$

36. $X \cup Y$

37. X'

38. Y'

39. $X' \cap Y'$

40. $X' \cap Z$

41. $Y \cap (X \cup Z)$

42. $X' \cap (Y' \cup Z)$

43. $(X \cap Y') \cup (Z' \cap Y')$

44. $(X \cap Y) \cup (X' \cap Z)$

45. In Example 6, what set do you get when you calculate $(A \cap B) \cup (A \cap B')$?

46. Explain in words why $(A \cap B) \cup (A \cap B') = A$.

Let U = {all students in this school}, M = {all students taking this course}, N = {all students taking accounting}, and P = {all students taking zoology}. Describe each set in words.

47. M'

48. $M \cup N$

49. $N \cap P$

50. $N' \cap P'$

51. Refer to the sets listed for Exercises 11–18. Which pairs of sets are disjoint?

52. Refer to the sets listed for Exercises 35–44. Which pairs are disjoint?

Refer to Example 8 in the text. Describe each set in Exercises 53–56 in words; then list the elements of each set.

53. B'

54. $A \cap B$

55. $(A \cap B)'$

56. $(C \cup D)'$

57. Let $A = \{1, 2, 3, \{3\}, \{1, 4, 7\}\}$. Answer each of the following as *true* or *false*.

a. $1 \in A$ **b.** $\{3\} \in A$ **c.** $\{2\} \in A$ **d.** $4 \in A$

e. $\{\{3\}\} \subset A$ **f.** $\{1, 4, 7\} \in A$ **g.** $\{1, 4, 7\} \subseteq A$

58. Let $B = \{a, b, c, \{d\}, \{e, f\}\}$. Answer each of the following as *true* or *false*.

a. $a \in B$ **b.** $\{b, c, d\} \subset B$ **c.** $\{d\} \in B$

d. $\{d\} \subseteq B$ **e.** $\{e, f\} \in B$ **f.** $\{a, \{e, f\}\} \subset B$

g. $\{e, f\} \subset B$

APPLICATIONS

Business and Economics

Mutual Funds **The tables in the next column show five of the largest holdings of four major mutual funds on January 31, 2010. *Sources: fidelity.com, franklintempleton.com, janus.com, vanguard.com.***

Vanguard 500	Fidelity New Millennium Fund
Exxon Mobil	Pfizer
Apple	Cisco Systems
General Electric	Wal-Mart Stores
IBM	Apple
JPMorgan Chase & Co.	JPMorgan Chase & Co.

Janus Perkins Large Cap Value	Templeton Large Cap Value Fund
Exxon Mobil	IBM
General Electric	General Electric
Wal-Mart Stores	Hewlett-Packard
AT&T	Home Depot
JPMorgan Chase & Co.	Aflac

Let U be the smallest possible set that includes all the corporations listed, and V, F, J, and T be the set of top holdings for each mutual fund, respectively. Find each set:

59. $V \cap J$

60. $V \cap (F \cup T)$

61. $(J \cup F)'$

62. $J' \cap T'$

63. **Sales Calls** Suppose that Kendra Gallegos has appointments with 9 potential customers. Kendra will be ecstatic if all 9 of these potential customers decide to make a purchase from her. Of course, in sales there are no guarantees. How many different sets of customers may place an order with Kendra? (*Hint:* Each set of customers is a subset of the original set of 9 customers.)

Life Sciences

Health **The following table shows some symptoms of an overactive thyroid and an underactive thyroid. *Source: The Merck Manual of Diagnosis and Therapy.***

Underactive Thyroid	Overactive Thyroid
Sleepiness, s	Insomnia, i
Dry hands, d	Moist hands, m
Intolerance of cold, c	Intolerance of heat, h
Goiter, g	Goiter, g

Let U be the smallest possible set that includes all the symptoms listed, N be the set of symptoms for an underactive thyroid, and O be the set of symptoms for an overactive thyroid. Find each set.

64. O'

65. N'

66. $N \cap O$

67. $N \cup O$

68. $N \cap O'$

Social Sciences

69. **APPLY IT** **Electoral College** U.S. presidential elections are decided by the Electoral College, in which each of the 50

states, plus the District of Columbia, gives all of its votes to a candidate.* Ignoring the number of votes each state has in the Electoral College, but including all possible combinations of states that could be won by either candidate, how many outcomes are possible in the Electoral College if there are two candidates? (*Hint:* The states that can be won by a candidate form a subset of all the states.)

General Interest

70. Musicians A concert featured a cellist, a flutist, a harpist, and a vocalist. Throughout the concert, different subsets of the four musicians performed together, with at least two musicians playing each piece. How many subsets of at least two are possible?

Television Cable Services **The following table lists some of the most popular cable television networks. Use this information for Exercises 71–76.** ***Source: The New York Times 2010 Almanac.***

Network	Subscribers (in millions)	Launch	Content
The Discovery Channel	98.0	1985	Nonfiction, nature, science
TNT	98.0	1988	Movies, sports, original programming
USA Network	97.5	1980	Sports, family entertainment
The Learning Channel (TLC)	97.3	1980	Original programming, family entertainment
TBS Superstation	97.3	1976	Movies, sports, original programming

List the elements of the following sets. For exercises 74–76, describe each set in words.

71. F, the set of networks that were launched before 1985.

72. G, the set of networks that feature sports.

73. H, the set of networks that have more than 97.6 million viewers.

*The exceptions are Maine and Nebraska, which allocate their electoral college votes according to the winner in each congressional district.

74. $F \cap H$ **75.** $G \cup H$ **76.** G'

77. Games In David Gale's game of Subset Takeaway, the object is for each player, at his or her turn, to pick a nonempty proper subset of a given set subject to the condition that no subset chosen earlier by either player can be a subset of the newly chosen set. The winner is the last person who can make a legal move. Consider the set $A = \{1, 2, 3\}$. Suppose Joe and Dorothy are playing the game and Dorothy goes first. If she chooses the proper subset $\{1\}$, then Joe cannot choose any subset that includes the element 1. Joe can, however, choose $\{2\}$ or $\{3\}$ or $\{2, 3\}$. Develop a strategy for Joe so that he can always win the game if Dorothy goes first. ***Source: Scientific American.***

States **In the following list of states, let A = {states whose name contains the letter e}, let B = {states with a population of more than 4,000,000}, and C = {states with an area greater than 40,000 square miles}.** ***Source: The New York Times 2010 Almanac.***

State	Population (1000s)	Area (sq. mi.)
Alabama	4662	52,419
Alaska	686	663,267
Colorado	4939	104,094
Florida	18,328	65,755
Hawaii	1288	10,931
Indiana	6377	36,418
Kentucky	4269	40,409
Maine	1316	35,385
Nebraska	1783	77,354
New Jersey	8683	8721

78. a. Describe in words the set $A \cup (B \cap C)'$.

b. List all elements in the set $A \cup (B \cap C)'$.

79. a. Describe in words the set $(A \cup B)' \cap C$.

b. List all elements in the set $(A \cup B)' \cap C$.

YOUR TURN ANSWERS

1. {Ohio, Oklahoma, Oregon}

2. True

3. $2^4 = 16$ subsets

4. $\{2, 4, 8\}$

5. $\{1, 3, 5, 6, 7, 9, 11, 12\}$

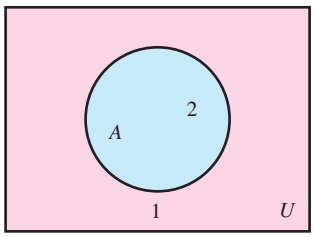
7.2 Applications of Venn Diagrams

APPLY IT

The responses to a survey of 100 households show that 76 have a DVD player, 21 have a Blue Ray player, and 12 have both. How many have neither a DVD player nor Blue Ray player?

In Example 3 we show how a Venn diagram can be used to sort out this information to answer the question.

Venn diagrams were used in the previous section to illustrate set union and intersection. The rectangular region of a Venn diagram represents the universal set U. Including only a single set A inside the universal set, as in Figure 4, divides U into two regions. Region 1



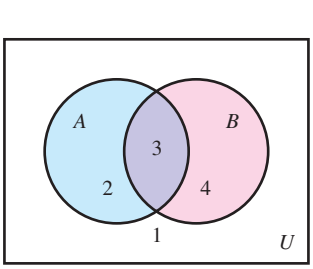
One set leads to 2 regions (numbering is arbitrary).

FIGURE 4

represents those elements of U outside set A (that is, the elements in A'), and region 2 represents those elements belonging to set A. (The numbering of these regions is arbitrary.)

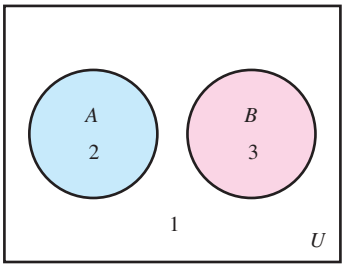
The Venn diagram in Figure 5(a) shows two sets inside U. These two sets divide the universal set into four regions. As labeled in Figure 5(a), region 1 represents the set whose elements are outside both set A and set B. Region 2 shows the set whose elements belong to A and not to B. Region 3 represents the set whose elements belong to both A and B. Which set is represented by region 4? (Again, the labeling is arbitrary.)

Two other situations can arise when representing two sets by Venn diagrams. If it is known that $A \cap B = \emptyset$, then the Venn diagram is drawn as in Figure 5(b). If it is known that $A \subseteq B$, then the Venn diagram is drawn as in Figure 5(c). For the material presented throughout this chapter we will only refer to Venn diagrams like the one in Figure 5(a), and note that some of the regions of the Venn diagram may be equal to the empty (or null) set.



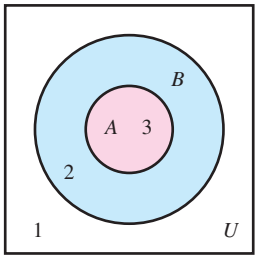
Two sets lead to 4 regions (numbering is arbitrary).

(a)



Two sets lead to 3 regions (numbering is arbitrary).

(b)



Two sets lead to 3 regions (numbering is arbitrary).

(c)

FIGURE 5

EXAMPLE 1 Venn Diagrams

Draw Venn diagrams similar to Figure 5(a) and shade the regions representing each set.

(a) $A' \cap B$

SOLUTION Set A' contains all the elements outside set A. As labeled in Figure 5(a), A' is represented by regions 1 and 4. Set B is represented by regions 3 and 4. The intersection of sets A' and B, the set $A' \cap B$, is given by the region common to both sets. The result is the set represented by region 4, which is blue in Figure 6. When looking for the intersection, remember to choose the area that is in one region *and* the other region.

In addition to the fact that region 4 in Figure 6 is $A' \cap B$, notice that region 1 is $A' \cap B'$, region 2 is $A \cap B'$, and region 3 is $A \cap B$.

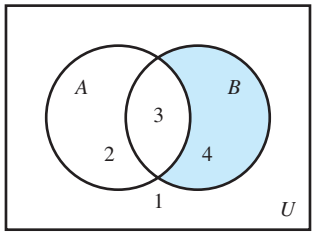


FIGURE 6

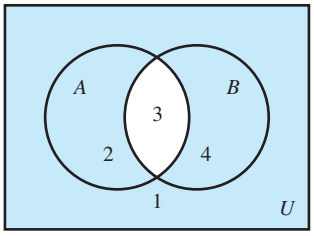


FIGURE 7

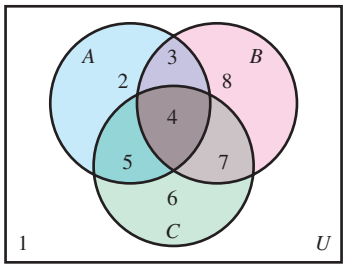
(b) $A' \cup B'$

SOLUTION Again, set A' is represented by regions 1 and 4, and set B' by regions 1 and 2. To find $A' \cup B'$, identify the region that represents the set of all elements in A', B', or both. The result, which is blue in Figure 7, includes regions 1, 2, and 4. When looking for the union, remember to choose the area that is in one region *or* the other region (or both).

YOUR TURN 1 Draw a Venn diagram and shade the region representing $A \cup B'$.

TRY YOUR TURN 1

Venn diagrams also can be drawn with three sets inside U. These three sets divide the universal set into eight regions, which can be numbered (arbitrarily) as in Figure 8.



Three sets lead to 8 regions.

FIGURE 8

EXAMPLE 2 Venn Diagram

In a Venn diagram, shade the region that represents $A' \cup (B \cap C')$.

SOLUTION First find $B \cap C'$. Set B is represented by regions 3, 4, 7, and 8, and set C' by regions 1, 2, 3, and 8. The overlap of these regions (regions 3 and 8) represents the set $B \cap C'$. Set A' is represented by regions 1, 6, 7, and 8. The union of the set represented by regions 3 and 8 and the set represented by regions 1, 6, 7, and 8 is the set represented by regions 1, 3, 6, 7, and 8, which are blue in Figure 9. TRY YOUR TURN 2

YOUR TURN 2 Draw a Venn diagram and shade the region representing $A' \cap (B \cup C)$.

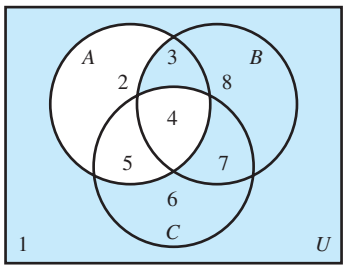


FIGURE 9

Applications

Venn diagrams can be used to analyze many applications, as illustrated in the following examples.

EXAMPLE 3 Entertainment Technology

A researcher collecting data on 100 households finds that

76 have a DVD player;

21 have a Blue Ray player; and

12 have both.

The researcher wants to answer the following questions.

(a) How many do not have a DVD player?

(b) How many have neither a DVD player nor a Blue Ray player?

(c) How many have a Blue Ray player but not a DVD player?

APPLY IT

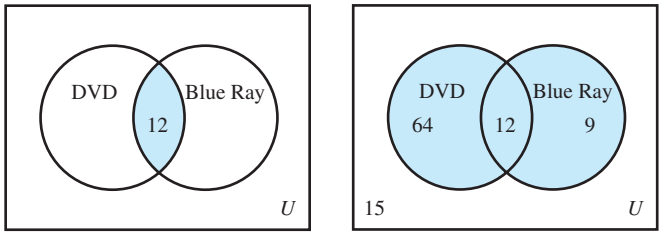
SOLUTION A Venn diagram like the one in Figure 10 will help sort out this information. In Figure 10(a), we put the number 12 in the region common to both a DVD player and a Blue Ray player, because 12 households have both. Of the 21 with a Blue Ray player, $21 - 12 = 9$ have no DVD player, so in Figure 10(b) we put 9 in the region for a Blue Ray player but no DVD player. Similarly, $76 - 12 = 64$ households have a DVD player but not

a Blue Ray player, so we put 64 in that region. Finally, the diagram shows that $100 - 64 - 12 - 9 = 15$ households have neither a DVD player nor a Blue Ray player. Now we can answer the questions:

(a) $15 + 9 = 24$ do not have a DVD player.

(b) 15 have neither.

(c) 9 have a Blue Ray player but not a DVD player.



(a) **(b)**

FIGURE 10

EXAMPLE 4 Magazines

A survey of 77 freshman business students at a large university produced the following results.

25 of the students read *Bloomberg Businessweek*;

19 read *The Wall Street Journal*;

27 do not read *Fortune*;

11 read *Bloomberg Businessweek* but not *The Wall Street Journal*;

11 read *The Wall Street Journal* and *Fortune*;

13 read *Bloomberg Businessweek* and *Fortune*;

9 read all three.

Use this information to answer the following questions.

(a) How many students read none of the publications?

(b) How many read only *Fortune*?

(c) How many read *Bloomberg Businessweek* and *The Wall Street Journal*, but not *Fortune*?

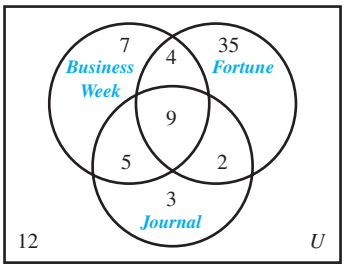


FIGURE 11

SOLUTION Since 9 students read all three publications, begin by placing 9 in the area that belongs to all three regions, as shown in Figure 11. Of the 13 students who read *Bloomberg Businessweek* and *Fortune*, 9 also read *The Wall Street Journal*. Therefore, only $13 - 9 = 4$ read just *Bloomberg Businessweek* and *Fortune*. Place the number 4 in the area of Figure 11 common only to *Bloomberg Businessweek* and *Fortune* readers.

In the same way, place $11 - 9 = 2$ in the region common only to *Fortune* and *The Wall Street Journal*. Of the 11 students who read *Bloomberg Businessweek* but not *The Wall Street Journal,* 4 read *Fortune*, so place $11 - 4 = 7$ in the region for those who read only *Bloomberg Businessweek*.

The data show that 25 students read *Bloomberg Businessweek*. However, $7 + 4 + 9 = 20$ readers have already been placed in the region representing *Bloomberg Businessweek*. The balance of this region will contain only $25 - 20 = 5$ students. These students read *Bloomberg Businessweek* and *The Wall Street Journal* but not *Fortune*. In the same way, $19 - (5 + 9 + 2) = 3$ students read only *The Wall Street Journal*.

Using the fact that 27 of the 77 students do not read *Fortune*, we know that 50 do read *Fortune*. We already have $4 + 9 + 2 = 15$ students in the region representing *Fortune*, leaving $50 - 15 = 35$ who read only *Fortune*.

A total of $7 + 4 + 35 + 5 + 9 + 2 + 3 = 65$ students are placed in the three circles in Figure 11. Since 77 students were surveyed, $77 - 65 = 12$ students read none of the three publications, and 12 is placed outside all three regions.

YOUR TURN 3 One hundred students were asked which fast food restaurants they had visited in the past month. The results are as follows:

47 ate at McDonald's;

46 ate at Taco Bell;

44 ate at Wendy's;

17 ate at McDonald's and Taco Bell;

19 ate at Taco Bell and Wendy's;

22 ate at Wendy's and McDonald's;

13 ate at all three.

Determine how many ate only at Taco Bell.

Now Figure 11 can be used to answer the questions asked above.

(a) There are 12 students who read none of the three publications.

(b) There are 35 students who read only *Fortune*.

(c) The overlap of the regions representing readers of *Bloomberg Businessweek* and *The Wall Street Journal* shows that 5 students read *Bloomberg Businessweek* and *The Wall Street Journal* but not *Fortune*.

TRY YOUR TURN 3

CAUTION A common error in solving problems of this type is to make a circle represent one set and another circle represent its complement. In Example 4, with one circle representing those who read *Bloomberg Businessweek*, we did not draw another for those who do not read *Bloomberg Businessweek*. An additional circle is not only unnecessary (because those not in one set are automatically in the other) but very confusing, because the region outside or inside both circles must be empty. Similarly, if a problem involves men and women, do not draw one circle for men and another for women. Draw one circle; if you label it "women," for example, then men are automatically those outside the circle.

EXAMPLE 5 Utility Maintenance

Jeff Friedman is a section chief for an electric utility company. The employees in his section cut down trees, climb poles, and splice wire. Friedman reported the following information to the management of the utility.

"Of the 100 employees in my section,

45 can cut trees;

50 can climb poles;

57 can splice wire;

22 can climb poles but can't cut trees;

20 can climb poles and splice wire;

25 can cut trees and splice wire;

14 can cut trees and splice wire but can't climb poles;

9 can't do any of the three (management trainees)."

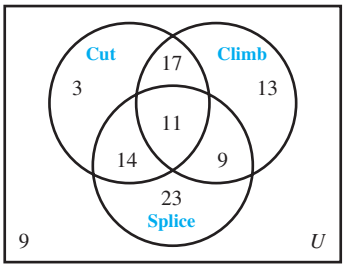


FIGURE 12

The data supplied by Friedman lead to the numbers shown in Figure 12. Add the numbers from all of the regions to get the total number of employees:

$$9 + 3 + 14 + 23 + 11 + 9 + 17 + 13 = 99.$$

Friedman claimed to have 100 employees, but his data indicated only 99. Management decided that Friedman didn't qualify as a section chief, and he was reassigned as a night-shift meter reader in Guam. (*Moral:* He should have taken this course.)

In all the examples above, we started with a piece of information specifying the relationship with all the categories. This is usually the best way to begin solving problems of this type.

As we saw in the previous section, we use the symbol $n(A)$ to indicate the *number* of elements in a finite set A. The following statement about the number of elements in the union of two sets will be used later in our study of probability.

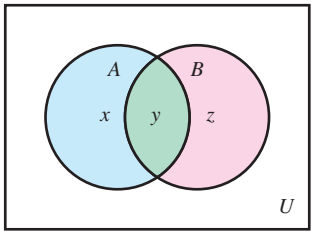


FIGURE 13

Union Rule for Sets

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

To prove this statement, let $x + y$ represent $n(A)$, y represent $n(A \cap B)$, and $y + z$ represent $n(B)$, as shown in Figure 13. Then

$$n(A \cup B) = x + y + z,$$

$$n(A) + n(B) - n(A \cap B) = (x + y) + (y + z) - y = x + y + z,$$

so

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

EXAMPLE 6 School Activities

A group of 10 students meet to plan a school function. All are majoring in accounting or economics or both. Five of the students are economics majors and 7 are majors in accounting. How many major in both subjects?

SOLUTION Let A represent the set of accounting majors and B represent the set of economics majors. Use the union rule, with $n(A) = 5$, $n(B) = 7$, and $n(A \cup B) = 10$. Find $n(A \cap B)$.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$10 = 5 + 7 - n(A \cap B),$$

so
$$n(A \cap B) = 5 + 7 - 10 = 2.$$

TRY YOUR TURN 4

When A and B are disjoint, then $n(A \cap B) = 0$, so the union rule simplifies to $n(A \cup B) = n(A) + n(B)$.

CAUTION The rule $n(A \cup B) = n(A) + n(B)$ is only valid when A and B are disjoint. When A and B are *not* disjoint, use the rule $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

YOUR TURN 4 A group of students are sitting in the lounge. All are texting or listening to music or both. Eleven are listening to music and 15 are texting. Eight are doing both. How many students are in the lounge?

EXAMPLE 7 Endangered Species

The following table gives the number of threatened and endangered animal species in the world as of January, 2010. ***Source: U.S. Fish and Wildlife Service.***

Endangered and Threatened Species			
	Endangered (E)	Threatened (T)	Totals
Amphibians and reptiles (A)	101	52	153
Arachnids and insects (I)	63	10	73
Birds (B)	256	23	279
Clams, crustaceans, corals and snails (C)	108	24	132
Fishes (F)	85	66	151
Mammals (M)	325	35	360
Totals	938	210	1148

Using the letters given in the table to denote each set, find the number of species in each of the following sets.

(a) $E \cap B$

SOLUTION The set $E \cap B$ consists of all species that are endangered *and* are birds. From the table, we see that there are 256 such species.

(b) $E \cup B$

SOLUTION The set $E \cup B$ consists of all species that are endangered *or* are birds. We include all 938 endangered species, plus the 23 bird species who are threatened but not endangered, for a total of 961. Alternatively, we could use the formula $n(E \cup B) = n(E) + n(B) - n(E \cap B) = 938 + 279 - 256 = 961$.

(c) $(F \cup M) \cap T'$

SOLUTION Begin with the set $F \cup M$, which is all species that are fish or mammals. This consists of the four categories with 85, 66, 325, and 35 species. Of this set, take those that are *not* threatened, for a total of $85 + 325 = 410$ species. This is the number of species of fish and mammals that are not threatened.

EXAMPLE 8 Online Activities

Suppose that a group of 150 students have done at least one of these activities online: purchasing an item, banking, and watching a video. In addition,

90 students have made an online purchase;

50 students have banked online;

70 students have watched an online video;

15 students have made an online purchase and watched an online video;

12 have banked online and watched an online video; and

10 students have done all three.

How many students have made an online purchase and banked online?

SOLUTION Let P represent the set of students who have made a purchase online, B the set who have banked online, and V the set who have watched a video online. Since 10 students did all three activities, begin by placing 10 in the area that belongs to all three regions, as shown in Figure 14. Of the 15 students who belong in sets P and V, 10 also belong to set B. Thus, only $15 - 10 = 5$ students were in the area of Figure 14 common only to sets P and V. Likewise, there are $12 - 10 = 2$ students who belong to only sets B and V. Since there are already $5 + 10 + 2 = 17$ students in set V, there are $70 - 17 = 53$ students in only set V.

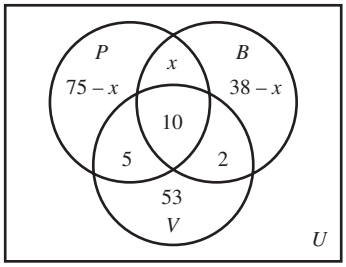


FIGURE 14

We cannot use the information about set P, since there are two regions in P for which we have no information. Similarly, we cannot use the information about set B. In such cases, we label a region with the variable x. Here we place x in the region common only to P and B, as shown in Figure 14.

Of the 90 students in set P, the number who are only in set P must be $90 - x - 10 - 5 = 75 - x$, and this expression is placed in the appropriate region in Figure 14. Similarly, the number who are in only set B is $50 - x - 10 - 2 = 38 - x$. Notice that because all 150 students participated in at least one activity, there are no elements in the region outside the three circles.

Now that the diagram is filled out, we can determine the value of x by recalling that the total number of students was 150. Thus,

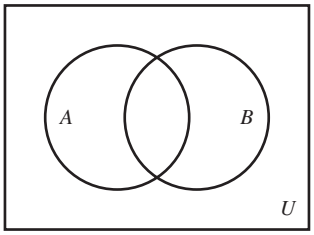
$$(75 - x) + 5 + x + 10 + (38 - x) + 2 + 53 = 150.$$

Simplifying, we have $183 - x = 150$, implying that $x = 33$. The number of students who made an online purchase and banked online is

$$33 + 10 = 43.$$

7.2 EXERCISES

Sketch a Venn diagram like the one in the figure, and use shading to show each set.



1. $B \cap A'$

2. $A \cup B'$

3. $A' \cup B$

4. $A' \cap B'$

5. $B' \cup (A' \cap B')$

6. $(A \cap B) \cup B'$

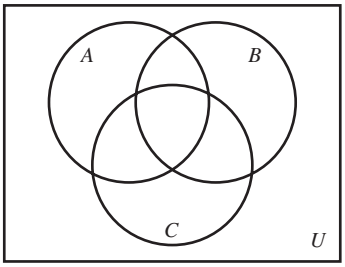
7. U'

8. $\emptyset'$

9. Three sets divide the universal set into at most ____ regions.

10. What does the notation $n(A)$ represent?

Sketch a Venn diagram like the one shown, and use shading to show each set.



11. $(A \cap B) \cap C$

12. $(A \cap C') \cup B$

13. $A \cap (B \cup C')$

14. $A' \cap (B \cap C)$

15. $(A' \cap B') \cap C'$

16. $(A \cap B') \cap C$

17. $(A \cap B') \cup C'$

18. $A' \cap (B' \cup C)$

19. $(A \cup B') \cap C$

20. $A' \cup (B' \cap C')$

Use the union rule to answer the following questions.

21. If $n(A) = 5$, $n(B) = 12$, and $n(A \cap B) = 4$, what is $n(A \cup B)$?

22. If $n(A) = 15$, $n(B) = 30$, and $n(A \cup B) = 33$, what is $n(A \cap B)$?

23. Suppose $n(B) = 9$, $n(A \cap B) = 5$, and $n(A \cup B) = 22$. What is $n(A)$?

24. Suppose $n(A \cap B) = 5$, $n(A \cup B) = 38$, and $n(A) = 13$. What is $n(B)$?

Draw a Venn diagram and use the given information to fill in the number of elements for each region.

25. $n(U) = 41, n(A) = 16, n(A \cap B) = 12, n(B') = 20$

26. $n(A) = 28, n(B) = 12, n(A \cup B) = 32, n(A') = 19$

27. $n(A \cup B) = 24, n(A \cap B) = 6, n(A) = 11,$
$n(A' \cup B') = 25$

28. $n(A') = 31, n(B) = 25, n(A' \cup B') = 46, n(A \cap B) = 12$

29. $n(A) = 28, n(B) = 34, n(C) = 25, n(A \cap B) = 14,$
$n(B \cap C) = 15, n(A \cap C) = 11, n(A \cap B \cap C) = 9,$
$n(U) = 59$

30. $n(A) = 54, n(A \cap B) = 22, n(A \cup B) = 85,$
$n(A \cap B \cap C) = 4, n(A \cap C) = 15, n(B \cap C) = 16,$
$n(C) = 44, n(B') = 63$

31. $n(A \cap B) = 6, n(A \cap B \cap C) = 4, n(A \cap C) = 7,$
$n(B \cap C) = 4, n(A \cap C') = 11, n(B \cap C') = 8,$
$n(C) = 15, n(A' \cap B' \cap C') = 5$

32. $n(A) = 13, n(A \cap B \cap C) = 4, n(A \cap C) = 6,$
$n(A \cap B') = 6, n(B \cap C) = 6, n(B \cap C') = 11,$
$n(B \cup C) = 22, n(A' \cap B' \cap C') = 5$

In Exercises 33–36, show that the statement is true by drawing Venn diagrams and shading the regions representing the sets on each side of the equals sign.*

33. $(A \cup B)' = A' \cap B'$

34. $(A \cap B)' = A' \cup B'$

35. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

36. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

37. Use the union rule of sets to prove that $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$. (*Hint:* Write $A \cup B \cup C$ as $A \cup (B \cup C)$ and use the formula from Exercise 35.)

*The statements in Exercises 33 and 34 are known as De Morgan's Laws. They are named for the English mathematician Augustus De Morgan (1806–1871). They are analogous to De Morgan's Laws for logic seen in the previous chapter.

Business and Economics

Use Venn diagrams to answer the following questions.

38. Cooking Preferences Jeff Friedman, of Example 5 in the text, was again reassigned, this time to the home economics department of the electric utility. He interviewed 140 people in a suburban shopping center to discover some of their cooking habits. He obtained the following results:

58 use microwave ovens;
63 use electric ranges;
58 use gas ranges;
19 use microwave ovens and electric ranges;
17 use microwave ovens and gas ranges;
4 use both gas and electric ranges;
1 uses all three;
2 use none of the three.

Should he be reassigned one more time? Why or why not?

39. Harvesting Fruit Toward the middle of the harvesting season, peaches for canning come in three types, early, late, and extra late, depending on the expected date of ripening. During a certain week, the following data were recorded at a fruit delivery station:

34 trucks went out carrying early peaches;
61 carried late peaches;
50 carried extra late;
25 carried early and late;
30 carried late and extra late;
8 carried early and extra late;
6 carried all three;
9 carried only figs (no peaches at all).

a. How many trucks carried only late variety peaches?

b. How many carried only extra late?

c. How many carried only one type of peach?

d. How many trucks (in all) went out during the week?

40. Cola Consumption Market research showed that the adult residents of a certain small town in Georgia fit the following categories of cola consumption. (We assume here that no one drinks both regular cola and diet cola.)

Age	Drink Regular Cola (R)	Drink Diet Cola (D)	Drink No Cola (N)	Totals
21–25 (Y)	40	15	15	70
26–35 (M)	30	30	20	80
Over 35 (O)	10	50	10	70
Totals	80	95	45	220

Using the letters given in the table, find the number of people in each set.

a. $Y \cap R$

b. $M \cap D$

c. $M \cup (D \cap Y)$

d. $Y' \cap (D \cup N)$

e. $O' \cup N$

f. $M' \cap (R' \cap N')$

g. Describe the set $M \cup (D \cap Y)$ in words.

41. Investment Habits The following table shows the results of a survey taken by a bank in a medium-sized town in Tennessee. The survey asked questions about the investment habits of bank customers. (We assume here that no one invests in more than one of type of investment.)

Age	Stocks (S)	Bonds (B)	Savings Accounts (A)	Totals
18–29 (Y)	6	2	15	23
30–49 (M)	14	5	14	33
50 or over (O)	32	20	12	64
Totals	52	27	41	120

Using the letters given in the table, find the number of people in each set.

a. $Y \cap B$ **b.** $M \cup A$ **c.** $Y \cap (S \cup B)$

d. $O' \cup (S \cup A)$ **e.** $(M' \cup O') \cap B$

f. Describe the set $Y \cap (S \cup B)$ in words.

42. Investment Survey Most mathematics professors love to invest their hard-earned money. A recent survey of 150 math professors revealed that

111 invested in stocks;
98 invested in bonds;
100 invested in certificates of deposit;
80 invested in stocks and bonds;
83 invested in bonds and certificates of deposit;
85 invested in stocks and certificates of deposit;
9 did not invest in any of the three.

How many mathematics professors invested in stocks and bonds and certificates of deposit?

Life Sciences

43. Genetics After a genetics experiment on 50 pea plants, the number of plants having certain characteristics was tallied, with the following results.

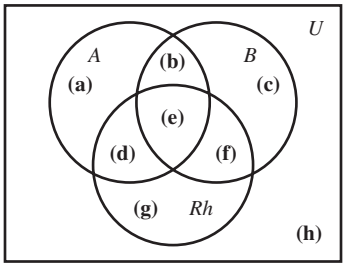
22 were tall;
25 had green peas;
39 had smooth peas;
9 were tall and had green peas;
20 had green peas and smooth peas;
6 had all three characteristics;
4 had none of the characteristics.

a. Find the number of plants that were tall and had smooth peas.

b. How many plants were tall and had peas that were neither smooth nor green?

c. How many plants were not tall but had peas that were smooth and green?

44. Blood Antigens Human blood can contain the A antigen, the B antigen, both the A and B antigens, or neither antigen. A third antigen, called the Rh antigen, is important in human reproduction, and again may or may not be present in an individual. Blood is called type A-positive if the individual has the A and Rh but not the B antigen. A person having only the A and B antigens is said to have type AB-negative blood. A person having only the Rh antigen has type O-positive blood. Other blood types are defined in a similar manner. Identify the blood types of the individuals in regions (a)–(h) below.



45. Blood Antigens (Use the diagram from Exercise 44.) In a certain hospital, the following data were recorded.

25 patients had the A antigen;
8 had the A and not the B antigen;
27 had the B antigen;
22 had the B and Rh antigens;
30 had the Rh antigen;
12 had none of the antigens;
16 had the A and Rh antigens;
15 had all three antigens.

How many patients

a. were represented? **b.** had exactly one antigen?

c. had exactly two antigens? **d.** had O-positive blood?

e. had AB-positive blood? **f.** had B-negative blood?

g. had O-negative blood? **h.** had A-positive blood?

46. Mortality The table lists the number of deaths in the United States during 2006 according to race and gender. Use this information and the letters given to find the number of people in each set. ***Source: National Vital Statistics Reports.***

	White (W)	Black (B)	American Indian (I)	Asian or Pacific Islander (A)
Female (F)	1,055,221	141,369	6407	21,325
Male (M)	1,022,328	148,602	7630	23,382

a. F **b.** $F \cap (I \cup A)$

c. $M \cup B$ **d.** $W' \cup I' \cup A'$

e. In words, describe the set in part b.

47. Hockey The table lists the number of head and neck injuries for 319 ice hockey players wearing either a full shield or half shield in the Canadian Inter-University Athletics Union during one season. Using the letters given in the table, find the number of injuries in each set. ***Source: JAMA.***

	Half Shield (*H*)	Full Shield (*F*)
Head and Face Injuries (A)	95	34
Concussions (B)	41	38
Neck Injuries (C)	9	7
Other Injuries (D)	202	150

a. $A \cap F$ **b.** $C \cap (H \cup F)$

c. $D \cup F$ **d.** $B' \cap C'$

Social Sciences

48. Military The number of female military personnel in 2009 is given in the following table. Use this information and the letters given to find the number of female military personnel in each set. ***Source: Department of Defense.***

	Army (*A*)	Air Force (*B*)	Navy (*C*)	Marines (*D*)	Totals
Officers (O)	14,322	12,097	7884	1202	35,505
Enlisted (E)	59,401	51,965	42,225	11,749	165,340
Cadets & Midshipmen (M)	688	922	920	0	2530
Totals	74,411	64,984	51,029	12,951	203,375

a. $A \cup B$ **b.** $E \cup (C \cup D)$ **c.** $O' \cap M'$

U.S. Population The projected U.S. population in 2020 (in millions) by age and race or ethnicity is given in the following table. Use this information in Exercises 49–54. *Source: U.S. Bureau of the Census.*

	Non-Hispanic White (*A*)	Hispanic (*B*)	Black (*C*)	Asian (*D*)	American Indian (*E*)
Under 45 (F)	110.6	37.6	30.2	13.1	2.2
45–64 (G)	55.3	10.3	9.9	4.3	0.6
65 and over (H)	41.4	4.7	5.0	2.2	0.3
Totals	207.3	52.6	45.1	19.6	3.1

Using the letters given in the table, find the number of people in each set.

49. $A \cap F$ **50.** $G \cup B$

51. $G \cup (C \cap H)$ **52.** $F \cap (B \cup H)$

53. $H \cup D$ **54.** $G' \cap (A' \cap C')$

Marital Status The following table gives the population breakdown (in thousands) of the U.S. population in 2008 based on marital status and race or ethnic origin. *Source: New York Times 2010 Almanac.*

	White (*W*)	Black (*B*)	Hispanic (*H*)	Asian or Pacific Islander (*A*)
Never married (N)	54,205	13,547	12,021	3518
Married (M)	106,517	9577	16,111	6741
Widowed (I)	11,968	1740	1068	507
Divorced/ separated (D)	23,046	4590	3477	665

Find the number of people in each set. Describe each set in words.

55. $N \cap (B \cup H)$

56. $(M \cup I) \cap A$

57. $(D \cup W) \cap A'$

58. $M' \cap (B \cup A)$

General Interest

59. Chinese New Year A survey of people attending a Lunar New Year celebration in Chinatown yielded the following results:

120 were women;
150 spoke Cantonese;
170 lit firecrackers;
108 of the men spoke Cantonese;
100 of the men did not light firecrackers;
18 of the non-Cantonese-speaking women lit firecrackers;
78 non-Cantonese-speaking men did not light firecrackers;
30 of the women who spoke Cantonese lit firecrackers.

a. How many attended?

b. How many of those who attended did not speak Cantonese?

c. How many women did not light firecrackers?

d. How many of those who lit firecrackers were Cantonese-speaking men?

60. Native American Ceremonies At a pow-wow in Arizona, 75 Native American families from all over the Southwest came to participate in the ceremonies. A coordinator of the pow-wow took a survey and found that

15 families brought food, costumes, and crafts;
25 families brought food and crafts;
42 families brought food;
35 families brought crafts;
14 families brought crafts but not costumes;
10 families brought none of the three items;
18 families brought costumes but not crafts.

a. How many families brought costumes and food?

b. How many families brought costumes?

c. How many families brought food, but not costumes?

d. How many families did not bring crafts?

e. How many families brought food or costumes?

61. Poultry Analysis A chicken farmer surveyed his flock with the following results. The farmer had

9 fat red roosters;
13 thin brown hens;
15 red roosters;
11 thin red chickens (hens and roosters);
17 red hens;
56 fat chickens (hens and roosters);
41 roosters;
48 hens.

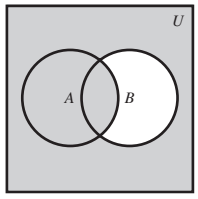
Assume all chickens are thin or fat, red or brown, and hens (female) or roosters (male). How many chickens were

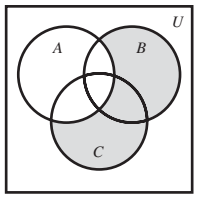
a. in the flock? **b.** red?

c. fat roosters? **d.** fat hens?

e. thin and brown? **f.** red and fat?

YOUR TURN ANSWERS

1. 

2. 

3. 23 **4.** 18

7.3 Introduction to Probability

APPLY IT **What is the probability that a randomly selected person in the United States is Hispanic or Black?**

After introducing probability, we will answer this question in Exercise 58.

If you go to a supermarket and buy 5 pounds of peaches at 99 cents per pound, you can easily find the *exact* price of your purchase: $4.95. On the other hand, the produce manager of the market is faced with the problem of ordering peaches. The manager may have a good estimate of the number of pounds of peaches that will be sold during the day, but it is impossible to predict the *exact* amount. The number of pounds that customers will purchase during a day is *random*: the quantity cannot be predicted exactly. A great many problems that come up in applications of mathematics involve random phenomena—those for which exact prediction is impossible. The best that we can do is determine the *probability* of the possible outcomes.

Sample Spaces

In probability, an **experiment** is an activity or occurrence with an observable result. Each repetition of an experiment is called a **trial**. The possible results of each trial are called **outcomes**. The set of all possible outcomes for an experiment is the **sample space** for that experiment. A sample space for the experiment of tossing a coin is made up of the outcomes heads (h) and tails (t). If S represents this sample space, then

$$S = \{h, t\}.$$

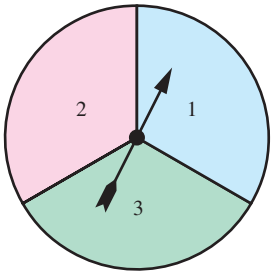


FIGURE 15

EXAMPLE 1 Sample Spaces

Give the sample space for each experiment.

(a) A spinner like the one in Figure 15 is spun.

SOLUTION The three outcomes are 1, 2, or 3, so the sample space is

$$\{1, 2, 3\}.$$

(b) For the purposes of a public opinion poll, respondents are classified as young, middle-aged, or senior, and as male or female.

SOLUTION A sample space for this poll could be written as a set of ordered pairs:

$$\{(\text{young, male}), (\text{young, female}), (\text{middle-aged, male}),$$
$$(\text{middle-aged, female}), (\text{senior, male}), (\text{senior, female})\}.$$

(c) An experiment consists of studying the numbers of boys and girls in families with exactly 3 children. Let b represent *boy* and g represent *girl*.

SOLUTION A three-child family can have 3 boys, written bbb, 3 girls, ggg, or various combinations, such as bgg. A sample space with four outcomes (not equally likely) is

$$S_1 = \{3 \text{ boys}, 2 \text{ boys and 1 girl}, 1 \text{ boy and 2 girls}, 3 \text{ girls}\}.$$

Notice that a family with 3 boys or 3 girls can occur in just one way, but a family of 2 boys and 1 girl or 1 boy and 2 girls can occur in more than one way. If the *order* of the births is considered, so that bgg is different from gbg or ggb, for example, another sample space is

$$S_2 = \{bbb, bbg, bgb, gbb, bgg, gbg, ggb, ggg\}.$$

The second sample space, S_2, has equally likely outcomes if we assume that boys and girls are equally likely. This assumption, while not quite true, is approximately true, so we will use it throughout this book. The outcomes in S_1 are not equally likely, since there is more than one way to get a family with 2 boys and 1 girl (bbg, bgb, or gbb) or a family with 2 girls and 1 boy (ggb, gbg, or bgg), but only one way to get 3 boys (bbb) or 3 girls (ggg). TRY YOUR TURN 1

YOUR TURN 1 Two coins are tossed, and a head or a tail is recorded for each coin. Give a sample space where each outcome is equally likely.

CAUTION An experiment may have more than one sample space, as shown in Example 1(c). The most convenient sample spaces have equally likely outcomes, but it is not always possible to choose such a sample space.

Events

An **event** is a subset of a sample space. If the sample space for tossing a coin is $S = \{h, t\}$, then one event is $E = \{h\}$, which represents the outcome "heads."

An ordinary die is a cube whose six different faces show the following numbers of dots: 1, 2, 3, 4, 5, and 6. If the die is fair (not "loaded" to favor certain faces over others), then any one of the faces is equally likely to come up when the die is rolled. The sample space for the experiment of rolling a single fair die is $S = \{1, 2, 3, 4, 5, 6\}$. Some possible events are listed below.

The die shows an even number: $E_1 = \{2, 4, 6\}$.

The die shows a 1: $E_2 = \{1\}$.

The die shows a number less than 5: $E_3 = \{1, 2, 3, 4\}$.

The die shows a multiple of 3: $E_4 = \{3, 6\}$.

Using the notation introduced earlier in this chapter, notice that $n(S) = 6, n(E_1) = 3$, $n(E_2) = 1, n(E_3) = 4$, and $n(E_4) = 2$.

EXAMPLE 2 Events

For the sample space S_2 in Example 1(c), write the following events.

(a) Event H: the family has exactly two girls

SOLUTION Families with three children can have exactly two girls with either bgg, gbg, or ggb, so event H is

$$H = \{bgg, gbg, ggb\}.$$

YOUR TURN 2 Two coins are tossed, and a head or a tail is recorded for each coin. Write the event E: the coins show exactly one head.

(b) Event K: the three children are the same sex

SOLUTION Two outcomes satisfy this condition: all boys or all girls.

$$K = \{bbb, ggg\}$$

(c) Event J: the family has three girls

SOLUTION Only ggg satisfies this condition, so

$$J = \{ggg\}.$$

TRY YOUR TURN 2

In Example 2(c), event J had only one possible outcome, ggg. Such an event, with only one possible outcome, is a **simple event**. If event E equals the sample space S, then E is called a **certain event**. If event $E = \emptyset$, then E is called an **impossible event**.

EXAMPLE 3 Events

Suppose a coin is flipped until both a head and a tail appear, or until the coin has been flipped four times, whichever comes first. Write each of the following events in set notation.

(a) The coin is flipped exactly three times.

SOLUTION This means that the first two flips of the coin did not include both a head and a tail, so they must both be heads or both be tails. Because the third flip is the last one, it must show the side of the coin not yet seen. Thus the event is

$$\{hht, tth\}.$$

(b) The coin is flipped at least three times.

SOLUTION In addition to the outcomes listed in part (a), there is also the possibility that the coin is flipped four times, which only happens when the first three flips are all heads or all tails. Thus the event is

$$\{hht, tth, hhhh, hhht, tttt, ttth\}.$$

(c) The coin is flipped at least two times.

SOLUTION This event consists of the entire sample space:

$$S = \{ht, th, hht, tth, hhhh, hhht, tttt, ttth\}.$$

This is an example of a certain event.

(d) The coin is flipped fewer than two times.

SOLUTION The coin cannot be flipped fewer than two times under the rules described, so the event is the empty set $\emptyset$. This is an example of an impossible event.

Since events are sets, we can use set operations to find unions, intersections, and complements of events. A summary of the set operations for events is given below.

Set Operations for Events

Let E and F be events for a sample space S.

$E \cap F$ occurs when both E **and** F occur;

$E \cup F$ occurs when E **or** F **or both** occur;

E' occurs when E does **not** occur.

EXAMPLE 4 Minimum-Wage Workers

A study of workers earning the minimum wage grouped such workers into various categories, which can be interpreted as events when a worker is selected at random. ***Source: Economic Policy Institute.*** Consider the following events:

E: worker is under 20;

F: worker is white;

G: worker is female.

Describe the following events in words.

(a) E'

SOLUTION E' is the event that the worker is 20 or over.

(b) $F \cap G'$

SOLUTION $F \cap G'$ is the event that the worker is white and not a female, that is, the worker is a white male.

(c) $E \cup G$

SOLUTION $E \cup G$ is the event that the worker is under 20 or is female. Note that this event includes all workers under 20, both male and female, and all female workers of any age. TRY YOUR TURN 3

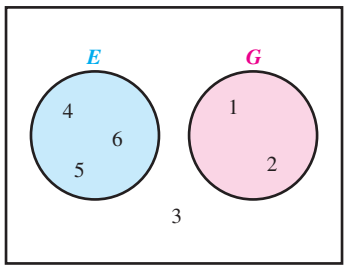
YOUR TURN 3 Describe the following event in words: $E' \cap F'$.

Two events that cannot both occur at the same time, such as rolling an even number and an odd number with a single roll of a die, are called *mutually exclusive events*.

Mutually Exclusive Events

Events E and F are **mutually exclusive events** if $E \cap \mathrm{F} = \emptyset$.

Any event E and its complement E' are mutually exclusive. By definition, mutually exclusive events are disjoint sets.



$E \cap G = \emptyset$

FIGURE 16

EXAMPLE 5 Mutually Exclusive Events

Let $S = \{1, 2, 3, 4, 5, 6\}$, the sample space for tossing a single die. Let $E = \{4, 5, 6\}$, and let $G = \{1, 2\}$. Then E and G are mutually exclusive events since they have no outcomes in common: $E \cap G = \emptyset$. See Figure 16.

Probability For sample spaces with *equally likely* outcomes, the probability of an event is defined as follows.

Basic Probability Principle

Let S be a sample space of equally likely outcomes, and let event E be a subset of S. Then the **probability** that event E occurs is

$$P(E) = \frac{n(E)}{n(S)}.$$

By this definition, the probability of an event is a number that indicates the relative likelihood of the event.

CAUTION The basic probability principle only applies when the outcomes are equally likely.

EXAMPLE 6 Basic Probabilities

Suppose a single fair die is rolled. Use the sample space $S = \{1, 2, 3, 4, 5, 6\}$ and give the probability of each event.

(a) E: the die shows an even number

SOLUTION Here, $E = \{2, 4, 6\}$, a set with three elements. Since S contains six elements,

$$P(E) = \frac{3}{6} = \frac{1}{2}.$$

(b) F: the die shows a number less than 10

SOLUTION Event F is a certain event, with

$$F = \{1, 2, 3, 4, 5, 6\},$$

so that

$$P(F) = \frac{6}{6} = 1.$$

(c) G: the die shows an 8

SOLUTION This event is impossible, so

$$P(G) = 0.$$

YOUR TURN 4 In Example 6, find the probability of event H: the die shows a number less than 5.

TRY YOUR TURN 4

A standard deck of 52 cards has four suits: hearts (♥), clubs (♣), diamonds (♦), and spades (♠), with 13 cards in each suit. The hearts and diamonds are red, and the spades and clubs are black. Each suit has an ace (A), a king (K), a queen (Q), a jack (J), and cards numbered from 2 to 10. The jack, queen, and king are called *face cards* and for many purposes can be thought of as having values 11, 12, and 13, respectively. The ace can be thought of as the low card (value 1) or the high card (value 14). See Figure 17. We will refer to this standard deck of cards often in our discussion of probability.

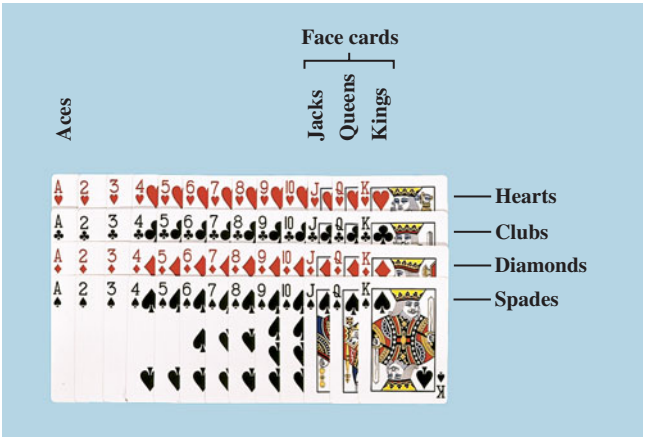


FIGURE 17

EXAMPLE 7 Playing Cards

If a single playing card is drawn at random from a standard 52-card deck, find the probability of each event.

(a) Drawing an ace

SOLUTION There are 4 aces in the deck. The event "drawing an ace" is

$$\{\text{heart ace, diamond ace, club ace, spade ace}\}.$$

Therefore,

$$P(\text{ace}) = \frac{4}{52} = \frac{1}{13}.$$

(b) Drawing a face card

SOLUTION Since there are 12 face cards (three in each of the four suits),

$$P(\text{face card}) = \frac{12}{52} = \frac{3}{13}.$$

(c) Drawing a spade

SOLUTION The deck contains 13 spades, so

$$P(\text{spade}) = \frac{13}{52} = \frac{1}{4}.$$

(d) Drawing a spade or a heart

SOLUTION Besides the 13 spades, the deck contains 13 hearts, so

$$P(\text{spade or heart}) = \frac{26}{52} = \frac{1}{2}.$$

YOUR TURN 5 Find the probability of drawing a jack or a king.

TRY YOUR TURN 5

In the preceding examples, the probability of each event was a number between 0 and 1. The same thing is true in general. Any event E is a subset of the sample space S, so $0 \le n(E) \le n(S)$. Since $P(E) = n(E)/n(S)$, it follows that $0 \le P(E) \le 1$. Note that a certain event has probability 1 and an impossible event has probability 0, as seen in Example 6.

For any event E, $\mathbf{0 \le P(E) \le 1}$.

Empirical Probability

In many real-life problems, it is not possible to establish exact probabilities for events. Instead, useful approximations are often found by drawing on past experience. The next example shows one approach to such **empirical probabilities**.

EXAMPLE 8 Injuries

The following table lists the estimated number of injuries in the United States associated with recreation equipment. ***Source: National Safety Council.***

Recreation Equipment Injuries	
Equipment	**Number of Injuries**
Bicycles	515,871
Skateboards	143,682
Trampolines	107,345
Playground climbing equipment	77,845
Swings or swing sets	59,144

Find the probability that a randomly selected person whose injury is associated with recreation equipment was hurt on a trampoline.

SOLUTION We first find the total number of injuries. Verify that the amounts in the table sum to 903,887. The probability is then found by dividing the number of people injured on trampolines by the total number of people injured. Thus,

$$P(\text{Trampolines}) = \frac{107{,}345}{903{,}887} \approx 0.1188.$$

7.3 EXERCISES

1. What is meant by a "fair" coin or die?
2. What is the sample space for an experiment?

Write sample spaces for the experiments in Exercises 3–10.

3. A month of the year is chosen for a wedding.
4. A day in April is selected for a bicycle race.
5. A student is asked how many points she earned on a recent 80-point test.
6. A person is asked the number of hours (to the nearest hour) he watched television yesterday.
7. The management of an oil company must decide whether to go ahead with a new oil shale plant or to cancel it.

8. A record is kept each day for three days about whether a particular stock goes up or down.

9. A coin is tossed, and a die is rolled.

10. A box contains five balls, numbered 1, 2, 3, 4, and 5. A ball is drawn at random, the number on it recorded, and the ball replaced. The box is shaken, a second ball is drawn, and its number is recorded.

11. Define an event.

12. What is a simple event?

For the experiments in Exercises 13–18, write out the sample space *S*, choosing an *S* with equally likely outcomes, if possible. Then give the value of $n(S)$ and tell whether the outcomes in *S* are equally likely. Finally, write the indicated events in set notation.

13. A committee of 2 people is selected at random from 5 executives: Alam, Bartolini, Chinn, Dickson, and Ellsberg.

a. Chinn is on the committee.

b. Dickson and Ellsberg are not both on the committee.

c. Both Alam and Chinn are on the committee.

14. Five states are being considered as the location for three new high-energy physics laboratories: California (CA), Colorado (CO), New Jersey (NJ), New York (NY), and Utah (UT). Three states will be chosen at random. Write elements of the sample space in the form (CA, CO, NJ).

a. All three states border an ocean.

b. Exactly two of the three states border an ocean.

c. Exactly one of the three states is west of the Mississippi River.

15. Slips of paper marked with the numbers 1, 2, 3, 4, and 5 are placed in a box. After being mixed, two slips are drawn simultaneously.

a. Both slips are marked with even numbers.

b. One slip is marked with an odd number and the other is marked with an even number.

c. Both slips are marked with the same number.

16. An unprepared student takes a three-question, true/false quiz in which he guesses the answers to all three questions, so each answer is equally likely to be correct or wrong.

a. The student gets three answers wrong.

b. The student gets exactly two answers correct.

c. The student gets only the first answer correct.

17. A coin is flipped until two heads appear, up to a maximum of four flips. (If three tails are flipped, the coin is still tossed a fourth time to complete the experiment).

a. The coin is tossed four times.

b. Exactly two heads are tossed.

c. No heads are tossed.

18. One jar contains four balls, labeled 1, 2, 3, and 4. A second jar contains five balls, labeled 1, 2, 3, 4, and 5. An experiment consists of taking one ball from the first jar, and then taking a ball from the second jar.

a. The number on the first ball is even.

b. The number on the second ball is even.

c. The sum of the numbers on the two balls is 5.

d. The sum of the numbers on the two balls is 1.

A single fair die is rolled. Find the probabilities of each event.

19. Getting a 2

20. Getting an odd number

21. Getting a number less than 5

22. Getting a number greater than 2

23. Getting a 3 or a 4

24. Getting any number except 3

A card is drawn from a well-shuffled deck of 52 cards. Find the probability of drawing the following.

25. A 9

26. A black card

27. A black 9

28. A heart

29. The 9 of hearts

30. A face card

31. A 2 or a queen

32. A black 7 or a red 8

33. A red card or a 10

34. A spade or a king

A jar contains 3 white, 4 orange, 5 yellow, and 8 black marbles. If a marble is drawn at random, find the probability that it is the following.

35. White

36. Orange

37. Yellow

38. Black

39. Not black

40. Orange or yellow

Which of Exercises 41–48 are examples of empirical probability?

41. The probability of heads on 5 consecutive tosses of a coin

42. The probability that a freshman entering college will graduate with a degree

43. The probability that a person is allergic to penicillin

44. The probability of drawing an ace from a standard deck of 52 cards

45. The probability that a person will get lung cancer from smoking cigarettes

46. A weather forecast that predicts a 70% chance of rain tomorrow

47. A gambler's claim that on a roll of a fair die, $P(\text{even}) = 1/2$

48. A surgeon's prediction that a patient has a 90% chance of a full recovery

49. The student sitting next to you in class concludes that the probability of the ceiling falling down on both of you before class ends is $1/2$, because there are two possible outcomes—the ceiling will fall or not fall. What is wrong with this reasoning?

50. The following puzzler was given on the *Car Talk* radio program. ***Source: Car Talk, Feb 24, 2001.***

"Three different numbers are chosen at random, and one is written on each of three slips of paper. The slips are then placed face down on the table. The objective is to choose the slip upon which is written the largest number. Here are the rules: You can turn

over any slip of paper and look at the amount written on it. If for any reason you think this is the largest, you're done; you keep it. Otherwise you discard it and turn over a second slip. Again, if you think this is the one with the biggest number, you keep that one and the game is over. If you don't, you discard that one too. . . . The chance of getting the highest number is one in three. Or is it? Is there a strategy by which you can improve the odds?"

The answer to the puzzler is that you can indeed improve the probability of getting the highest number by the following strategy. Pick one of the slips of paper, and after looking at the number, throw it away. Then pick a second slip; if it has a larger number than the first slip, stop. If not, pick the third slip. Find the probability of winning with this strategy.*

APPLICATIONS

Business and Economics

51. Survey of Workers The management of a firm wishes to check on the opinions of its assembly line workers. Before the workers are interviewed, they are divided into various categories. Define events E, F, and G as follows.

E: worker is female
F: worker has worked less than 5 years
G: worker contributes to a voluntary retirement plan

Describe each event in words.

a. E' **b.** $E \cap F$ **c.** $E \cup G'$

d. F' **e.** $F \cup G$ **f.** $F' \cap G'$

52. Research Funding In 2008, funding for research and development in the United States totaled $397.6 billion dollars. Support came from various sources, as shown in the following table. *Source: National Science Foundation.*

Source	Amount (in billions of dollars)
Federal government	103.7
State and local government	3.5
Industry	267.8
Academic institutions	10.6
Other	12.0

Find the probability that funds for a particular project came from each source.

a. Federal government

b. Industry

c. Academic institutions

53. Investment Survey Exercise 42 of the previous section presented a survey of 150 mathematics professors. Use the information given in that exercise to find each probability.

a. A randomly chosen professor invested in stocks and bonds and certificates of deposit.

b. A randomly chosen professor invested in only bonds.

*This is a special case of the famous Googol problem. For more details, see "Recognizing the Maximum of a Sequence" by John P. Gilbert and Frederick Mosteller, *Journal of the American Statistical Association*, Vol. 61, No. 313, March 1966, pp. 35–73.

54. Labor Force The 2008 and the 2018 (projected) civilian labor forces by age are given in the following table. *Source: U.S. Department of Labor.*

Age (in years)	2008 (in millions)	2018 (in millions)
16 to 24	22.0	21.1
25 to 54	104.4	105.9
55 and over	27.9	39.8
Total	154.3	166.8

a. In 2008, find the probability that a member of the civilian labor force is age 55 or older.

b. In 2018, find the probability that a member of the civilian labor force is age 55 or over.

c. What do these projections imply about the future civilian labor force?

Life Sciences

55. Medical Survey For a medical experiment, people are classified as to whether they smoke, have a family history of heart disease, or are overweight. Define events E, F, and G as follows.

E: person smokes
F: person has a family history of heart disease
G: person is overweight

Describe each event in words.

a. G' **b.** $F \cap G$ **c.** $E \cup G'$

56. Medical Survey Refer to Exercise 55. Describe each event in words.

a. $E \cup F$ **b.** $E' \cap F$ **c.** $F' \cup G'$

57. Causes of Death There were 2,424,059 U.S. deaths in 2007. They are listed according to cause in the following table. If a randomly selected person died in 2007, use this information to find the following probabilities. *Source: Centers for Disease Control and Prevention.*

Cause	Number of Deaths
Heart Disease	615,651
Cancer	560,187
Cerebrovascular disease	133,990
Chronic lower respiratory disease	129,311
Accidents	117,075
Alzheimer's disease	74,944
Diabetes mellitus	70,905
Influenza and pneumonia	52,847
All other causes	669,149

a. The probability that the cause of death was heart disease

b. The probability that the cause of death was cancer or heart disease

c. The probability that the cause of death was not an accident and was not diabetes mellitus

Social Sciences

58. APPLY IT U.S. Population The projected U.S. population (in thousands) by race in 2020 and 2050 is given in the table. *Source: Bureau of the Census.*

Race	2020	2050
White	207,393	207,901
Hispanic	52,652	96,508
Black	41,538	53,555
Asian and Pacific Islander	18,557	32,432
Other	2602	3535

Find the probability that a randomly selected person in the given year is of the race specified.

a. Hispanic in 2020 **b.** Hispanic in 2050

c. Black in 2020 **d.** Black in 2050

59. Congressional Service The following table gives the number of years of service of senators in the 111th Congress of the United States of America. Find the probability that a randomly selected senator of the 111th Congress served 20–29 years when Congress convened. *Source: Roll Call.*

Years of Service	Number of Senators
0–9	48
10–19	27
20–29	17
30–39	6
40 or more	2

60. Civil War Estimates of the Union Army's strength and losses for the battle of Gettysburg are given in the following table, where *strength* is the number of soldiers immediately preceding the battle and *loss* indicates a soldier who was killed, wounded, captured, or missing. ***Source: Regimental Strengths and Losses of Gettysburg.***

Unit	Strength	Loss
I Corps (Reynolds)	12,222	6059
II Corps (Hancock)	11,347	4369
III Corps (Sickles)	10,675	4211
V Corps (Sykes)	10,907	2187
VI Corps (Sedgwick)	13,596	242
XI Corps (Howard)	9188	3801
XII Corps (Slocum)	9788	1082
Cavalry (Pleasonton)	11,851	610
Artillery (Tyler)	2376	242
Total	91,950	22,803

a. Find the probability that a randomly selected union soldier was from the XI Corps.

b. Find the probability that a soldier was lost in the battle.

c. Find the probability that a I Corps soldier was lost in the battle.

d. Which group had the highest probability of not being lost in the battle?

e. Which group had the highest probability of loss?

f. Explain why these probabilities vary.

61. Civil War Estimates of the Confederate Army's strength and losses for the battle of Gettysburg are given in the following table, where *strength* is the number of soldiers immediately preceding the battle and *loss* indicates a soldier who was killed, wounded, captured, or missing. ***Source: Regimental Strengths and Losses at Gettysburg.***

Unit	Strength	Loss
I Corps (Longstreet)	20,706	7661
II Corps (Ewell)	20,666	6603
III Corps (Hill)	22,083	8007
Cavalry (Stuart)	6621	286
Total	70,076	22,557

a. Find the probability that a randomly selected confederate soldier was from the III Corps.

b. Find the probability that a confederate soldier was lost in the battle.

c. Find the probability that a I Corps soldier was lost in the battle.

d. Which group had the highest probability of not being lost in the battle?

e. Which group had the highest probability of loss?

General Interest

62. Native American Ceremonies Exercise 60 of the previous section presented a survey of families participating in a pow-wow in Arizona. Use the information given in that exercise to find each probability.

a. A randomly chosen family brought costumes and food.

b. A randomly chosen family brought crafts, but neither food nor costumes.

c. A randomly chosen family brought food or costumes.

63. Chinese New Year Exercise 59 of the previous section presented a survey of people attending a Lunar New Year celebration in Chinatown. Use the information given in that exercise to find each of the following probabilities.

a. A randomly chosen attendee speaks Cantonese.

b. A randomly chosen attendee does not speak Cantonese.

c. A randomly chosen attendee was a woman that did not light a firecracker.

YOUR TURN ANSWERS

1. $S = \{HH, HT, TH, TT\}$ **2.** $E = \{HT, TH\}$

3. $E' \cap F'$ is the event that the worker is 20 or over and is not white.

4. $P(H) = \frac{4}{6} = \frac{2}{3}$. **5.** $P(\text{jack or a king}) = 8/52 = 2/13$.

7.4 Basic Concepts of Probability

APPLY IT **What is the probability that a dollar of advertising in the United States is spent on magazines or newspapers?**

We will determine this probability in Example 8. But first we need to develop additional rules for calculating probability, beginning with the probability of a union of two events.

The Union Rule

To determine the probability of the union of two events E and F in a sample space S, use the union rule for sets,

$$n(E \cup F) = n(E) + n(F) - n(E \cap F),$$

which was proved in Section 7.2. Assuming that the events in the sample space S are equally likely, divide both sides by $n(S)$, so that

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$
$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Although our derivation is valid only for sample spaces with equally likely events, the result is valid for any events E and F from any sample space, and is called the **union rule for probability**.

Union Rule for Probability

For any events E and F from a sample space S,

$$\boldsymbol{P(E \cup F) = P(E) + P(F) - P(E \cap F).}$$

EXAMPLE 1 Probabilities with Playing Cards

If a single card is drawn from an ordinary deck of cards, find the probability that it will be a red or a face card.

SOLUTION Let R represent the event "red card" and F the event "face card." There are 26 red cards in the deck, so $P(R) = 26/52$. There are 12 face cards in the deck, so $P(F) = 12/52$. Since there are 6 red face cards in the deck, $P(R \cap F) = 6/52$. By the union rule, the probability of the card being red or a face card is

$$P(R \cup F) = P(R) + P(F) - P(R \cap F)$$
$$= \frac{26}{52} + \frac{12}{52} - \frac{6}{52} = \frac{32}{52} = \frac{8}{13}.$$

YOUR TURN 1 In Example 1, find the probability of an ace or a club.

TRY YOUR TURN 1

EXAMPLE 2 Probabilities with Dice

Suppose two fair dice are rolled. Find each probability.

(a) The first die shows a 2, or the sum of the results is 6 or 7.

SOLUTION The sample space for the throw of two dice is shown in Figure 18 on the next page, where 1-1 represents the event "the first die shows a 1 and the second die shows a 1," 1-2 represents "the first die shows a 1 and the second die shows a 2," and so on. Let A represent the event "the first die shows a 2," and B represent the event "the sum of the results is 6 or 7." These events are indicated in Figure 18. From the diagram, event A has 6 elements, B has 11 elements, the intersection of A and B has 2 elements, and the sample space has 36 elements. Thus,

$$P(A) = \frac{6}{36}, \quad P(B) = \frac{11}{36}, \quad \text{and} \quad P(A \cap B) = \frac{2}{36}.$$

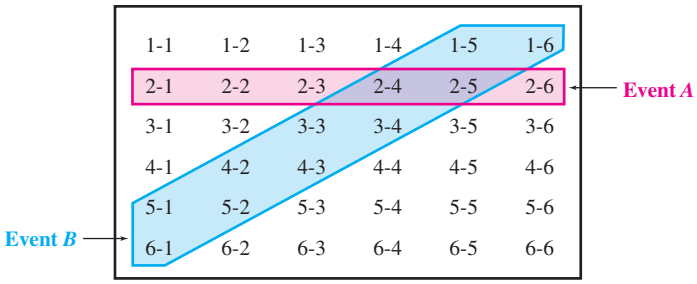


FIGURE 18

By the union rule,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{6}{36} + \frac{11}{36} - \frac{2}{36} = \frac{15}{36} = \frac{5}{12}.$$

(b) The sum of the results is 11, or the second die shows a 5.

SOLUTION $P(\text{sum is } 11) = 2/36$, $P(\text{second die shows a } 5) = 6/36$, and $P(\text{sum is 11 and second die shows a } 5) = 1/36$, so

$$P(\text{sum is 11 or second die shows a 5}) = \frac{2}{36} + \frac{6}{36} - \frac{1}{36} = \frac{7}{36}.$$

YOUR TURN 2 In Example 2, find the probability that the sum is 8, or both die show the same number.

TRY YOUR TURN 2

CAUTION You may wonder why we did not use $S = \{2, 3, 4, 5, \ldots, 12\}$ as the sample space in Example 2. Remember, we prefer to use a sample space with equally likely outcomes. The outcomes in set S above are not equally likely—a sum of 2 can occur in just one way, a sum of 3 in two ways, a sum of 4 in three ways, and so on, as shown in Figure 18.

If events E and F are mutually exclusive, then $E \cap F = \emptyset$ by definition; hence, $P(E \cap F) = 0$. In this case the union rule simplifies to $P(E \cup F) = P(E) + P(F)$.

CAUTION The rule $P(E \cup F) = P(E) + P(F)$ is only valid when E and F are mutually exclusive. When E and F are *not* mutually exclusive, use the rule $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

The Complement Rule

By the definition of E', for any event E from a sample space S,

$$E \cup E' = S \quad \text{and} \quad E \cap E' = \emptyset.$$

Since $E \cap E' = \emptyset$, events E and E' are mutually exclusive, so that

$$P(E \cup E') = P(E) + P(E').$$

However, $E \cup E' = S$, the sample space, and $P(S) = 1$. Thus

$$P(E \cup E') = P(E) + P(E') = 1.$$

Rearranging these terms gives the following useful rule for complements.

Complement Rule

$$\boldsymbol{P(E) = 1 - P(E')} \quad \text{and} \quad \boldsymbol{P(E') = 1 - P(E)}.$$

EXAMPLE 3 Complement Rule

If a fair die is rolled, what is the probability that any number but 5 will come up?

SOLUTION If E is the event that 5 comes up, then E' is the event that any number but 5 comes up. Since $P(E) = 1/6$, we have $P(E') = 1 - 1/6 = 5/6$.

EXAMPLE 4 Complement Rule

If two fair dice are rolled, find the probability that the sum of the numbers rolled is greater than 3. Refer to Figure 18.

SOLUTION To calculate this probability directly, we must find the probabilities that the sum is 4, 5, 6, 7, 8, 9, 10, 11, or 12 and then add them. It is much simpler to first find the probability of the complement, the event that the sum is less than or equal to 3.

$$\begin{aligned} P(\text{sum} \le 3) &= P(\text{sum is } 2) + P(\text{sum is } 3) \\ &= \frac{1}{36} + \frac{2}{36} \\ &= \frac{3}{36} = \frac{1}{12} \end{aligned}$$

Now use the fact that $P(E) = 1 - P(E')$ to get

$$\begin{aligned} P(\text{sum} > 3) &= 1 - P(\text{sum} \le 3) \\ &= 1 - \frac{1}{12} = \frac{11}{12}. \end{aligned}$$

YOUR TURN 3 Find the probability that when two fair dice are rolled, the sum is less than 11.

TRY YOUR TURN 3

Odds Sometimes probability statements are given in terms of **odds**, a comparison of $P(E)$ with $P(E')$. For example, suppose $P(E) = 4/5$. Then $P(E') = 1 - 4/5 = 1/5$. These probabilities predict that E will occur 4 out of 5 times and E' will occur 1 out of 5 times. Then we say the *odds in favor* of E are 4 to 1.

Odds

The **odds in favor** of an event E are defined as the ratio of $P(E)$ to $P(E')$, or

$$\frac{P(E)}{P(E')}, \text{ where } P(E') \neq 0.$$

EXAMPLE 5 Odds in Favor of Rain

Suppose the weather forecaster says that the probability of rain tomorrow is $1/3$. Find the odds in favor of rain tomorrow.

SOLUTION Let E be the event "rain tomorrow." Then E' is the event "no rain tomorrow." Since $P(E) = 1/3$, $P(E') = 2/3$. By the definition of odds, the odds in favor of rain are

$$\frac{1/3}{2/3} = \frac{1}{2}, \quad \text{written} \quad 1 \text{ to } 2, \text{ or } 1:2.$$

On the other hand, the odds that it will *not* rain, or the *odds against* rain, are

$$\frac{2/3}{1/3} = \frac{2}{1}, \quad \text{written} \quad 2 \text{ to } 1, \text{ or } 2:1.$$

YOUR TURN 4 If the probability of snow tomorrow is $3/10$, find the odds in favor of snow tomorrow.

TRY YOUR TURN 4

If the odds in favor of an event are, say, 3 to 5, then the probability of the event is $3/8$, while the probability of the complement of the event is $5/8$. (Odds of 3 to 5 indicate

3 outcomes in favor of the event out of a total of 8 possible outcomes.) This example suggests the following generalization.

If the odds favoring event E are m to n, then

$$P(E) = \frac{m}{m+n} \quad \text{and} \quad P(E') = \frac{n}{m+n}.$$

EXAMPLE 6 Package Delivery

The odds that a particular package will be delivered on time are 25 to 2.

(a) Find the probability that the package will be delivered on time.

SOLUTION Odds of 25 to 2 show 25 favorable chances out of $25 + 2 = 27$ chances altogether:

$$P(\text{package will be delivered on time}) = \frac{25}{25+2} = \frac{25}{27}.$$

(b) Find the odds against the package being delivered on time.

SOLUTION Using the complement rule, there is a $2/27$ chance that the package will not be delivered on time. So the odds against the package being delivered on time are

$$\frac{P(\text{package will not be delivered on time})}{P(\text{package will be delivered on time})} = \frac{2/27}{25/27} = \frac{2}{25},$$

or 2 : 25.

YOUR TURN 5 If the odds that a package will be delivered on time are 17 to 3, find the probability that the package will not be delivered on time.

TRY YOUR TURN 5

EXAMPLE 7 Odds in Horse Racing

If the odds in favor of a particular horse's winning a race are 5 to 7, what is the probability that the horse will win the race?

SOLUTION The odds indicate chances of 5 out of 12 ($5 + 7 = 12$) that the horse will win, so

$$P(\text{winning}) = \frac{5}{12}.$$

Race tracks generally give odds *against* a horse winning. In this case, the track would give the odds as 7 to 5. Of course, race tracks, casinos, and other gambling establishments need to give odds that are more favorable to the house than those representing the actual probabilities, because they need to make a profit.

YOUR TURN 6 If the odds against a particular horse winning a race are 7 to 3, what is the probability the horse will win the race?

TRY YOUR TURN 6

Probability Distribution

A table listing each possible outcome of an experiment and its corresponding probability is called a **probability distribution.** The assignment of probabilities may be done in any reasonable way (on an empirical basis, as in the next example, or by theoretical reasoning, as in Section 7.3), provided that it satisfies the following conditions.

Properties of Probability

Let S be a sample space consisting of n distinct outcomes, $s_1, s_2, \ldots, s_n$. An acceptable probability assignment consists of assigning to each outcome s_i a number p_i (the probability of s_i) according to these rules.

1. The probability of each outcome is a number between 0 and 1.

$$0 \le p_1 \le 1, \quad 0 \le p_2 \le 1, \ldots, \quad 0 \le p_n \le 1$$

2. The sum of the probabilities of all possible outcomes is 1.

$$p_1 + p_2 + p_3 + \cdots + p_n = 1$$

EXAMPLE 8 Advertising Volume

The following table lists U.S. advertising volume in millions of dollars by medium in 2008. *Source: Advertising Age.*

Advertising Volume	
Medium	**Expenditures**
Television	65,137
Magazines	28,580
Newspapers	25,059
Internet	9727
Radio	9501
Outdoor	3964

(a) Construct a probability distribution for the probability that a dollar of advertising is spent on each medium.

SOLUTION We first find the total spent and then divide the amount spent on each medium by the total. Verify that the amounts in the table sum to 141,968. The probability that a dollar is spent on newspapers, for example, is $P(\text{newspapers}) = 25{,}059/141{,}968 \approx 0.1765$. Similarly, we could divide each amount by 141,968, with the results (rounded to four decimal places) shown in the following table.

Advertising Volume	
Medium	**Probabilities**
Television	0.4588
Magazines	0.2013
Newspapers	0.1765
Internet	0.0685
Radio	0.0669
Outdoor	0.0279

Verify that this distribution satisfies the conditions for probability. Each probability is between 0 and 1, so the first condition holds. The probabilities in this table sum to 0.9999. In theory, to satisfy the second condition, they should total 1.0000, but this does not always occur when the individual numbers are rounded.

(b) Find the probability that a dollar of advertising in the United States is spent on newspapers or magazines.

APPLY IT

SOLUTION The categories in the table are mutually exclusive simple events. Thus, to find the probability that an advertising dollar is spent on newspapers or magazines, we use the union rule to calculate

$$P(\text{newspapers or magazines}) = 0.1765 + 0.2013 = 0.3778.$$

We could get this same result by summing the amount spent on newspapers and magazines, and dividing the total by 141,968.

Thus, more than a third of all advertising dollars are spent on these two media, a figure that should be of interest to both advertisers and the owners of the various media.

Probability distributions are discussed further in the next chapter.

EXAMPLE 9 Clothing

Susan is a college student who receives heavy sweaters from her aunt at the first sign of cold weather. Susan has determined that the probability that a sweater is the wrong size is 0.47, the probability that it is a loud color is 0.59, and the probability that it is both the wrong size and a loud color is 0.31.

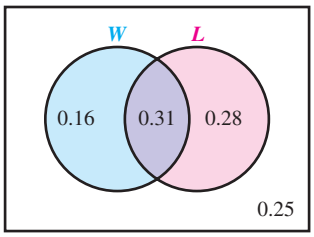


FIGURE 19

(a) Find the probability that the sweater is the correct size and not a loud color.

SOLUTION Let W represent the event "wrong size," and L represent "loud color." Place the given information on a Venn diagram, starting with 0.31 in the intersection of the regions W and L (see Figure 19). As stated earlier, event W has probability 0.47. Since 0.31 has already been placed inside the intersection of W and L,

$$0.47 - 0.31 = 0.16$$

goes inside region W, but outside the intersection of W and L, that is, in the region $W \cap L'$. In the same way,

$$0.59 - 0.31 = 0.28$$

goes inside the region for L, and outside the overlap, that is, in the region $L \cap W'$.

Using regions W and L, the event we want is $W' \cap L'$. From the Venn diagram in Figure 19, the labeled regions have a total probability of

$$0.16 + 0.31 + 0.28 = 0.75.$$

Since the entire region of the Venn diagram must have probability 1, the region outside W and L, or $W' \cap L'$, has probability

$$1 - 0.75 = 0.25.$$

The probability is 0.25 that the sweater is the correct size and not a loud color.

(b) Find the probability that the sweater is the correct size or is not loud.

SOLUTION The corresponding region, $W' \cup L'$, has probability

$$0.25 + 0.16 + 0.28 = 0.69.$$

7.4 EXERCISES

1. Define mutually exclusive events in your own words.

2. Explain the union rule for mutually exclusive events.

Decide whether the events in Exercises 3–8 are mutually exclusive.

3. Owning a dog and owning an MP3 player

4. Being a business major and being from Texas

5. Being retired and being 70 years old

6. Being a teenager and being 70 years old

7. Being one of the ten tallest people in the United States and being under 4 feet tall

8. Being male and being a nurse

Two dice are rolled. Find the probabilities of rolling the given sums.

9. **a.** 2 **b.** 4 **c.** 5 **d.** 6

10. **a.** 8 **b.** 9 **c.** 10 **d.** 13

11. **a.** 9 or more **b.** Less than 7
c. Between 5 and 8 (exclusive)

12. **a.** Not more than 5 **b.** Not less than 8
c. Between 3 and 7 (exclusive)

Two dice are rolled. Find the probabilities of the following events.

13. The first die is 3 or the sum is 8.

14. The second die is 5 or the sum is 10.

One card is drawn from an ordinary deck of 52 cards. Find the probabilities of drawing the following cards.

15. **a.** A 9 or 10
b. A red card or a 3
c. A 9 or a black 10
d. A heart or a black card
e. A face card or a diamond

16. **a.** Less than a 4 (count aces as ones)
b. A diamond or a 7
c. A black card or an ace
d. A heart or a jack
e. A red card or a face card

Kristi Perez invites 13 relatives to a party: her mother, 2 aunts, 3 uncles, 2 brothers, 1 male cousin, and 4 female cousins. If the

chances of any one guest arriving first are equally likely, find the probabilities that the first guest to arrive is as follows.

17. **a.** A brother or an uncle **b.** A brother or a cousin
c. A brother or her mother

18. **a.** An uncle or a cousin **b.** A male or a cousin
c. A female or a cousin

The numbers 1, 2, 3, 4, and 5 are written on slips of paper, and 2 slips are drawn at random one at a time without replacement. Find the probabilities in Exercises 19 and 20.

19. **a.** The sum of the numbers is 9.
b. The sum of the numbers is 5 or less.
c. The first number is 2 or the sum is 6.

20. **a.** Both numbers are even.
b. One of the numbers is even or greater than 3.
c. The sum is 5 or the second number is 2.

Use Venn diagrams to work Exercises 21 and 22.

21. Suppose $P(E) = 0.26$, $P(F) = 0.41$, and $P(E \cap F) = 0.16$. Find the following.
a. $P(E \cup F)$ **b.** $P(E' \cap F)$
c. $P(E \cap F')$ **d.** $P(E' \cup F')$

22. Let $P(Z) = 0.42$, $P(Y) = 0.35$, and $P(Z \cup Y) = 0.59$. Find each probability.
a. $P(Z' \cap Y')$ **b.** $P(Z' \cup Y')$
c. $P(Z' \cup Y)$ **d.** $P(Z \cap Y')$

23. Three unusual dice, A, B, and C, are constructed such that die A has the numbers 3, 3, 4, 4, 8, 8; die B has the numbers 1, 1, 5, 5, 9, 9; and die C has the numbers 2, 2, 6, 6, 7, 7.
a. If dice A and B are rolled, find the probability that B beats A, that is, the number that appears on die B is greater than the number that appears on die A.
b. If dice B and C are rolled, find the probability that C beats B.
c. If dice A and C are rolled, find the probability that A beats C.
d. Which die is better? Explain.

24. In the "Ask Marilyn" column of *Parade* magazine, a reader wrote about the following game: You and I each roll a die. If your die is higher than mine, you win. Otherwise, I win. The reader thought that the probability that each player wins is $1/2$. Is this correct? If not, what is the probability that each player wins? ***Source: Parade magazine.***

25. Define what is meant by odds.

26. On page 134 of Roger Staubach's autobiography, *First Down, Lifetime to Go*, Staubach makes the following statement regarding his experience in Vietnam: ***Source: First Down, Lifetime to Go.***

> "Odds against a direct hit are very low but when your life is in danger, you don't worry too much about the odds."

Is this wording consistent with our definition of odds, for and against? How could it have been said so as to be technically correct?

A single fair die is rolled. Find the odds in favor of getting the results in Exercises 27–30.

27. 3 **28.** 4, 5, or 6

29. 2, 3, 4, or 5 **30.** Some number less than 6

31. A marble is drawn from a box containing 3 yellow, 4 white, and 11 blue marbles. Find the odds in favor of drawing the following.
a. A yellow marble **b.** A blue marble
c. A white marble **d.** Not drawing a white marble

32. Two dice are rolled. Find the odds of rolling the following. (Refer to Figure 18.)
a. A sum of 3 **b.** A sum of 7 or 11
c. A sum less than 5 **d.** Not a sum of 6

33. What is a probability distribution?

34. What conditions must hold for a probability distribution to be acceptable?

An experiment is conducted for which the sample space is $S = \{s_1, s_2, s_3, s_4, s_5\}$. Which of the probability assignments in Exercises 35–40 are possible for this experiment? If an assignment is not possible, tell why.

35.

Outcomes	s_1	s_2	s_3	s_4	s_5
Probabilities	0.09	0.32	0.21	0.25	0.13

36.

Outcomes	s_1	s_2	s_3	s_4	s_5
Probabilities	0.92	0.03	0	0.02	0.03

37.

Outcomes	s_1	s_2	s_3	s_4	s_5
Probabilities	1/3	1/4	1/6	1/8	1/10

38.

Outcomes	s_1	s_2	s_3	s_4	s_5
Probabilities	1/5	1/3	1/4	1/5	1/10

39.

Outcomes	s_1	s_2	s_3	s_4	s_5
Probabilities	0.64	−0.08	0.30	0.12	0.02

40.

Outcomes	s_1	s_2	s_3	s_4	s_5
Probabilities	0.05	0.35	0.5	0.2	−0.3

One way to solve a probability problem is to repeat the experiment many times, keeping track of the results. Then the probability can be approximated using the basic definition of the probability of an event E: $P(E) = n(E)/n(S)$, where E occurs $n(E)$ times out of $n(S)$ trials of an experiment. This is called the Monte Carlo method of finding probabilities. If physically repeating the experiment is too tedious, it may be simulated using a random-number generator, available on most computers and scientific or graphing calculators. To simulate a coin toss or the roll of a die on the TI-84 Plus, change the setting to fixed decimal mode with 0 digits displayed, and enter `rand` or

`rand*6+.5`**, respectively. For a coin toss, interpret 0 as a head and 1 as a tail. In either case, the** `ENTER` **key can be pressed repeatedly to perform multiple simulations.**

41. Suppose two dice are rolled. Use the Monte Carlo method with at least 50 repetitions to approximate the following probabilities. Compare with the results of Exercise 11.

a. P(the sum is 9 or more)

b. P(the sum is less than 7)

42. Suppose two dice are rolled. Use the Monte Carlo method with at least 50 repetitions to approximate the following probabilities. Compare with the results of Exercise 12.

a. P(the sum is not more than 5)

b. P(the sum is not less than 8)

43. Suppose three dice are rolled. Use the Monte Carlo method with at least 100 repetitions to approximate the following probabilities.

a. P(the sum is 5 or less)

b. P(neither a 1 nor a 6 is rolled)

44. Suppose a coin is tossed 5 times. Use the Monte Carlo method with at least 50 repetitions to approximate the following probabilities.

a. P(exactly 4 heads)

b. P(2 heads and 3 tails)

45. The following description of the classic "Linda problem" appeared in the *New Yorker:* "In this experiment, subjects are told, 'Linda is thirty-one years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice and also participated in antinuclear demonstrations.' They are then asked to rank the probability of several possible descriptions of Linda today. Two of them are 'bank teller' and 'bank teller and active in the feminist movement.'" Many people rank the second event as more likely. Explain why this violates basic concepts of probability. ***Source: New Yorker.***

46. You are given $P(A \cup B) = 0.7$ and $P(A \cup B') = 0.9$. Determine $P(A)$. Choose one of the following. ***Source: Society of Actuaries.***

a. 0.2 **b.** 0.3 **c.** 0.4 **d.** 0.6 **e.** 0.8

APPLICATIONS

Business and Economics

47. Defective Merchandise Suppose that 8% of a certain batch of calculators have a defective case, and that 11% have defective batteries. Also, 3% have both a defective case and defective batteries. A calculator is selected from the batch at random. Find the probability that the calculator has a good case and good batteries.

48. Profit The probability that a company will make a profit this year is 0.74.

a. Find the probability that the company will not make a profit this year.

b. Find the odds against the company making a profit.

49. Credit Charges The table shows the probabilities of a person accumulating specific amounts of credit card charges over a 12-month period. Find the probabilities that a person's total charges during the period are the following.

a. \$500 or more **b.** Less than \$1000

c. \$500 to \$2999 **d.** \$3000 or more

Charges	Probability
Under \$100	0.21
\$100–\$499	0.17
\$500–\$999	0.16
\$1000–\$1999	0.15
\$2000–\$2999	0.12
\$3000–\$4999	0.08
\$5000–\$9999	0.07
\$10,000 or more	0.04

50. Employment The table shows the projected probabilities of a worker employed by different occupational groups in 2018. ***Source: U.S. Department of Labor.***

Occupation	Probability
Management and business	0.1047
Professional	0.2182
Service	0.2024
Sales	0.1015
Office and administrative support	0.1560
Farming, fishing, forestry	0.0061
Construction	0.0531
Production	0.0585
Other	0.0995

If a worker in 2018 is selected at random, find the following.

a. The probability that the worker is in sales or service.

b. The probability that the worker is not in construction.

c. The odds in favor of the worker being in production.

51. Labor Force The following table gives the 2018 projected civilian labor force probability distribution by age and gender. ***Source: U.S. Department of Labor.***

Age	Male	Female	Total
16–24	0.066	0.061	0.127
25–54	0.343	0.291	0.634
55 and over	0.122	0.117	0.239
Total	0.531	0.469	1.000

Find the probability that a randomly selected worker is the following.

a. Female and 16 to 24 years old

b. 16 to 54 years old

c. Male or 25 to 54 years old

d. Female or 16 to 24 years old

Life Sciences

52. Body Types A study on body types gave the following results: 45% were short; 25% were short and overweight; and 24% were tall and not overweight. Find the probabilities that a person is the following.

a. Overweight

b. Short, but not overweight

c. Tall and overweight

53. Color Blindness Color blindness is an inherited characteristic that is more common in males than in females. If M represents male and C represents red-green color blindness, we use the relative frequencies of the incidences of males and red-green color blindness as probabilities to get

$$P(C) = 0.039,\ P(M \cap C) = 0.035,\ P(M \cup C) = 0.491.$$

Source: Parsons' Diseases of the Eye.

Find the following probabilities.

a. $P(C')$ **b.** $P(M)$ **c.** $P(M')$

d. $P(M' \cap C')$ **e.** $P(C \cap M')$ **f.** $P(C \cup M')$

54. Genetics Gregor Mendel, an Austrian monk, was the first to use probability in the study of genetics. In an effort to understand the mechanism of character transmittal from one generation to the next in plants, he counted the number of occurrences of various characteristics. Mendel found that the flower color in certain pea plants obeyed this scheme:

Pure red crossed with pure white produces red.

From its parents, the red offspring received genes for both red (R) and white (W), but in this case red is *dominant* and white *recessive*, so the offspring exhibits the color red. However, the offspring still carries both genes, and when two such offspring are crossed, several things can happen in the third generation. The table below, which is called a *Punnet square*, shows the equally likely outcomes.

		Second Parent	
		R	***W***
First Parent	*R*	*RR*	*RW*
	W	*WR*	*WW*

Use the fact that red is dominant over white to find the following. Assume that there are an equal number of red and white genes in the population.

a. P(a flower is red)

b. P(a flower is white)

55. Genetics Mendel found no dominance in snapdragons, with one red gene and one white gene producing pink-flowered offspring. These second-generation pinks, however, still carry one red and one white gene, and when they are crossed, the next generation still yields the Punnet square from Exercise 54. Find each probability.

a. P(red)

b. P(pink)

c. P(white)

(Mendel verified these probability ratios experimentally and did the same for many characteristics other than flower color. His work, published in 1866, was not recognized until 1890.)

56. Genetics In most animals and plants, it is very unusual for the number of main parts of the organism (such as arms, legs, toes, or flower petals) to vary from generation to generation. Some species, however, have *meristic variability,* in which the number of certain body parts varies from generation to generation. One researcher studied the front feet of certain guinea pigs and produced the following probabilities. ***Source: Genetics.***

$$P(\text{only four toes, all perfect}) = 0.77$$
$$P(\text{one imperfect toe and four good ones}) = 0.13$$
$$P(\text{exactly five good toes}) = 0.10$$

Find the probability of each event.

a. No more than four good toes

b. Five toes, whether perfect or not

57. Doctor Visit The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist. Choose one of the following. (*Hint:* Use the union rule for probability.) ***Source: Society of Actuaries.***

a. 0.05 **b.** 0.12 **c.** 0.18

d. 0.25 **e.** 0.35

58. Shoulder Injuries Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist. Choose one of the following. (*Hint:* Use the union rule for probability, and let $x = P$(patient visits a physical therapist).) ***Source: Society of Actuaries.***

a. 0.26 **b.** 0.38 **c.** 0.40

d. 0.48 **e.** 0.62

59. Health Plan An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are 1/4, 1/3, and 5/12, respectively. Determine the probability that a randomly chosen employee will choose no supplementary coverage. Choose one of the following. (*Hint:* Draw a Venn diagram with three sets, and let $x = P(A \cap B)$. Use the fact that 4 of the 8 regions in the Venn diagram have a probability of 0.) ***Source: Society of Actuaries.***

a. 0 **b.** 47/144 **c.** 1/2

d. 97/144 **e.** 7/9

Social Sciences

60. Presidential Candidates In 2002, *The New York Times* columnist William Safire gave the following odds against various prominent Democrats receiving their party's presidential nomination in 2004.

Al Gore: 2 to 1
Tom Daschle: 4 to 1
John Kerry: 4 to 1
Chris Dodd: 4 to 1
Joe Lieberman: 5 to 1
Joe Biden: 5 to 1
Pat Leahy: 6 to 1
Russell Feingold: 8 to 1
John Edwards: 9 to 1
Dick Gephardt: 15 to 1

John Allen Paulos observed that there is something wrong with those odds. Translate these odds into probabilities of winning the nomination, and then explain why these are not possible. ***Sources: The New York Times and ABC News.***

61. Earnings The following data were gathered for 130 adult U.S. workers: 55 were women; 3 women earned more than $40,000; and 62 men earned $40,000 or less. Find the probability that an individual is

a. a woman earning $40,000 or less;

b. a man earning more than $40,000;

c. a man or is earning more than $40,000;

d. a woman or is earning $40,000 or less.

62. Expenditures for Music A survey of 100 people about their music expenditures gave the following information: 38 bought rock music; 20 were teenagers who bought rock music; and 26 were teenagers. Find the probabilities that a person is

a. a teenager who buys nonrock music;

b. someone who buys rock music or is a teenager;

c. not a teenager;

d. not a teenager, but a buyer of rock music.

63. Refugees In a refugee camp in southern Mexico, it was found that 90% of the refugees came to escape political oppression, 80% came to escape abject poverty, and 70% came to escape both. What is the probability that a refugee in the camp was not poor nor seeking political asylum?

64. Community Activities At the first meeting of a committee to plan a local Lunar New Year celebration, the persons attending are 3 Chinese men, 4 Chinese women, 3 Vietnamese women, 2 Vietnamese men, 4 Korean women, and 2 Korean men. A chairperson is selected at random. Find the probabilities that the chairperson is the following.

a. Chinese

b. Korean or a woman

c. A man or Vietnamese

d. Chinese or Vietnamese

e. Korean and a woman

65. Elections If the odds that a given candidate will win an election are 3 to 2, what is the probability that the candidate will lose?

66. Military There were 203,375 female military personnel in 2009 in various ranks and military branches, as listed in the table. ***Source: Department of Defense.***

	Army (*A*)	Air Force (*B*)	Navy (*C*)	Marines (*D*)
Officers (O)	14,322	12,097	7884	1202
Enlisted (E)	59,401	51,965	42,225	11,749
Cadets & Midshipmen (M)	688	922	920	0

a. Convert the numbers in the table to probabilities.

b. Find the probability that a randomly selected woman is in the Army.

c. Find the probability that a randomly selected woman is an officer in the Navy or Marine Corps.

d. $P(A \cup B)$

e. $P(E \cup (C \cup D))$

67. Perceptions of Threat Research has been carried out to measure the amount of intolerance that citizens of Russia have for left-wing Communists and right-wing Fascists, as indicated in the table below. Note that the numbers are given as percents and each row sums to 100 (except for rounding). ***Source: Political Research Quarterly.***

Russia	None at All	Don't Know	Not Very Much	Somewhat	Extremely
Left-Wing Communists	47.8	6.7	31.0	10.5	4.1
Right-Wing Fascists	3.0	3.2	7.1	27.1	59.5

a. Find the probability that a randomly chosen citizen of Russia would be somewhat or extremely intolerant of right-wing Fascists.

b. Find the probability that a randomly chosen citizen of Russia would be completely tolerant of left-wing Communists.

c. Compare your answers to parts a and b and provide possible reasons for these numbers.

68. Perceptions of Threat Research has been carried out to measure the amount of intolerance that U.S. citizens have for left-wing Communists and right-wing Fascists, as indicated in the table. Note that the numbers are given as percents and each row sums to 100 (except for rounding). ***Source: Political Research Quarterly.***

United States	None at All	Don't Know	Not Very Much	Somewhat	Extremely
Left-Wing Communists	13.0	2.7	33.0	34.2	17.1
Right-Wing Fascists	10.1	3.3	20.7	43.1	22.9

a. Find the probability that a randomly chosen U.S. citizen would have at least some intolerance of right-wing Fascists.

b. Find the probability that a randomly chosen U.S. citizen would have at least some intolerance of left-wing Communists.

c. Compare your answers to parts a and b and provide possible reasons for these numbers.

d. Compare these answers to the answers to Exercise 67.

General Interest

69. Olympics In recent winter Olympics, each part of the women's figure skating program has 12 judges, but the scores of only 9 of the judges are randomly selected for the final results. As we will see in the next chapter, there are 220 possible ways for the 9 judges whose scores are counted to be selected. *The New York Times* examined those 220 possibilities for the short program in the 2006 Olympics, based on the published scores of the judges, and listed what the results would have been for each, as shown below. ***Source: The New York Times.***

a. The winner of the short program was Sasha Cohen. For a random combination of 9 judges, what is the probability of that outcome?

b. The second place finisher in the short program was Irina Slutskaya. For a random combination of 9 judges, what is the probability of that outcome?

c. The third place finisher in the short program was Shizuka Arakawa. For a random combination of 9 judges, what is the probability of that outcome? Do not include outcomes that include a tie.

Outcome	1. Slutskaya 2. Cohen 3. Arakawa	1. Slutskaya 2. Arakawa 3. Cohen	1. Slutskaya 2, 3. Arakawa and Cohen tied	1. Cohen 2. Slutskaya 3. Arakawa	1. Cohen 2. Arakawa 3. Slutskaya
Number of Possible Judging Combinations	92	33	3	67	25

70. Book of Odds The following table gives the probabilities that a particular event will occur. Convert each probability to the odds in favor of the event. ***Source: The Book of Odds.***

Event	Probability for the Event
An NFL pass will be intercepted.	0.03
A U.S. president owned a dog during his term in office.	0.65
A woman owns a pair of high heels.	0.61
An adult smokes.	0.21
A flight is cancelled.	0.02

71. Book of Odds The following table gives the odds that a particular event will occur. Convert each odd to the probability that the event will occur. ***Source: The Book of Odds.***

Event	Odds for the Event
A powerball entry will win the jackpot.	1 to 195,199,999
An adult will be struck by lightning during a year.	1 to 835,499
An adult will file for personal bankruptcy during a year.	1 to 157.6
A person collects stamps.	1 to 59.32

YOUR TURN ANSWERS

1. $16/52 = 4/13$ **2.** $10/36 = 5/18$ **3.** $33/36 = 11/12$

4. 3 to 7 or 3:7 **5.** $3/20$ **6.** $3/10$

7.5 Conditional Probability; Independent Events

APPLY IT **What is the probability that a broker who uses research picks stocks that go up?**

The manager for a brokerage firm has noticed that some of the firm's stockbrokers have selected stocks based on the firm's research, while other brokers tend to follow their own instincts. To see whether the research department performs better than the brokers' instincts, the manager surveyed 100 brokers, with results as shown in the following table.

Results of Stockbroker Survey			
	Picked Stocks That Went Up (A)	Didn't Pick Stocks That Went Up (A')	Totals
Used Research (B)	30	15	45
Didn't Use Research (B')	30	25	55
Totals	60	40	100

Letting A represent the event "picked stocks that went up," and letting B represent the event "used research," we can find the following probabilities.

$$P(A) = \frac{60}{100} = 0.6 \qquad P(A') = \frac{40}{100} = 0.4$$

$$P(B) = \frac{45}{100} = 0.45 \qquad P(B') = \frac{55}{100} = 0.55$$

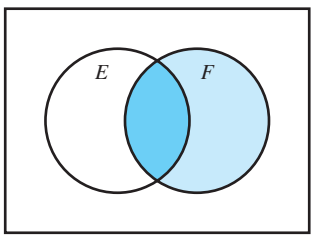
APPLY IT

To answer the question asked at the beginning of this section, suppose we want to find the probability that a broker using research will pick stocks that go up. From the table, of the 45 brokers who use research, 30 picked stocks that went up, with

$$P(\text{broker who uses research picks stocks that go up}) = \frac{30}{45} \approx 0.6667.$$

This is a different number than the probability that a broker picks stocks that go up, 0.6, since we have additional information (the broker uses research) that has *reduced the sample space*. In other words, we found the probability that a broker picks stocks that go up, A, given the additional information that the broker uses research, B. This is called the *conditional probability* of event A, given that event B has occurred. It is written $P(A \mid B)$ and read as "the probability of A given B." In this example,

$$P(A|B) = \frac{30}{45}.$$



Event F has a total of $n(F)$ elements.

FIGURE 20

To generalize this result, assume that E and F are two events for a particular experiment and that all events in the sample space S are equally likely. We want to find $P(E \mid F)$, the probability that E occurs given F has occurred. Since we assume that F has occurred, reduce the sample space to F: look only at the elements inside F. See Figure 20. Of those $n(F)$ elements, there are $n(E \cap F)$ elements where E also occurs. This makes

$$P(E|F) = \frac{n(E \cap F)}{n(F)}.$$

This equation can also be written as the quotient of two probabilities. Divide numerator and denominator by $n(S)$ to get

$$P(E|F) = \frac{n(E \cap F)/n(S)}{n(F)/n(S)} = \frac{P(E \cap F)}{P(F)}.$$

This last result motivates the definition of conditional probability.

Conditional Probability

The **conditional probability** of event E given event F, written $P(E|F)$, is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \quad \text{where } P(F) \neq 0.$$

Although the definition of conditional probability was motivated by an example with equally likely outcomes, it is valid in all cases. However, for *equally likely outcomes*, conditional probability can be found by directly applying the definition, or by first reducing the sample space to event F, and then finding the number of outcomes in F that are also in event E. Thus,

$$P(E|F) = \frac{n(E \cap F)}{n(F)}.$$

In the preceding example, the conditional probability could have also been found using the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{30/100}{45/100} = \frac{30}{45} = \frac{2}{3}.$$

EXAMPLE 1 Stocks

Use the information given in the chart at the beginning of this section to find the following probabilities.

(a) $P(B|A)$

SOLUTION This represents the probability that the broker used research, given that the broker picked stocks that went up. Reduce the sample space to A. Then find $n(A \cap B)$ and $n(A)$.

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{n(A \cap B)}{n(A)} = \frac{30}{60} = \frac{1}{2}$$

If a broker picked stocks that went up, then the probability is $1/2$ that the broker used research.

(b) $P(A'|B)$

SOLUTION In words, this is the probability that a broker picks stocks that do not go up, even though he used research.

$$P(A'|B) = \frac{n(A' \cap B)}{n(B)} = \frac{15}{45} = \frac{1}{3}$$

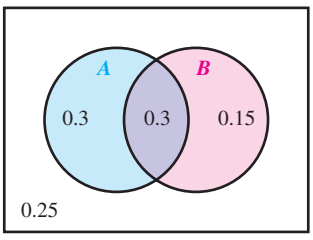
(c) $P(B'|A')$

SOLUTION Here, we want the probability that a broker who picked stocks that did not go up did not use research.

$$P(B'|A') = \frac{n(B' \cap A')}{n(A')} = \frac{25}{40} = \frac{5}{8}$$

TRY YOUR TURN 1

YOUR TURN 1 In Example 1, find $P(A|B')$.



$P(A) = 0.3 + 0.3 = 0.6$

FIGURE 21

Venn diagrams are useful for illustrating problems in conditional probability. A Venn diagram for Example 1, in which the probabilities are used to indicate the number in the set defined by each region, is shown in Figure 21. In the diagram, $P(B|A)$ is found by reducing the sample space to just set A. Then $P(B|A)$ is the ratio of the number in that part of set B that is also in A to the number in set A, or $0.3/0.6 = 0.5$.

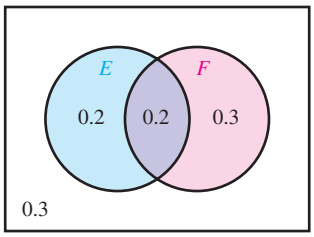


FIGURE 22

YOUR TURN 2 Given $P(E) = 0.56$, $P(F) = 0.64$, and $P(E \cup F) = 0.80$, find $P(E \mid F)$.

EXAMPLE 2 Conditional Probabilities

Given $P(E) = 0.4$, $P(F) = 0.5$, and $P(E \cup F) = 0.7$, find $P(E|F)$.

SOLUTION Find $P(E \cap F)$ first. By the union rule,

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(E \cap F) \\ 0.7 &= 0.4 + 0.5 - P(E \cap F) \\ P(E \cap F) &= 0.2. \end{aligned}$$

$P(E|F)$ is the ratio of the probability of that part of E that is in F to the probability of F, or

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.2}{0.5} = \frac{2}{5}.$$

TRY YOUR TURN 2

The Venn diagram in Figure 22 illustrates Example 2.

EXAMPLE 3 Tossing Coins

Two fair coins were tossed, and it is known that at least one was a head. Find the probability that both were heads.

SOLUTION At first glance, the answer may appear to be $1/2$, but this is not the case. The sample space has four equally likely outcomes, $S = \{hh, ht, th, tt\}$. Because of the condition that at least one coin was a head, the sample space is reduced to $\{hh, ht, th\}$. Since only one outcome in this reduced sample space is 2 heads,

$$P(\text{2 heads} \mid \text{at least 1 head}) = \frac{1}{3}.$$

Alternatively, we could use the conditional probability definition. Define two events:

$$E_1 = \text{at least 1 head} = \{hh, ht, th\}$$

and

$$E_2 = \text{2 heads} = \{hh\}.$$

YOUR TURN 3 In Example 3, find the probability of exactly one head, given that there is at least one tail.

Since there are four equally likely outcomes, $P(E_1) = 3/4$ and $P(E_1 \cap E_2) = 1/4$. Therefore,

$$P(E_2|E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

TRY YOUR TURN 3

EXAMPLE 4 Playing Cards

Two cards are drawn from a standard deck, one after another without replacement. Find the probability that the second card is red, given that the first card is red.

SOLUTION According to the conditional probability formula,

$$P(\text{second card is red} \mid \text{first card is red}) = \frac{P(\text{second card is red and the first card is red})}{P(\text{first card is red})}.$$

We will soon see how to compute probabilities such as the one in the numerator. But there is a much simpler way to calculate this conditional probability. We only need to observe that with one red card gone, there are 51 cards left, 25 of which are red, so

$$P(\text{second card is red} \mid \text{first card is red}) = \frac{25}{51}.$$

It is important not to confuse $P(A|B)$ with $P(B|A)$. For example, in a criminal trial, a prosecutor may point out to the jury that the probability of the defendant's DNA profile matching that of a sample taken at the scene of the crime, given that the defendant is innocent, $P(D|I)$, is very small. What the jury must decide, however, is the probability that the defendant is innocent, given that the defendant's DNA profile matches the sample, $P(I|D)$. Confusing the two is an error sometimes called "the prosecutor's fallacy," and the 1990 conviction of a rape suspect in England was overturned by a panel of judges, who ordered a retrial, because the fallacy made the original trial unfair. ***Source: New Scientist.***

In the next section, we will see how to compute $P(A|B)$ when we know $P(B|A)$.

Product Rule

If $P(E) \neq 0$ and $P(F) \neq 0$, then the definition of conditional probability shows that

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \quad \text{and} \quad P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

Using the fact that $P(E \cap F) = P(F \cap E)$, and solving each of these equations for $P(E \cap F)$, we obtain the following rule.

Product Rule of Probability

If E and F are events, then $P(E \cap F)$ may be found by either of these formulas.

$$P(E \cap F) = P(F) \cdot P(E|F) \quad \text{or} \quad P(E \cap F) = P(E) \cdot P(F|E)$$

The product rule gives a method for finding the probability that events E and F both occur, as illustrated by the next few examples.

EXAMPLE 5 Business Majors

In a class with $2/5$ women and $3/5$ men, 25% of the women are business majors. Find the probability that a student chosen from the class at random is a female business major.

SOLUTION Let B and W represent the events "business major" and "woman," respectively. We want to find $P(B \cap W)$. By the product rule,

$$P(B \cap W) = P(W) \cdot P(B|W).$$

Using the given information, $P(W) = 2/5 = 0.4$ and $P(B|W) = 0.25$. Thus,

$$P(B \cap W) = 0.4(0.25) = 0.10.$$

YOUR TURN 4 At a local college, $4/5$ of the students live on campus. Of those who live on campus, 25% have cars on campus. Find the probability that a student lives on campus and has a car.

TRY YOUR TURN 4

The next examples show how a tree diagram is used with the product rule to find the probability of a sequence of events.

EXAMPLE 6 Advertising

A company needs to hire a new director of advertising. It has decided to try to hire either person A or B, who are assistant advertising directors for its major competitor. To decide between A and B, the company does research on the campaigns managed by either A or B (no campaign is managed by both) and finds that A is in charge of twice as many advertising campaigns as B. Also, A's campaigns have satisfactory results 3 out of 4 times, while B's campaigns have satisfactory results only 2 out of 5 times. Suppose one of the competitor's advertising campaigns (managed by A or B) is selected randomly.

We can represent this situation using a tree diagram as follows. Let A denote the event "Person A manages the job" and B the event "person B manages the job." Notice that A and

B are complementary events. Since A does twice as many jobs as B, we have $P(A) = 2/3$ and $P(B) = 1/3$, as noted on the first-stage branches of the tree in Figure 23.

Let S be the event "satisfactory results" and U the event "unsatisfactory results." When A manages the job, the probability of satisfactory results is $3/4$ and of unsatisfactory results $1/4$ as noted on the second-stage branches. Similarly, the probabilities when B manages the job are noted on the remaining second-stage branches. The composite branches labeled 1 to 4 represent the four mutually exclusive possibilities for the managing and outcome of the selected campaign.

(a) Find the probability that A is in charge of the selected campaign and that it produces satisfactory results.

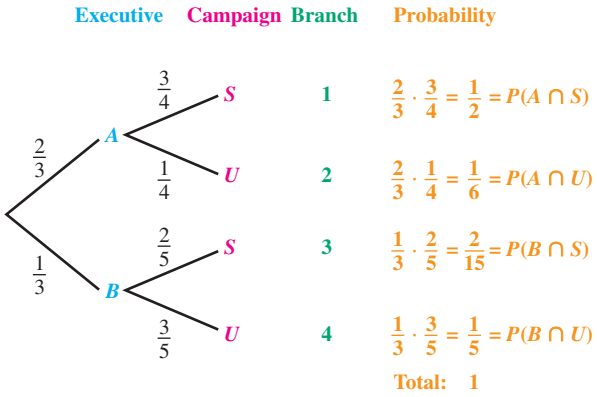


FIGURE 23

SOLUTION We are asked to find $P(A \cap S)$. We know that when A does the job, the probability of success is $3/4$, that is, $P(S|A) = 3/4$. Hence, by the product rule,

$$P(A \cap S) = P(A) \cdot P(S|A) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}.$$

The event $A \cap S$ is represented by branch 1 of the tree, and, as we have just seen, its probability is the product of the probabilities of the pieces that make up that branch.

(b) Find the probability that B runs the campaign and that it produces satisfactory results.

SOLUTION We must find $P(B \cap S)$. The event is represented by branch 3 of the tree, and, as before, its probability is the product of the probabilities of the pieces of that branch:

$$P(B \cap S) = P(B) \cdot P(S|B) = \frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}.$$

(c) What is the probability that the selected campaign is satisfactory?

SOLUTION The event S is the union of the mutually exclusive events $A \cap S$ and $B \cap S$, which are represented by branches 1 and 3 of the diagram. By the union rule,

$$P(S) = P(A \cap S) + P(B \cap S) = \frac{1}{2} + \frac{2}{15} = \frac{19}{30}.$$

Thus, the probability of an event that appears on several branches is the sum of the probabilities of each of these branches.

(d) What is the probability that the selected campaign is unsatisfactory?

SOLUTION $P(U)$ can be read from branches 2 and 4 of the tree.

$$P(U) = \frac{1}{6} + \frac{1}{5} = \frac{11}{30}$$

Alternatively, since U is the complement of S,

$$P(U) = 1 - P(S) = 1 - \frac{19}{30} = \frac{11}{30}.$$

YOUR TURN 5 In Example 6, what is the probability that A is in charge of the selected campaign and that it produces unsatisfactory results?

TRY YOUR TURN 5

EXAMPLE 7 Environmental Inspections

The Environmental Protection Agency is considering inspecting 6 plants for environmental compliance: 3 in Chicago, 2 in Los Angeles, and 1 in New York. Due to a lack of inspectors, they decide to inspect two plants selected at random, one this month and one next month, with each plant equally likely to be selected, but no plant selected twice. What is the probability that 1 Chicago plant and 1 Los Angeles plant are selected?

SOLUTION A tree diagram showing the various possible outcomes is given in Figure 24. In this diagram, the events of inspecting a plant in Chicago, Los Angeles, and New York are represented by C, LA, and NY, respectively. For the first inspection, $P(\text{C first}) = 3/6 = 1/2$ because 3 of the 6 plants are in Chicago, and all plants are equally likely to be selected. Likewise, $P(\text{LA first}) = 1/3$ and $P(\text{NY first}) = 1/6$.

For the second inspection, we first note that one plant has been inspected and, therefore, removed from the list, leaving 5 plants. For example, $P(\text{LA second} \mid \text{C first}) = 2/5$ since 2 of the 5 remaining plants are in Los Angeles. The remaining second inspection probabilities are calculated in the same manner.

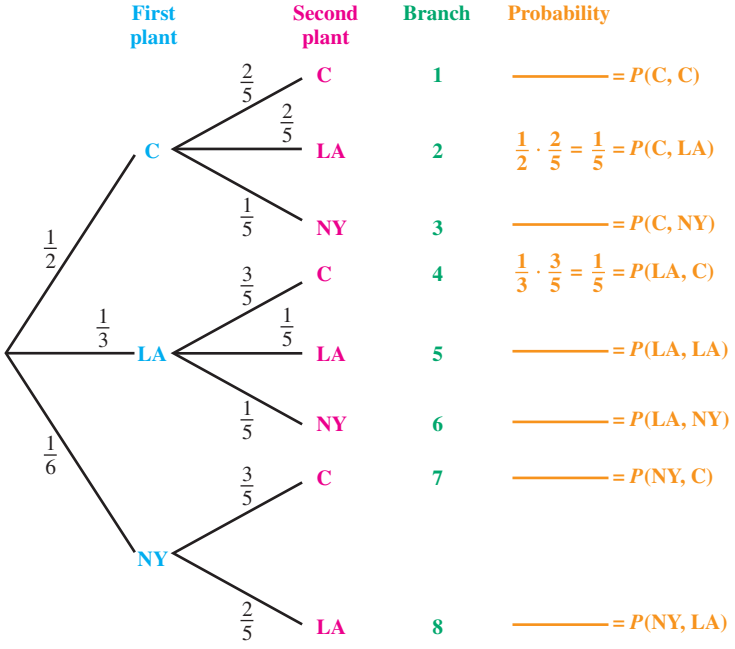


FIGURE 24

We want to find the probability of selecting exactly 1 Chicago plant and 1 Los Angeles plant. This event can occur in two ways: inspecting Chicago this month and Los Angeles next month (branch 2 of the tree diagram), or inspecting Los Angeles this month and Chicago next month (branch 4). For branch 2,

$$P(\text{C first}) \cdot P(\text{LA second} \mid \text{C first}) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}.$$

For branch 4, where Los Angeles is inspected first,

$$P(\text{LA first}) \cdot P(\text{C second} \mid \text{LA first}) = \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}.$$

Since the two events are mutually exclusive, the final probability is the sum of these two probabilities.

$$\begin{aligned} P(1\text{ C}, 1\text{ LA}) &= P(\text{C first}) \cdot P(\text{LA second} \mid \text{C first}) \\ &\quad + P(\text{LA first}) \cdot P(\text{C second} \mid \text{LA first}) \\ &= \frac{2}{5} \end{aligned}$$

YOUR TURN 6 In Example 7, what is the probability that 1 New York plant and 1 Chicago plant are selected?

TRY YOUR TURN 6

FOR REVIEW

You may wish to refer to the picture of a deck of cards shown in Figure 17 (Section 7.3) and the description accompanying it.

The product rule is often used with *stochastic processes*, which are mathematical models that evolve over time in a probabilistic manner. For example, selecting factories at random for inspection is such a process, in which the probabilities change with each successive selection.

EXAMPLE 8 Playing Cards

Two cards are drawn from a standard deck, one after another without replacement.

(a) Find the probability that the first card is a heart and the second card is red.

SOLUTION Start with the tree diagram in Figure 25. On the first draw, since there are 13 hearts among the 52 cards, the probability of drawing a heart is $13/52 = 1/4$. On the second draw, since a (red) heart has been drawn already, there are 25 red cards in the remaining 51 cards. Thus, the probability of drawing a red card on the second draw, given that the first is a heart, is $25/51$. By the product rule of probability,

$$\begin{aligned} &P(\text{heart first and red second}) \\ &\quad = P(\text{heart first}) \cdot P(\text{red second} \mid \text{heart first}) \\ &\quad = \frac{1}{4} \cdot \frac{25}{51} = \frac{25}{204} \approx 0.123. \end{aligned}$$

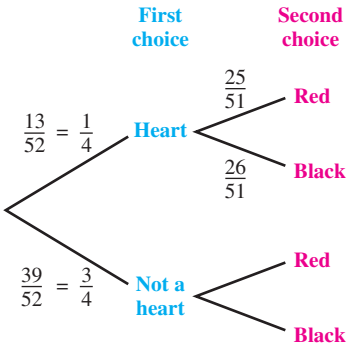


FIGURE 25

(b) Find the probability that the second card is red.

SOLUTION To solve this, we need to fill out the bottom branch of the tree diagram in Figure 25. Unfortunately, if the first card is not a heart, it is not clear how to find the probability that the second card is red, because it depends upon whether the first card is red or black. One way to solve this problem would be to divide the bottom branch into two separate branches: diamond and black card (club or spade).

There is a simpler way, however, since we don't care whether or not the first card is a heart, as we did in part (a). Instead, we'll consider whether the first card is red or black and then do the same for the second card. The result, with the corresponding probabilities, is in Figure 26. The probability that the second card is red is found by multiplying the probabilities along the two branches and adding.

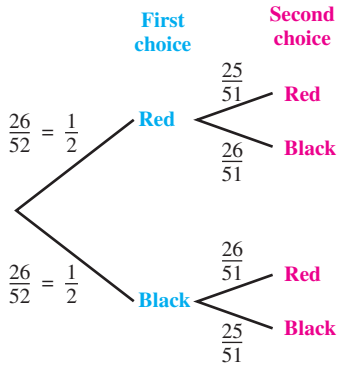


FIGURE 26

$$\begin{aligned} P(\text{red second}) &= \frac{1}{2} \cdot \frac{25}{51} + \frac{1}{2} \cdot \frac{26}{51} \\ &= \frac{1}{2} \end{aligned}$$

The probability is $1/2$, exactly the same as the probability that any card is red. If we know nothing about the first card, there is no reason for the probability of the second card to be anything other than $1/2$.

Independent Events

Suppose, in Example 8(a), that we draw the two cards *with* replacement rather than without replacement (that is, we put the first card back before drawing the second card). If the first card is a heart, then the probability of drawing a red card on the second draw is $26/52$, rather than $25/51$, because there are still 52 cards in the deck, 26 of them red. In this case, $P(\text{red second} \mid \text{heart first})$ is the same as $P(\text{red second})$. The value of the second card is not affected by the value of the first card. We say that the event that the second card is red is *independent* of the event that the first card is a heart since the knowledge of the first card does not influence what happens to the second card. On the other hand, when we draw without replacement, the events that the first card is a heart and that the second card is red are *dependent* events. The fact that the first card is a heart means there is one less red card in the deck, influencing the probability that the second card is red.

As another example, consider tossing a fair coin twice. If the first toss shows heads, the probability that the next toss is heads is still $1/2$. Coin tosses are independent events, since the outcome of one toss does not influence the outcome of the next toss. Similarly, rolls of a fair die are independent events.

On the other hand, the events "the milk is old" and "the milk is sour" are dependent events; if the milk is old, there is an increased chance that it is sour. Also, in the example at the beginning of this section, the events A (broker picked stocks that went up) and B (broker used research) are dependent events, because information about the use of research affected the probability of picking stocks that go up. That is, $P(A|B)$ is different from $P(A)$.

If events E and F are independent, then the knowledge that E has occurred gives no (probability) information about the occurrence or nonoccurrence of event F. That is, $P(F)$ is exactly the same as $P(F|E)$, or

$$P(F|E) = P(F).$$

This, in fact, is the formal definition of independent events.

Independent Events

Events E and F are **independent events** if

$$\boldsymbol{P(F|E) = P(F)} \quad \text{or} \quad \boldsymbol{P(E|F) = P(E)}.$$

If the events are not independent, they are **dependent events**.

When E and F are independent events, then $P(F|E) = P(F)$ and the product rule becomes

$$P(E \cap F) = P(E) \cdot P(F|E) = P(E) \cdot P(F).$$

Conversely, if this equation holds, then it follows that $P(F) = P(F|E)$. Consequently, we have this useful fact:

Product Rule for Independent Events

Events E and F are independent events if and only if

$$\boldsymbol{P(E \cap F) = P(E) \cdot P(F)}.$$

EXAMPLE 9 Calculator

A calculator requires a keystroke assembly and a logic circuit. Assume that 99% of the keystroke assemblies are satisfactory and 97% of the logic circuits are satisfactory. Find the probability that a finished calculator will be satisfactory.

SOLUTION If the failure of a keystroke assembly and the failure of a logic circuit are independent events, then

$$\begin{aligned} &P(\text{satisfactory calculator}) \\ &\quad = P(\text{satisfactory keystroke assembly}) \cdot P(\text{satisfactory logic circuit}) \\ &\quad = (0.99)(0.97) \approx 0.96. \end{aligned}$$

(The probability of a defective calculator is $1 - 0.96 = 0.04$.) TRY YOUR TURN 7

YOUR TURN 7 The probability that you roll a five on a single die is $1/6$. Find the probability you roll two five's in a row.

CAUTION It is common for students to confuse the ideas of *mutually exclusive* events and *independent* events. Events E and F are mutually exclusive if $E \cap F = \emptyset$. For example, if a family has exactly one child, the only possible outcomes are $B = \{\text{boy}\}$ and $G = \{\text{girl}\}$. These two events are mutually exclusive. The events are *not* independent, however, since $P(G|B) = 0$ (if a family with only one child has a boy, the probability it has a girl is then 0). Since $P(G|B) \neq P(G)$, the events are not independent.

Of all the families with exactly two children, the events $G_1 = \{\text{first child is a girl}\}$ and $G_2 = \{\text{second child is a girl}\}$ are independent, since $P(G_2|G_1)$ equals $P(G_2)$. However, G_1 and G_2 are not mutually exclusive, since $G_1 \cap G_2 = \{\text{both children are girls}\} \neq \emptyset$.

To show that two events E and F are independent, show that $P(F|E) = P(F)$ or that $P(E|F) = P(E)$ or that $P(E \cap F) = P(E) \cdot P(F)$. Another way is to observe that knowledge of one outcome does not influence the probability of the other outcome, as we did for coin tosses.

NOTE In some cases, it may not be apparent from the physical description of the problem whether two events are independent or not. For example, it is not obvious whether the event that a baseball player gets a hit tomorrow is independent of the event that he got a hit today. In such cases, it is necessary to calculate whether $P(F|E) = P(F)$, or, equivalently, whether $P(E \cap F) = P(E) \cdot P(F)$.

EXAMPLE 10 Snow in Manhattan

On a typical January day in Manhattan the probability of snow is 0.10, the probability of a traffic jam is 0.80, and the probability of snow or a traffic jam (or both) is 0.82. Are the event "it snows" and the event "a traffic jam occurs" independent?

SOLUTION Let S represent the event "it snows" and T represent the event "a traffic jam occurs." We must determine whether

$$P(T|S) = P(T) \qquad \text{or} \qquad P(S|T) = P(S).$$

We know $P(S) = 0.10$, $P(T) = 0.8$, and $P(S \cup T) = 0.82$. We can use the union rule (or a Venn diagram) to find $P(S \cap T) = 0.08$, $P(T|S) = 0.8$, and $P(S|T) = 0.1$. Since

$$P(T|S) = P(T) = 0.8 \qquad \text{and} \qquad P(S|T) = P(S) = 0.1,$$

the events "it snows" and "a traffic jam occurs" are independent. TRY YOUR TURN 8

YOUR TURN 8 The probability that you do your math homework is 0.8, the probability that you do your history assignment is 0.7, and the probability of you doing your math homework or your history assignment is 0.9. Are the events "do your math homework" and "do your history assignment" independent?

Although we showed $P(T|S) = P(T)$ and $P(S|T) = P(S)$ in Example 10, only one of these results is needed to establish independence. It is also important to note that independence of events does not necessarily follow intuition; it is established from the mathematical definition of independence.

7.5 EXERCISES

If a single fair die is rolled, find the probabilities of the following results.

1. A 2, given that the number rolled was odd
2. A 4, given that the number rolled was even
3. An even number, given that the number rolled was 6
4. An odd number, given that the number rolled was 6.

If two fair dice are rolled, find the probabilities of the following results.

5. A sum of 8, given that the sum is greater than 7
6. A sum of 6, given that the roll was a "double" (two identical numbers)
7. A double, given that the sum was 9
8. A double, given that the sum was 8.

If two cards are drawn without replacement from an ordinary deck, find the probabilities of the following results.

9. The second is a heart, given that the first is a heart.
10. The second is black, given that the first is a spade.
11. The second is a face card, given that the first is a jack.
12. The second is an ace, given that the first is not an ace.
13. A jack and a 10 are drawn.
14. An ace and a 4 are drawn.
15. Two black cards are drawn.
16. Two hearts are drawn.
17. In your own words, explain how to find the conditional probability $P(E|F)$.
18. In your own words, define independent events.

Decide whether each of the following pairs of events are dependent or independent.

19. A red and a green die are rolled. A is the event that the red die comes up even, and B is the event that the green die comes up even.
20. C is the event that it rains more than 10 days in Chicago next June, and D is the event that it rains more than 15 days.
21. E is the event that a resident of Texas lives in Dallas, and F is the event that a resident of Texas lives in either Dallas or Houston.
22. A coin is flipped. G is the event that today is Tuesday, and H is the event that the coin comes up heads.

In the previous section, we described an experiment in which the numbers 1, 2, 3, 4, and 5 are written on slips of paper, and 2 slips are drawn at random one at a time without replacement. Find each probability in Exercises 23 and 24.

23. The probability that the first number is 3, given the following.

a. The sum is 7.

b. The sum is 8.

24. The probability that the sum is 8, given the following.

a. The first number is 5.

b. The first number is 4.

25. Suppose two dice are rolled. Let A be the event that the sum of the two dice is 7. Find an event B related to numbers on the dice such that A and B are

a. independent;

b. dependent.

26. Your friend asks you to explain how the product rule for independent events differs from the product rule for dependent events. How would you respond?

27. Another friend asks you to explain how to tell whether two events are dependent or independent. How would you reply? (Use your own words.)

28. A student reasons that the probability in Example 3 of both coins being heads is just the probability that the other coin is a head, that is, $1/2$. Explain why this reasoning is wrong.

29. Let A and B be independent events with $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{5}$. Find $P(A \cap B)$ and $P(A \cup B)$.

30. If A and B are events such that $P(A) = 0.5$ and $P(A \cup B) = 0.7$, find $P(B)$ when

a. A and B are mutually exclusive;

b. A and B are independent.

31. The following problem, submitted by Daniel Hahn of Blairstown, Iowa, appeared in the "Ask Marilyn" column of *Parade* magazine. ***Source: Parade magazine.***

"You discover two booths at a carnival. Each is tended by an honest man with a pair of covered coin shakers. In each shaker is a single coin, and you are allowed to bet upon the chance that both coins in that booth's shakers are heads after the man in the booth shakes them, does an inspection, and can tell you that at least one of the shakers contains a head. The difference is that the man in the first booth always looks inside both of his shakers, whereas the man in the second booth looks inside only one of the shakers. Where will you stand the best chance?"

32. The following question was posed in *Chance News* by Craig Fox and Yoval Rotenstrich. You are playing a game in which a fair coin is flipped and a fair die is rolled. You win a prize if both the coin comes up heads and a 6 is rolled on the die. Now suppose the coin is tossed and the die is rolled, but you are not allowed to see either result. You are told, however, that either the head or the 6 occurred. You are then offered the chance to cancel the game and play a new game in which a die is rolled (there is no coin), and you win a prize if a 6 is rolled. ***Source: Chance News.***

a. Is it to your advantage to switch to the new game, or to stick with the original game? Answer this question by calculating your probability of winning in each case.

b. Many people erroneously think that it's better to stick with the original game. Discuss why this answer might seem intuitive, but why it is wrong.

33. Suppose a male defendant in a court trial has a mustache, beard, tattoo, and an earring. Suppose, also, that an eyewitness has identified the perpetrator as someone with these characteristics. If the respective probabilities for the male population in this region are 0.35, 0.30, 0.10, and 0.05, is it fair to multiply these probabilities together to conclude that the probability that a person having these characteristics is 0.000525, or 21 in 40,000, and thus decide that the defendant must be guilty?

34. In a two-child family, if we assume that the probabilities of a male child and a female child are each 0.5, are the events *all children are the same sex* and *at most one male* independent? Are they independent for a three-child family?

35. Laura Johnson, a game show contestant, could win one of two prizes: a shiny new Porsche or a shiny new penny. Laura is given two boxes of marbles. The first box has 50 pink marbles in it and the second box has 50 blue marbles in it. The game show host will pick someone from the audience to be blindfolded and then draw a marble from one of the two boxes. If a pink marble is drawn, she wins the Porsche. Otherwise, Laura wins the penny. Can Laura increase her chances of winning by redistributing some of the marbles from one box to the other? Explain. ***Source: Car Talk.***

APPLICATIONS

Business and Economics

Banking **The Midtown Bank has found that most customers at the tellers' windows either cash a check or make a deposit. The following table indicates the transactions for one teller for one day.**

	Cash Check	No Check	Totals
Make Deposit	60	20	80
No Deposit	30	10	40
Totals	90	30	120

Letting C represent "cashing a check" and D represent "making a deposit," express each probability in words and find its value.

36. $P(C|D)$ **37.** $P(D'|C)$ **38.** $P(C'|D')$

39. $P(C'|D)$ **40.** $P[(C \cap D)']$

41. **Airline Delays** In February 2010, the major U.S. airline with the fewest delays was United Airlines, for which 77.3% of their flights arrived on time. Assume that the event that a given flight arrives on time is independent of the event that another flight arrives on time. ***Source: U.S. Department of Transportation.***

a. Katie O'Connor plans to take four separate flights for her publisher next month on United Airlines. Assuming that the airline has the same on-time performance as in February 2010, what is the probability that all four flights arrive on time?

b. Discuss how realistic it is to assume that the on-time arrivals of the different flights are independent.

42. **Backup Computers** Corporations where a computer is essential to day-to-day operations, such as banks, often have a second backup computer in case the main computer fails. Suppose there is a 0.003 chance that the main computer will fail in a given time

period and a 0.005 chance that the backup computer will fail while the main computer is being repaired. Assume these failures represent independent events, and find the fraction of the time that the corporation can assume it will have computer service. How realistic is our assumption of independence?

43. ATM Transactions Among users of automated teller machines (ATMs), 92% use ATMs to withdraw cash, and 32% use them to check their account balance. Suppose that 96% use ATMs to either withdraw cash or check their account balance (or both). Given a woman who uses an ATM to check her account balance, what is the probability that she also uses an ATM to get cash? *Source: Chicago Tribune.*

Quality Control **A bicycle factory runs two assembly lines, A and B. If 95% of line A's products pass inspection, while only 85% of line B's products pass inspection, and 60% of the factory's bikes come off assembly line A (the rest off B), find the probabilities that one of the factory's bikes did not pass inspection and came off the following.**

44. Assembly line A

45. Assembly line B

46. Find the probability that one of the factory's bikes did not pass inspection.

Life Sciences

47. Genetics Both of a certain pea plant's parents had a gene for red and a gene for white flowers. (See Exercise 54 in Section 7.4.) If the offspring has red flowers, find the probability that it combined a gene for red and a gene for white (rather than 2 for red).

48. Medical Experiment A medical experiment showed that the probability that a new medicine is effective is 0.75, the probability that a patient will have a certain side effect is 0.4, and the probability that both events occur is 0.3. Decide whether these events are dependent or independent.

Genetics **Assuming that boy and girl babies are equally likely, fill in the remaining probabilities on the tree diagram below and use that information to find the probability that a family with three children has all girls, given the following.**

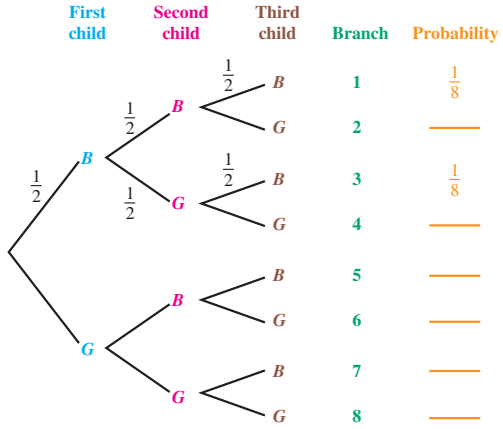
49. The first is a girl.

50. The third is a girl.

51. The second is a girl.

52. At least 2 are girls.

53. At least 1 is a girl.



Color Blindness **The following table shows frequencies for red-green color blindness, where M represents "person is male" and C represents "person is color-blind." Use this table to find the following probabilities. (See Exercise 53, Section 7.4.)**

	M	M'	Totals
C	0.035	0.004	0.039
C'	0.452	0.509	0.961
Totals	0.487	0.513	1.000

54. $P(M)$

55. $P(C)$

56. $P(M \cap C)$

57. $P(M \cup C)$

58. $P(M|C)$

59. $P(C|M)$

60. $P(M'|C)$

61. Are the events C and M, described above, dependent? What does this mean?

62. Color Blindness A scientist wishes to determine whether there is a relationship between color blindness (C) and deafness (D).

a. Suppose the scientist found the probabilities listed in the table. What should the findings be? (See Exercises 54–61.)

b. Explain what your answer tells us about color blindness and deafness.

	D	D'	Totals
C	0.0008	0.0392	0.0400
C'	0.0192	0.9408	0.9600
Totals	0.0200	0.9800	1.0000

63. Overweight According to a recent report, 68.3% of men and 64.1% of women in the United States were overweight. Given that 49.3% of Americans are men and 50.7% are women, find the probability that a randomly selected American fits the following description. *Source: JAMA.*

a. An overweight man

b. Overweight

c. Are the events "male" and "overweight" independent?

Hockey **The table below lists the number of head and neck injuries for 319 ice hockey players' exposures wearing either a full shield or half shield in the Canadian Inter-University Athletics Union. *Source: JAMA.***

For a randomly selected injury, find each probability.

64. $P(A)$

65. $P(C|F)$

66. $P(A|H)$

67. $P(B'|H')$

	Half Shield (H)	Full Shield (F)	Totals
Head and Face Injuries (A)	95	34	129
Concussions (B)	41	38	79
Neck Injuries (C)	9	7	16
Other Injuries (D)	202	150	352
Totals	347	229	576

68. Are the events A and H independent events?*

69. Blood Pressure A doctor is studying the relationship between blood pressure and heartbeat abnormalities in her patients. She tests a random sample of her patients and notes their blood pressures (high, low, or normal) and their heartbeats (regular or irregular). She finds that:

(i) 14% have high blood pressure.

(ii) 22% have low blood pressure.

(iii) 15% have an irregular heartbeat.

(iv) Of those with an irregular heartbeat, one-third have high blood pressure.

(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure? Choose one of the following. (*Hint:* Make a table similar to the one for Exercises 54–61.) ***Source: Society of Actuaries.***

a. 2% **b.** 5% **c.** 8% **d.** 9% **e.** 20%

70. Breast Cancer To explain why the chance of a woman getting breast cancer in the next year goes up each year, while the chance of a woman getting breast cancer in her lifetime goes down, Ruma Falk made the following analogy. Suppose you are looking for a letter that you may have lost. You have 8 drawers in your desk. There is a probability of 0.1 that the letter is in any one of the 8 drawers and a probability of 0.2 that the letter is not in any of the drawers. ***Source: Chance News.***

a. What is the probability that the letter is in drawer 1?

b. Given that the letter is not in drawer 1, what is the probability that the letter is in drawer 2?

c. Given that the letter is not in drawer 1 or 2, what is the probability that the letter is in drawer 3?

d. Given that the letter is not in drawers 1–7, what is the probability that the letter is in drawer 8?

e. Based on your answers to parts a–d, what is happening to the probability that the letter is in the next drawer?

f. What is the probability that the letter is in some drawer?

g. Given that the letter is not in drawer 1, what is the probability that the letter is in some drawer?

h. Given that the letter is not in drawer 1 or 2, what is the probability that the letter is in some drawer?

i. Given that the letter is not in drawers 1–7, what is the probability that the letter is in some drawer?

j. Based on your answers to parts f–i, what is happening to the probability that the letter is in some drawer?

71. Twins A 1920 study of 17,798 pairs of twins found that 5844 consisted of two males, 5612 consisted of two females, and 6342 consisted of a male and a female. Of course, all of the mixed-gender pairs were not identical twins. The same-gender pairs may or may not have been identical twins. The goal here is to use the data to estimate p, the probability that a pair of twins is identical.

a. Use the data to find the proportion of twins who were male.

b. Denoting your answer from part a by $P(B)$, show that the probability that a pair of twins is male is

$$pP(B) + (1 - p)(P(B))^2.$$

(*Hint:* Draw a tree diagram. The first set of branches should be identical and not identical. Then note that if one member of a pair of identical twins is male, the other must also be male.)

c. Using your answers from parts a and b, plus the fact that 5844 of the 17,798 twin pairs were male, find an estimate for the value of p.

d. Find an expression, similar to the one in part b, for the probability that a pair of twins is female.

e. Using your answers from parts a and d, plus the fact that 5612 of the 17,798 twin pairs were female, find an estimate for the value of p.

f. Find an expression, similar to those in parts b and d, for the probability that a pair of twins consists of one male and one female.

g. Using your answers from parts a and f, plus the fact that 6342 of the 17,798 twin pairs consisted of one male and one female, find an estimate for the value of p.

*We are assuming here and in other exercises that the events consist entirely of the numbers given in the table. If the numbers are interpreted as a sample of all people fitting the description of the events, then testing for independence is more complicated, requiring a technique from statistics known as a *contingency table*.

Social Sciences

72. Working Women A survey has shown that 52% of the women in a certain community work outside the home. Of these women, 64% are married, while 86% of the women who do not work outside the home are married. Find the probabilities that a woman in that community can be categorized as follows.

a. Married

b. A single woman working outside the home

73. Cigarette Smokers The following table gives a recent estimate (in millions) of the smoking status among persons 25 years of age and over and their highest level of education. ***Source: National Health Interview Survey.***

Education	Current Smoker	Former Smoker	Non-Smoker	Total
Less than a high school diploma	7.90	6.66	14.12	28.68
High school diploma or GED	14.38	13.09	25.70	53.17
Some college	12.41	13.55	28.65	54.61
Bachelor's degree or higher	4.97	12.87	38.34	56.18
Total	39.66	46.17	106.81	192.64

a. Find the probability that a person is a current smoker.

b. Find the probability that a person has less than a high school diploma.

c. Find the probability that a person is a current smoker and has less than a high school diploma.

d. Find the probability that a person is a current smoker, given that the person has less than a high school diploma.

e. Are the events "current smoker" and "less than a high school diploma" independent events?

Physical Sciences

74. Rain Forecasts In a letter to the journal *Nature*, Robert A. J. Matthews gives the following table of outcomes of forecast and weather over 1000 1-hour walks, based on the United Kingdom's Meteorological office's 83% accuracy in 24-hour forecasts. ***Source: Nature.***

	Rain	No Rain	Totals
Forecast of Rain	66	156	222
Forecast of No Rain	14	764	778
Totals	80	920	1000

a. Verify that the probability that the forecast called for rain, given that there was rain, is indeed 83%. Also verify that the probability that the forecast called for no rain, given that there was no rain, is also 83%.

b. Calculate the probability that there was rain, given that the forecast called for rain.

c. Calculate the probability that there was no rain, given that the forecast called for no rain.

d. Observe that your answer to part c is higher than 83% and that your answer to part b is much lower. Discuss which figure best describes the accuracy of the weather forecast in recommending whether or not you should carry an umbrella.

75. Earthquakes There are seven geologic faults (and possibly more) capable of generating a magnitude 6.7 earthquake in the region around San Francisco. Their probabilities of rupturing by the year 2032 are 27%, 21%, 11%, 10%, 4%, 3%, and 3%. ***Source: Science News.***

a. Calculate the probability that at least one of these faults erupts by the year 2032, assuming that these are independent events.

b. Scientists forecast a 62% chance of an earthquake with magnitude at least 6.7 in the region around San Francisco by the year 2032. Compare this with your answer from part a. Consider the realism of the assumption of independence. Also consider the role of roundoff. For example, the probability of 10% for one of the faults is presumably rounded to the nearest percent, with the actual probability between 9.5% and 10.5%.

76. Reliability The probability that a key component of a space rocket will fail is 0.03.

a. How many such components must be used as backups to ensure that the probability of at least one of the components working is 0.999999 or more?

b. Is it reasonable to assume independence here?

General Interest

77. Titanic The table at the bottom of the page lists the number of passengers who were on the Titanic and the number of passengers who survived, according to class of ticket. Use this information to determine the following (round answers to four decimal places). ***Source: Mathematics Teacher.***

a. What is the probability that a randomly selected passenger was second class?

b. What is the overall probability of surviving?

c. What is the probability of a first-class passenger surviving?

d. What is the probability of a child who was also in the third class surviving?

e. Given that the survivor is from first class, what is the probability that she was a woman?

f. Given that a male has survived, what is the probability that he was in third class?

g. Are the events third-class survival and male survival independent events? What does this imply?

78. Real Estate A real estate agent trying to sell you an attractive beachfront house claims that it will not collapse unless it is subjected simultaneously to extremely high winds and extremely high waves. According to weather service records, there is a 0.001 probability of extremely high winds, and the same for extremely high waves. The real estate agent claims, therefore, that the probability of both occurring is $(0.001)(0.001) = 0.000001$. What is wrong with the agent's reasoning?

79. Age and Loans Suppose 20% of the population are 65 or over, 26% of those 65 or over have loans, and 53% of those under 65 have loans. Find the probabilities that a person fits into the following categories.

a. 65 or over and has a loan

b. Has a loan

80. Women Joggers In a certain area, 15% of the population are joggers and 40% of the joggers are women. If 55% of those who do not jog are women, find the probabilities that

	Children		Women		Men		Totals	
	On	Survived	On	Survived	On	Survived	On	Survived
First Class	6	6	144	140	175	57	325	203
Second Class	24	24	165	76	168	14	357	114
Third Class	79	27	93	80	462	75	634	182
Totals	109	57	402	296	805	146	1316	499

an individual from that community fits the following descriptions.

a. A woman jogger

b. A man who is not a jogger

c. A woman

d. Are the events that a person is a woman and a person is a jogger independent? Explain.

81. Diet Soft Drinks Two-thirds of the population are on a diet at least occasionally. Of this group, 4/5 drink diet soft drinks, while 1/2 of the rest of the (nondieting) population drink diet soft drinks. Find the probabilities that a person fits into the following categories.

a. Drinks diet soft drinks

b. Diets, but does not drink diet soft drinks

82. Driver's License Test The Motor Vehicle Department has found that the probability of a person passing the test for a driver's license on the first try is 0.75. The probability that an individual who fails on the first test will pass on the second try is 0.80, and the probability that an individual who fails the first and second tests will pass the third time is 0.70. Find the probabilities that an individual will do the following.

a. Fail both the first and second tests

b. Fail three times in a row

c. Require at least two tries

83. Ballooning A pair of mathematicians in a hot air balloon were told that there are four independent burners, any one of which is sufficient to keep the balloon aloft. If the probability of any one burner failing during a flight is 0.001, what is the probability that the balloon will crash due to all four burners failing?

84. Speeding Tickets A smooth-talking young man has a 1/3 probability of talking a policeman out of giving him a speeding ticket. The probability that he is stopped for speeding during a given weekend is 1/2. Find the probabilities of the events in parts a and b.

a. He will receive no speeding tickets on a given weekend.

b. He will receive no speeding tickets on 3 consecutive weekends.

c. We have assumed that what happens on the second or third weekend is the same as what happened on the first weekend. Is this realistic? Will driving habits remain the same after getting a ticket?

85. Luxury Cars In one area, 4% of the population drive luxury cars. However, 17% of the CPAs drive luxury cars. Are the events "person drives a luxury car" and "person is a CPA" independent?

86. Studying A teacher has found that the probability that a student studies for a test is 0.60, the probability that a student gets a good grade on a test is 0.70, and the probability that both occur is 0.52.

a. Are these events independent?

b. Given that a student studies, find the probability that the student gets a good grade.

c. Given that a student gets a good grade, find the probability that the student studied.

87. Basketball A basketball player is fouled and now faces a one-and-one free throw situation. She shoots the first free throw. If she misses it, she scores 0 points. If she makes the first free throw, she gets to shoot a second free throw. If she misses the second free throw, she scores only one point for the first shot. If she makes the second free throw, she scores two points (one for each made shot). ***Source: Mathematics Teacher.***

a. If her free-throwing percentage for this season is 60%, calculate the probability that she scores 0 points, 1 point, or 2 points.

b. Determine the free-throwing percentage necessary for the probability of scoring 0 points to be the same as the probability of scoring 2 points.* (*Hint:* Let p be the free-throwing percentage, and then solve $p^2 = 1 - p$ for p. Use only the positive value of p.)

88. Basketball The same player from Exercise 87 is now shooting two free throws (that is, she gets to shoot a second free throw whether she makes or misses the first shot.) Assume her free-throwing percentage is still 60%. Find the probability she scores 0 points, 1 point, or 2 points. ***Source: Mathematics Teacher.***

89. Football A football coach whose team is 14 points behind needs two touchdowns to win. Each touchdown is worth 6 points. After a touchdown, the coach can choose either a 1-point kick, which is almost certain to succeed, or a 2-point conversion, which is roughly half as likely to succeed. After the first touchdown, the coach must decide whether to go for 1 or 2 points. If the 2-point conversion is successful, the almost certain 1-point kick after the second touchdown will win the game. If the 2-point conversion fails, the team can try another 2-point conversion after the second touchdown to tie. Some coaches, however, prefer to go for the almost certain 1-point kick after the first touchdown, hoping that the momentum will help them get a 2-point conversion after the second touchdown and win the game. They fear that an unsuccessful 2-point conversion after the first touchdown will discourage the team, which can then at best tie. ***Source: The Mathematics Teacher.***

a. Draw a tree diagram for the 1-point kick after the first touchdown and the 2-point conversion after the second touchdown. Letting the probability of success for the 1-point kick and the 2-point conversion be k and r, respectively, show that

$$P(\text{win}) = kr,$$
$$P(\text{tie}) = r(1 - k), \quad \text{and}$$
$$P(\text{lose}) = 1 - r.$$

b. Consider the case of trying for a 2-point conversion after the first touchdown. If it succeeds, try a 1-point kick after the second touchdown. If the 2-point conversion fails, try

* The solution is the reciprocal of the number $\frac{1+\sqrt{5}}{2}$, known as the golden ratio or the divine proportion. This number has great significance in architecture, science, and mathematics.

another one after the second touchdown. Draw a tree diagram and use it to show that

$$P(\text{win}) = kr,$$
$$P(\text{tie}) = r(2 - k - r), \quad \text{and}$$
$$P(\text{lose}) = (1 - r)^2.$$

c. What can you say about the probability of winning under each strategy?

d. Given that $r < 1$, which strategy has a smaller probability of losing? What does this tell you about the value of the two strategies?

YOUR TURN ANSWERS

1. $30/55 = 6/11$ **2.** 0.625 **3.** $2/3$
4. $1/5$ **5.** $1/6$ **6.** $1/5$
7. $1/36$ **8.** No

7.6 Bayes' Theorem

APPLY IT **What is the probability that a particular defective item was produced by a new machine operator?**

This question will be answered in Example 2 using Bayes' theorem, discussed in this section.

Suppose the probability that an applicant is hired, *given the applicant is qualified*, is known. The manager might also be interested in the probability that the applicant is qualified, *given the applicant was hired*. More generally, if $P(E|F)$ is known for two events E and F, then $P(F|E)$ can be found using a tree diagram. Since $P(E|F)$ is known, the first outcome is either F or F'. Then for each of these outcomes, either E or E' occurs, as shown in Figure 27.

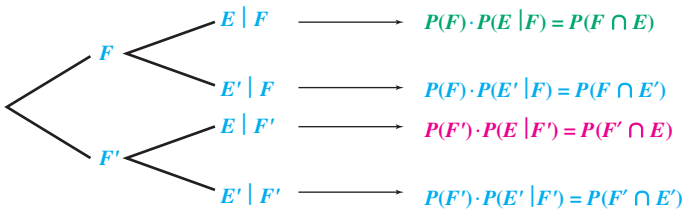


FIGURE 27

The four cases have the probabilities shown on the right. Notice $P(E \cap F)$ is the first case and $P(E)$ is the sum of the first and third cases in the tree diagram. By the definition of conditional probability,

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')}.$$

This result is a special case of Bayes' theorem, which is generalized later in this section.

Bayes' Theorem (Special Case)

$$P(F|E) = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')}$$

EXAMPLE 1 Worker Errors

For a fixed length of time, the probability of a worker error on a certain production line is 0.1, the probability that an accident will occur when there is a worker error is 0.3, and the probability that an accident will occur when there is no worker error is 0.2. Find the probability of a worker error if there is an accident.

YOUR TURN 1 The probability that a student will pass a math exam is 0.8 if he or she attends the review session and 0.65 if he or she does not attend the review session. Sixty percent of the students attend the review session. What is the probability that, given a student passed, the student attended the review session?

SOLUTION Let E represent the event of an accident, and let F represent the event of worker error. From the information given,

$$P(F) = 0.1, \qquad P(E|F) = 0.3, \qquad \text{and} \qquad P(E|F') = 0.2.$$

These probabilities are shown on the tree diagram in Figure 28.

Find $P(F|E)$ by dividing the probability that both E and F occur, given by branch 1, by the probability that E occurs, given by the sum of branches 1 and 3.

$$P(F|E) = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')}$$
$$= \frac{(0.1)(0.3)}{(0.1)(0.3) + (0.9)(0.2)} = \frac{0.03}{0.21} = \frac{1}{7} \approx 0.1429$$

TRY YOUR TURN 1

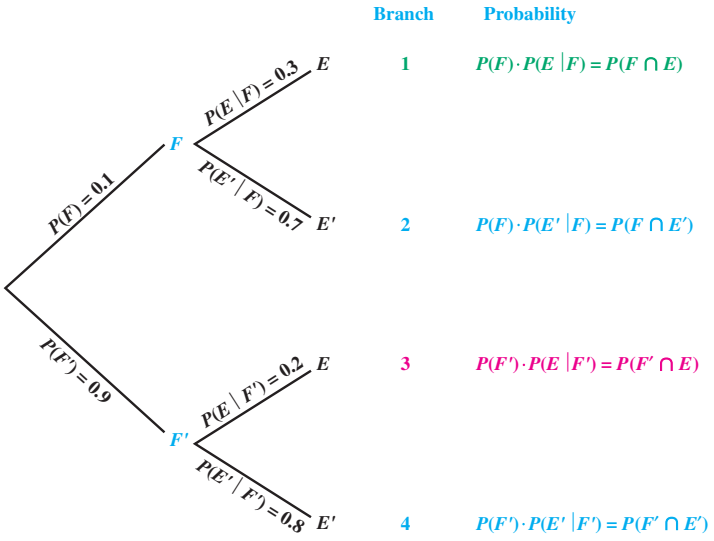


FIGURE 28

The special case of Bayes' theorem can be generalized to more than two events with the tree diagram in Figure 29. This diagram shows the paths that can produce an event E. We assume that the events $F_1, F_2, \ldots, F_n$ are mutually exclusive events (that is, disjoint events) whose union is the sample space, and that E is an event that has occurred. See Figure 30.

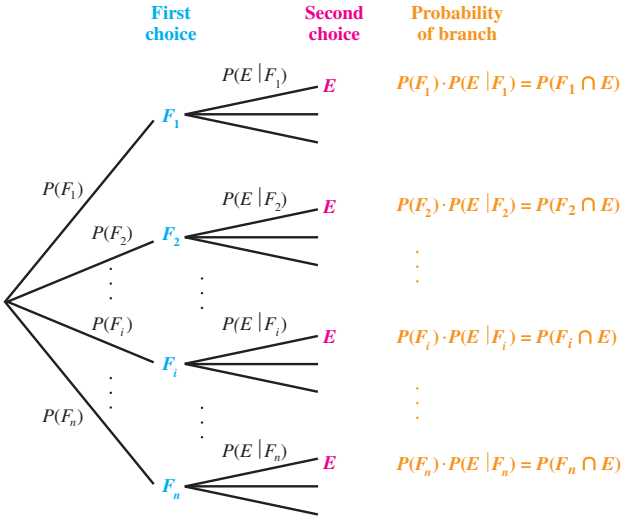


FIGURE 29

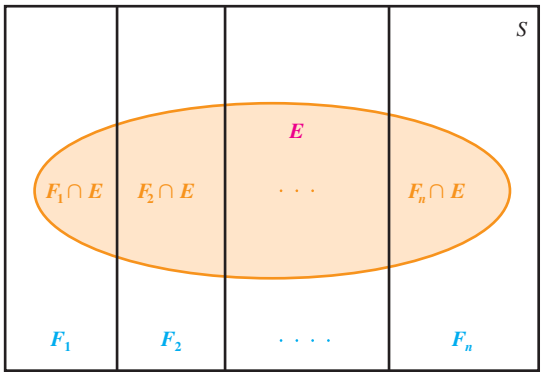


FIGURE 30

The probability $P(F_i|E)$, where $1 \le i \le n$, can be found by dividing the probability for the branch containing $P(E|F_i)$ by the sum of the probabilities of all the branches producing event E.

Bayes' Theorem

$$P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + \cdots + P(F_n) \cdot P(E|F_n)}$$

This result is known as **Bayes' theorem**, after the Reverend Thomas Bayes (1702–1761), whose paper on probability was published about three years after his death.

The statement of Bayes' theorem can be daunting. Actually, it is easier to remember the formula by thinking of the tree diagram that produced it. Go through the following steps.

Using Bayes' Theorem

1. Start a tree diagram with branches representing $F_1, F_2, \ldots, F_n$. Label each branch with its corresponding probability.
2. From the end of each of these branches, draw a branch for event E. Label this branch with the probability of getting to it, $P(E|F_i)$.
3. You now have n different paths that result in event E. Next to each path, put its probability—the product of the probabilities that the first branch occurs, $P(F_i)$, and that the second branch occurs, $P(E|F_i)$; that is, the product $P(F_i) \cdot P(E|F_i)$, which equals $P(F_i \cap E)$.
4. $P(F_i|E)$ is found by dividing the probability of the branch for F_i by the sum of the probabilities of all the branches producing event E.

EXAMPLE 2 Machine Operators

Based on past experience, a company knows that an experienced machine operator (one or more years of experience) will produce a defective item 1% of the time. Operators with some experience (up to one year) have a 2.5% defect rate, and new operators have a 6% defect rate. At any one time, the company has 60% experienced operators, 30% with some experience, and 10% new operators. Find the probability that a particular defective item was produced by a new operator.

APPLY IT

SOLUTION Let E represent the event "item is defective," F_1 represent "item was made by an experienced operator," F_2 represent "item was made by an operator with some experience," and F_3 represent "item was made by a new operator." Then

$$\begin{aligned} P(F_1) &= 0.60 & P(E|F_1) &= 0.01 \\ P(F_2) &= 0.30 & P(E|F_2) &= 0.025 \\ P(F_3) &= 0.10 & P(E|F_3) &= 0.06. \end{aligned}$$

We need to find $P(F_3|E)$, the probability that an item was produced by a new operator, given that it is defective. First, draw a tree diagram using the given information, as in Figure 31. The steps leading to event E are shown in red.

Find $P(F_3|E)$ with the bottom branch of the tree in Figure 31: Divide the probability for this branch by the sum of the probabilities of all the branches leading to E, or

$$P(F_3|E) = \frac{0.10(0.06)}{0.60(0.01) + 0.30(0.025) + 0.10(0.06)} = \frac{0.006}{0.0195} = \frac{4}{13} \approx 0.3077.$$

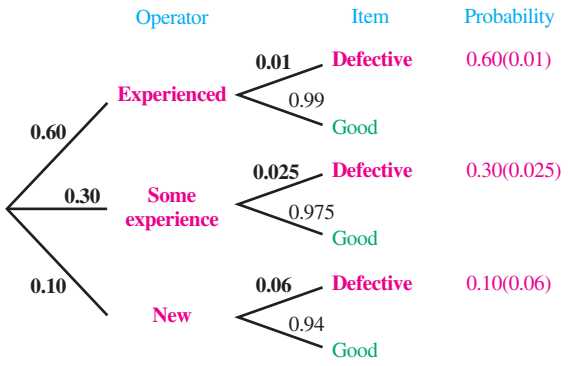


FIGURE 31

YOUR TURN 2 The English department at a small college has found that 12% of freshmen test into English I, 68% test into English II, and 20% test into English III. Eighty percent of students in English I will seek help from the writing center, 40% of those in English II, and 11% of those in English III. Given that a student received help from the writing center, find the probability that the student is in English I.

In a similar way, the probability that the defective item was produced by an operator with some experience is

$$P(F_2|E) = \frac{0.30(0.025)}{0.60(0.01) + 0.30(0.025) + 0.10(0.06)} = \frac{0.0075}{0.0195} = \frac{5}{13} \approx 0.3846.$$

Finally, the probability that the defective item was produced by an experienced operator is $P(F_1|E) = 4/13 \approx 0.3077$. Check that $P(F_1|E) + P(F_2|E) + P(F_3|E) = 1$ (that is, the defective item was made by *someone*). TRY YOUR TURN 2

EXAMPLE 3 Manufacturing

A manufacturer buys items from six different suppliers. The fraction of the total number of items obtained from each supplier, along with the probability that an item purchased from that supplier is defective, are shown in the following table.

Manufacturing Supplies		
Supplier	Fraction of Total Supplied	Probability of Defect
1	0.05	0.04
2	0.12	0.02
3	0.16	0.07
4	0.23	0.01
5	0.35	0.03
6	0.09	0.05

Find the probability that a defective item came from supplier 5.

SOLUTION Let F_1 be the event that an item came from supplier 1, with F_2, F_3, F_4, F_5, and F_6 defined in a similar manner. Let E be the event that an item is defective. We want to find $P(F_5|E)$. Use the probabilities in the table above to prepare a tree diagram, or work with the rows of the table to get

$$P(F_5|E) = \frac{(0.35)(0.03)}{(0.05)(0.04) + (0.12)(0.02) + (0.16)(0.07) + (0.23)(0.01) + (0.35)(0.03) + (0.09)(0.05)}$$

$$= \frac{0.0105}{0.0329} \approx 0.319.$$

There is about a 32% chance that a defective item came from supplier 5. Even though supplier 5 has only 3% defectives, his probability of being "guilty" is relatively high, about 32%, because of the large fraction supplied by 5.

CAUTION Notice that the 0.04 in the upper right of the previous table represents the probability of a defective item *given* that the item came from supplier 1. In contrast, the probability of 0.035 in the table for Exercises 54–61 of the previous section represents the probability that a person is color-blind *and* male. The tables in this section represent probability in a different way than those of the previous section. Tables that you encounter outside of this course might represent probability in either way. You can usually tell what is intended by the context, but be careful!

7.6 EXERCISES

For two events M and N, $P(M) = 0.4$, $P(N|M) = 0.3$, and $P(N|M') = 0.4$. Find the following.

1. $P(M|N)$ **2.** $P(M'|N)$

For mutually exclusive events R_1, R_2, and R_3, we have $P(R_1) = 0.15$, $P(R_2) = 0.55$, and $P(R_3) = 0.30$. Also, $P(Q|R_1) = 0.40$, $P(Q|R_2) = 0.20$, and $P(Q|R_3) = 0.70$. Find the following.

3. $P(R_1|Q)$ **4.** $P(R_2|Q)$

5. $P(R_3|Q)$ **6.** $P(R_1'|Q)$

Suppose you have three jars with the following contents: 2 black balls and 1 white ball in the first, 1 black ball and 2 white balls in the second, and 1 black ball and 1 white ball in the third. One jar is to be selected, and then 1 ball is to be drawn from the selected jar. If the probabilities of selecting the first, second, or third jar are 1/2, 1/3, and 1/6, respectively, find the probabilities that if a white ball is drawn, it came from the following jars.

7. The second jar **8.** The third jar

APPLICATIONS

Business and Economics

9. Employment Test A manufacturing firm finds that 70% of its new hires turn out to be good workers and 30% become poor workers. All current workers are given a reasoning test. Of the good workers, 85% pass it; 35% of the poor workers pass it. Assume that these figures will hold true in the future. If the company makes the test part of its hiring procedure and only hires people who meet the previous requirements and pass the test, what percent of the new hires will turn out to be good workers?

Job Qualifications **Of all the people applying for a certain job, 75% are qualified and 25% are not. The personnel manager claims that she approves qualified people 85% of the time; she approves an unqualified person 20% of the time. Find each probability.**

10. A person is qualified if he or she was approved by the manager.

11. A person is unqualified if he or she was approved by the manager.

Quality Control **A building contractor buys 70% of his cement from supplier A and 30% from supplier B. A total of 90% of the bags from A arrive undamaged, while 95% of the bags from B arrive undamaged. Give the probabilities that a damaged bag is from the following sources.**

12. Supplier A **13.** Supplier B

Appliance Reliability **Companies A, B, and C produce 15%, 40%, and 45%, respectively, of the major appliances sold in a certain area. In that area, 1% of the company A appliances, $1\frac{1}{2}$% of the company B appliances, and 2% of the company C appliances need service within the first year. Suppose a defective appliance is chosen at random; find the probabilities that it was manufactured by the following companies.**

14. Company A **15.** Company B

Television Advertising **On a given weekend in the fall, a tire company can buy television advertising time for a college football game, a baseball game, or a professional football game. If the company sponsors the college football game, there is a 70% chance of a high rating, a 50% chance if they sponsor a baseball game, and a 60% chance if they sponsor a professional football game. The probabilities of the company sponsoring these various games are 0.5, 0.2, and 0.3, respectively. Suppose the company does get a high rating; find the probabilities that it sponsored the following.**

16. A college football game

17. A professional football game

18. Auto Insurance An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of Driver	Probability of Accident	Portion of Company's Insured Drivers
16–20	0.06	0.08
21–30	0.03	0.15
31–65	0.02	0.49
66–99	0.04	0.28

A randomly selected driver that the company insures has an accident. Calculate the probability that the driver was age 16–20. Choose one of the following. ***Source: Society of Actuaries.***

a. 0.13 **b.** 0.16 **c.** 0.19 **d.** 0.23 **e.** 0.40

19. Life Insurance An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policyholder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year. A policyholder dies in the next year. What is the probability that the deceased policyholder was ultra-preferred? Choose one of the following. *Source: Society of Actuaries.*

a. 0.0001 **b.** 0.0010 **c.** 0.0071 **d.** 0.0141 **e.** 0.2817

20. Automobile Collisions An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of Driver	Percentage of All Drivers	Probability of at Least One Collision
Teen	8%	0.15
Young Adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver? Choose one of the following. *Source: Society of Actuaries.*

a. 0.06 **b.** 0.16 **c.** 0.19 **d.** 0.22 **e.** 0.25

21. Shipping Errors The following information pertains to three shipping terminals operated by Krag Corp. *Source: CPA Examination.*

Terminal	Percentage of Cargo Handled	Percentage of Error
Land	50	2
Air	40	4
Sea	10	14

Krag's internal auditor randomly selects one set of shipping documents, ascertaining that the set selected contains an error. Which of the following gives the probability that the error occurred in the Land Terminal?

a. 0.02 **b.** 0.10

c. 0.25 **d.** 0.50

22. Mortgage Defaults A bank finds that the relationship between mortgage defaults and the size of the down payment is given by the following table.

Down Payment	Number of Mortgages with This Down Payment	Probability of Default
5%	1260	0.06
10%	700	0.04
20%	560	0.02
25%	280	0.01

a. If a default occurs, what is the probability that it is on a mortgage with a 5% down payment?

b. What is the probability that a mortgage that is paid to maturity has a 10% down payment?

Life Sciences

23. Colorectal Cancer Researchers found that only one out of 24 physicians could give the correct answer to the following problem: "The probability of colorectal cancer can be given as 0.3%. If a person has colorectal cancer, the probability that the hemoccult test is positive is 50%. If a person does not have colorectal cancer, the probability that he still tests positive is 3%. What is the probability that a person who tests positive actually has colorectal cancer?" What is the correct answer? *Source: Science.*

24. Hepatitis Blood Test The probability that a person with certain symptoms has hepatitis is 0.8. The blood test used to confirm this diagnosis gives positive results for 90% of people with the disease and 5% of those without the disease. What is the probability that an individual who has the symptoms and who reacts positively to the test actually has hepatitis?

25. Sensitivity and Specificity The *sensitivity* of a medical test is defined as the probability that a test will be positive given that a person has a disease, written $P(T^+|D^+)$. The *specificity* of a test is defined as the probability that a test will be negative given that the person does not have the disease, written $P(T^-|D^-)$. For example, the sensitivity and specificity for breast cancer during a mammography exam are approximately 79.6% and 90.2%, respectively. *Source: National Cancer Institute.*

a. It is estimated that 0.5% of U.S. women under the age 40 have breast cancer. Find the probability that a woman under 40 who tests positive during a mammography exam actually has breast cancer.

b. Given that a woman under 40 tests negative during a mammography exam, find the probability that she does not have breast cancer.

c. According to the National Cancer Institute, failure to diagnose breast cancer is the most common cause of medical malpractice litigation. Given a woman under 40 tests negative for breast cancer, find the probability that she does have breast cancer.

d. It is estimated that 1.5% of U.S. women over the age of 50 have breast cancer. Find the probability that a woman over 50 who tests positive actually has breast cancer.

26. Test for HIV Clinical studies have demonstrated that rapid HIV tests have a sensitivity (probability that a test will be positive given that a person has the disease) of approximately 99.9% and a specificity (probability that a test will be negative given that the

person does not have the disease) of approximately 99.8%. *Source: Centers for Disease Control and Prevention*

a. In some HIV clinics, the prevalence of HIV was high, about 5%. Find the probability a person actually has HIV, given that the test came back positive from one of these clinics.

b. In other clinics, like a family planning clinic, the prevalence of HIV was low, about 0.1%. Find the probability that a person actually has HIV, given that the test came back positive from one of these clinics.

c. The answers in parts a and b are significantly different. Explain why.

27. Smokers A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker. Choose one of the following. (*Hint:* Let $x = P$(a nonsmoker dies).) *Source: Society of Actuaries.*

a. 0.20 **b.** 0.25 **c.** 0.35 **d.** 0.42 **e.** 0.57

28. Emergency Room Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;

(ii) 30% of the emergency room patients were serious;

(iii) the rest of the emergency room patients were stable;

(iv) 40% of the critical patients died;

(v) 10% of the serious patients died; and

(vi) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival? Choose one of the following. *Source: Society of Actuaries.*

a. 0.06 **b.** 0.29 **c.** 0.30 **d.** 0.39 **e.** 0.64

29. Blood Test A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease, given that the test indicates the presence of the disease. Choose one of the following. *Source: Society of Actuaries.*

a. 0.324 **b.** 0.657 **c.** 0.945 **d.** 0.950 **e.** 0.995

30. Circulation The probability that a randomly chosen male has a circulation problem is 0.25. Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem. What is the conditional probability that a male has a circulation problem, given that he is a smoker? Choose one of the following. *Source: Society of Actuaries.*

a. 1/4 **b.** 1/3 **c.** 2/5 **d.** 1/2 **e.** 2/3

Social Sciences

31. Alcohol Abstinence The Harvard School of Public Health completed a study on alcohol consumption on college campuses in 2001. They concluded that 20.7% of women attending all-women colleges abstained from alcohol, compared to 18.6% of women attending coeducational colleges. Approximately 4.7% of women college students attend all-women schools. *Source: Harvard School of Public Health.*

a. What is the probability that a randomly selected female student abstains from alcohol?

b. If a randomly selected female student abstains from alcohol, what is the probability she attends a coeducational college?

32. Murder During the murder trial of O. J. Simpson, Alan Dershowitz, an advisor to the defense team, stated on television that only about 0.1% of men who batter their wives actually murder them. Statistician I. J. Good observed that even if, given that a husband is a batterer, the probability he is guilty of murdering his wife is 0.001, what we really want to know is the probability that the husband is guilty, given that the wife was murdered. Good estimates the probability of a battered wife being murdered, given that her husband is not guilty, as 0.001. The probability that she is murdered if her husband is guilty is 1, of course. Using these numbers and Dershowitz's 0.001 probability of the husband being guilty, find the probability that the husband is guilty, given that the wife was murdered. *Source: Nature.*

Never-Married Adults by Age Group **The following tables give the proportion of men and of women 18 and older in each age group in 2009, as well as the proportion in each group who have never been married.** *Source: U.S Census Bureau.*

Age	Men Proportion of Population	Proportion Never Married
18–24	0.132	0.901
25–34	0.186	0.488
35–44	0.186	0.204
45–64	0.348	0.118
65 or over	0.148	0.044

Age	Women Proportion of Population	Proportion Never Married
18–24	0.121	0.825
25–34	0.172	0.366
35–44	0.178	0.147
45–64	0.345	0.092
65 or over	0.184	0.040

33. Find the probability that a randomly selected man who has never married is between 35 and 44 years old (inclusive).

34. Find the probability that a randomly selected woman who has been married is between 18 and 24 (inclusive).

35. Find the probability that a randomly selected woman who has never been married is between 45 and 64 (inclusive).

36. Seat Belt Effectiveness A 2009 federal study showed that 63.8% of occupants involved in a fatal car crash wore seat belts. Of those in a fatal crash who wore seat belts, 2% were ejected from the vehicle. For those not wearing seat belts, 36% were ejected from the vehicle. ***Source: National Highway Traffic Safety Administration.***

a. Find the probability that a randomly selected person in a fatal car crash who was ejected from the vehicle was wearing a seat belt.

b. Find the probability that a randomly selected person in a fatal car crash who was not ejected from the vehicle was not wearing a seat belt.

Smokers by Age Group **The following table gives the proportion of U.S. adults in each age group in 2008, as well as the proportion in each group who smoke.** ***Source: Centers for Disease Control and Prevention.***

Age	Proportion of Population	Proportion that Smoke
18 – 44 years	0.49	0.23
45 – 64 years	0.34	0.22
64 – 74 years	0.09	0.12
75 years and over	0.08	0.06

37. Find the probability that a randomly selected adult who smokes is between 18 and 44 years of age (inclusive).

38. Find the probability that a randomly selected adult who does not smoke is between 45 and 64 years of age (inclusive).

General Interest

39. Terrorists John Allen Paulos has pointed out a problem with massive, untargeted wiretaps. To illustrate the problem, he supposes that one out of every million Americans has terrorist ties. Furthermore, he supposes that the terrorist profile is 99% accurate, so that if a person has terrorist ties, the profile will pick them up 99% of the time, and if the person does not have terrorist ties, the profile will accidentally pick them up only 1% of the time. Given that the profile has picked up a person, what is the probability that the person actually has terrorist ties? Discuss how your answer affects your opinion on domestic wiretapping. ***Source: Who's Counting.***

40. Three Prisoners The famous "problem of three prisoners" is as follows.

Three men, A, B, and C, were in jail. A knew that one of them was to be set free and the other two were to be executed. But he didn't know who was the one to be spared. To the jailer who did know, A said, "Since two out of the three will be executed, it is certain that either B or C will be, at least. You will give me no information about my own chances if you give me the name of one man, B or C, who is going to be executed." Accepting this argument after some thinking, the jailer said "B will be executed." Thereupon A felt happier because now either he or C would go free, so his chance had increased from 1/3 to 1/2. ***Source: Cognition.***

a. Assume that initially each of the prisoners is equally likely to be set free. Assume also that if both B and C are to be executed, the jailer is equally likely to name either B or C. Show that A is wrong, and that his probability of being freed, given that the jailer says B will be executed, is still 1/3.

b. Now assume that initially the probabilities of A, B, and C being freed are 1/4, 1/4, and 1/2, respectively. As in part a, assume also that if both B and C are to be executed, the jailer is equally likely to name either B or C. Now show that A's probability of being freed, given that the jailer says B will be executed, actually drops to 1/5. Discuss the reasonableness of this answer, and why this result might violate someone's intuition.

YOUR TURN ANSWERS

1. 0.6486 **2.** 0.2462

7 CHAPTER REVIEW

SUMMARY

We began this chapter by introducing sets, which are collections of objects. We introduced the following set operations:

- complement (A' is the set of elements not in A),
- intersection ($A \cap B$ is the set of elements belonging to both set A and set B), and
- union ($A \cup B$ is the set of elements belonging to either set A or set B or both).

We used tree diagrams and Venn diagrams to define and study concepts in set operations as well as in probability. We introduced the following terms:

- experiment (an activity or occurrence with an observable result),
- trial (a repetition of an experiment),
- outcome (a result of a trial),
- sample space (the set of all possible outcomes for an experiment), and
- event (a subset of a sample space).

We investigated how to compute various probabilities and we explored some of the properties of probability. In particular, we studied the following concepts:

- empirical probability (based on how frequently an event actually occurred),
- conditional probability (in which some other event is assumed to have occurred),

- odds (an alternative way of expressing probability),
- independent events (in which the occurrence of one event does not affect the probability of another), and
- Bayes' theorem (used to calculate certain types of conditional probability).

Throughout the chapter, many applications of probability were introduced and analyzed. In the next two chapters, we will employ these techniques to further our study into the fields of probability and statistics.

Sets Summary

Number of Subsets A set of k distinct elements has 2^k subsets.

Disjoint Sets If sets A and B are disjoint, then

$$A \cap B = \emptyset \quad \text{and} \quad n(A \cap B) = 0.$$

Union Rule for Sets For any sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Probability Summary

Basic Probability Principle Let S be a sample space of equally likely outcomes, and let event E be a subset of S. Then the probability that event E occurs is

$$P(E) = \frac{n(E)}{n(S)}.$$

Mutually Exclusive Events If E and F are mutually exclusive events,

$$E \cap F = \emptyset \quad \text{and} \quad P(E \cap F) = 0.$$

Union Rule For any events E and F from a sample space S,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Complement Rule $P(E) = 1 - P(E') \quad \text{and} \quad P(E') = 1 - P(E)$

Odds The odds in favor of event E are $\dfrac{P(E)}{P(E')}$, where $P(E') \neq 0$.

If the odds favoring event E are m to n, then

$$P(E) = \frac{m}{m+n} \quad \text{and} \quad P(E') = \frac{n}{m+n}.$$

Properties of Probability

1. For any event E in sample space S, $0 \leq P(E) \leq 1$.
2. The sum of the probabilities of all possible distinct outcomes is 1.

Conditional Probability The conditional probability of event E, given that event F has occurred, is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \quad \text{where } P(F) \neq 0.$$

For equally likely outcomes, conditional probability is found by reducing the sample space to event F; then

$$P(E|F) = \frac{n(E \cap F)}{n(F)}.$$

Product Rule of Probability If E and F are events, then $P(E \cap F)$ may be found by either of these formulas.

$$P(E \cap F) = P(F) \cdot P(E|F) \quad \text{or} \quad P(E \cap F) = P(E) \cdot P(F|E)$$

Independent Events If E and F are independent events,

$$P(E|F) = P(E), \quad P(F|E) = P(F), \quad \text{and} \quad P(E \cap F) = P(E) \cdot P(F).$$

Bayes' Theorem

$$P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + \cdots + P(F_n) \cdot P(E|F_n)}$$

KEY TERMS

7.1
set
element (member)
empty set (or null set)
set-builder notation
universal set
subset
tree diagram
Venn diagram
complement
intersection
disjoint sets
union

7.2
union rule for sets

7.3
experiment
trial
outcome
sample space
event
simple event
certain event
impossible event
mutually exclusive events
probability
empirical probability

7.4
union rule for probability
odds
probability distribution

7.5
conditional probability
product rule
independent events
dependent events

7.6
Bayes' theorem

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. A set is a subset of itself.

2. A set has more subsets than it has elements.

3. The union of two sets always has more elements than either set.

4. The intersection of two sets always has fewer elements than either set.

5. The number of elements in the union of two sets can be found by adding the number of elements in each set.

6. The probability of an event is always at least 0 and no larger than 1.

7. The probability of the union of two events can be found by adding the probability of each event.

8. The probability of drawing the Queen of Hearts from a deck of cards is an example of empirical probability.

9. If two events are mutually exclusive, then they are independent.

10. The probability of two independent events can be found by multiplying the probabilities of each event.

11. The probability of an event E given an event F is the same as the probability of F given E.

12. Bayes' theorem can be useful for calculating conditional probability.

PRACTICE AND EXPLORATIONS

Write true or false for each statement.

13. $9 \in \{8, 4, -3, -9, 6\}$

14. $4 \notin \{3, 9, 7\}$

15. $2 \notin \{0, 1, 2, 3, 4\}$

16. $0 \in \{0, 1, 2, 3, 4\}$

17. $\{3, 4, 5\} \subseteq \{2, 3, 4, 5, 6\}$

18. $\{1, 2, 5, 8\} \subseteq \{1, 2, 5, 10, 11\}$

19. $\{3, 6, 9, 10\} \subseteq \{3, 9, 11, 13\}$

20. $\emptyset \subseteq \{1\}$

21. $\{2, 8\} \not\subseteq \{2, 4, 6, 8\}$

22. $0 \subseteq \emptyset$

In Exercises 23–32, let $U = \{\mathbf{a, b, c, d, e, f, g, h}\}$, $K = \{\mathbf{c, d, e, f, h}\}$, and $R = \{\mathbf{a, c, d, g}\}$. Find the following.

23. The number of subsets of K

24. The number of subsets of R

25. K'

26. R'

27. $K \cap R$

28. $K \cup R$

29. $(K \cap R)'$

30. $(K \cup R)'$

31. $\emptyset'$

32. U'

In Exercises 33–38, let

$U =$ {all employees of the K. O. Brown Company};
$A =$ {employees in the accounting department};
$B =$ {employees in the sales department};
$C =$ {female employees};
$D =$ {employees with an MBA degree}.

Describe each set in words.

33. $A \cap C$

34. $B \cap D$

35. $A \cup D$

36. $A' \cap D$

37. $B' \cap C'$

38. $(B \cup C)'$

Draw a Venn diagram and shade each set.

39. $A \cup B'$

40. $A' \cap B$

41. $(A \cap B) \cup C$

42. $(A \cup B)' \cap C$

Write the sample space S for each experiment, choosing an S with equally likely outcomes, if possible.

43. Rolling a die

44. Drawing a card from a deck containing only the 13 spades

45. Measuring the weight of a person to the nearest half pound (the scale will not measure more than 300 lb)

46. Tossing a coin 4 times

A jar contains 5 balls labeled 3, 5, 7, 9, and 11, respectively, while a second jar contains 4 red and 2 green balls. An experiment consists of pulling 1 ball from each jar, in turn. In Exercises 47–50, write each set using set notation.

47. The sample space

48. Event E: the number on the first ball is greater than 5

49. Event F: the second ball is green

50. Are the outcomes in the sample space in Exercise 47 equally likely?

In Exercises 51–58, find the probability of each event when a single card is drawn from an ordinary deck.

51. A heart

52. A red queen

53. A face card or a heart

54. Black or a face card

55. Red, given that it is a queen

56. A jack, given that it is a face card

57. A face card, given that it is a king

58. A king, given that it is not a face card

59. Describe what is meant by disjoint sets.

60. Describe what is meant by mutually exclusive events.

61. How are disjoint sets and mutually exclusive events related?

62. Define independent events.

63. Are independent events always mutually exclusive? Are they ever mutually exclusive?

64. An uproar has raged since September 1990 over the answer to a puzzle published in *Parade* magazine, a supplement of the Sunday newspaper. In the "Ask Marilyn" column, Marilyn vos Savant answered the following question: "Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car; behind the others, goats. You pick a door, say number 1, and the host, who knows what's behind the other doors, opens another door, say number 3, which has a goat. He then says to you, 'Do you want to pick door number 2?' Is it to your advantage to take the switch?"

Ms. vos Savant estimates that she has since received some 10,000 letters; most of them, including many from mathematicians and statisticians, disagreed with her answer. Her answer has been debated by both professionals and amateurs and tested in classes at all levels from grade school to graduate school. But by performing the experiment repeatedly, it can be shown that vos Savant's answer was correct. Find the probabilities of getting the car if you switch or do not switch, and then answer the question yourself. (*Hint:* Consider the sample space.) ***Source: Parade Magazine.***

Find the odds in favor of a card drawn from an ordinary deck being the following.

65. A club

66. A black jack

67. A red face card or a queen

68. An ace or a club

Find the probabilities of getting the following sums when two fair dice are rolled.

69. 8

70. 0

71. At least 10

72. No more than 5

73. An odd number greater than 8

74. 12, given that the sum is greater than 10

75. 7, given that at least one die shows a 4

76. At least 9, given that at least one die shows a 5

77. Suppose $P(E) = 0.51$, $P(F) = 0.37$, and $P(E \cap F) = 0.22$. Find the following.

a. $P(E \cup F)$ **b.** $P(E \cap F')$ **c.** $P(E' \cup F)$

d. $P(E' \cap F')$

78. An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Calculate the number of blue balls in the second urn. Choose one of the following. ***Source: Society of Actuaries.***

a. 4 **b.** 20 **c.** 24 **d.** 44 **e.** 64

79. Box A contains 5 red balls and 1 black ball; box B contains 2 red balls and 3 black balls. A box is chosen, and a ball is selected from it. The probability of choosing box A is 3/8. If the selected ball is black, what is the probability that it came from box A?

80. Find the probability that the ball in Exercise 79 came from box B, given that it is red.

APPLICATIONS

Business and Economics

Appliance Repairs **Of the appliance repair shops listed in the phone book, 80% are competent and 20% are not. A competent shop can repair an appliance correctly 95% of the time; an incompetent shop can repair an appliance correctly 55% of the time. Suppose an appliance was repaired correctly. Find the probabilities that it was repaired by the following.**

81. A competent shop

82. An incompetent shop

Suppose an appliance was repaired incorrectly. Find the probabilities that it was repaired by the following.

83. A competent shop

84. An incompetent shop

85. Find the probability that an appliance brought to a shop chosen at random is repaired correctly.

86. Are the events that a repair shop is competent and that the repair is done correctly independent? Explain.

87. Sales A company sells printers and copiers. Let E be the event "a customer buys a printer," and let F be the event "a customer buys a copier." Write the following using $\cap$, $\cup$, or $'$ as necessary.

a. A customer buys neither machine.

b. A customer buys at least one of the machines.

88. Defective Items A sample shipment of five hair dryers is chosen at random. The probability of exactly 0, 1, 2, 3, 4, or 5 hair dryers being defective is given in the following table.

Number Defective	0	1	2	3	4	5
Probability	0.34	0.26	0.18	0.12	0.07	0.03

Find the probabilities that the following numbers of hair dryers are defective.

a. No more than 3

b. At least 3

89. Defective Items A manufacturer buys items from four different suppliers. The fraction of the total number of items that is obtained from each supplier, along with the probability that an item purchased from that supplier is defective, is shown in the table below.

Supplier	Fraction of Total Supplied	Probability of Defective
1	0.17	0.01
2	0.39	0.02
3	0.35	0.05
4	0.09	0.03

a. Find the probability that a randomly selected item is defective.

b. Find the probability that a defective item came from supplier 4.

c. Find the probability that a defective item came from supplier 2.

d. Are the events that an item came from supplier 4 and that the item is defective independent? Explain.

90. Car Buyers The table shows the results of a survey of buyers of a certain model of car.

Car Type	Satisfied	Not Satisfied	Totals
New	300	100	
Used	450		600
Totals		250	

a. Complete the table.

b. How many buyers were surveyed?

c. How many bought a new car and were satisfied?

d. How many were not satisfied?

e. How many bought used cars?

f. How many of those who were not satisfied had purchased a used car?

g. Rewrite the event stated in part f using the expression "given that."

h. Find the probability of the outcome in parts f and g.

i. Find the probability that a used-car buyer is not satisfied.

j. You should have different answers in parts h and i. Explain why.

k. Are the events that a car is new and that the customer is satisfied independent? Explain.

91. Auto Insurance An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.

(ii) 70% of the customers insure more than one car.

(iii) 20% of the customers insure a sports car.

(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car. Choose one of the following. (*Hint:* Draw a tree diagram, and let x be the probability that a customer who insures exactly one car insures a sports car.) ***Source: Society of Actuaries.***

a. 0.13 **b.** 0.21 **c.** 0.24 **d.** 0.25 **e.** 0.30

92. Auto Insurance An auto insurance company has 10,000 policyholders. Each policyholder is classified as:

(i) young or old;

(ii) male or female; and

(iii) married or single.

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single? Choose one of the following. ***Source: Society of Actuaries.***

a. 280 **b.** 423 **c.** 486 **d.** 880 **e.** 896

93. Auto Insurance An actuary studying the insurance preferences of automobile owners makes the following conclusions:

(i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.

(ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.

(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.

What is the probability that an automobile owner purchases neither collision nor disability coverage? Choose one of the following. ***Source: Society of Actuaries.***

a. 0.18 **b.** 0.33 **c.** 0.48 **d.** 0.67 **e.** 0.82

94. Insurance An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of these policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year. Choose one of the following. ***Source: Society of Actuaries.***

a. 20 **b.** 29 **c.** 41 **d.** 53 **e.** 70

Life Sciences

95. Sickle Cell Anemia The square on the next page shows the four possible (equally likely) combinations when both parents are carriers of the sickle cell anemia trait. Each carrier parent has normal cells (N) and trait cells (T).

a. Complete the table.

b. If the disease occurs only when two trait cells combine, find the probability that a child born to these parents will have sickle cell anemia.

c. The child will carry the trait but not have the disease if a normal cell combines with a trait cell. Find this probability.

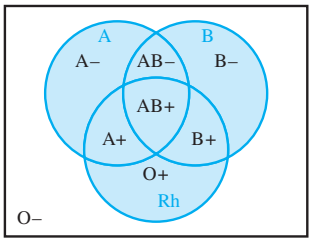
d. Find the probability that the child is neither a carrier nor has the disease.

		Second Parent	
		N_2	T_2
First Parent	N_1		N_1T_2
	T_1		

96. Blood Antigens In Exercise 44 of Section 7.2, we described the eight types of human blood. The percentage of the population having each type is as follows:

O^+: 38%; O^-: 8%; A^+: 32%; A^-: 7%;
B^+: 9%; B^-: 2%; AB^+: 3%; AB^-: 1%.

When a person receives a blood transfusion, it is important that the blood be compatible, which means that it introduces no new antigens into the recipient's blood. The following diagram helps illustrate what blood types are compatible. ***Source: The Mathematics Teacher.***



The universal blood type is O^-, since it has none of the additional antigens. The circles labeled A, B, and Rh contain blood types with the A antigen, B antigen, and Rh antigen, respectively. A person with O^- blood can only be transfused with O^- blood, because any other type would introduce a new antigen. Thus the probability that blood from a random donor is compatible is just 8%. A person with AB^+ blood already has all antigens, so the probability that blood from a random donor is compatible is 100%. Find the probability that blood from a random donor is compatible with a person with each blood type.

a. O^+ **b.** A^+ **c.** B^+

d. A^- **e.** B^- **f.** AB^-

97. Risk Factors An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is 1/3. What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A? Choose one of the following. ***Source: Society of Actuaries.***

a. 0.280 **b.** 0.311 **c.** 0.467 **d.** 0.484 **e.** 0.700

Social Sciences

98. Elections In the 2008 presidential election, over 130 million people voted, of which 46.3% were male and 53.7% were female. Of the male voters, 49% voted for Barack Obama and 48% voted for John McCain. Of the female voters, 56% voted for Obama and 43% voted for McCain. ***Source: U.S. Census and The New York Times.***

a. Find the percentage of voters who voted for Obama.

b. Find the probability that a randomly selected voter for Obama was male.

c. Find the probability that a randomly selected voter for Obama was female.

99. Television Viewing Habits A telephone survey of television viewers revealed the following information:

20 watch situation comedies;
19 watch game shows;
27 watch movies;
19 watch movies but not game shows;
15 watch situation comedies but not game shows;
10 watch both situation comedies and movies;
3 watch all three;
7 watch none of these.

a. How many viewers were interviewed?

b. How many viewers watch comedies and movies but not game shows?

c. How many viewers watch only movies?

d. How many viewers do not watch movies?

100. Randomized Response Method for Getting Honest Answers to Sensitive Questions There are many personal questions that most people would rather not answer. In fact, when a person is asked such a question, a common response is to provide a false answer. In 1965, Stanley Warner developed a method to assure that the identity of an individual who answers a question remains anonymous, thus assuring an honest answer to the question. For this method, instead of one sensitive question being asked to a person, there are two questions, one sensitive and one nonsensitive, and which question the person answers depends on a randomized procedure. The interviewer, who doesn't know which question a person is answering, simply records the answer given as either "Yes" or "No." In this way, there is no way of knowing to which question the person answered "Yes" or "No." Then, using conditional probability, we can estimate the percentage of people who answer "Yes" to the sensitive question. For example, suppose that the two questions are

A: Does your birth year end in an odd digit? (Nonsensitive)
B: Have you ever intentionally cheated on an examination? (Sensitive)

We already know that $P(\text{"Yes"} \mid \text{Question } A) = 1/2$, since half the years are even and half are odd. The answer we seek is $P(\text{"Yes"} \mid \text{Question } B)$. In this experiment, a student is asked to flip a coin and answer question *A* if the coin comes up heads and otherwise answer question *B*. Note that the interviewer does not know the outcome of the coin flip or which question is being answered. The percentage of students answering "Yes" is used to approximate $P(\text{"Yes"})$, which is then used to estimate the percentage of students who have cheated on an examination. ***Source: Journal of the American Statistical Association.***

a. Use the fact that the event "Yes" is the union of the event "Yes and Question *A*" with the event "Yes and Question *B*" to prove that

$$P(\text{"Yes"} \mid \text{Question } B) = \frac{P(\text{"Yes"}) - P(\text{"Yes"} \mid \text{Question } A) \cdot P(\text{Question } A)}{P(\text{Question } B)}.$$

b. If this technique is tried on 100 subjects and 60 answered "Yes," what is the approximated probability that a person randomly selected from the group has intentionally cheated on an examination?

101. Police Lineup To illustrate the difficulties with eyewitness identifications from police lineups, John Allen Paulos considers a "lineup" of three pennies, in which we know that two are fair (innocent) and the third (the culprit) has a 75% probability of landing heads. The probability of picking the culprit by chance is, of course, 1/3. Suppose we observe three heads in a row on one of the pennies. If we then guess that this penny is the culprit, what is the probability that we're right? ***Source: Who's Counting.***

102. SIDS On July 15, 2005, a panel in England ruled that Roy Meadow, a renowned expert on child abuse and co-founder of London's Royal College of Paediatrics and Child Health, should be erased from the register of physicians in Britain for his faulty statistics at the trial of Sally Clark, who was convicted of murdering her first two babies. Meadow testified at the trial that the probability of a baby dying of sudden death syndrome (SIDS) is 1/8543. He then calculated that the probability of two babies in a family dying of SIDS is $(1/8543)^2 \approx 1/73{,}000{,}000$. With such a small probability of both babies dying of SIDS, he concluded that the babies were instead murdered. What assumption did Meadow make in doing this calculation? Discuss reasons why this assumption may be invalid. (*Note:* Clark spent three years in prison before her conviction was reversed.) ***Source: Science.***

Physical Sciences

103. Earthquake It has been reported that government scientists have predicted that the odds for a major earthquake occurring in the San Francisco Bay area during the next 30 years are 9 to 1. What is the probability that a major earthquake will occur during the next 30 years in San Francisco? ***Source: The San Francisco Chronicle.***

General Interest

104. Making a First Down A first down is desirable in football—it guarantees four more plays by the team making it, assuming no score or turnover occurs in the plays. After getting a first down, a team can get another by advancing the ball at least 10 yards. During the four plays given by a first down, a team's position will be indicated by a phrase such as "third and 4," which means that the team has already had two of its four plays, and that 4 more yards are needed to get 10 yards necessary for another first down. An article in a management journal offers the following results for 189 games for a particular National Football League season. "Trials" represents the number of times a team tried to make a first down, given that it was currently playing either a third or a fourth down. Here, n represents the number of yards still needed for a first down. ***Source: Management Science.***

n	Trials	Successes	Probability of Making First Down with n Yards to Go
1	543	388	
2	327	186	
3	356	146	
4	302	97	
5	336	91	

a. Complete the table.

b. Why is the sum of the answers in the table not equal to 1?

105. States Of the 50 United States, the following is true:

24 are west of the Mississippi River (western states);*
22 had populations less than 3.6 million in the 2010 census (small states);
26 begin with the letters A through M (early states);
9 are large late (beginning with the letters N through Z) eastern states;
14 are small western states;
11 are small early states;
7 are small early western states.

a. How many western states had populations more than 3.6 million in the 2010 census and begin with the letters N through Z?

b. How many states east of the Mississippi had populations more than 3.6 million in the 2010 census?

106. Music Country-western songs often emphasize three basic themes: love, prison, and trucks. A survey of the local country-western radio station produced the following data:

12 songs were about a truckdriver who was in love while in prison;
13 were about a prisoner in love;
28 were about a person in love;
18 were about a truckdriver in love;
33 were about people not in prison;
18 were about prisoners;
15 were about truckdrivers who were in prison;
16 were about truckdrivers who were not in prison.

a. How many songs were surveyed?

Find the number of songs about

b. truckdrivers;

c. prisoners who are not truckdrivers or in love;

d. prisoners who are in love but are not truckdrivers;

e. prisoners who are not truckdrivers;

f. people not in love.

107. Gambling The following puzzle was featured on the Puzzler part of the radio program *Car Talk* on February 23, 2002. A con man puts three cards in a bag; one card is green on both sides, one is red on both sides, and the third is green on one side and red on the other. He lets you pick one card out of the bag and put it on a table, so you can see that a red side is face up, but neither of you can see the other side. He offers to bet you even money that the other side is also red. In other words, if you bet $1, you lose if the other side is red but get back $2 if the other side is green. Is this a good bet? What is the probability that the other side is red? ***Source: Car Talk.***

108. Missiles In his novel *Debt of Honor*, Tom Clancy writes the following:

"There were ten target points—missile silos, the intelligence data said, and it pleased the Colonel [Zacharias] to be eliminating the hateful things, even though the price of that was the lives of other

*We count here states such as Minnesota, which has more than half of its area to the west of the Mississippi.

men. There were only three of them [bombers], and his bomber, like the others, carried only eight weapons [smart bombs]. The total number of weapons carried for the mission was only twenty-four, with two designated for each silo, and Zacharias's last four for the last target. Two bombs each. Every bomb had a 95% probability of hitting within four meters of the aim point, pretty good numbers really, except that this sort of mission had precisely no margin for error. Even the paper probability was less than half a percent chance of a double miss, but that number times ten targets meant a 5% chance that [at least] one missile would survive, and that could not be tolerated." ***Source: Debt of Honor.***

Determine whether the calculations in this quote are correct by the following steps.

a. Given that each bomb had a 95% probability of hitting the missile silo on which it was dropped, and that two bombs were dropped on each silo, what is the probability of a double miss?

b. What is the probability that a specific silo was destroyed (that is, that at least one bomb of the two bombs struck the silo)?

c. What is the probability that all ten silos were destroyed?

d. What is the probability that at least one silo survived? Does this agree with the quote?

e. What assumptions need to be made for the calculations in parts a through d to be valid? Discuss whether these assumptions seem reasonable.

109. Viewing Habits A survey of a group's viewing habits over the last year revealed the following information:

(i) 28% watched gymnastics;

(ii) 29% watched baseball;

(iii) 19% watched soccer;

(iv) 14% watched gymnastics and baseball;

(v) 12% watched baseball and soccer;

(vi) 10% watched gymnastics and soccer;

(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year. Choose one of the following. ***Source: Society of Actuaries.***

a. 24 **b.** 36 **c.** 41 **d.** 52 **e.** 60

EXTENDED APPLICATION

MEDICAL DIAGNOSIS

When a patient is examined, information (typically incomplete) is obtained about his or her state of health. Probability theory provides a mathematical model appropriate for this situation, as well as a procedure for quantitatively interpreting such partial information to arrive at a reasonable diagnosis. ***Source: Some Mathematical Models in Biology.***

To develop a model, we list the states of health that can be distinguished in such a way that the patient can be in one and only one state at the time of the examination. For each state of health H, we associate a number, $P(H)$, between 0 and 1 such that the sum of all these numbers is 1. This number $P(H)$ represents the probability, before examination, that a patient is in the state of health H, and $P(H)$ may be chosen subjectively from medical experience, using any information available prior to the examination. The probability may be most conveniently established from clinical records; that is, a mean probability is established for patients in general, although the number would vary from patient to patient. Of course, the more information that is brought to bear in establishing $P(H)$, the better the diagnosis.

For example, limiting the discussion to the condition of a patient's heart, suppose there are exactly three states of health, with probabilities as follows.

	State of Health, H	$P(H)$
H_1	Patient has a normal heart	0.8
H_2	Patient has minor heart irregularities	0.15
H_3	Patient has a severe heart condition	0.05

Having selected $P(H)$, the information from the examination is processed. First, the results of the examination must be classified. The examination itself consists of observing the state of a number of characteristics of the patient. Let us assume that the examination for a heart condition consists of a stethoscope examination and a cardiogram. The outcome of such an examination, C, might be one of the following:

C_1 = stethoscope shows normal heart and cardiogram shows normal heart;

C_2 = stethoscope shows normal heart and cardiogram shows minor irregularities;

and so on.

It remains to assess for each state of health H the conditional probability $P(C|H)$ of each examination outcome C using only the knowledge that a patient is in a given state of health. (This may be based on the medical knowledge and clinical experience of the doctor.) The conditional probabilities $P(C|H)$ will not vary from patient to patient (although they should be reviewed periodically), so that they may be built into a diagnostic system.

Suppose the result of the examination is C_1. Let us assume the following probabilities:

$$P(C_1|H_1) = 0.9,$$
$$P(C_1|H_2) = 0.4,$$
$$P(C_1|H_3) = 0.1.$$

Now, for a given patient, the appropriate probability associated with each state of health H, after examination, is $P(H|C)$, where C is the outcome of the examination. This can be calculated by using Bayes' theorem. For example, to find $P(H_1|C_1)$—that is, the probability that the patient has a normal heart given that the examination showed a normal stethoscope examination and a normal cardiogram—we use Bayes' theorem as follows:

$$P(H_1|C_1) = \frac{P(C_1|H_1)P(H_1)}{P(C_1|H_1)P(H_1) + P(C_1|H_2)P(H_2) + P(C_1|H_3)P(H_3)}$$
$$= \frac{(0.9)(0.8)}{(0.9)(0.8) + (0.4)(0.15) + (0.1)(0.05)} \approx 0.92.$$

Hence, the probability is about 0.92 that the patient has a normal heart on the basis of the examination results. This means that in 8 out of 100 patients, some abnormality will be present and not be detected by the stethoscope or the cardiogram.

EXERCISES

1. Find $P(H_2|C_1)$.

2. Assuming the following probabilities, find $P(H_1|C_2)$.

$$P(C_2|H_1) = 0.2 \qquad P(C_2|H_2) = 0.8 \qquad P(C_2|H_3) = 0.3$$

3. Assuming the probabilities of Exercise 2, find $P(H_3|C_2)$.

DIRECTIONS FOR GROUP PROJECT

Find an article on medical decision making from a medical journal and develop a doctor–patient scenario for that particular decision. Then create a role-playing activity where the doctor and nurse present the various options and the mathematics associated with making such a decision to a patient. Make sure to present the mathematics in a manner that the average patient might understand. (Hint: Many leading medical journals include articles on medical decision making. One particular journal that certainly includes such research is Medical Decision Making.)

8 Counting Principles; Further Probability Topics

8.1 The Multiplication Principle; Permutations

8.2 Combinations

8.3 Probability Applications of Counting Principles

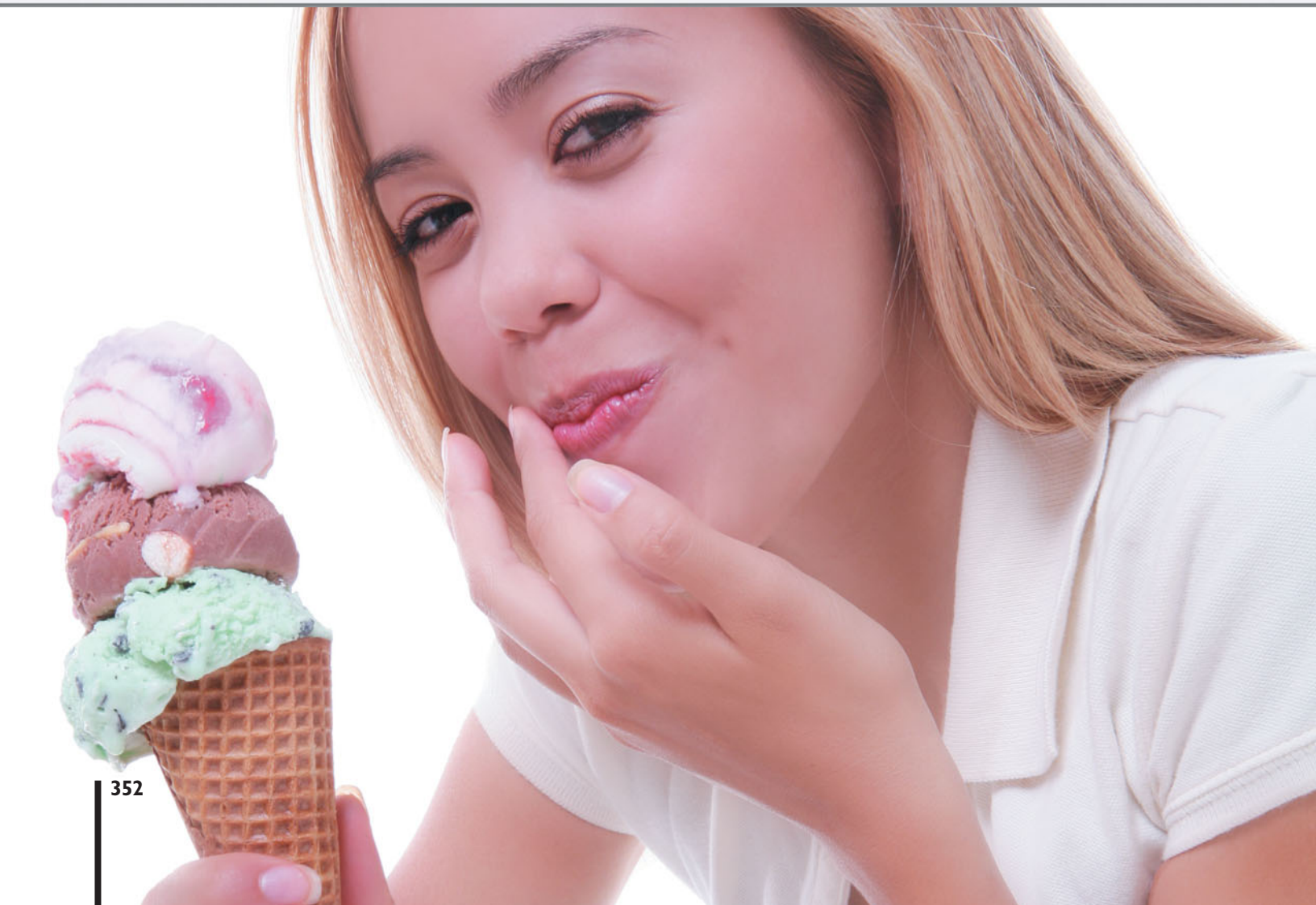
8.4 Binomial Probability

8.5 Probability Distributions; Expected Value

Chapter 8 Review

Extended Application: Optimal Inventory for a Service Truck

If you have 31 ice cream flavors available, how many different three-scoop cones can you make? The answer, which is surprisingly large, involves counting permutations or combinations, the subject of the first two sections in this chapter. The counting formulas we will develop have important applications in probability theory.



If we flip two coins and record heads or tails for each coin, we can list the four possible outcomes in the sample space as $S = \{hh, ht, th, tt\}$. But what if we flip 10 coins? As we shall see, there are over 1000 possible outcomes, which makes a list of the sample space impractical. Fortunately, there are methods for counting the outcomes in a set without actually listing them. We introduce these methods in the first two sections, and then we use this approach in the third section to find probabilities. In the fourth section, we introduce binomial experiments, which consist of repeated independent trials with only two possible outcomes, and develop a formula for binomial probabilities. The final section continues the discussion of probability distributions that we began in Chapter 7.

8.1 The Multiplication Principle; Permutations

APPLY IT **In how many ways can seven panelists be seated in a row of seven chairs?**
Before we answer this question in Example 8, we will begin with a simpler example.

Suppose a breakfast consists of one pastry, for which you can choose a bagel, a muffin, or a donut, and one beverage, for which you can choose coffee or juice. How many different breakfasts can you have? For each of the 3 pastries there are 2 different beverages, or a total of $3 \cdot 2 = 6$ different breakfasts, as shown in Figure 1. This example illustrates a general principle of counting, called the **multiplication principle.**

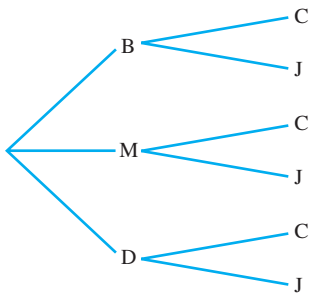


FIGURE 1

Multiplication Principle

Suppose n choices must be made, with

m_1 ways to make choice 1,
m_2 ways to make choice 2,

and so on, with

m_n ways to make choice n.

Then there are

$$m_1 \cdot m_2 \cdot \cdots \cdot m_n$$

different ways to make the entire sequence of choices.

EXAMPLE 1 Combination Lock

A certain combination lock can be set to open to any 3-letter sequence.

(a) How many sequences are possible?

SOLUTION Since there are 26 letters in the alphabet, there are 26 choices for each of the 3 letters. By the multiplication principle, there are $26 \cdot 26 \cdot 26 = 17{,}576$ different sequences.

(b) How many sequences are possible if no letter is repeated?

SOLUTION There are 26 choices for the first letter. It cannot be used again, so there are 25 choices for the second letter and then 24 choices for the third letter. Consequently, the number of such sequences is $26 \cdot 25 \cdot 24 = 15{,}600$. TRY YOUR TURN 1

YOUR TURN 1 A combination lock can be set to open to any 4-digit sequence. How many sequences are possible? How many sequences are possible if no digit is repeated?

EXAMPLE 2 Morse Code

Morse code uses a sequence of dots and dashes to represent letters and words. How many sequences are possible with at most 3 symbols?

SOLUTION "At most 3" means "1 or 2 or 3" here. Each symbol may be either a dot or a dash. Thus the following number of sequences are possible in each case.

Morse Code	
Number of Symbols	**Number of Sequences**
1	**2**
2	$\mathbf{2 \cdot 2} = 4$
3	$\mathbf{2 \cdot 2 \cdot 2} = 8$

Altogether, $2 + 4 + 8 = 14$ different sequences are possible.

EXAMPLE 3 I Ching

FIGURE 2

An ancient Chinese philosophical work known as the *I Ching (Book of Changes)* is often used as an oracle from which people can seek and obtain advice. The philosophy describes the duality of the universe in terms of two primary forces: *yin* (passive, dark, receptive) and *yang* (active, light, creative). Figure 2 shows the traditional symbol for yin and yang. The yin energy can also be represented by a broken line (– –) and the yang by a solid line (—). These lines are written on top of one another in groups of three, known as *trigrams.* For example, the trigram ☱ is called *Tui,* the Joyous, and has the image of a lake.

(a) How many trigrams are there altogether?

SOLUTION Think of choosing between the 2 types of lines for each of the 3 positions in the trigram. There will be 2 choices for each position, so there are $2 \cdot 2 \cdot 2 = 8$ different trigrams.

(b) The trigrams are grouped together, one on top of the other, in pairs known as *hexagrams.* Each hexagram represents one aspect of the *I Ching* philosophy. How many hexagrams are there?

SOLUTION For each position in the hexagram there are 8 possible trigrams, giving $8 \cdot 8 = 64$ hexagrams.

EXAMPLE 4 Books

A teacher has 5 different books that he wishes to arrange side by side. How many different arrangements are possible?

SOLUTION Five choices will be made, one for each space that will hold a book. Any of the 5 books could be chosen for the first space. There are 4 choices for the second space, since 1 book has already been placed in the first space; there are 3 choices for the third space, and so on. By the multiplication principle, the number of different possible arrangements is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

TRY YOUR TURN 2

YOUR TURN 2 A teacher is lining up 8 students for a spelling bee. How many different line-ups are possible?

FOR REVIEW

The natural numbers, also referred to as the positive integers, are the numbers 1, 2, 3, 4, etc.

The use of the multiplication principle often leads to products such as $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, the product of all the natural numbers from 5 down to 1. If n is a natural number, the symbol $n!$ (read "n *factorial*") denotes the product of all the natural numbers from n down to 1. If $n = 1$, this formula is understood to give $1! = 1$.

Factorial Notation

For any natural number n,

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

Also, by definition,

$$0! = 1.$$

With this symbol, the product $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be written as 5!. Also, $3! = 3 \cdot 2 \cdot 1 = 6$. The definition of $n!$ could be used to show that $n[(n-1)]! = n!$ for all natural numbers $n \geq 2$. It is helpful if this result also holds for $n = 1$. This can happen only if 0! equals 1, as defined above.

As n gets large, $n!$ grows very quickly. For example, $5! = 120$, while $10! = 3{,}628{,}800$. Most calculators can be used to determine $n!$ for small values of n. A calculator with a 10-digit display and scientific notation capability will usually give the exact value of $n!$ for $n \leq 13$, and approximate values of $n!$ for $14 \leq n \leq 69$. The value of 70! is approximately 1.198×10^{100}, which is too large for most calculators. To get a sense of how large 70! is, suppose a computer counted the numbers from 1 to 70! at a rate of 1 billion numbers per second. If the computer started when the universe began, by now it would have calculated only a tiny fraction of the total.

TECHNOLOGY NOTE

On many graphing calculators, the factorial of a number is accessible through a menu. On the TI-84 Plus, for example, this menu is found by pressing the MATH key, selecting PRB (for probability), and then selecting the !.

EXAMPLE 5 Books

Suppose the teacher in Example 4 wishes to place only 3 of the 5 books on his desk. How many arrangements of 3 books are possible?

SOLUTION The teacher again has 5 ways to fill the first space, 4 ways to fill the second space, and 3 ways to fill the third. Since he wants to use only 3 books, only 3 spaces can be filled (3 events) instead of 5, for $5 \cdot 4 \cdot 3 = 60$ arrangements. TRY YOUR TURN 3

YOUR TURN 3 A teacher wishes to place 5 out of 8 different books on her shelf. How many arrangements of 5 books are possible?

Permutations

The answer 60 in Example 5 is called the number of *permutations* of 5 things taken 3 at a time. A **permutation** of r (where $r \geq 1$) elements from a set of n elements is any specific ordering or arrangement, *without repetition*, of the r elements. Each rearrangement of the r elements is a different permutation. The number of permutations of n things taken r at a time (with $r \leq n$) is written $P(n, r)$.* Based on the work in Example 5,

$$P(5, 3) = 5 \cdot 4 \cdot 3 = 60.$$

Factorial notation can be used to express this product as follows.

$$5 \cdot 4 \cdot 3 = 5 \cdot 4 \cdot 3 \cdot \frac{2 \cdot 1}{2 \cdot 1} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

This example illustrates the general rule of permutations, which can be stated as follows.

Permutations

If $P(n, r)$ (where $r \leq n$) is the number of permutations of n elements taken r at a time, then

$$P(n, r) = \frac{n!}{(n-r)!}.$$

*An alternate notation for $P(n, r)$ is ${}_nP_r$.

CAUTION The letter P here represents *permutations*, not *probability*. In probability notation, the quantity in parentheses describes an *event*. In permutations notation, the quantity in parentheses always comprises *two numbers*.

The proof of the permutations rule follows the discussion in Example 5. There are n ways to choose the first of the r elements, $n - 1$ ways to choose the second, and $n - r + 1$ ways to choose the rth element, so that

$$P(n, r) = n(n - 1)(n - 2)\cdots(n - r + 1).$$

Now multiply on the right by $(n - r)!/(n - r)!$.

$$\begin{aligned} P(n, r) &= n(n - 1)(n - 2)\cdots(n - r + 1) \cdot \frac{(n - r)!}{(n - r)!} \\ &= \frac{n(n - 1)(n - 2)\cdots(n - r + 1)(n - r)!}{(n - r)!} \\ &= \frac{n!}{(n - r)!} \end{aligned}$$

Because we defined 0! equal to 1, the formula for permutations gives the special case

$$P(n, n) = \frac{n!}{(n - n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!.$$

This result also follows from the multiplication principle, because $P(n, n)$ gives the number of permutations of n objects, and there are n choices for the first object, $n - 1$ for the second, and so on, down to just 1 choice for the last object. Example 4 illustrated this idea.

$P(n, n)$
The number of permutations of a set with n elements is $n!$; that is $P(n, n) = n!$.

EXAMPLE 6 Permutations of Letters

Find the following.

(a) The number of permutations of the letters A, B, and C.

SOLUTION By the formula $P(n, n) = n!$, with $n = 3$,

$$P(3, 3) = 3! = 3 \cdot 2 \cdot 1 = 6.$$

The 6 *permutations* (or *arrangements*) are ABC, ACB, BAC, BCA, CAB, CBA.

(b) The number of permutations if just 2 of the letters A, B, and C are to be used

SOLUTION Find $P(3, 2)$.

$$P(3, 2) = \frac{3!}{(3 - 2)!} = \frac{3!}{1!} = 3! = 6$$

This result is exactly the same answer as in part (a). This is because, in the case of $P(3, 3)$, after the first 2 choices are made, the third is already determined, as shown in the table below.

Permutations of Three Letters						
First Two Letters	AB	AC	BA	BC	CA	CB
Third Letter	C	B	C	A	B	A

YOUR TURN 4 Find the number of permutations of the letters L, M, N, O, P, and Q, if just three of the letters are to be used.

TRY YOUR TURN 4

To find $P(n, r)$, we can use either the permutations formula or direct application of the multiplication principle, as the following example shows.

EXAMPLE 7 Politics

In a recent election, eight candidates sought the Democratic nomination for president. In how many ways could voters rank their first, second, and third choices?

Method 1 Calculating By Hand

SOLUTION This is the same as finding the number of permutations of 8 elements taken 3 at a time. Since there are 3 choices to be made, the multiplication principle gives $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$. Alternatively, use the permutations formula to get

$$P(8, 3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot \mathbf{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\mathbf{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = 8 \cdot 7 \cdot 6 = 336.$$

Method 2 Graphing Calculator

Graphing calculators have the capacity to compute permutations. For example, on a TI-84 Plus, $P(8, 3)$ can be calculated by inputting 8 followed by `nPr` (found in the `MATH-PRB` menu), and a 3 yielding 336, as shown in Figure 3.

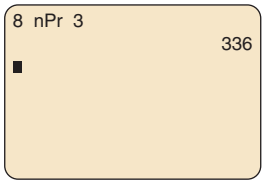


FIGURE 3

Method 3 Spreadsheet

Spreadsheets can also compute permutations. For example, in Microsoft Excel, $P(8, 3)$ can be calculated by inputting 8 and 3 in cells, say, A1 and B1, and then typing "`=FACT(A1)/FACT(A1-B1)`" in cell C1 or, for that matter, any other cell.

CAUTION When calculating the number of permutations with the formula, do not try to cancel unlike factorials. For example,

$$\frac{8!}{4!} \neq 2! = 2 \cdot 1 = 2.$$

$$\frac{8!}{4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \mathbf{4 \cdot 3 \cdot 2 \cdot 1}}{\mathbf{4 \cdot 3 \cdot 2 \cdot 1}} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680.$$

Always write out the factors first, then cancel where appropriate.

EXAMPLE 8 Television Panel

A televised talk show will include 4 women and 3 men as panelists.

(a) In how many ways can the panelists be seated in a row of 7 chairs?

APPLY IT

SOLUTION Find $P(7, 7)$, the total number of ways to seat 7 panelists in 7 chairs.

$$P(7, 7) = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

There are 5040 ways to seat the 7 panelists.

(b) In how many ways can the panelists be seated if the men and women are to be alternated?

SOLUTION Use the multiplication principle. In order to alternate men and women, a woman must be seated in the first chair (since there are 4 women and only 3 men), any of the

men next, and so on. Thus there are 4 ways to fill the first seat, 3 ways to fill the second seat, 3 ways to fill the third seat (with any of the 3 remaining women), and so on. This gives

$$\underset{W_1}{4} \cdot \underset{M_1}{3} \cdot \underset{W_2}{3} \cdot \underset{M_2}{2} \cdot \underset{W_3}{2} \cdot \underset{M_3}{1} \cdot \underset{W_4}{1} = 144$$

ways to seat the panelists.

(c) In how many ways can the panelists be seated if the men must sit together, and the women must also sit together?

SOLUTION Use the multiplication principle. We first must decide how to arrange the two groups (men and women). There are 2! ways of doing this. Next, there are 4! ways of arranging the women and 3! ways of arranging the men, for a total of

$$2!\,4!\,3! = 2 \cdot 24 \cdot 6 = 288$$

ways.

(d) In how many ways can one woman and one man from the panel be selected?

SOLUTION There are 4 ways to pick the woman and 3 ways to pick the man, for a total of

$$4 \cdot 3 = 12$$

ways.

YOUR TURN 5 Two freshmen, 2 sophomores, 2 juniors, and 3 seniors are on a panel. In how many ways can the panelists be seated if each class must sit together?

TRY YOUR TURN 5

If the n objects in a permutation are not all distinguishable—that is, if there are n_1 of type 1, n_2 of type 2, and so on for r different types, then the number of **distinguishable permutations** is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

For example, suppose we want to find the number of permutations of the numbers 1, 1, 4, 4, 4. We cannot distinguish between the two 1's or among the three 4's, so using 5! would give too many distinguishable arrangements. Since the two 1's are indistinguishable and account for 2! of the permutations, we divide 5! by 2!. Similarly, we also divide by 3! to account for the three indistinguishable 4's. This gives

$$\frac{5!}{2!\,3!} = 10$$

permutations.

EXAMPLE 9 Mississippi

In how many ways can the letters in the word *Mississippi* be arranged?

SOLUTION This word contains 1 m, 4 i's, 4 s's, and 2 p's. To use the formula, let $n = 11$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, and $n_4 = 2$ to get

$$\frac{11!}{1!\,4!\,4!\,2!} = 34{,}650$$

arrangements.

YOUR TURN 6 In how many ways can the letters in the word *Tennessee* be arranged?

TRY YOUR TURN 6

NOTE If Example 9 had asked for the number of ways that the letters in a word with 11 *different* letters could be arranged, the answer would be 11! = 39,916,800.

EXAMPLE 10 Yogurt

A student buys 3 cherry yogurts, 2 raspberry yogurts, and 2 blueberry yogurts. She puts them in her dormitory refrigerator to eat one a day for the next week. Assuming yogurts of

YOUR TURN 7 A student has 4 pairs of identical blue socks, 5 pairs of identical brown socks, 3 pairs of identical black socks, and 2 pairs of identical white socks. In how many ways can he select socks to wear for the next two weeks?

the same flavor are indistinguishable, in how many ways can she select yogurts to eat for the next week?

SOLUTION This problem is again one of distinguishable permutations. The 7 yogurts can be selected in 7! ways, but since the 3 cherry, 2 raspberry, and 2 blueberry yogurts are indistinguishable, the total number of distinguishable orders in which the yogurts can be selected is

$$\frac{7!}{3!\,2!\,2!} = 210.$$

TRY YOUR TURN 7

8.1 EXERCISES

In Exercises 1–12, evaluate the factorial or permutation.

1. 6!

2. 7!

3. 15!

4. 16!

5. $P(13, 2)$

6. $P(12, 3)$

7. $P(38, 17)$

8. $P(33, 19)$

9. $P(n, 0)$

10. $P(n, n)$

11. $P(n, 1)$

12. $P(n, n-1)$

13. How many different types of homes are available if a builder offers a choice of 6 basic plans, 3 roof styles, and 2 exterior finishes?

14. A menu offers a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

15. A couple has narrowed down the choice of a name for their new baby to 4 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

16. In a club with 16 members, how many ways can a slate of 3 officers consisting of president, vice-president, and secretary/treasurer be chosen?

17. Define *permutation* in your own words.

18. Explain the difference between *distinguishable* and *indistinguishable* permutations.

19. In Example 6, there are six 3-letter permutations of the letters A, B, and C. How many 3-letter subsets (unordered groups of letters) are there?

20. In Example 6, how many unordered 2-letter subsets of the letters A, B, and C are there?

21. Find the number of distinguishable permutations of the letters in each word.

a. initial **b.** little **c.** decreed

22. A printer has 5 A's, 4 B's, 2 C's, and 2 D's. How many different "words" are possible that use all these letters? (A "word" does not have to have any meaning here.)

23. Wing has different books to arrange on a shelf: 4 blue, 3 green, and 2 red.

a. In how many ways can the books be arranged on a shelf?

b. If books of the same color are to be grouped together, how many arrangements are possible?

c. In how many distinguishable ways can the books be arranged if books of the same color are identical but need not be grouped together?

d. In how many ways can you select 3 books, one of each color, if the order in which the books are selected does not matter?

e. In how many ways can you select 3 books, one of each color, if the order in which the books are selected matters?

24. A child has a set of differently shaped plastic objects. There are 3 pyramids, 4 cubes, and 7 spheres.

a. In how many ways can she arrange the objects in a row if each is a different color?

b. How many arrangements are possible if objects of the same shape must be grouped together and each object is a different color?

c. In how many distinguishable ways can the objects be arranged in a row if objects of the same shape are also the same color but need not be grouped together?

d. In how many ways can you select 3 objects, one of each shape, if the order in which the objects are selected does not matter and each object is a different color?

e. In how many ways can you select 3 objects, one of each shape, if the order in which the objects are selected matters and each object is a different color?

25. If you already knew the value of 9!, how could you find the value of 10! quickly?

26. Given that 450! is approximately equal to $1.7333687 \times 10^{1000}$ (to 8 digits of accuracy), find 451! to 7 digits of accuracy.

27. When calculating $n!$, the number of ending zeros in the answer can be determined prior to calculating the actual number by finding the number of times 5 can be factored from $n!$. For example, 7! only has one 5 occurring in its calculation, and so there is only one ending zero in 5040. The number 10! has two 5's (one from the 5 and one from the 10) and so there must be two ending zeros in the answer 3,628,800. Use this idea to determine the number of zeros that occur in the following factorials, and then explain why this works.

a. 13! **b.** 27! **c.** 75!

28. Because of the view screen, calculators only show a fixed number of digits, often 10 digits. Thus, an approximation of a

number will be shown by only including the 10 largest place values of the number. Using the ideas from the previous exercise, determine if the following numbers are correct or if they are incorrect by checking if they have the correct number of ending zeros. (*Note:* Just because a number has the correct number of zeros does not imply that it is correct.)

a. $12! = 479{,}001{,}610$

b. $23! = 25{,}852{,}016{,}740{,}000{,}000{,}000{,}000$

c. $15! = 1{,}307{,}643{,}680{,}000$

d. $14! = 87{,}178{,}291{,}200$

29. Some students find it puzzling that $0! = 1$, and think that $0!$ should equal 0. If this were true, what would be the value of $P(4, 4)$ using the permutations formula?

APPLICATIONS

Business and Economics

30. Automobile Manufacturing An automobile manufacturer produces 8 models, each available in 7 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

31. Marketing In a recent marketing campaign, Olive Garden Italian Restaurant® offered a "Never-Ending Pasta Bowl." The customer could order an array of pasta dishes, selecting from 7 types of pasta and 6 types of sauce, including 2 with meat.

a. If the customer selects one pasta type and one sauce type, how many different "pasta bowls" can a customer order?

b. How many different "pasta bowls" can a customer order without meat?

32. Investments Natalie Graham's financial advisor has given her a list of 9 potential investments and has asked her to select and rank her favorite five. In how many different ways can she do this?

33. Scheduling A local television station has eleven slots for commercials during a special broadcast. Six restaurants and 5 stores have bought slots for the broadcast.

a. In how many ways can the commercials be arranged?

b. In how many ways can the commercials be arranged so that the restaurants are grouped together and the stores are grouped together?

c. In how many ways can the commercials be arranged so that the restaurant and store commercials are alternating?

Life Sciences

34. Drug Sequencing Twelve drugs have been found to be effective in the treatment of a disease. It is believed that the sequence in which the drugs are administered is important in the effectiveness of the treatment. In how many different sequences can 5 of the 12 drugs be administered?

35. Insect Classification A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

36. Science Conference At an annual college science conference, student presentations are scheduled one after another in the afternoon session. This year, 5 students are presenting in biology, 5 students are presenting in chemistry, and 2 students are presenting in physics.

a. In how many ways can the presentations be scheduled?

b. In how many ways can the presentations be scheduled so that each subject is grouped together?

c. In how many ways can the presentations be scheduled if the conference must begin and end with a physics presentation?

Social Sciences

37. Social Science Experiment In an experiment on social interaction, 6 people will sit in 6 seats in a row. In how many ways can this be done?

38. Election Ballots In an election with 3 candidates for one office and 6 candidates for another office, how many different ballots may be printed?

General Interest

39. Baseball Teams A baseball team has 19 players. How many 9-player batting orders are possible?

40. Union Elections A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

41. Programming Music A concert to raise money for an economics prize is to consist of 5 works: 2 overtures, 2 sonatas, and a piano concerto.

a. In how many ways can the program be arranged?

b. In how many ways can the program be arranged if an overture must come first?

42. Programming Music A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if

a. they begin with a traditional piece?

b. an original piece will be played last?

43. Radio Station Call Letters How many different 4-letter radio station call letters can be made if

a. the first letter must be K or W and no letter may be repeated?

b. repeats are allowed, but the first letter is K or W?

c. the first letter is K or W, there are no repeats, and the last letter is R?

44. Telephone Numbers How many 7-digit telephone numbers are possible if the first digit cannot be zero and

a. only odd digits may be used?

b. the telephone number must be a multiple of 10 (that is, it must end in zero)?

c. the telephone number must be a multiple of 100?

d. the first 3 digits are 481?

e. no repetitions are allowed?

Telephone Area Codes **Several years ago, the United States began running out of telephone numbers. Telephone companies introduced new area codes as numbers were used up, and eventually almost all area codes were used up.**

45. a. Until recently, all area codes had a 0 or 1 as the middle digit, and the first digit could not be 0 or 1. How many area

codes are there with this arrangement? How many telephone numbers does the current 7-digit sequence permit per area code? (The 3-digit sequence that follows the area code cannot start with 0 or 1. Assume there are no other restrictions.)

b. The actual number of area codes under the previous system was 152. Explain the discrepancy between this number and your answer to part a.

46. The shortage of area codes was avoided by removing the restriction on the second digit. (This resulted in problems for some older equipment, which used the second digit to determine that a long-distance call was being made.) How many area codes are available under the new system?

47. License Plates For many years, the state of California used 3 letters followed by 3 digits on its automobile license plates.

a. How many different license plates are possible with this arrangement?

b. When the state ran out of new numbers, the order was reversed to 3 digits followed by 3 letters. How many new license plate numbers were then possible?

c. Several years ago, the numbers described in b were also used up. The state then issued plates with 1 letter followed by 3 digits and then 3 letters. How many new license plate numbers will this provide?

48. Social Security Numbers A social security number has 9 digits. How many social security numbers are there? The U.S. population in 2010 was about 309 million. Is it possible for every U.S. resident to have a unique social security number? (Assume no restrictions.)

49. Postal Zip Codes The U.S. Postal Service currently uses 5-digit zip codes in most areas. How many zip codes are possible if there are no restrictions on the digits used? How many would be possible if the first number could not be 0?

50. Postal Zip Codes The U.S. Postal Service is encouraging the use of 9-digit zip codes in some areas, adding 4 digits after the usual 5-digit code. How many such zip codes are possible with no restrictions?

51. Games The game of Sets uses a special deck of cards. Each card has either one, two, or three identical shapes, all of the same color and style. There are three possible shapes: squiggle, diamond, and oval. There are three possible colors: green, purple, and red. There are three possible styles: solid, shaded, or outline. The deck consists of all possible combinations of shape, color, style, and number of shapes. How many cards are in the deck? ***Source: Sets.***

52. Games In the game of Scattergories, the players take 12 turns. In each turn, a 20-sided die is rolled; each side has a letter. The players must then fill in 12 categories (e.g., vegetable, city, etc.) with a word beginning with the letter rolled. Considering that a game consists of 12 rolls of the 20-sided die, and that rolling the same side more than once is allowed, how many possible games are there? ***Source: Milton Bradley.***

53. Games The game of Twenty Questions consists of asking 20 questions to determine a person, place, or thing that the other person is thinking of. The first question, which is always "Is it an animal, vegetable, or mineral?" has three possible answers. All the other questions must be answered "Yes" or "No." How many possible objects can be distinguished in this game, assuming that all 20 questions are asked? Are 20 questions enough?

54. Traveling Salesman In the famous Traveling Salesman Problem, a salesman starts in any one of a set of cities, visits every city in the set once, and returns to the starting city. He would like to complete this circuit with the shortest possible distance.

a. Suppose the salesman has 10 cities to visit. Given that it does not matter what city he starts in, how many different circuits can he take?

b. The salesman decides to check all the different paths in part a to see which is shortest, but realizes that a circuit has the same distance whichever direction it is traveled. How many different circuits must he check?

c. Suppose the salesman has 70 cities to visit. Would it be feasible to have a computer check all the different circuits? Explain your reasoning.

55. Circular Permutations Circular permutations arise in applications involving arrangements around a closed loop, as in the previous exercise. Here are two examples.

a. A ferris wheel has 20 seats. How many ways can 20 students arrange themselves on the ferris wheel if each student takes a different seat? We consider two arrangements to be identical if they differ only by rotations of the wheel.

b. A necklace is to be strung with 15 beads, each of a different color. In how many ways can the beads be arranged? We consider two arrangements to be identical if they differ only by rotations of the necklace or by flipping the necklace over. (*Hint:* If every arrangement is counted twice, the correct number of arrangements can be found by dividing by 2.)

YOUR TURN ANSWERS

1. 10,000; 5040 **2.** 40,320 **3.** 6720 **4.** 120
5. 1152 **6.** 3780 **7.** 2,522,520

8.2 Combinations

APPLY IT **In how many ways can a manager select 4 employees for promotion from 12 eligible employees?**

As we shall see in Example 5, permutations alone cannot be used to answer this question, but combinations will provide the answer.

In the previous section, we saw that there are 60 ways that a teacher can arrange 3 of 5 different books on his desk. That is, there are 60 permutations of 5 books taken 3 at a time.

Suppose now that the teacher does not wish to arrange the books on his desk but rather wishes to choose, without regard to order, any 3 of the 5 books for a book sale to raise money for his school. In how many ways can this be done?

At first glance, we might say 60 again, but this is incorrect. The number 60 counts all possible *arrangements* of 3 books chosen from 5. The following 6 arrangements, however, would all lead to the same set of 3 books being given to the book sale.

mystery-biography-textbook	biography-textbook-mystery
mystery-textbook-biography	textbook-biography-mystery
biography-mystery-textbook	textbook-mystery-biography

The list shows 6 different *arrangements* of 3 books, but only one *subset* of 3 books. A subset of items listed *without regard to order* is called a **combination**. The number of combinations of 5 things taken 3 at a time is written $C(5, 3)$.* Since they are subsets, combinations are *not ordered.*

To evaluate $C(5, 3)$, start with the $5 \cdot 4 \cdot 3$ *permutations* of 5 things taken 3 at a time. Since combinations are not ordered, find the number of combinations by dividing the number of permutations by the number of ways each group of 3 can be ordered; that is, divide by 3!.

$$C(5, 3) = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

There are 10 ways that the teacher can choose 3 books for the book sale.

Generalizing this discussion gives the following formula for the number of combinations of n elements taken r at a time:

$$\boldsymbol{C(n, r) = \frac{P(n, r)}{r!}}.$$

Another version of this formula is found as follows.

$$\begin{aligned} C(n, r) &= \frac{\boldsymbol{P(n, r)}}{r!} \\ &= \frac{\boldsymbol{n!}}{\boldsymbol{(n - r)!}} \cdot \frac{1}{r!} \qquad P(n, r) = \frac{\boldsymbol{n!}}{\boldsymbol{(n - r)!}}. \\ &= \frac{n!}{(n - r)!\, r!} \end{aligned}$$

The steps above lead to the following result.

Combinations

If $C(n, r)$, denotes the number of combinations of n elements taken r at a time, where $r \le n$, then

$$\boldsymbol{C(n, r) = \frac{n!}{(n - r)!\, r!}}.$$

EXAMPLE 1 Committees

How many committees of 3 people can be formed from a group of 8 people?

Method 1 Calculating By Hand

SOLUTION A committee is an unordered group, so use the combinations formula for $C(8, 3)$.

$$C(8, 3) = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6 \cdot \boldsymbol{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\boldsymbol{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

*Other common notations for $C(n, r)$ are ${}_nC_r$, C_r^n, and $\binom{n}{r}$.

Method 2 Graphing Calculator

Graphing calculators have the capacity to compute combinations. For example, on a TI-84 Plus, $C(8, 3)$ can be calculated by inputting 8 followed by `nCr` (found in the `MATH-PRB` menu) and a 3 yielding 56, as shown in Figure 4.

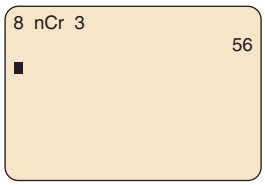


FIGURE 4

Method 3 Spreadsheet

Spreadsheets can also compute combinations. For example, in Microsoft Excel, $C(8, 3)$ can be calculated by inputting 8 and 3 in cells, say, A1 and B1, and then typing "`=FACT(A1)/(FACT(A1-B1)*FACT(B1))`" in cell C1 or, for that matter, any other cell. The command "`= MULTINOMIAL(5, 3)`" also gives this answer.

YOUR TURN 1 How many committees of 4 people can be formed from a group of 10 people?

TRY YOUR TURN 1

Example 1 shows an alternative way to compute $C(n, r)$. Take r or $n - r$, whichever is smaller. Write the factorial of this number in the denominator. In the numerator, write out a sufficient number of factors of $n!$ so there is one factor in the numerator for each factor in the denominator. For example, to calculate $C(8, 3)$ or $C(8, 5)$ write

$$\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$

The factors that are omitted (written in color in Example 1) cancel out of the numerator and denominator, so need not be included.

Notice from the previous discussion that $C(8, 3) = C(8, 5)$. (See Exercise 25 for a generalization of this idea.) One interpretation of this fact is that the number of ways to form a committee of 3 people chosen from a group of 8 is the same as the number of ways to choose the 5 people who are not on the committee.

Notice that this is *not* true with permutations: $P(8, 3) \neq P(8, 5)$.

EXAMPLE 2 Lawyers

Three lawyers are to be selected from a group of 30 to work on a special project.

(a) In how many different ways can the lawyers be selected?

SOLUTION Here we wish to know the number of 3-element combinations that can be formed from a set of 30 elements. (We want combinations, not permutations, since order within the group of 3 doesn't matter.)

$$C(30, 3) = \frac{30!}{27!3!} = \frac{30 \cdot 29 \cdot 28 \cdot 27!}{27! \cdot 3 \cdot 2 \cdot 1} \qquad 30! = 30 \cdot 29 \cdot 28 \cdot 27!$$

$$= \frac{30 \cdot 29 \cdot 28}{3 \cdot 2 \cdot 1}$$

$$= 4060$$

There are 4060 ways to select the project group.

(b) In how many ways can the group of 3 be selected if a certain lawyer must work on the project?

SOLUTION Since 1 lawyer already has been selected for the project, the problem is reduced to selecting 2 more from the remaining 29 lawyers.

$$C(29, 2) = \frac{29!}{27!\,2!} = \frac{29 \cdot 28 \cdot 27!}{27! \cdot 2 \cdot 1} = \frac{29 \cdot 28}{2 \cdot 1} = 29 \cdot 14 = 406$$

In this case, the project group can be selected in 406 ways.

(c) In how many ways can a nonempty group of at most 3 lawyers be selected from these 30 lawyers?

SOLUTION Here, by "at most 3" we mean "1 or 2 or 3." (The number 0 is excluded because the group is nonempty.) Find the number of ways for each case.

Case	Number of Ways
1	$C(30, 1) = \frac{30!}{29!\,1!} = \frac{30 \cdot 29!}{29!\,(1)} = 30$
2	$C(30, 2) = \frac{30!}{28!\,2!} = \frac{30 \cdot 29 \cdot 28!}{28! \cdot 2 \cdot 1} = 435$
3	$C(30, 3) = \frac{30!}{27!\,3!} = \frac{30 \cdot 29 \cdot 28 \cdot 27!}{27! \cdot 3 \cdot 2 \cdot 1} = 4060$

The total number of ways to select at most 3 lawyers will be the sum

$$30 + 435 + 4060 = 4525.$$

YOUR TURN 2 From a class of 15 students, a group of 3 or 4 students will be selected to work on a special project. In how many ways can a group of 3 or 4 students be selected?

TRY YOUR TURN 2

EXAMPLE 3 Sales

A salesman has 10 accounts in a certain city.

(a) In how many ways can he select 3 accounts to call on?

SOLUTION Within a selection of 3 accounts, the arrangement of the calls is not important, so there are

$$C(10, 3) = \frac{10!}{7!\,3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

ways he can make a selection of 3 accounts.

(b) In how many ways can he select at least 8 of the 10 accounts to use in preparing a report?

SOLUTION "At least 8" means "8 or more," which is "8 or 9 or 10." First find the number of ways to choose in each case.

Case	Number of Ways
8	$C(10, 8) = \frac{10!}{2!\,8!} = \frac{10 \cdot 9}{2 \cdot 1} = 45$
9	$C(10, 9) = \frac{10!}{1!\,9!} = \frac{10}{1} = 10$
10	$C(10, 10) = \frac{10!}{0!\,10!} = 1$

He can select at least 8 of the 10 accounts in $45 + 10 + 1 = 56$ ways.

FOR REVIEW

Notice in Example 3 that to calculate the number of ways to select 8 or 9 or 10 accounts, we added the three numbers found. The union rule for sets from Chapter 7 says that when A and B are disjoint sets, the number of elements in A or B is the number of elements in A plus the number in B.

CAUTION When we are making a first decision *and* a second decision, we *multiply* to find the total number of ways. When we are making a decision in which the first choice *or* the second choice are valid choices, we *add* to find the total number of ways.

The formulas for permutations and combinations given in this section and in the previous section will be very useful in solving probability problems in the next section. Any difficulty in using these formulas usually comes from being unable to differentiate between them. Both permutations and combinations give the number of ways to choose r objects from a set of n objects. The differences between permutations and combinations are outlined in the following table.

Permutations	Combinations
Different orderings or arrangements of the r objects are different permutations.	Each choice or subset of r objects gives one combination. Order within the group of r objects does not matter.
$P(n, r) = \dfrac{n!}{(n-r)!}$	$C(n, r) = \dfrac{n!}{(n-r)!\,r!}$
Clue words: arrangement, schedule, order	Clue words: group, committee, set, sample
Order matters!	**Order does not matter!**

In the next examples, concentrate on recognizing which formula should be applied.

EXAMPLE 4 Permutations and Combinations

For each problem, tell whether permutations or combinations should be used to solve the problem.

(a) How many 4-digit code numbers are possible if no digits are repeated?

SOLUTION Since changing the order of the 4 digits results in a different code, use permutations.

(b) A sample of 3 light bulbs is randomly selected from a batch of 15. How many different samples are possible?

SOLUTION The order in which the 3 light bulbs are selected is not important. The sample is unchanged if the items are rearranged, so combinations should be used.

(c) In a baseball conference with 8 teams, how many games must be played so that each team plays every other team exactly once?

SOLUTION Selection of 2 teams for a game is an *unordered* subset of 2 from the set of 8 teams. Use combinations again.

(d) In how many ways can 4 patients be assigned to 6 different hospital rooms so that each patient has a private room?

SOLUTION The room assignments are an *ordered* selection of 4 rooms from the 6 rooms. Exchanging the rooms of any 2 patients within a selection of 4 rooms gives a different assignment, so permutations should be used. TRY YOUR TURN 3

YOUR TURN 3 Solve the problems in Example 4.

EXAMPLE 5 Promotions

A manager must select 4 employees for promotion; 12 employees are eligible.

(a) In how many ways can the 4 be chosen?

SOLUTION Since there is no reason to differentiate among the 4 who are selected, use combinations.

$$C(12, 4) = \frac{12!}{8!\,4!} = 495$$

(b) In how many ways can 4 employees be chosen (from 12) to be placed in 4 different jobs?

SOLUTION In this case, once a group of 4 is selected, they can be assigned in many different ways (or arrangements) to the 4 jobs. Therefore, this problem requires permutations.

$$P(12, 4) = \frac{12!}{8!} = 11{,}880$$

TRY YOUR TURN 4

APPLY IT

YOUR TURN 4 In how many ways can a committee of 3 be chosen from a group of 20 students? In how many ways can three officers (president, treasurer, and secretary) be selected from a group of 20 students?

FOR REVIEW

Example 6 involves a standard deck of 52 playing cards, as shown in Figure 17 in Chapter 7. Recall the discussion that accompanies the photograph.

EXAMPLE 6 Playing Cards

Five cards are dealt from a standard 52-card deck.

(a) How many such hands have only face cards?

SOLUTION The face cards are the king, queen, and jack of each suit. Since there are 4 suits, there are 12 face cards. The arrangement of the 5 cards is not important, so use combinations to get

$$C(12, 5) = \frac{12!}{7!\,5!} = 792.$$

(b) How many such hands have a full house of aces and eights (3 aces and 2 eights)?

SOLUTION The arrangement of the 3 aces or the 2 eights does not matter, so we use combinations. There are $C(4, 3)$ ways to get 3 aces from the four aces in the deck, and $C(4, 2)$ ways to get 2 eights. By the multiplication principle we get

$$C(4, 3) \cdot C(4, 2) = 4 \cdot 6 = 24.$$

(c) How many such hands have exactly 2 hearts?

SOLUTION There are 13 hearts in the deck, so the 2 hearts will be selected from those 13 cards. The other 3 cards must come from the remaining 39 cards that are not hearts. Use combinations and the multiplication principle to get

$$C(\mathbf{13}, \mathbf{2}) \cdot C(\mathbf{39}, \mathbf{3}) = 78 \cdot 9139 = 712{,}842.$$

Notice that the two numbers in red in the combinations add up to 52, the total number of cards, and the two numbers in blue add up to 5, the number of cards in a hand.

(d) How many such hands have cards of a single suit?

SOLUTION Since the arrangement of the 5 cards is not important, use combinations. The total number of ways that 5 cards of a particular suit of 13 cards can occur is $C(13, 5)$. There are four different suits, so the multiplication principle gives

$$4 \cdot C(13, 5) = 4 \cdot 1287 = 5148$$

ways to deal 5 cards of the same suit. TRY YOUR TURN 5

YOUR TURN 5 How many five card hands have exactly 2 aces?

As Example 6 shows, often both combinations and the multiplication principle must be used in the same problem.

EXAMPLE 7 Soup

To illustrate the differences between permutations and combinations in another way, suppose 2 cans of soup are to be selected from 4 cans on a shelf: noodle (N), bean (B), mushroom (M), and tomato (T). As shown in Figure 5(a), there are 12 ways to select 2 cans from the 4 cans if the order matters (if noodle first and bean second is considered different from bean, then

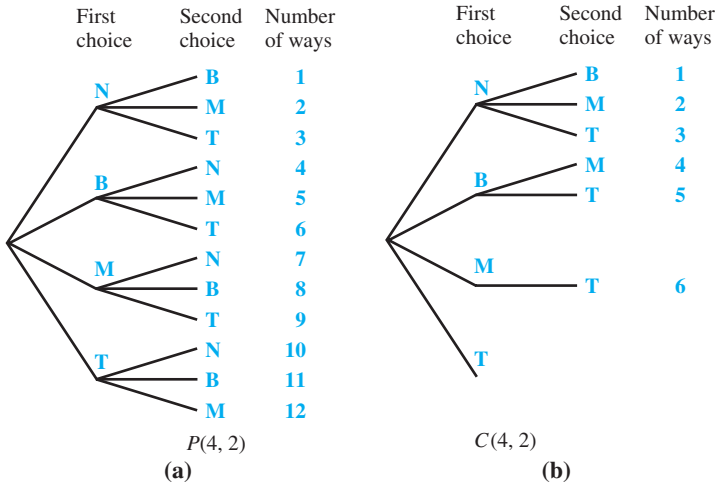


FIGURE 5

noodle, for example). On the other hand, if order is unimportant, then there are 6 ways to choose 2 cans of soup from the 4, as illustrated in Figure 5(b).

CAUTION It should be stressed that not all counting problems lend themselves to either permutations or combinations. Whenever a tree diagram or the multiplication principle can be used directly, it's often best to use it.

8.2 EXERCISES

1. Define combinations in your own words.

2. Explain the difference between a permutation and a combination.

Evaluate each combination.

3. $C(8, 3)$

4. $C(12, 5)$

5. $C(44, 20)$

6. $C(40, 18)$

7. $C(n, 0)$

8. $C(n, n)$

9. $C(n, 1)$

10. $C(n, n - 1)$

11. In how many ways can a hand of 6 clubs be chosen from an ordinary deck?

12. In how many ways can a hand of 6 red cards be chosen from an ordinary deck?

13. Five cards are marked with the numbers 1, 2, 3, 4, and 5, then shuffled, and 2 cards are drawn.

a. How many different 2-card combinations are possible?

b. How many 2-card hands contain a number less than 3?

14. An economics club has 31 members.

a. If a committee of 4 is to be selected, in how many ways can the selection be made?

b. In how many ways can a committee of at least 1 and at most 3 be selected?

15. Use a tree diagram for the following.

a. Find the number of ways 2 letters can be chosen from the set {L, M, N} if order is important and repetition is allowed.

b. Reconsider part a if no repeats are allowed.

c. Find the number of combinations of 3 elements taken 2 at a time. Does this answer differ from part a or b?

16. Repeat Exercise 15 using the set {L, M, N, P}.

In Exercises 17–24, decide whether each exercise involves permutations or combinations, and then solve the problem.

17. In a club with 9 male and 11 female members, how many 5-member committees can be chosen that have

a. all men?

b. all women?

c. 3 men and 2 women?

18. In Exercise 17, how many committees can be selected that have

a. at least 4 women?

b. no more than 2 men?

19. In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

20. A group of 3 students is to be selected from a group of 14 students to take part in a class in cell biology.

a. In how many ways can this be done?

b. In how many ways can the group who will not take part be chosen?

21. Marbles are being drawn without replacement from a bag containing 16 marbles.

a. How many samples of 2 marbles can be drawn?

b. How many samples of 4 marbles can be drawn?

c. If the bag contains 3 yellow, 4 white, and 9 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

22. There are 7 rotten apples in a crate of 26 apples.

a. How many samples of 3 apples can be drawn from the crate?

b. How many samples of 3 could be drawn in which all 3 are rotten?

c. How many samples of 3 could be drawn in which there are two good apples and one rotten one?

23. A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

a. all black?

b. all red?

c. all yellow?

d. 2 black and 1 red?

e. 2 black and 1 yellow?

f. 2 yellow and 1 black?

g. 2 red and 1 yellow?

24. In how many ways can 5 out of 9 plants be arranged in a row on a windowsill?

25. Show that $C(n, r) = C(n, n - r)$.

26. Padlocks with digit dials are often referred to as "combination locks." According to the mathematical definition of combination, is this an accurate description? Explain.

27. The following problem was posed on National Public Radio's *Weekend Edition*: In how many points can 6 circles intersect? ***Source: National Public Radio.***

a. Find the answer for 6 circles.

b. Find the general answer for n circles.

28. How many different dominoes can be formed from the numbers 0…6? (*Hint:* A domino may have the same number of dots on both halves of it or it may have a different number of dots on each half.)

APPLICATIONS

Business and Economics

29. Work Assignments From a group of 8 newly hired office assistants, 3 are selected. Each of these 3 assistants will be assigned to a different manager. In how many ways can they be selected and assigned?

30. Assembly Line Sampling Five items are to be randomly selected from the first 50 items on an assembly line to determine the defect rate. How many different samples of 5 items can be chosen?

31. Sales Schedules A salesperson has the names of 6 prospects.

a. In how many ways can she arrange her schedule if she calls on all 6?

b. In how many ways can she arrange her schedule if she can call on only 4 of the 6?

32. Worker Grievances A group of 9 workers decides to send a delegation of 3 to their supervisor to discuss their grievances.

a. How many delegations are possible?

b. If it is decided that a particular worker must be in the delegation, how many different delegations are possible?

c. If there are 4 women and 5 men in the group, how many delegations would include at least 1 woman?

33. Hamburger Variety Hamburger Hut sells regular hamburgers as well as a larger burger. Either type can include cheese, relish, lettuce, tomato, mustard, or catsup.

a. How many different hamburgers can be ordered with exactly three extras?

b. How many different regular hamburgers can be ordered with exactly three extras?

c. How many different regular hamburgers can be ordered with at least five extras?

34. Ice Cream Flavors Baskin-Robbins advertises that it has 31 flavors of ice cream.

a. How many different double-scoop cones can be made? Assume that the order of the scoops matters.

b. How many different triple-scoop cones can be made?

c. How many different double-scoop cones can be made if order doesn't matter?

Life Sciences

35. Research Participants From a group of 16 smokers and 22 nonsmokers, a researcher wants to randomly select 8 smokers and 8 nonsmokers for a study. In how many ways can the study group be selected?

36. Plant Hardiness In an experiment on plant hardiness, a researcher gathers 6 wheat plants, 3 barley plants, and 2 rye plants. She wishes to select 4 plants at random.

a. In how many ways can this be done?

b. In how many ways can this be done if exactly 2 wheat plants must be included?

Social Sciences

37. Legislative Committee A legislative committee consists of 5 Democrats and 4 Republicans. A delegation of 3 is to be selected to visit a small Pacific island republic.

a. How many different delegations are possible?

b. How many delegations would have all Democrats?

c. How many delegations would have 2 Democrats and 1 Republican?

d. How many delegations would include at least 1 Republican?

38. Political Committee From 10 names on a ballot, 4 will be elected to a political party committee. In how many ways can the committee of 4 be formed if each person will have a different responsibility, and different assignments of responsibility are considered different committees?

39. Judges When Paul Martinek, publisher of *Lawyers Weekly USA*, was a guest on the television news program *The O'Reilly Factor*, he discussed a decision by a three-judge panel, chosen at random from judges in the Ninth Circuit in California. The judges had ruled that the mandatory recitation of the Pledge of Allegience is unconstitutional because of the phrase "under God." According to Martinek, "Because there are 45 judges in the Ninth Circuit, there are 3000 different combinations of three-judge panels." Is this true? If not, what is the correct number? ***Source: Mathematics Teacher.***

General Interest

40. Bridge How many different 13-card bridge hands can be selected from an ordinary deck?

41. Poker Five cards are chosen from an ordinary deck to form a hand in poker. In how many ways is it possible to get the following results?

a. 4 queens **b.** No face card

c. Exactly 2 face cards **d.** At least 2 face cards

e. 1 heart, 2 diamonds, and 2 clubs

42. Poker In poker, a flush consists of 5 cards with the same suit, such as 5 diamonds.

a. Find the number of ways of getting a flush consisting of cards with values from 5 to 10 by listing all the possibilities.

b. Find the number of ways of getting a flush consisting of cards with values from 5 to 10 by using combinations.

43. Baseball If a baseball coach has 5 good hitters and 4 poor hitters on the bench and chooses 3 players at random, in how many ways can he choose at least 2 good hitters?

44. Softball The coach of the Morton Valley Softball Team has 6 good hitters and 8 poor hitters. He chooses 3 hitters at random.

a. In how many ways can he choose 2 good hitters and 1 poor hitter?

b. In how many ways can he choose 3 good hitters?

c. In how many ways can he choose at least 2 good hitters?

45. Flower Selection Five orchids from a collection of 20 are to be selected for a flower show.

a. In how many ways can this be done?

b. In how many ways can the 5 be selected if 2 special plants from the 20 must be included?

46. Lottery A state lottery game requires that you pick 6 different numbers from 1 to 99. If you pick all 6 winning numbers, you win the jackpot.

a. How many ways are there to choose 6 numbers if order is not important?

b. How many ways are there to choose 6 numbers if order matters?

47. Lottery In Exercise 46, if you pick 5 of the 6 numbers correctly, you win $250,000. In how many ways can you pick exactly 5 of the 6 winning numbers without regard to order?

48. Committees Suppose that out of 19 members of a club, two committees are to be formed. A nominating committee is to consist of 7 members, and a public relations committee is to consist of 5 members. No one can be on both committees.

a. Calculate the number of ways that the two committees can be formed, assuming that the nominating committee is formed first.

b. Calculate the number of ways that the two committees can be formed, assuming that the public relations committee is formed first. Verify that this answer is the same as that of part a.

c. Suppose the 7 members of the nominating committee wear red T-shirts, the 5 members of the public relations committee wear yellow T-shirts, and the remaining members of the club wear white T-shirts. A photographer lines up the members of the club to take a picture, but the picture is so blurry that people wearing the same color T-shirt are indistinguishable. In how many distinguishable ways can the club members line up? Explain why this answer is the same as the answers to parts a and b.

49. Committee A small department of 5 people decides to form a hiring committee. The only restriction on the size of the committee is that it must have at least 2 members.

a. Calculate the number of different committees possible by adding up the number of committees of different sizes.

b. Calculate the number of different committees possible by taking the total number of subsets of the 5 members and subtracting the number of committees that are invalid because they have too few members.

50. License Plates Officials from a particular state are considering a new type of license plate consisting of three letters followed by three numbers. If the letters cannot be repeated and must be in alphabetical order, calculate the number of possible distinct license plates.

51. Passwords A certain website requires users to log on using a security password.

a. If passwords must consist of six letters, followed by a single digit, determine the total number of possible distinct passwords.

b. If passwords must consist of six non-repetitive letters, followed by a single digit, determine the total number of possible distinct passwords.

52. Pizza Varieties A television commercial for Little Caesars pizza announced that with the purchase of two pizzas, one could receive free any combination of up to five toppings on each pizza. The commercial shows a young child waiting in line at Little Caesars who calculates that there are 1,048,576 possibilities for the toppings on the two pizzas. ***Source: The Mathematics Teacher.***

a. Verify the child's calculation. Use the fact that Little Caesars has 11 toppings to choose from. Assume that the order of the two pizzas matters; that is, if the first pizza has combination 1 and the second pizza has combination 2, that is different from combination 2 on the first pizza and combination 1 on the second.

b. In a letter to *The Mathematics Teacher*, Joseph F. Heiser argued that the two combinations described in part a should be counted as the same, so the child has actually overcounted. Give the number of possibilities if the order of the two pizzas doesn't matter.

53. Pizza In an ad for Pizza Hut, Jessica Simpson explains to the Muppets that there are more than 6 million possibilities for their 4forAll Pizza. Griffin Weber and Glenn Weber wrote an article explaining that the number of possibilities is far more than 6 million, as described below. ***Source: The College Mathematics Journal.***

a. Each pizza can have up to 3 toppings, out of 17 possible choices, or can be one of four specialty pizzas. Calculate the number of different pizzas possible.

b. Out of the total possible pizzas calculated in the first part of this exercise, a 4forAll Pizza consists of four pizzas in a box. Keeping in mind that the four pizzas could all be different, or there could be two or three different pizzas in the box, or all four pizzas could be the same, calculate the total number of 4forAll Pizzas possible.

c. The article considers another way of counting the number in part b. Suppose that only 8 pizzas were available, and they were listed in a row with lines separating each type, as in the following diagram:

$$\mathrm{A}|\mathrm{B}|\mathrm{C}|\mathrm{D}|\mathrm{E}|\mathrm{F}|\mathrm{G}|\mathrm{H}.$$

A person orders 4 pizzas by placing 4 X's in the desired places on the diagram, after which the letters can be ignored. For example, an order for 2 of A, 1 of C, and 1 of G would look like the following diagram.

$$\mathrm{XX}||\mathrm{X}|||\mathrm{X}|$$

The number of ways this can be done is then the number of ways of arranging 11 objects, 4 of which are X and the other 7 of which are vertical lines, or

$$C(11, 4) = 330.$$

Use similar reasoning to verify the answer to part b.

54. Cereal The Post Corporation once introduced the cereal, *Create a Crunch*™, in which the consumers could combine ingredients to create their own unique cereal. Each box contained 8 packets of food goods. There were four types of cereal: Frosted Alpha Bits®, Cocoa Pebbles®, Fruity Pebbles®, and

Honey Comb®. Also included in the box were four "Add-Ins": granola, blue rice cereal, marshmallows, and sprinkles.

a. What is the total number of breakfasts that could be made if a breakfast is defined as any one or more cereals or add-ins?

b. If Sara Ouellette chose to mix one type of cereal with one add-in, how many different breakfasts could she make?

c. If Kristen Schmitt chose to mix two types of cereal with three add-ins, how many different breakfasts could she make?

d. If Matthew Pastier chose to mix at least one type of cereal with at least one type of add-in, how many breakfasts could he make?

e. If Inga Moffitt's favorite cereal is Fruity Pebbles®, how many different cereals could she make if each of her mixtures must include this cereal?

55. Appetizers Applebee's restaurant recently advertised "Ultimate Trios," where the customer was able to pick any three trio-sized appetizers from a list of nine options. The ad claimed that there were over 200 combinations.

a. If each customer's selection must be a different item, how many meal combinations are possible?

b. If each customer can select the same item two or even three times in each trio, how many different trios are possible?

c. Using the answers to parts a and b, discuss that restaurant's claim.

d. Two of the trio items, the Buffalo chicken wings and the boneless Buffalo wings, each have five different sauce options. This implies that there are actually 17 different trio choices that are available to a customer. In this scenario, how many different trio meal combinations are possible? (Assume that each of the three selected items is different.)

e. How many different trios are possible if 2 of the items are different flavored boneless Buffalo wings? (Assume that the third item is not a boneless Buffalo wing.)

56. Football Writer Gregg Easterbrook, discussing ESPN's unsuccessful attempt to predict the winners for the six National Football League (NFL) divisions and the six wild-card slots, claimed that there were 180 different ways to make this forecast. Reader Milton Eisner wrote in to tell him that the actual number is much larger. To make the calculation, note that at the time the NFL consisted of two conferences, each of which consisted of three divisions. Five of the divisions had five teams, while the other had six. There was one winner from each of the six divisions, plus three wild-card slots from each of the two conferences. How many ways could the six division winners and six wild-card slots have been chosen? ***Source: Slate Magazine.***

57. Music In the opera *Amahl and the Night Visitors*, the shepherds sing a chorus involving 18 different names, a challenge for singers trying to remember the names in the correct order. (Two of the three authors of this textbook have sung this chorus in public.)

a. In how many ways can the names be arranged?

b. Not all the arrangements of names in part a could be sung, because 10 of the names have 3 syllables, 4 have 2 syllables, and 4 have 4 syllables. Of the 6 lines in the chorus, 4 lines consist of a 3-syllable name repeated, followed by a 2-syllable and then a 4-syllable name (e.g., Emily, Emily, Michael, Bartholomew), and 2 lines consist of a 3-syllable name repeated, followed by two more 3-syllable names (e.g., Josephine, Josephine, Angela, Jeremy). No names are repeated except where we've indicated. (If you think this is confusing, you should try memorizing the chorus.) How many arrangements of the names could fit this pattern?

58. Olympics In recent Winter Olympics, there were 12 judges for each part of the women's figure skating program, but the scores of only 9 of the judges were randomly selected for the final results. ***Source: The New York Times.***

a. In how many ways can the 9 judges whose scores are counted be selected?

b. Women's figure skating consists of a short program and a long program, with different judges for each part. How many different sets of judges' scores are possible for the entire event?

YOUR TURN ANSWERS

1. 210 **2.** 1820 **3. a.** 5040 **b.** 455 **c.** 28 **d.** 360 **4.** 1140; 6840 **5.** 103,776

8.3 Probability Applications of Counting Principles

APPLY IT

If 3 engines are tested from a shipping container packed with 12 diesel engines, 2 of which are defective, what is the probability that at least 1 of the defective engines will be found (in which case the container will not be shipped)?

This problem, which is solved in Example 3, could theoretically be solved with a tree diagram, but it would require a tree with a large number of branches. Many of the probability problems involving *dependent* events that were solved earlier by using tree diagrams can also be solved by using permutations or combinations. Permutations and combinations are especially helpful when the numbers involved are large.

To compare the method of using permutations or combinations with the method of tree diagrams used in Section 7.5, the first example repeats Example 7 from that section.

EXAMPLE 1 Environmental Inspections

The Environmental Protection Agency is considering inspecting 6 plants for environmental compliance: 3 in Chicago, 2 in Los Angeles, and 1 in New York. Due to a lack of inspectors, they decide to inspect 2 plants selected at random, 1 this month and 1 next month, with each plant equally likely to be selected, but no plant is selected twice. What is the probability that 1 Chicago plant and 1 Los Angeles plant are selected?

SOLUTION

Method 1 Multiplication Principle

To find the probability, we use the probability fraction, $P(E) = n(E)/n(S)$, where E is the event that 1 Chicago plant and 1 Los Angeles plant is selected and S is the sample space. We will use the multiplication principle since the plants are selected one at a time, and the first is inspected this month while the second is inspected next month.

FOR REVIEW

The use of combinations to solve probability problems depends on the basic probability principle introduced in Section 7.3 and repeated here:

Let S be a sample space with equally likely outcomes, and let event E be a subset of S. Then the probability that event E occurs, written $P(E)$, is

$$P(E) = \frac{n(E)}{n(S)},$$

where $n(E)$ and $n(S)$ represent the number of elements in sets E and S.

To calculate the numerator, we find the number of elements in E. There are two ways to select a Chicago plant and a Los Angeles plant: Chicago first and Los Angeles second, or Los Angeles first and Chicago second. The Chicago plant can be selected from the 3 Chicago plants in $C(3, 1)$ ways, and the Los Angeles plant can be selected from the 2 Los Angeles plants in $C(2, 1)$ ways. Using the multiplication principle, we can select a Chicago plant then a Los Angeles plant in $C(3, 1) \cdot C(2, 1)$ ways, and a Los Angeles plant then a Chicago plant in $C(2, 1) \cdot C(3, 1)$ ways. By the union rule, we can select one Chicago plant and one Los Angeles plant in

$$C(3, 1) \cdot C(2, 1) + C(2, 1) \cdot C(3, 1) \text{ ways,}$$

giving the numerator of the probability fraction.

For the denominator, we calculate the number of elements in the sample space. There are 6 ways to select the first plant and 5 ways to select the second, for a total of $6 \cdot 5$ ways. The required probability is

$$\begin{aligned} \text{P(1 C and 1 LA)} &= \frac{C(3, 1) \cdot C(2, 1) + C(2, 1) \cdot C(3, 1)}{6 \cdot 5} \\ &= \frac{3 \cdot 2 + 2 \cdot 3}{6 \cdot 5} = \frac{12}{30} = \frac{2}{5}. \end{aligned}$$

This agrees with the answer found in Example 7 of Section 7.5.

Method 2 Combinations

This example can be solved more simply by observing that the probability that 1 Chicago plant and 1 Los Angeles plant are selected should not depend upon the order in which the plants are selected, so we may use combinations. The numerator is simply the number of ways of selecting 1 Chicago plant out of 3 Chicago plants and 1 Los Angeles plant out of 2 Los Angeles plants. The denominator is just the number of ways of selecting 2 plants out of 6. Then

$$P(\text{1 C and 1 LA}) = \frac{C(3, 1) \cdot C(2, 1)}{C(6, 2)} = \frac{6}{15} = \frac{2}{5}.$$

This helps explain why combinations tend to be used more often than permutations in probability. Even if order matters in the original problem, it is sometimes possible to ignore order and use combinations. Be careful to do this only when the final result does not depend on the order of events. Order often does matter. (If you don't believe this, try getting dressed tomorrow morning and then taking your shower.)

Method 3 Tree Diagram

In Section 7.5, we found this probability using the tree diagram shown in Figure 6 on the next page. Two of the branches correspond to drawing 1 Chicago plant and 1 Los Angeles plant. The probability for each branch is calculated by multiplying the probabilities along

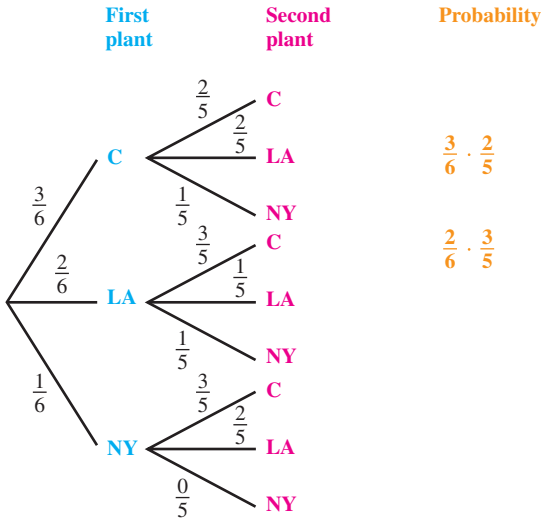


FIGURE 6

the branch, as we did in the previous chapter. The resulting probabilities for the two branches are then added, giving the result

$$P(1 \text{ C and } 1 \text{ LA}) = \frac{3}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{3}{5} = \frac{2}{5}.$$

YOUR TURN 1 In Example 1, what is the probability that 1 New York plant and 1 Chicago plant are selected?

TRY YOUR TURN 1

CAUTION The problems in the first two sections of this chapter asked how many ways a certain operation can be done. The problems in this section ask what is the probability that a certain event occurs; the solution involves answering questions about how many ways the event and the operation can be done.

- If a problem asks how many ways something can be done, the answer must be a nonnegative integer.
- If a problem asks for a probability, the answer must be a number between 0 and 1.

EXAMPLE 2 Nursing

From a group of 22 nurses, 4 are to be selected to present a list of grievances to management.

(a) In how many ways can this be done?

SOLUTION Four nurses from a group of 22 can be selected in $C(22, 4)$ ways. (Use combinations, since the group of 4 is an unordered set.)

$$C(22, 4) = \frac{22!}{18!\,4!} = \frac{(22)(21)(20)(19)}{(4)(3)(2)(1)} = 7315$$

There are 7315 ways to choose 4 people from 22.

(b) One of the nurses is Lori Hales. Find the probability that Hales will be among the 4 selected.

SOLUTION The probability that Hales will be selected is given by $n(E)/n(S)$, where E is the event that the chosen group includes Hales, and S is the sample space for the experiment of choosing a group of 4. There is only $C(1, 1) = 1$ way to choose Hales. The number of ways that the other 3 nurses can be chosen from the remaining 21 nurses is

$$C(21, 3) = \frac{21!}{18!\,3!} = 1330.$$

The probability that Hales will be one of the 4 chosen is

$$P(\text{Hales is chosen}) = \frac{n(E)}{n(S)} = \frac{C(\mathbf{1}, \mathbf{1}) \cdot C(\mathbf{21}, \mathbf{3})}{C(\mathbf{22}, \mathbf{4})} = \frac{1330}{7315} \approx 0.1818.$$

Notice that the two numbers in red in the numerator, 1 and 21, add up to the number in red in the denominator, 22. This indicates that the 22 nurses have been split into two groups, one of size 1 (Hales) and the other of size 21 (the other nurses). Similarly, the green numbers indicate that the 4 nurses chosen consist of two groups of size 1 (Hales) and size 3 (the other nurses chosen).

YOUR TURN 2 If 8 of the 22 nurses are men in Example 2, what is the probability that exactly 2 men are among the 4 nurses selected?

(c) Find the probability that Hales will not be selected.

SOLUTION The probability that Hales will not be chosen is $1 - 0.1818 = 0.8182$.

TRY YOUR TURN 2

EXAMPLE 3 Diesel Engines

When shipping diesel engines abroad, it is common to pack 12 engines in one container. Suppose that a company has received complaints from its customers that many of the engines arrive in nonworking condition. To help solve this problem, the company decides to make a spot check of containers after loading. The company will test 3 engines from a container at random; if any of the 3 are nonworking, the container will not be shipped until each engine in it is checked. Suppose a given container has 2 nonworking engines. Find the probability that the container will not be shipped.

APPLY IT

SOLUTION The container will not be shipped if the sample of 3 engines contains 1 or 2 defective engines. If $P(1 \text{ defective})$ represents the probability of exactly 1 defective engine in the sample, then

$$P(\text{not shipping}) = P(1 \text{ defective}) + P(2 \text{ defective}).$$

There are $C(12, 3)$ ways to choose the 3 engines for testing:

$$C(12, 3) = \frac{12!}{9!\,3!} = 220.$$

There are $C(2, 1)$ ways of choosing 1 defective engine from the 2 in the container, and for each of these ways, there are $C(10, 2)$ ways of choosing 2 good engines from among the 10 in the container. By the multiplication principle, there are

$$C(2, 1) \cdot C(10, 2) = \frac{2!}{1!\,1!} \cdot \frac{10!}{8!\,2!} = 2 \cdot 45 = 90$$

ways of choosing a sample of 3 engines containing 1 defective engine with

$$P(1 \text{ defective}) = \frac{90}{220} = \frac{9}{22}.$$

There are $C(2, 2)$ ways of choosing 2 defective engines from the 2 defective engines in the container, and $C(10, 1)$ ways of choosing 1 good engine from among the 10 good engines, for

$$C(2, 2) \cdot C(10, 1) = 1 \cdot 10 = 10$$

ways of choosing a sample of 3 engines containing 2 defective engines. Finally,

$$P(2 \text{ defective}) = \frac{10}{220} = \frac{1}{22}$$

and

$$\begin{aligned} P(\text{not shipping}) &= P(1 \text{ defective}) + P(2 \text{ defective}) \\ &= \frac{9}{22} + \frac{1}{22} = \frac{10}{22} \approx 0.4545. \end{aligned}$$

YOUR TURN 3 Suppose the container in Example 3 has 4 nonworking engines. Find the probability that the container will not be shipped.

FOR REVIEW

Recall from Section 7.4 that if E and E' are complements, then $P(E') = 1 - P(E)$. In Example 3, the event "0 defective in the sample" is the complement of the event "1 or 2 defective in the sample," since there are only 0 or 1 or 2 defective engines possible in the sample of 3 engines.

Notice that the probability is $1 - 0.4545 = 0.5455$ that the container will be shipped, even though it has 2 defective engines. The management must decide whether this probability is acceptable; if not, it may be necessary to test more than 3 engines from a container.

TRY YOUR TURN 3

Observe that in Example 3, the complement of finding 1 or 2 defective engines is finding 0 defective engines. Then instead of finding the sum $P(1 \text{ defective}) + P(2 \text{ defective})$, the result in Example 3 could be found as $1 - P(0 \text{ defective})$.

$$\begin{aligned} P(\text{not shipping}) &= 1 - P(0 \text{ defective in sample}) \\ &= 1 - \frac{C(2,0) \cdot C(10,3)}{C(12,3)} \\ &= 1 - \frac{1(120)}{220} \\ &= 1 - \frac{120}{220} = \frac{100}{220} \approx 0.4545 \end{aligned}$$

EXAMPLE 4 Poker

In a common form of the card game *poker*, a hand of 5 cards is dealt to each player from a deck of 52 cards. There are a total of

$$C(52,5) = \frac{52!}{47!\,5!} = 2{,}598{,}960$$

such hands possible. Find the probability of getting each of the following hands.

(a) A hand containing only hearts, called a *heart flush*

SOLUTION There are 13 hearts in a deck, with

$$C(13,5) = \frac{13!}{8!\,5!} = \frac{(13)(12)(11)(10)(9)}{(5)(4)(3)(2)(1)} = 1287$$

different hands containing only hearts. The probability of a heart flush is

$$P(\text{heart flush}) = \frac{C(13,5) \cdot C(39,0)}{C(52,5)} = \frac{1287}{2{,}598{,}960} \approx 0.0004952.$$

You don't really need the $C(39,0)$, since this just equals 1, but it might help to remind you that you are choosing none of the 39 cards that remain after the hearts are removed.

(b) A flush of any suit (5 cards of the same suit)

SOLUTION There are 4 suits in a deck, so

$$P(\text{flush}) = 4 \cdot P(\text{heart flush}) = 4 \cdot 0.0004952 \approx 0.001981.$$

(c) A full house of aces and eights (3 aces and 2 eights)

SOLUTION There are $C(4,3)$ ways to choose 3 aces from among the 4 in the deck, and $C(4,2)$ ways to choose 2 eights.

$$P(3 \text{ aces, } 2 \text{ eights}) = \frac{C(4,3) \cdot C(4,2) \cdot C(44,0)}{C(52,5)} = \frac{4 \cdot 6 \cdot 1}{2{,}598{,}960} \approx 0.000009234$$

(d) Any full house (3 cards of one value, 2 of another)

SOLUTION

Method 1 Standard Procedure

The 13 values in a deck give 13 choices for the first value. As in part (c), there are $C(4,3)$ ways to choose the 3 cards from among the 4 cards that have that value. This leaves 12 choices for the second value (order *is* important here, since a full house of 3

aces and 2 eights is not the same as a full house of 3 eights and 2 aces). From the 4 cards that have the second value, there are $C(4, 2)$ ways to choose 2. The probability of any full house is then

$$P(\text{full house}) = \frac{13 \cdot C(4, 3) \cdot 12 \cdot C(4, 2)}{2{,}598{,}960} \approx 0.001441.$$

Method 2 Alternative Procedure

As an alternative way of counting the numerator, first count the number of different values in the hand.* Since there are 13 values from which to choose, and we need 2 different values (one for the set of 3 cards and one for the set of 2), there are $C(13, 2)$ ways to choose the values. Next, of the two values chosen, select the value for which there are 3 cards, which can be done $C(2, 1)$ ways. This automatically determines that the other value is the one for which there are 2 cards. Next, choose the suits for each value. For the value with 3 cards, there are $C(4, 3)$ values of the suits, and for the value with 2 cards, there are $C(4, 2)$ values. Putting this all together,

$$P(\text{full house}) = \frac{C(13, 2) \cdot C(2, 1) \cdot C(4, 3) \cdot C(4, 2)}{2{,}598{,}960} \approx 0.001441.$$

YOUR TURN 4 In Example 4, what is the probability of a hand containing two pairs, one of aces and the other of kings? (This hand contains 2 aces, 2 kings, and a fifth card that is neither an ace nor a king.)

TRY YOUR TURN 4

EXAMPLE 5 Letters

Each of the letters w, y, o, m, i, n, and g is placed on a separate slip of paper. A slip is pulled out, and its letter is recorded in the order in which the slip was drawn. This is done four times.

(a) If the slip is not replaced after the letter is recorded, find the probability that the word "wing" is formed.

SOLUTION The sample space contains all possible arrangements of the seven letters, taken four at a time. Since order matters, use *permutations* to find the number of arrangements in the sample space.

$$P(7, 4) = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$$

Since there is only one way that the word "wing" can be formed, the required probability is $1/840 \approx 0.001190$.

YOUR TURN 5 Find the probability that the word "now" is formed if 3 slips are chosen without replacement in Example 5. Find the probability that the word "now" is formed if 3 slips are chosen with replacement.

(b) If the slip is replaced after the letter is recorded, find the probability that the word "wing" is formed.

SOLUTION Since the letters can be repeated, there are 7 possible outcomes for each draw of the slip. To calculate the number of arrangements in the sample space, use the *multiplication principle*. The number of arrangements in the sample space is $7^4 = 2401$, and the required probability is $1/2401 \approx 0.0004165$.

TRY YOUR TURN 5

EXAMPLE 6 Birthdays

Suppose a group of n people is in a room. Find the probability that at least 2 of the people have the same birthday.

SOLUTION "Same birthday" refers to the month and the day, not necessarily the same year. Also, ignore leap years, and assume that each day in the year is equally likely as a birthday. To see how to proceed, we first look at the case in which $n = 5$ and find the probability that *no 2 people* from among 5 people have the same birthday. There are 365 different birthdays possible for the first of the 5 people, 364 for the second (so that the people have different birthdays), 363 for the third, and so on. The number of ways the

*We learned this approach from Professor Peter Grassi of Hofstra University.

5 people can have different birthdays is thus the number of permutations of 365 days taken 5 at a time or

$$P(365, 5) = 365 \cdot 364 \cdot 363 \cdot 362 \cdot 361.$$

The number of ways that 5 people can have the same birthday or different birthdays is

$$365 \cdot 365 \cdot 365 \cdot 365 \cdot 365 = (365)^5.$$

Finally, the *probability* that none of the 5 people have the same birthday is

$$\frac{P(365, 5)}{365^5} = \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365 \cdot 365 \cdot 365 \cdot 365 \cdot 365} \approx 0.9729.$$

The probability that at least 2 of the 5 people *do* have the same birthday is $1 - 0.9729 = 0.0271$.

Now this result can be extended to more than 5 people. Generalizing, the probability that no 2 people among n people have the same birthday is

$$\frac{P(365, n)}{365^n}.$$

The probability that at least 2 of the n people *do* have the same birthday is

$$1 - \frac{P(365, n)}{365^n}.$$

The following table shows this probability for various values of n.

Number of People, n	Probability That Two Have the Same Birthday
5	0.0271
10	0.1169
15	0.2529
20	0.4114
22	0.4757
23	0.5073
25	0.5687
30	0.7063
35	0.8144
40	0.8912
50	0.9704
366	1

The probability that 2 people among 23 have the same birthday is 0.5073, a little more than half. Many people are surprised at this result; it seems that a larger number of people should be required.

TECHNOLOGY NOTE

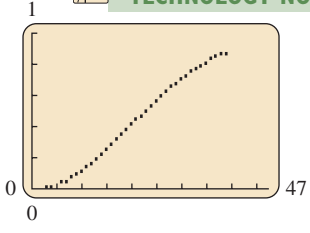


FIGURE 7

Using a graphing calculator, we can graph the probability formula in the previous example as a function of n, but care must be taken that the graphing calculator evaluates the function at integer points. Figure 7 was produced on a TI-84 Plus by letting $Y_1 = 1 - (365\ nPr\ X)/365\wedge X$ on $0 \le x \le 47$. (This domain ensures integer values for x.) Notice that the graph does not extend past $x = 39$. This is because $P(365, n)$ and 365^n are too large for the calculator when $n \ge 40$.

An alternative way of doing the calculations that does not run into such large numbers is based on the concept of conditional probability. The probability that the first person's birthday does not match any so far is 365/365. The probability that the second person's birthday does not match the first's is 364/365. The probability that the third person's birthday does not match the first's or the

second's is 363/365. By the product rule of probability, the probability that none of the first 3 people have matching birthdays is

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}.$$

Similarly, the probability that no two people in a group of 40 have the same birthday is

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \cdots \cdot \frac{326}{365}.$$

This probability can be calculated (and then subtracted from 1 to get the probability we seek) without overflowing the calculator by multiplying each fraction times the next, rather then trying to compute the entire numerator and the entire denominator. The calculations are somewhat tedious to do by hand but can be programmed on a graphing calculator or computer.

As we saw in Examples 1 and 4(d), probability can sometimes be calculated in more than one way. We now look at one more example of this.

EXAMPLE 7 Fruit

Ray and Nate are arranging a row of fruit at random on a table. They have 5 apples, 6 oranges, and 7 lemons. What is the probability that all fruit of the same kind are together?

SOLUTION

Method 1
Distinguishable Permutations

Ray can't tell individual pieces of fruit of the same kind apart. All apples look the same to him, as do all oranges and all lemons. So in the denominator of the probability, he calculates the number of distinguishable ways to arrange the 18 pieces of fruit, given that all apples are indistinguishable, as are all oranges and all lemons.

$$\frac{18!}{5!\,6!\,7!} = 14{,}702{,}688$$

As for the numerator, the only choice is how to arrange the 3 kinds of fruit, for which there are $3! = 6$ ways. Thus

$$P(\text{all fruit of the same kind are together}) = \frac{6}{14{,}702{,}688} \approx 4.081 \times 10^{-7}.$$

Method 2
Permutations

Nate has better eyesight than Ray and can tell the individual pieces of fruit apart. So in the denominator of the probability, he calculates the number of ways to arrange the 18 pieces of fruit, which is

$$18! \approx 6.4024 \times 10^{15}.$$

For the numerator, he first must choose how to arrange the 3 kinds of fruit, for which there are 3! ways. Then there are 5! ways to arrange the apples, 6! ways to arrange the oranges, and 7! ways to arrange the lemons, for a total number of possibilities of

$$3!\,5!\,6!\,7! = 2{,}612{,}736{,}000.$$

Therefore,

$$P(\text{all fruit of the same kind are together}) = \frac{2{,}612{,}736{,}000}{6.4024 \times 10^{15}} \approx 4.081 \times 10^{-7}.$$

The results for Method 1 and Method 2 are the same. The probability does not depend on whether a person can distinguish individual pieces of the same kind of fruit.

YOUR TURN 6 If Ray and Nate arrange 2 kiwis, 3 apricots, 4 pineapples, and 5 coconuts in a row at random, what is the probability that all fruit of the same kind are together?

TRY YOUR TURN 6

8.3 EXERCISES

A basket contains 7 red apples and 4 yellow apples. A sample of 3 apples is drawn. Find the probabilities that the sample contains the following.

1. All red apples

2. All yellow apples

3. 2 yellow and 1 red apple

4. More red than yellow apples

In a club with 9 male and 11 female members, a 5-member committee will be randomly chosen. Find the probability that the committee contains the following.

5. All men

6. All women

7. 3 men and 2 women

8. 2 men and 3 women

9. At least 4 women

10. No more than 2 men

Two cards are drawn at random from an ordinary deck of 52 cards.

11. How many 2-card hands are possible?

Find the probability that the 2-card hand described above contains the following.

12. 2 aces

13. At least 1 ace

14. All spades

15. 2 cards of the same suit

16. Only face cards

17. No face cards

18. No card higher than 8 (count ace as 1)

Twenty-six slips of paper are each marked with a different letter of the alphabet and placed in a basket. A slip is pulled out, its letter recorded (in the order in which the slip was drawn), and the slip is replaced. This is done 5 times. Find the probabilities that the following "words" are formed.

19. Chuck

20. A word that starts with "p"

21. A word with no repetition of letters

22. A word that contains no "*x*," "*y*," or "*z*"

23. Discuss the relative merits of using tree diagrams versus combinations to solve probability problems. When would each approach be most appropriate?

24. Several examples in this section used the rule $P(E') = 1 - P(E)$. Explain the advantage (especially in Example 6) of using this rule.

For Exercises 25–28, refer to Example 6 in this section.

25. A total of 43 men have served as president through 2010.* Set up the probability that, if 43 men were selected at random, at least 2 have the same birthday.†

26. Set up the probability that at least 2 of the 100 U.S. senators have the same birthday.

27. What is the probability that at least 2 of the 435 members of the House of Representatives have the same birthday?

28. Argue that the probability that in a group of n people *exactly one pair* have the same birthday is

$$C(n, 2) \cdot \frac{P(365, n-1)}{365^n}.$$

29. After studying all night for a final exam, a bleary-eyed student randomly grabs 2 socks from a drawer containing 9 black, 6 brown, and 2 blue socks, all mixed together. What is the probability that she grabs a matched pair?

30. Three crows, 4 blue jays, and 5 starlings sit in a random order on a section of telephone wire. Find the probability that birds of a feather flock together, that is, that all birds of the same type are sitting together.

31. If the letters l, i, t, t, l, and e are chosen at random, what is the probability that they spell the word "little"?

32. If the letters M, i, s, s, i, s, s, i, p, p, and i are chosen at random, what is the probability that they spell the word "Mississippi"?

33. An elevator has 4 passengers and stops at 7 floors. It is equally likely that a person will get off at any one of the 7 floors. Find the probability that at least 2 passengers leave at the same floor. (*Hint:* Compare this with the birthday problem.)

34. On National Public Radio, the *Weekend Edition* program posed the following probability problem: Given a certain number of balls, of which some are blue, pick 5 at random. The probability that all 5 are blue is 1/2. Determine the original number of balls and decide how many were blue. ***Source: Weekend Edition.***

35. A reader wrote to the "Ask Marilyn" column in *Parade* magazine, "You have six envelopes to pick from. Two-thirds (that is, four) are empty. One-third (that is, two) contain a $100 bill. You're told to choose 2 envelopes at random. Which is more likely: (1) that you'll get at least one $100 bill, or (2) that you'll get no $100 bill at all?" Find the two probabilities. ***Source: Parade Magazine.***

APPLICATIONS

Business and Economics

Quality Control **A shipment of 11 printers contains 2 that are defective. Find the probability that a sample of the following sizes, drawn from the 11, will not contain a defective printer.**

36. 1

37. 2

38. 3

39. 4

*Although Obama is the 44th President, the 22nd and 24th Presidents were the same man: Grover Cleveland.

†In fact, James Polk and Warren Harding were both born on November 2.

Refer to Example 3. The managers feel that the probability of 0.5455 that a container will be shipped even though it contains 2 defective engines is too high. They decide to increase the sample size chosen. Find the probabilities that a container will be shipped even though it contains 2 defective engines, if the sample size is increased to the following.

40. 4 **41.** 5

42. Sales Presentations Heidi Shadix and Boyd Shepherd are among 9 representatives making presentations at the annual sales meeting. The presentations are randomly ordered. Find the probability that Heidi is the first presenter and Boyd is the last presenter.

43. Sales Schedule Dan LaChapelle has the name of 6 prospects, including a customer in Scottsdale. He randomly arranges his schedule to call on only 4 of the 6 prospects. Find the probability that the customer from Scottsdale is not called upon.

Social Sciences

44. Election Ballots Five names are put on a ballot in a randomly selected order. What is the probability that they are not in alphabetical order?

45. Native American Council At the first meeting of a committee to plan a Northern California pow-wow, there were 3 women and 3 men from the Miwok tribe, 2 men and 3 women from the Hoopa tribe, and 4 women and 5 men from the Pomo tribe. If the ceremony subcouncil consists of 5 people and is randomly selected, find the probabilities that the subcouncil contains the following:

a. 3 men and 2 women;

b. exactly 3 Miwoks and 2 Pomos;

c. 2 Miwoks, 2 Hoopas, and a Pomo;

d. 2 Miwoks, 2 Hoopas, and 2 Pomos;

e. more women than men;

f. exactly 3 Hoopas;

g. at least 2 Pomos.

46. Education A school in Bangkok requires that students take an entrance examination. After the examination, there is a drawing in which 5 students are randomly selected from each group of 40 for automatic acceptance into the school, regardless of their performance on the examination. The drawing consists of placing 35 red and 5 green pieces of paper into a box. Each student picks a piece of paper from the box and then does not return the piece of paper to the box. The 5 lucky students who pick the green pieces are automatically accepted into the school. *Source: Mathematics Teacher.*

a. What is the probability that the first person wins automatic acceptance?

b. What is the probability that the last person wins automatic acceptance?

c. If the students are chosen by the order of their seating, does this give the student who goes first a better chance of winning than the second, third, . . . person? (*Hint:* Imagine that the 40 pieces of paper have been mixed up and laid in a row so that the first student picks the first piece of paper, the second student picks the second piece of paper, and so on.)

47. Education At a conference promoting excellence in education for African Americans in Detroit, special-edition books were selected to be given away in contests. There were 9 books written by Langston Hughes, 5 books by James Baldwin, and 7 books by Toni Morrison. The judge of one contest selected 6 books at random for prizes. Find the probabilities that the selection consisted of the following.

a. 3 Hughes and 3 Morrison books

b. Exactly 4 Baldwin books

c. 2 Hughes, 3 Baldwin, and 1 Morrison book

d. At least 4 Hughes books

e. Exactly 4 books written by males (Morrison is female)

f. No more than 2 books written by Baldwin

General Interest

Poker Find the probabilities of the following hands at poker. Assume aces are either high or low.

48. Royal flush (5 highest cards of a single suit)

49. Straight flush (5 in a row in a single suit, but not a royal flush)

50. Four of a kind (4 cards of the same value)

51. Straight (5 cards in a row, not all of the same suit), with ace either high or low

52. Three of a kind (3 cards of one value, with the other cards of two different values)

53. Two pairs (2 cards of one value, 2 of another value, and 1 of a third value)

54. One pair (2 cards of one value, with the other cards of three different values)

Bridge A bridge hand is made up of 13 cards from a deck of 52. Find the probabilities that a hand chosen at random contains the following.

55. Only hearts

56. At least 3 aces

57. Exactly 2 aces and exactly 2 kings

58. 6 of one suit, 4 of another, and 3 of another

Texas Hold'Em In a version of poker called Texas Hold'Em, each player has 2 cards, and by the end of the round an additional 5 cards are on the table, shared by all the players. (For more about poker hands, see Example 4 and Exercises 48–54.) Each player's hand consists of the best 5 cards out of the 7 cards available to that player. For example, if a player holds 2 kings, and on the table there is one king and 4 cards with 4 other values, then the player has three of a kind. It's possible that the player has an even better hand. Perhaps five of the seven cards are of the same suit, making a flush, or the other 4 cards are queen, jack, 10, and 9, so the player has a straight. For Exercises 59–64, calculate the probability of each hand in Texas Hold'Em, but for simplicity, ignore the possibility that the cards might form an even better hand.

59. One pair (2 cards of one value, with the other cards of five different values)

60. Two pairs (2 cards of one value, 2 of another, with the other cards of three different values)

61. Three of a kind (3 cards of one value, with the other cards of four different values)

62. Four of a kind (4 cards of one value, with the other cards of three different values)

63. Flush (at least 5 cards of the same suit)

64. Full house (3 cards of one value and two of another. Careful: The two unused cards could be a pair of another value or two different cards of other values. Also, the 7 cards could consist of 3 cards of one value, 3 of second value, and one card of a third value. We won't consider the case of 3 cards of one value and 4 of another, because even though this forms a full house, it also forms four of a kind, which is better.)

65. Suppose you are playing Texas Hold'Em and you've just received your two cards. You observe that they are both hearts. You get excited, because if at least 3 of the 5 cards that are on the table are hearts, you'll have a flush, which means that you'll likely win the round. Given that your two cards are hearts, and you know nothing of any other cards, what is the probability that you'll have a flush by the time all 5 cards are on the table?

66. Lottery In the previous section, we found the number of ways to pick 6 different numbers from 1 to 99 in a state lottery. Assuming order is unimportant, what is the probability of picking all 6 numbers correctly to win the big prize?

67. Lottery In Exercise 66, what is the probability of picking exactly 5 of the 6 numbers correctly?

68. Lottery An article in *The New York Times* discussing the odds of winning the lottery stated, "And who cares if a game-theory professor once calculated the odds of winning as equal to a poker player's chance of drawing four royal flushes in a row, all in spades—then getting up from the card table and meeting four strangers, all with the same birthday?" Calculate this probability. Does this probability seem comparable to the odds of winning the lottery? (Ignore February 29 as a birthday, and assume that all four strangers have the same birthday as each other, not necessarily the same as the poker player.) ***Source: The New York Times Magazine.***

69. Lottery A reader wrote to the "Ask Marilyn" column in *Parade* magazine, "A dozen glazed doughnuts are riding on the answer to this question: Are the odds of winning in a lotto drawing higher when picking 6 numbers out of 49 or when picking 5 numbers out of 52?" Calculate each probability to answer the question. ***Source: Parade Magazine.***

70. Lottery On October 22, 2005, the Powerball Lottery had a record jackpot of $340 million, which was a record at the time. To enter the lottery, 5 numbers are picked between 1 and 55, plus a bonus number between 1 and 42. All 6 numbers must be correct to win the jackpot.

a. What is the probability of winning the jackpot with a single ticket?

b. In an article for the *Minneapolis Star Tribune*, mathematician Douglas Arnold was quoted as saying, "If you were to select a group of Powerball numbers every minute for 138 years, you would have about a 50 percent chance of picking the winning Powerball ticket." Calculate the actual probability, using an estimate of 365.25 for the number of days in the year. (Arnold later told *Chance News* that this was an "off-the-top-of-my-head calculation" made when a reporter called.) ***Sources: Minneapolis Star Tribune and Chance News.***

71. Canadian Lottery In June 2004, Canada introduced a change in its lottery that violated the usual convention that the smaller the probability of an event, the bigger the prize. In this lottery, participants have to guess six numbers from 1 to 49. Six numbers between 1 and 49 are then drawn at random, plus a seventh "bonus number." ***Source: Chance.***

a. A fifth prize of $10 goes to those who correctly guess exactly three of the six numbers, but do not guess the bonus number. Find the probability of winning fifth prize.

b. A sixth prize of $5 goes to those who correctly guess exactly two of the six numbers plus the bonus number. Find the probability of winning sixth prize, and compare this with the probability of winning fifth prize.

72. Barbie A controversy arose in 1992 over the Teen Talk Barbie doll, each of which was programmed with four sayings randomly picked from a set of 270 sayings. The controversy was over the saying, "Math class is tough," which some felt gave a negative message toward girls doing well in math. In an interview with *Science*, a spokeswoman for Mattel, the makers of Barbie, said that "There's a less than 1% chance you're going to get a doll that says math class is tough." Is this figure correct? If not, give the correct figure. ***Source: Science.***

73. Football During the 1988 college football season, the Big Eight Conference ended the season in a "perfect progression," as shown in the following table. ***Source: The American Statistician.***

Won	Lost	Team
7	0	Nebraska (NU)
6	1	Oklahoma (OU)
5	2	Oklahoma State (OSU)
4	3	Colorado (CU)
3	4	Iowa State (ISU)
2	5	Missouri (MU)
1	6	Kansas (KU)
0	7	Kansas State (KSU)

Someone wondered what the probability of such an outcome might be.

a. How many games do the 8 teams play?

b. Assuming no ties, how many different outcomes are there for all the games together?

c. In how many ways could the 8 teams end in a perfect progression?

d. Assuming that each team had an equally likely probability of winning each game, find the probability of a perfect progression with 8 teams.

e. Find a general expression for the probability of a perfect progression in an n-team league with the same assumptions.

74. Unluckiest Fan During the 2009 season, the Washington Nationals baseball team won 59 games and lost 103 games. Season ticket holder Stephen Krupin reported in an interview that he watched the team lose all 19 games that he attended that season. The interviewer speculated that this must be a record for bad luck. ***Source: Chance News.***

a. Based on the full 2009 season record, calculate the probability that a person would attend 19 Washington Nationals games and the Nationals would lose all 19 games.

b. However, Mr. Krupin only attended home games. The Nationals had 33 wins and 48 losses at home in 2009. Calculate the probability that a person would attend 19 Washington Nationals home games and the Nationals would lose all 19 games.

75. Bingo Bingo has become popular in the United States, and it is an efficient way for many organizations to raise money. The bingo card has 5 rows and 5 columns of numbers from 1 to 75, with the center given as a free cell. Balls showing one of the 75 numbers are picked at random from a container. If the drawn number appears on a player's card, then the player covers the number. In general, the winner is the person who first has a card with an entire row, column, or diagonal covered. ***Source: Mathematics Teacher.***

a. Find the probability that a person will win bingo after just four numbers are called.

b. An L occurs when the first column and the bottom row are both covered. Find the probability that an L will occur in the fewest number of calls.

c. An X-out occurs when both diagonals are covered. Find the probability that an X-out occurs in the fewest number of calls.

d. If bingo cards are constructed so that column one has 5 of the numbers from 1 to 15, column two has 5 of the numbers from 16 to 30, column three has 4 of the numbers from 31 to 45, column four has 5 of the numbers from 46 to 60, and column five has 5 of the numbers from 61 to 75, how many different bingo cards could be constructed? (*Hint:* Order matters!)

76. Suppose a box contains 3 red and 3 blue balls. A ball is selected at random and removed, without observing its color. The box now either contains 3 red and 2 blue balls or 2 red and 3 blue balls. ***Source: 40 Puzzles and Problems in Probability and Mathematical Statistics.***

a. Nate removes a ball at random from the box, observes its color, and puts the ball back. He performs this experiment a total of 6 times, and each time the ball is blue. What is the probability that a red ball was initially removed from the box? (*Hint:* Use Bayes' Theorem.)

b. Ray removes a ball at random from the box, observes its color, and puts the ball back. He performs this experiment a total of 80 times. Out of these, the ball was blue 44 times and red 36 times. What is the probability that a red ball was initially removed from the box?

c. Many people intuitively think that Nate's experiment gives more convincing evidence than Ray's experiment that a red ball was removed. Explain why this is wrong.

YOUR TURN ANSWERS

1. 1/5 **2.** 0.3483 **3.** 0.7455 **4.** 0.0006095
5. 1/210; 1/343 **6.** 9.514×10^{-6}

8.4 Binomial Probability

APPLY IT **What is the probability that 3 out of 6 randomly selected college students attend more than one institution during their college career?**
We will calculate this probability in Example 2.

The question above involves an experiment that is repeated 6 times. Many probability problems are concerned with experiments in which an event is repeated many times. Other examples include finding the probability of getting 7 heads in 8 tosses of a coin, of hitting a target 6 times out of 6, and of finding 1 defective item in a sample of 15 items. Probability problems of this kind are called **Bernoulli trials** problems, or **Bernoulli processes**, named after the Swiss mathematician Jakob Bernoulli (1654–1705), who is well known for his work in probability theory. In each case, some outcome is designated a success and any other outcome is considered a failure. This labeling is arbitrary and does not necessarily have anything to do with real success or failure. Thus, if the probability of a success in a single trial is p, the probability of failure will be $1 - p$. A Bernoulli trials problem, or **binomial experiment**, must satisfy the following conditions.

Binomial Experiment

1. The same experiment is repeated a fixed number of times.
2. There are only two possible outcomes, success and failure.
3. The repeated trials are independent, so that the probability of success remains the same for each trial.

EXAMPLE 1 Sleep

The chance that an American falls asleep with the TV on at least three nights a week is 1/4. Suppose a researcher selects 5 Americans at random and is interested in the probability that all 5 are "TV sleepers." ***Source: Harper's Magazine.***

FOR REVIEW

Recall that if A and B are independent events,

$$P(A \text{ and } B) = P(A)P(B).$$

SOLUTION Here the experiment, selecting a person, is repeated 5 times. If selecting a TV sleeper is labeled a success, then getting a "non-TV sleeper" is labeled a failure. The 5 trials are almost independent. There is a very slight dependence; if, for example, the first person selected is a TV sleeper, then there is one less TV sleeper to choose from when we select the next person (assuming we never select the same person twice). When selecting a small sample out of a large population, however, the probability changes negligibly, so researchers consider such trials to be independent. Thus, the probability that all 5 in our sample are sleepers is

$$\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \left(\frac{1}{4}\right)^5 \approx 0.0009766.$$

Now suppose the problem in Example 1 is changed to that of finding the probability that exactly 4 of the 5 people in the sample are TV sleepers. This outcome can occur in more than one way, as shown below, where s represents a success (a TV sleeper) and f represents a failure (a non-TV sleeper).

outcome 1: $s\ s\ s\ s\ f$
outcome 2: $s\ s\ s\ f\ s$
outcome 3: $s\ s\ f\ s\ s$
outcome 4: $s\ f\ s\ s\ s$
outcome 5: $f\ s\ s\ s\ s$

Keep in mind that since the probability of success is 1/4, the probability of failure is $1 - 1/4 = 3/4$. The probability, then, of each of these 5 outcomes is

$$\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right).$$

Since the 5 outcomes represent mutually exclusive events, add the 5 identical probabilities, which is equivalent to multiplying the above probability by 5. The result is

$$P(4 \text{ of the 5 people are TV sleepers}) = 5\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) = \frac{15}{4^5} \approx 0.01465.$$

In the same way, we can compute the probability of selecting 3 TV sleepers in our sample of 5. The probability of any one way of achieving 3 successes and 2 failures will be

$$\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2.$$

Rather than list all the ways of achieving 3 successes out of 5 trials, we will count this number using combinations. The number of ways to select 3 elements out of a set of 5 is $C(5, 3) = 5!/(2!\,3!) = 10$, giving

$$P(3 \text{ of the 5 people are TV sleepers}) = 10\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{90}{4^5} \approx 0.08789.$$

A similar argument works in the general case.

Binomial Probability

If p is the probability of success in a single trial of a binomial experiment, the probability of x successes and $n - x$ failures in n independent repeated trials of the experiment, known as **binomial probability**, is

$$\boldsymbol{P(x \text{ successes in } n \text{ trials}) = C(n, x)\cdot p^x\cdot(1-p)^{n-x}.}$$

EXAMPLE 2 College Students

A recent survey found that 59% of college students attend more than one institution during their college career. Suppose a sample of 6 students is chosen. Assuming that each student's college attendance pattern is independent of the others, find the probability of each of the following. ***Source: The New York Times.***

(a) Exactly 3 of the 6 students attend more than one institution.

APPLY IT

SOLUTION Think of the 6 students chosen as 6 independent trials. A success occurs if the student attends more than one institution. Then this is a binomial experiment with $n = 6$ and $p = P(\text{attend more than one institution}) = 0.59$. To find the probability that exactly 3 students attend more than one institution, let $x = 3$ and use the binomial probability formula.

$$\begin{aligned} P(\text{exactly 3 of 6 students}) &= C(6, 3)(0.59)^3(1 - 0.59)^{6-3} \\ &= 20(0.59)^3(0.41)^3 \\ &= 20(0.2054)(0.06892) \\ &\approx 0.2831 \end{aligned}$$

(b) None of the 6 students attend more than one institution.

YOUR TURN 1 Find the probability that exactly 2 of the 6 students in Example 2 attend more than 1 institution.

SOLUTION Let $x = 0$.

$$\begin{aligned} P(\text{exactly 0 of 6 students}) &= C(6, 0)(0.59)^0(1 - 0.59)^6 \\ &= 1(1)(0.41)^6 \approx 0.00475 \end{aligned}$$

TRY YOUR TURN 1

EXAMPLE 3 Coin Toss

Find the probability of getting exactly 7 heads in 8 tosses of a fair coin.

SOLUTION The probability of success (getting a head in a single toss) is $1/2$. The probability of a failure (getting a tail) is $1 - 1/2 = 1/2$. Thus,

$$P(7 \text{ heads in 8 tosses}) = C(8, 7)\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^1 = 8\left(\frac{1}{2}\right)^8 = 0.03125.$$

YOUR TURN 2 In Example 3, find the probability of getting exactly 4 heads in 8 tosses of a fair coin.

TRY YOUR TURN 2

EXAMPLE 4 Defective Items

Assuming that selection of items for a sample can be treated as independent trials, and that the probability that any 1 item is defective is 0.01, find the following.

(a) The probability of 1 defective item in a random sample of 15 items from a production line

SOLUTION Here, a "success" is a defective item. Since selecting each item for the sample is assumed to be an independent trial, the binomial probability formula applies. The probability of success (a defective item) is 0.01, while the probability of failure (an acceptable item) is 0.99. This makes

$$\begin{aligned} P(1 \text{ defective in 15 items}) &= C(15, 1)(0.01)^1(0.99)^{14} \\ &= 15(0.01)(0.99)^{14} \\ &\approx 0.1303. \end{aligned}$$

(b) The probability of at most 1 defective item in a random sample of 15 items from a production line

SOLUTION "At most 1" means 0 defective items or 1 defective item. Since 0 defective items is equivalent to 15 acceptable items,

$$P(0 \text{ defective}) = (0.99)^{15} \approx 0.8601.$$

YOUR TURN 3 In Example 4, find the probability of 2 or 3 defective items in a random sample of 15 items from a production line.

Use the union rule, noting that 0 defective and 1 defective are mutually exclusive events, to get

$$\begin{aligned} P(\text{at most 1 defective}) &= P(0 \text{ defective}) + P(1 \text{ defective}) \\ &\approx 0.8601 + 0.1303 \\ &= 0.9904. \end{aligned}$$

TRY YOUR TURN 3

EXAMPLE 5 Checkout Scanners

The Federal Trade Commission (FTC) monitors pricing accuracy to ensure that consumers are charged the correct price at the checkout. According to the FTC, 29% of stores that use checkout scanners do not accurately charge customers. ***Source: Federal Trade Commission.***

(a) If you shop at 3 stores that use checkout scanners, what is the probability that you will be incorrectly charged in at least one store?

SOLUTION We can treat this as a binomial experiment, letting $n = 3$ and $p = 0.29$. We need to find the probability of "at least 1" incorrect charge, which means 1 or 2 or 3 incorrect charges. To make our calculation simpler, we will use the complement. We will find the probability of being charged incorrectly in none of the 3 stores, that is, $P(0 \text{ incorrect charges})$, and then find $1 - P(0 \text{ incorrect charges})$.

$$\begin{aligned} P(0 \text{ incorrect charges}) &= C(3, 0)(0.29)^0(0.71)^3 \\ &= 1(1)(0.357911) \approx 0.3579 \\ P(\text{at least one}) &= 1 - P(0 \text{ incorrect charges}) \\ &\approx 1 - 0.3579 = 0.6421 \end{aligned}$$

(b) If you shop at 3 stores that use checkout scanners, what is the probability that you will be incorrectly charged in at most one store?

SOLUTION "At most one" means 0 or 1, so

$$\begin{aligned} P(0 \text{ or } 1) &= P(0) + P(1) \\ &= C(3, 0)(0.29)^0(0.71)^3 + C(3, 1)(0.29)^1(0.71)^2 \\ &= 1(1)(0.357911) + 3(0.29)(0.5041) \approx 0.7965. \end{aligned}$$

YOUR TURN 4 In Example 5, if you shop at 4 stores that use checkout scanners, find the probability that you will be charged incorrectly in at least one store.

TRY YOUR TURN 4

The triangular array of numbers shown below is called **Pascal's triangle** in honor of the French mathematician Blaise Pascal (1623–1662), who was one of the first to use it extensively. The triangle was known long before Pascal's time and appears in Chinese and Islamic manuscripts from the eleventh century.

Pascal's Triangle

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

The array provides a quick way to find binomial probabilities. The nth row of the triangle, where $n = 0, 1, 2, 3, \ldots$, gives the coefficients $C(n, r)$ for $r = 0, 1, 2, 3, \ldots, n$. For example, for $n = 4$, $1 = C(4, 0)$, $4 = C(4, 1)$, $6 = C(4, 2)$, and so on. Each number in the triangle is the sum of the two numbers directly above it. For example, in the row for

$n = 4$, 1 is the sum of 1, the only number above it, 4 is the sum of 1 and 3, 6 is the sum of 3 and 3, and so on. Adding in this way gives the sixth row:

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1.$$

Notice that Pascal's triangle tells us, for example, that $C(4, 1) + C(4, 2) = C(5, 2)$ (that is, $4 + 6 = 10$). Using the combinations formula, it can be shown that, in general, $C(n, r) + C(n, r + 1) = C(n + 1, r + 1)$. This is left as an exercise.

EXAMPLE 6 Pascal's Triangle

Use Pascal's triangle to find the probability in Example 5 that if you shop at 6 stores that use checkout scanners, at least 3 will charge you incorrectly

SOLUTION The probability of success is 0.29. Since at least 3 means 3, 4, 5, or 6,

$$\begin{aligned} P(\text{at least } 3) &= P(3) + P(4) + P(5) + P(6) \\ &= C(6, 3)(0.29)^3(0.71)^3 + C(6, 4)(0.29)^4(0.71)^2 \\ &\quad + C(6, 5)(0.29)^5(0.71)^1 + C(6, 6)(0.29)^6(0.71)^0. \end{aligned}$$

Use the sixth row of Pascal's triangle for the combinations to get

$$\begin{aligned} P(\text{at least } 3) &= 20(0.29)^3(0.71)^3 + 15(0.29)^4(0.71)^2 \\ &\quad + 6(0.29)^5(0.71)^1 + 1(0.29)^6(0.71)^0 \\ &= 0.1746 + 0.0535 + 0.0087 + 0.0006 \\ &= 0.2374. \end{aligned}$$

YOUR TURN 5 In Example 5, find the probability that if you shop at 6 stores with checkout scanners, at most 3 stores will charge you incorrectly.

TRY YOUR TURN 5

EXAMPLE 7 Independent Jury

If each member of a 9-person jury acts independently of each other and makes the correct determination of guilt or innocence with probability 0.65, find the probability that the majority of jurors will reach a correct verdict. ***Source: Frontiers in Economics.***

Method 1 Calculating By Hand

SOLUTION Since the jurors in this particular situation act independently, we can treat this as a binomial experiment. Thus, the probability that the majority of the jurors will reach the correct verdict is given by

$$\begin{aligned} P(\text{at least } 5) &= C(9, 5)(0.65)^5(0.35)^4 + C(9, 6)(0.65)^6(0.35)^3 \\ &\quad + C(9, 7)(0.65)^7(0.35)^2 + C(9, 8)(0.65)^8(0.35)^1 + C(9, 9)(0.65)^9 \\ &= 0.2194 + 0.2716 + 0.2162 + 0.1004 + 0.0207 \\ &= 0.8283. \end{aligned}$$

Method 2 Graphing Calculator

Some graphing calculators provide binomial probabilities. On a TI-84 Plus, for example, the command `binompdf(9, .65, 5)`, found in the `DISTR` menu, gives 0.21939, which is the probability that $x = 5$. Alternatively, the command `binomcdf(9, .65, 4)` gives 0.17172 as the probability that 4 or fewer jurors will make the correct decision. Subtract 0.17172 from 1 to get 0.82828 as the probability that the majority of the jurors will make the correct decision. This value rounds to 0.8283, which is in agreement with Method 1. Often, graphing calculators are more accurate than calculations by hand due to the accumulation of rounding errors when doing successive calculations by hand.

Method 3 Spreadsheet

Some spreadsheets also provide binomial probabilities. In Microsoft Excel, for example, the command "`=BINOMDIST(5, 9, .65, 0)`" gives 0.21939, which is the probability that $x = 5$. Alternatively, the command "`=BINOMDIST (4, 9, .65, 1)`" gives 0.17172 as the probability that 4 or fewer jurors will make the correct decision. Subtract 0.17172 from 1 to get 0.82828 as the probability that the majority of the jurors will make the correct decision. This value agrees with the value found in Methods 1 and 2.

8.4 EXERCISES

Suppose that a family has 5 children. Also, suppose that the probability of having a girl is 1/2. Find the probabilities that the family has the following children.

1. Exactly 2 girls and 3 boys

2. Exactly 3 girls and 2 boys

3. No girls

4. No boys

5. At least 4 girls

6. At least 3 boys

7. No more than 3 boys

8. No more than 4 girls

A die is rolled 12 times. Find the probabilities of rolling the following.

9. Exactly 12 ones

10. Exactly 6 ones

11. Exactly 1 one

12. Exactly 2 ones

13. No more than 3 ones

14. No more than 1 one

A certain unfair coin lands on heads 1/4 of the time. This coin is tossed 6 times. Find the probabilities of getting the following.

15. All heads

16. Exactly 3 heads

17. No more than 3 heads

18. At least 3 heads

19. How do you identify a probability problem that involves a binomial experiment?

20. How is Pascal's triangle used to find probabilities?

21. Using the definition of combination in Section 8.2, prove that

$$C(n, r) + C(n, r + 1) = C(n + 1, r + 1)$$

(This is the formula underlying Pascal's triangle.)

In Exercises 22 and 23, argue that the use of binomial probabilities is not applicable and, thus, the probabilities that are computed are not correct.

22. In England, a woman was found guilty of smothering her two infant children. Much of the Crown's case against the lady was based on the testimony from a pediatrician who indicated that the chances of two crib deaths occurring in both siblings was only about 1 in 73 million. This number was calculated by assuming that the probability of a single crib death is 1 in 8543 and the probability of two crib deaths is 1 in 8543^2 (i.e., binomial). (See Chapter 7 Review Exercise 102.) ***Source: Science.***

23. A contemporary radio station in Boston has a contest in which a caller is asked his or her date of birth. If the caller's date of birth, including the day, month, and year of birth, matches a predetermined date, the caller wins $1 million. Assuming that there were 36,525 days in the twentieth century and the contest was run 51 times on consecutive days, the probability that the grand prize will be won is

$$1 - \left(1 - \frac{1}{36{,}525}\right)^{51} \approx 0.0014.$$

Source: Chance News.

APPLICATIONS

Business and Economics

Management **The survey discussed in Example 5 also found that customers are charged incorrectly for 1 out of every 30 items, on average. Suppose a customer purchases 15 items. Find the following probabilities.**

24. A customer is charged incorrectly on 3 items.

25. A customer is not charged incorrectly for any item.

26. A customer is charged incorrectly on at least one item.

27. A customer is charged incorrectly on at least 2 items.

28. A customer is charged incorrectly on at most 2 items.

Credit Cards **A survey of consumer finance found that 25.4% of credit-card-holding families hardly ever pay off the balance. Suppose a random sample of 20 credit-card-holding families is taken. Find the probabilities of each of the following results.** ***Source: Statistical Abstract of the United States.***

29. Exactly 6 families hardly ever pay off the balance.

30. Exactly 9 families hardly ever pay off the balance.

31. At least 4 families hardly ever pay off the balance.

32. At most 5 families hardly ever pay off the balance.

Personnel Screening **A company gives prospective workers a 6-question, multiple-choice test. Each question has 5 possible answers, so that there is a 1/5 or 20% chance of answering a question correctly just by guessing. Find the probabilities of getting the following results by chance.**

33. Exactly 2 correct answers

34. No correct answers

35. At least 4 correct answers

36. No more than 3 correct answers

Quality Control **A factory tests a random sample of 20 transistors for defects. The probability that a particular transistor will be defective has been established by past experience as 0.05.**

37. What is the probability that there are no defective transistors in the sample?

38. What is the probability that the number of defective transistors in the sample is at most 2?

39. Quality Control The probability that a certain machine turns out a defective item is 0.05. Find the probabilities that in a run of 75 items, the following results are obtained.

a. Exactly 5 defective items

b. No defective items

c. At least 1 defective item

40. Survey Results A company is taking a survey to find out whether people like its product. Its last survey indicated that 70% of the population like the product. Based on that, in a sample of 58 people, find the probabilities of the following.

a. All 58 like the product.

b. From 28 to 30 (inclusive) like the product.

41. Pecans Pecan producers blow air through the pecans so that the lighter ones are blown out. The lighter-weight pecans are generally bad and the heavier ones tend to be better. These "blow outs" and "good nuts" are often sold to tourists along the highway. Suppose 60% of the "blow outs" are good, and 80% of the "good nuts" are good. ***Source: Irvin R. Hentzel.***

a. What is the probability that if you crack and check 20 "good nuts" you will find 8 bad ones?

b. What is the probability that if you crack and check 20 "blow outs" you will find 8 bad ones?

c. If we assume that 70% of the roadside stands sell "good nuts," and that out of 20 nuts we find 8 that are bad, what is the probability that the nuts are "blow outs"?

42. Hurricane Insurance A company prices its hurricane insurance using the following assumptions:

(i) In any calendar year, there can be at most one hurricane.

(ii) In any calendar year, the probability of a hurricane is 0.05.

(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period. Choose one of the following. ***Source: Society of Actuaries.***

a. 0.06 **b.** 0.19 **c.** 0.38 **d.** 0.62 **e.** 0.92

Life Sciences

Breast Cancer **A recent study found that 85% of breast-cancer cases are detectable by mammogram. Suppose a random sample of 15 women with breast cancer are given mammograms. Find the probability of each of the following results, assuming that detection in the cases is independent. *Source: Harper's Index.***

43. All of the cases are detectable.

44. None of the cases are detectable.

45. Not all cases are detectable.

46. More than half of the cases are detectable.

Births of Twins **The probability that a birth will result in twins is 0.012. Assuming independence (perhaps not a valid assumption), what are the probabilities that out of 100 births in a hospital, there will be the following numbers of sets of twins?**

47. Exactly 2 sets of twins

48. At most 2 sets of twins

49. Effects of Radiation The probability of a mutation of a given gene under a dose of 1 roentgen of radiation is approximately 2.5×10^{-7}. What is the probability that in 10,000 genes, at least 1 mutation occurs?

50. Flu Inoculations A flu vaccine has a probability of 80% of preventing a person who is inoculated from getting the flu. A county health office inoculates 83 people. Find the probabilities of the following.

a. Exactly 10 of the people inoculated get the flu.

b. No more than 4 of the people inoculated get the flu.

c. None of the people inoculated get the flu.

51. Color Blindness The probability that a male will be color-blind is 0.042. Find the probabilities that in a group of 53 men, the following will be true.

a. Exactly 5 are color-blind.

b. No more than 5 are color-blind.

c. At least 1 is color-blind.

52. Pharmacology In placebo-controlled trials of Pravachol®, a drug that is prescribed to lower cholesterol, 7.3% of the patients who were taking the drug experienced nausea/ vomiting, whereas 7.1% of the patients who were taking the placebo experienced nausea/vomiting. ***Source: Bristol-Myers Squibb Company.***

a. If 100 patients who are taking Pravachol® are selected, what is the probability that 10 or more will experience nausea/ vomiting?

b. If a second group of 100 patients receives a placebo, what is the probability that 10 or more will experience nausea/vomiting?

c. Since 7.3% is larger than 7.1%, do you believe that the Pravachol® causes more people to experience nausea/ vomiting than a placebo? Explain.

53. Genetic Fingerprinting The use of DNA has become an integral part of many court cases. When DNA is extracted from cells and body fluids, genetic information is represented by bands of information, which look similar to a bar code at a grocery store. It is generally accepted that in unrelated people, the probability of a particular band matching is 1 in 4. ***Source: University of Exeter.***

a. If 5 bands are compared in unrelated people, what is the probability that all 5 of the bands match? (Express your answer in terms of "1 chance in ?".)

b. If 20 bands are compared in unrelated people, what is the probability that all 20 of the bands match? (Express your answer in terms of "1 chance in ?".)

c. If 20 bands are compared in unrelated people, what is the probability that 16 or more bands match? (Express your answer in terms of "1 chance in ?".)

d. If you were deciding paternity and there were 16 matches out of 20 bands compared, would you believe that the person being tested was the father? Explain.

54. Salmonella According to *The Salt Lake Tribune*, the Coffee Garden in Salt Lake City ran into trouble because of their four-egg quiche:

"A Salt Lake County Health Department inspector paid a visit recently and pointed out that research by the Food and Drug Administration indicates that one in four eggs carries *Salmonella* bacterium, so restaurants should never use more than three eggs when preparing quiche.

The manager on duty wondered aloud if simply throwing out three eggs from each dozen and using the remaining nine in four-egg quiches would serve the same purpose.

The inspector wasn't sure, but she said she would research it." ***Source: The Salt Lake Tribune.***

a. Assuming that one in four eggs carries *Salmonella*, and that the event that any one egg is infected is independent of whether any other egg is infected, find the probability that at least one of the eggs in a four-egg quiche carries *Salmonella.*

b. Repeat part a for a three-egg quiche.

c. Discuss whether the assumption of independence is justified.

d. Discuss whether the inspector's reasoning makes sense.

55. Herbal Remedies According to Dr. Peter A.G.M. De Smet of the Netherlands, "If an herb caused an adverse reaction in 1 in 1,000 users, a traditional healer would have to treat 4,800 patients with that herb (i.e., one new patient every single working day for more than 18 years) to have a 95 percent chance of observing the reaction in more than one user." Verify this calculation by finding the probability of observing more than one reaction in 4800 patients, given that 1 in 1000 has a reaction. ***Source: The New England Journal of Medicine.***

56. Vaccines A hospital receives 1/5 of its flu vaccine shipments from Company X and the remainder of its shipments from other companies. Each shipment contains a very large number of vaccine vials. For Company X's shipments, 10% of the vials are ineffective. For every other company, 2% of the vials are ineffective. The hospital tests 30 randomly selected vials from a shipment and finds that one vial is ineffective. What is the probability that this shipment came from Company X? Choose one of the following. (*Hint:* Find the probability that one out of 30 vials is ineffective, given that the shipment came from Company X and that the shipment came from other companies. Then use Bayes' theorem.) ***Source: Society of Actuaries.***

a. 0.10 **b.** 0.14 **c.** 0.37 **d.** 0.63 **e.** 0.86

57. Health Study A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups? Choose one of the following. ***Source: Society of Actuaries.***

a. 0.096 **b.** 0.192 **c.** 0.235 **d.** 0.376 **e.** 0.469

Social Sciences

58. Women Working A recent study found that 60% of working mothers would prefer to work part-time if money were not a concern. Find the probability that if 10 working mothers are selected at random, that at least 3 of them would prefer to work part-time. ***Source: Pew Research Center.***

Volunteering **A recent survey found that 83% of first-year college students were involved in volunteer work at least occasionally. Suppose a random sample of 12 college students is taken. Find the probabilities of each of the following results.** ***Source: The New York Times.***

59. Exactly 7 students volunteered at least occasionally.

60. Exactly 9 students volunteered at least occasionally.

61. At least 9 students volunteered at least occasionally.

62. At most 9 students volunteered at least occasionally.

63. Minority Enrollment According to the U.S. Department of Education, 32.2% of all students enrolled in degree-granting institutions (those that grant associate's or higher degrees) belong to minorities. Find the probabilities of the following results in a random sample of 10 students enrolled in degree-granting institutions. ***Source: National Center for Education Statistics.***

a. Exactly 2 belong to a minority.

b. Three or fewer belong to a minority.

c. Exactly 5 do not belong to a minority.

d. Six or more do not belong to a minority.

64. Cheating According to a poll conducted by *U.S. News and World Report*, 84% of college students believe they need to cheat to get ahead in the world today. ***Source: U.S. News and World Report.***

a. Do the results of this poll indicate that 84% of all college students cheat? Explain.

b. If this result is accurate and 100 college students are asked if they believe that cheating is necessary to get ahead in the world, what is the probability that 90 or more of the students will answer affirmatively to the question?

65. Education A study by Cleveland Clinic tracked a cohort of very-low-birth-weight infants for twenty years. The results of the study indicated that 74% of the very-low-birth-weight babies graduated from high school during this time period. The study also reported that 83% of the comparison group of normal-birth-weight babies graduated from high school during the same period. ***Source: The New England Journal of Medicine.***

a. If 40 very-low-birth-weight babies were tracked through high school, what is the probability that at least 30 will graduate from high school by age 20?

b. If 40 babies from the comparison group were tracked through high school, what is the probability that at least 30 will graduate from high school by age 20?

66. War Dead A newspaper article questioned whether soldiers and marines from some states bear greater risks in Afghanistan and Iraq than those from others. Out of 644,066 troops deployed as of the time of the article, 1174 had been killed, for a probability of being killed of $p = 1174/644{,}066$. Assume the deaths are independent.* ***Source: Valley News.***

a. Vermont had 9 deaths out of 1613 troops deployed. Find the probability of at least this many deaths.

*For further statistical analysis, see http://www.dartmouth.edu/~chance/ForWiki/GregComments.pdf.

b. Massachusetts had 28 deaths out of 7146 troops deployed. Find the probability of at least this many deaths.

c. Florida had 54 deaths out of 62,572 troops deployed. Find the probability of at most this many deaths.

d. Discuss why the assumption of independence may be questionable.

67. Sports In many sports championships, such as the World Series in baseball and the Stanley Cup final series in hockey, the winner is the first team to win four games. For this exercise, assume that each game is independent of the others, with a constant probability p that one specified team (say, the National League team) wins.

a. Find the probability that the series lasts for four, five, six, and seven games when $p = 0.5$. (*Hint:* Suppose the National League wins the series, so they must win the last game. Consider how the previous games might come out. Then consider the probability that the American League wins.)

b. Morrison and Schmittlein have found that the Stanley Cup finals can be described by letting $p = 0.73$ be the probability that the better team wins each game. Find the probability that the series lasts for four, five, six, and seven games. ***Source: Chance.***

c. Some have argued that the assumption of independence does not apply. Discuss this issue. ***Source: Mathematics Magazine.***

YOUR TURN ANSWERS

1. 0.1475 **2.** 0.2734 **3.** 0.009617
4. 0.7459 **5.** 0.9372

8.5 Probability Distributions; Expected Value

APPLY IT **What is the expected payback for someone who buys one ticket in a raffle?**
In Example 3, we will calculate the expected payback or expected value of this raffle.

We shall see that the *expected value* of a probability distribution is a type of average. Probability distributions were introduced briefly in Chapter 7 on Sets and Probability. Now we take a more complete look at probability distributions. A probability distribution depends on the idea of a *random variable*, so we begin with that.

Random Variables

When researchers carry out an experiment it is necessary to quantify the possible outcomes of the experiment. This process will enable the researcher to recognize individual outcomes and analyze the data. The most common way to keep track of the individual outcomes of the experiment is to assign a numerical value to each of the different possible outcomes of the experiment. For example, if a coin is tossed 2 times, the possible outcomes are: *hh*, *ht*, *th*, and *tt*. For each of these possible outcomes, we could record the number of heads. Then the outcome, which we will label x, is one of the numbers 0, 1, or 2. Of course, we could have used other numbers, like 00, 01, 10, and 11, to indicate these same outcomes, but the values of 0, 1, and 2 are simpler and provide an immediate description of the exact outcome of the experiment. Notice that using this random variable also gives us a way to readily know how many tails occurred in the experiment. Thus, in some sense, the values of x are random, so x is called a **random variable.**

Random Variable

A **random variable** is a function that assigns a real number to each outcome of an experiment.

Probability Distribution

A table that lists the possible values of a random variable, together with the corresponding probabilities, is called a **probability distribution**. The sum of the probabilities in a probability distribution must always equal 1. (The sum in some distributions may vary slightly from 1 because of rounding.)

EXAMPLE 1 Computer Monitors

A shipment of 12 computer monitors contains 3 broken monitors. A shipping manager checks a sample of four monitors to see if any are broken. Give the probability distribution for the number of broken monitors that the shipping manager finds.

SOLUTION Let x represent the random variable "number of broken monitors found by the manager." Since there are 3 broken monitors, the possible values of x are 0, 1, 2, and 3. We can calculate the probability of each x using the methods of Section 8.3. There are 3 broken monitors, and 9 unbroken monitors, so the number of ways of choosing 0 broken monitors (which implies 4 unbroken monitors) is $C(3, 0) \cdot C(9, 4)$. The number of ways of choosing a sample of 4 monitors is $C(12, 4)$. Therefore, the probability of choosing 0 broken monitors is

$$P(0) = \frac{C(3,0) \cdot C(9,4)}{C(12,4)} = \frac{1\left(\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}\right)}{\left(\frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1}\right)} = \frac{126}{495} = \frac{14}{55}.$$

Similarly, the probability of choosing 1 broken monitor is

$$P(1) = \frac{C(3,1) \cdot C(9,3)}{C(12,4)} = \frac{3 \cdot 84}{495} = \frac{252}{495} = \frac{28}{55}.$$

The probability of choosing 2 broken monitors is

$$P(2) = \frac{C(3,2) \cdot C(9,2)}{C(12,4)} = \frac{3 \cdot 36}{495} = \frac{108}{495} = \frac{12}{55}.$$

The probability of choosing 3 broken monitors is

$$P(3) = \frac{C(3,3) \cdot C(9,1)}{C(12,4)} = \frac{1 \cdot 9}{495} = \frac{9}{495} = \frac{1}{55}.$$

YOUR TURN 1 Suppose the inspector in Example 1 chose only two monitors to inspect. Find the probability distribution for the number of broken monitors.

The results can be put in a table, called a probability distribution.

Probability Distribution of Broken Monitors in Sample				
x	0	1	2	3
$P(x)$	14/55	28/55	12/55	1/55

TRY YOUR TURN 1

Instead of writing the probability distribution as a table, we could write the same information as a set of ordered pairs:

$$\{(0, 14/55), (1, 28/55), (2, 12/55), (3, 1/55)\}.$$

There is just one probability for each value of the random variable. Thus, a probability distribution defines a function, called a **probability distribution function**, or simply a **probability function**. We shall use the terms "probability distribution" and "probability function" interchangeably.

The information in a probability distribution is often displayed graphically as a special kind of bar graph called a **histogram**. The bars of a histogram all have the same width, usually 1. (The widths might be different from 1 when the values of the random variable are not consecutive integers.) The heights of the bars are determined by the probabilities. A histogram for the data in Example 1 is given in Figure 8. A histogram shows important characteristics of a distribution that may not be readily apparent in tabular form, such as the relative sizes of the probabilities and any symmetry in the distribution.

The area of the bar above $x = 0$ in Figure 8 is the product of 1 and 14/55, or $1 \cdot 14/55 = 14/55$. Since each bar has a width of 1, its area is equal to the probability that corresponds to that value of x. The probability that a particular value will occur is thus given by the area of the appropriate bar of the graph. For example, the probability that one or more monitors is broken is the sum of the areas for $x = 1$, $x = 2$, and $x = 3$. This area, shown in pink in Figure 9, corresponds to 41/55 of the total area, since

$$\begin{aligned} P(x \geq 1) &= P(x = 1) + P(x = 2) + P(x = 3) \\ &= 28/55 + 12/55 + 1/55 = 41/55. \end{aligned}$$

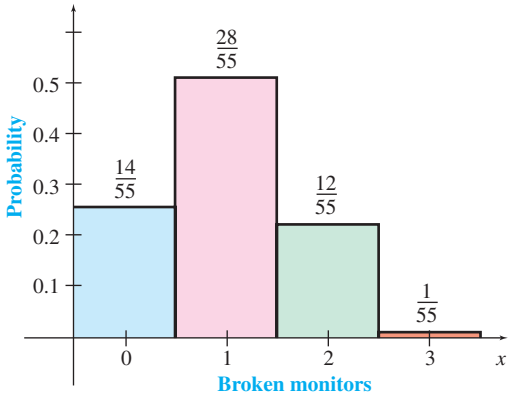


FIGURE 8

FIGURE 9

EXAMPLE 2 Probability Distributions

(a) Give the probability distribution for the number of heads showing when two coins are tossed.

SOLUTION Let x represent the random variable "number of heads." Then x can take on the values 0, 1, or 2. Now find the probability of each outcome. To find the probability of 0, 1, or 2 heads, we can either use binomial probability, or notice that there are 4 outcomes in the sample space: $\{hh, ht, th, tt\}$.The results are shown in the table with Figure 10.

Probability Distribution of Heads			
x	0	1	2
$P(x)$	1/4	1/2	1/4

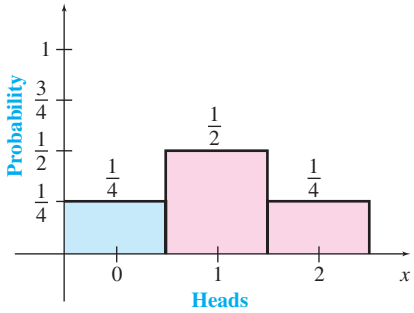


FIGURE 10

(b) Draw a histogram for the distribution in the table. Find the probability that at least one coin comes up heads.

SOLUTION The histogram is shown in Figure 10. The portion in pink represents

$$P(x \geq 1) = P(x = 1) + P(x = 2)$$
$$= \frac{3}{4}.$$

TRY YOUR TURN 2

YOUR TURN 2 Find the probability distribution and draw a histogram for the number of tails showing when three coins are tossed.

Expected Value

In working with probability distributions, it is useful to have a concept of the typical or average value that the random variable takes on. In Example 2, for instance, it seems reasonable that, on the average, one head shows when two coins are tossed. This does not tell what will happen the next time we toss two coins; we may get two heads, or we may get none. If we tossed two coins many times, however, we would expect that, in the long run, we would average about one head for each toss of two coins.

A way to solve such problems in general is to imagine flipping two coins 4 times. Based on the probability distribution in Example 2, we would expect that 1 of the 4 times we would get 0

heads, 2 of the 4 times we would get 1 head, and 1 of the 4 times we would get 2 heads. The total number of heads we would get, then, is

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 1 = 4.$$

The expected numbers of heads per toss is found by dividing the total number of heads by the total number of tosses, or

$$\frac{0 \cdot 1 + 1 \cdot 2 + 2 \cdot 1}{4} = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1.$$

Notice that the expected number of heads turns out to be the sum of the three values of the random variable x multiplied by their corresponding probabilities. We can use this idea to define the *expected value* of a random variable as follows.

Expected Value

Suppose the random variable x can take on the n values $x_1, x_2, x_3, \ldots, x_n$. Also, suppose the probabilities that these values occur are, respectively, $p_1, p_2, p_3, \ldots, p_n$. Then the **expected value** of the random variable is

$$E(x) = x_1p_1 + x_2p_2 + x_3p_3 + \cdots + x_np_n.$$

EXAMPLE 3 Computer Monitors

In Example 1, find the expected number of broken monitors that the shipping manager finds.

SOLUTION Using the values in the first table in this section and the definition of expected value, we find that

$$E(x) = 0 \cdot \frac{14}{55} + 1 \cdot \frac{28}{55} + 2 \cdot \frac{12}{55} + 3 \cdot \frac{1}{55} = 1.$$

On average, the shipping manager will find 1 broken monitor in the sample of 4. On reflection, this seems natural; 3 of the 12 monitors, or $1/4$ of the total, are broken. We should expect, then, that $1/4$ of the sample of 4 monitors are broken.

Physically, the expected value of a probability distribution represents a balance point. If we think of the histogram in Figure 8 as a series of weights with magnitudes represented by the heights of the bars, then the system would balance if supported at the point corresponding to the expected value.

EXAMPLE 4 Symphony Orchestra

Suppose a local symphony decides to raise money by raffling an HD television worth $400, a dinner for two worth $80, and 2 CDs worth $20 each. A total of 2000 tickets are sold at $1 each. Find the expected payback for a person who buys one ticket in the raffle.

APPLY IT

Method 1
Direct Calculation

SOLUTION

Here the random variable represents the possible amounts of payback, where payback = amount won − cost of ticket. The payback of the person winning the television is $400 (amount won) − $1 (cost of ticket) = $399. The payback for each losing ticket is $0 − $1 = −$1.

The paybacks of the various prizes, as well as their respective probabilities, are shown in the table on the next page. The probability of winning $19 is $2/2000$ because there are 2 prizes worth $20. We have not reduced the fractions in order to keep all the denominators equal. Because there are 4 winning tickets, there are 1996 losing tickets, so the probability of winning −$1 is $1996/2000$.

Probability Distribution of Prize Winnings				
x	\$399	\$79	\$19	−\$1
$P(x)$	1/2000	1/2000	2/2000	1996/2000

The expected payback for a person buying one ticket is

$$399\left(\frac{1}{2000}\right) + 79\left(\frac{1}{2000}\right) + 19\left(\frac{2}{2000}\right) + (-1)\left(\frac{1996}{2000}\right) = -\frac{1480}{2000}$$
$$= -0.74.$$

On average, a person buying one ticket in the raffle will lose \$0.74, or 74¢.

It is not possible to lose 74¢ in this raffle: either you lose \$1, or you win a prize worth \$400, \$80, or \$20, minus the \$1 you pay to play. But if you bought tickets in many such raffles over a long period of time, you would lose 74¢ per ticket on average. It is important to note that the expected value of a random variable may be a number that can never occur in any one trial of the experiment.

Method 2 Alternate Procedure

An alternative way to compute expected value in this and other examples is to calculate the expected amount won and then subtract the cost of the ticket afterward. The amount won is either \$400 (with probability 1/2000), \$80 (with probability 1/2000), \$20 (with probability 2/2000), or \$0 (with probability 1996/2000). The expected payback for a person buying one ticket is then

$$400\left(\frac{1}{2000}\right) + 80\left(\frac{1}{2000}\right) + 20\left(\frac{2}{2000}\right) + 0\left(\frac{1996}{2000}\right) - 1 = -\frac{1480}{2000}$$
$$= -0.74.$$

YOUR TURN 3 Suppose that there is a \$5 raffle with prizes worth \$1000, \$500, and \$250. Suppose 1000 tickets are sold and you purchase a ticket. Find your expected payback for this raffle.

TRY YOUR TURN 3

EXAMPLE 5 Friendly Wager

Each day Donna and Mary toss a coin to see who buys coffee (\$1.20 a cup). One tosses and the other calls the outcome. If the person who calls the outcome is correct, the other buys the coffee; otherwise the caller pays. Find Donna's expected payback.

SOLUTION Assume that an honest coin is used, that Mary tosses the coin, and that Donna calls the outcome. The possible results and corresponding probabilities are shown below.

	Possible Results			
Result of Toss	Heads	Heads	Tails	Tails
Call	Heads	Tails	Heads	Tails
Caller Wins?	Yes	No	No	Yes
Probability	1/4	1/4	1/4	1/4

Donna wins a \$1.20 cup of coffee whenever the results and calls match, and she loses a \$1.20 cup when there is no match. Her expected payback is

$$(1.20)\left(\frac{1}{4}\right) + (-1.20)\left(\frac{1}{4}\right) + (-1.20)\left(\frac{1}{4}\right) + (1.20)\left(\frac{1}{4}\right) = 0.$$

This implies that, over the long run, Donna neither wins nor loses.

A game with an expected value of 0 (such as the one in Example 5) is called a **fair game**. Casinos do not offer fair games. If they did, they would win (on average) \$0, and have a hard time paying the help! Casino games have expected winnings for the house that vary from 1.5 cents per dollar to 60 cents per dollar. Exercises 47–52 at the end of the section ask you to find the expected payback for certain games of chance.

The idea of expected value can be very useful in decision making, as shown by the next example.

EXAMPLE 6 Life Insurance

At age 50, you receive a letter from Mutual of Mauritania Insurance Company. According to the letter, you must tell the company immediately which of the following two options you will choose: take $20,000 at age 60 (if you are alive, $0 otherwise) or $30,000 at age 70 (again, if you are alive, $0 otherwise). Based *only* on the idea of expected value, which should you choose?

SOLUTION Life insurance companies have constructed elaborate tables showing the probability of a person living a given number of years into the future. From a recent such table, the probability of living from age 50 to 60 is 0.88, while the probability of living from age 50 to 70 is 0.64. The expected values of the two options are given below.

$$\text{First option: } (20{,}000)(0.88) + (0)(0.12) = 17{,}600$$
$$\text{Second option: } (30{,}000)(0.64) + (0)(0.36) = 19{,}200$$

Based strictly on expected values, choose the second option.

EXAMPLE 7 Bachelor's Degrees

According to the National Center for Education Statistics, 78.7% of those earning bachelor's degrees in education in the United States in 2006–2007 were female. Suppose 5 holders of bachelor's degrees in education from 2006 to 2007 are picked at random. ***Source: National Center for Education Statistics.***

(a) Find the probability distribution for the number that are female.

SOLUTION We first note that each of the 5 people in the sample is either female (with probability 0.787) or male (with probability 0.213). As in the previous section, we may assume that the probability for each member of the sample is independent of that of any other. Such a situation is described by binomial probability with $n = 5$ and $p = 0.787$, for which we use the binomial probability formula

$$P(x \text{ successes in } n \text{ trials}) = C(n, x) \cdot p^x \cdot (1 - p)^{n-x},$$

where x is the number of females in the sample. For example, the probability of 0 females is

$$P(x = 0) = C(5, 0)(0.787)^0(0.213)^5 \approx 0.0004.$$

Similarly, we could calculate the probability that x is any value from 0 to 5, resulting in the probability distribution below (with all probabilities rounded to four places).

Probability Distribution of Female Education Graduate						
x	0	1	2	3	4	5
$P(x)$	0.0004	0.0081	0.0599	0.2211	0.4086	0.3019

(b) Find the expected number of females in the sample of 5 people.

SOLUTION Using the formula for expected value, we have

$$E(x) = 0(0.0004) + 1(0.0081) + 2(0.0599) + 3(0.2211) + 4(0.4086) + 5(0.3019) = 3.935.$$

On average, 3.935 of the people in the sample of 5 will be female.

YOUR TURN 4 In the same survey quoted in Example 7, 81.6% of those earning bachelor's degrees in engineering were male. Suppose 5 holders of bachelor's degrees in engineering were picked at random, find the expected number of male engineers.

TRY YOUR TURN 4

There is another way to get the answer in part (b) of the previous example. Because 78.7% of those earning bachelor's degrees in education in the United States in 2006–2007 are female, it is reasonable to expect 78.7% of our sample to be female. Thus, 78.7% of 5 is $5(0.787) = 3.935$. Notice that what we have done is to multiply n by p. It can be shown that this method always gives the expected value for binomial probability.

Expected Value for Binomial Probability

For binomial probability, $E(x) = np$. In other words, the expected number of successes is the number of trials times the probability of success in each trial.

EXAMPLE 8 Female Children

Suppose a family has 3 children.

(a) Find the probability distribution for the number of girls.

SOLUTION Assuming girls and boys are equally likely, the probability distribution is binomial with $n = 3$ and $p = 1/2$. Letting x be the number of girls in the formula for binomial probability, we find, for example,

$$P(x = 0) = C(3, 0)\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

The other values are found similarly, and the results are shown in the following table.

Probability Distribution of Number of Girls				
x	0	1	2	3
$P(x)$	1/8	3/8	3/8	1/8

We can verify this by noticing that in the sample space S of all 3-child families, there are eight equally likely outcomes: $S = \{ggg, ggb, gbg, gbb, bgg, bgb, bbg, bbb\}$. One of the outcomes has 0 girls, three have 1 girl, three have 2 girls, and one has 3 girls.

(b) Find the expected number of girls in a 3-child family using the distribution from part (a).

SOLUTION Using the formula for expected value, we have

$$\text{Expected number of girls} = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$
$$= \frac{12}{8} = 1.5.$$

On average, a 3-child family will have 1.5 girls. This result agrees with our intuition that, on average, half the children born will be girls.

(c) Find the expected number of girls in a 3-child family using the formula for expected value for binomial probability.

SOLUTION Using the formula $E(x) = np$ with $n = 3$ and $p = 1/2$, we have

$$\text{Expected number of girls} = 3\left(\frac{1}{2}\right) = 1.5.$$

This agrees with our answer from part (b), as it must. TRY YOUR TURN 5

YOUR TURN 5 Find the expected number of girls in a family of a dozen children.

8.5 EXERCISES

For each experiment described below, let x determine a random variable, and use your knowledge of probability to prepare a probability distribution.

1. Four coins are tossed, and the number of heads is noted.
2. Two dice are rolled, and the total number of points is recorded.
3. Three cards are drawn from a deck. The number of aces is counted.
4. Two balls are drawn from a bag in which there are 4 white balls and 2 black balls. The number of black balls is counted.

Draw a histogram for the following, and shade the region that gives the indicated probability.

5. Exercise 1; $P(x \le 2)$
6. Exercise 2; $P(x \ge 11)$

7. Exercise 3; $P(\text{at least one ace})$

8. Exercise 4; $P(\text{at least one black ball})$

Find the expected value for each random variable.

9.

x	2	3	4	5
$P(x)$	0.1	0.4	0.3	0.2

10.

y	4	6	8	10
$P(y)$	0.4	0.4	0.05	0.15

11.

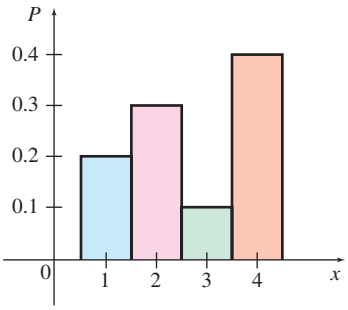
z	9	12	15	18	21
$P(z)$	0.14	0.22	0.38	0.19	0.07

12.

x	30	32	36	38	44
$P(x)$	0.31	0.29	0.26	0.09	0.05

Find the expected value for the random variable x having the probability function shown in each graph.

13.



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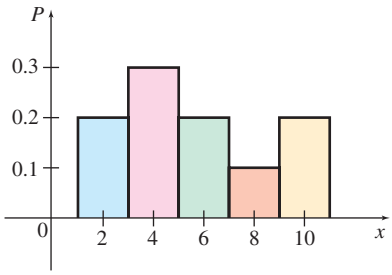

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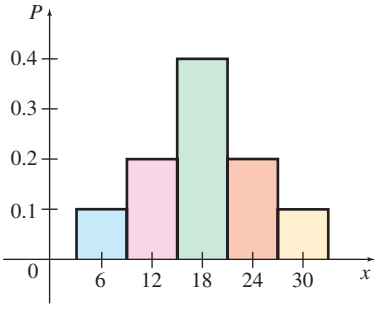

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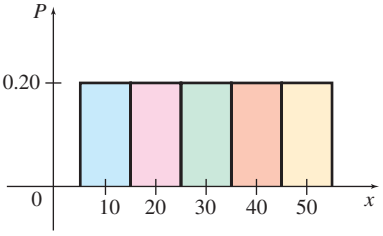


17. For the game in Example 5, find Mary's expected payback. Is it a fair game?

18. Suppose one day Mary brings a 2-headed coin and uses it to toss for the coffee. Since Mary tosses, Donna calls.

a. Is this still a fair game?

b. What is Donna's expected payback if she calls heads?

c. What is Donna's expected payback if she calls tails?

Solve each exercise. Many of these exercises require the use of combinations.

19. Suppose 3 marbles are drawn from a bag containing 3 yellow and 4 white marbles.

a. Draw a histogram for the number of yellow marbles in the sample.

b. What is the expected number of yellow marbles in the sample?

20. Suppose 5 apples in a barrel of 25 apples are known to be rotten.

a. Draw a histogram for the number of rotten apples in a sample of 2 apples.

b. What is the expected number of rotten apples in a sample of 2 apples?

21. Suppose a die is rolled 4 times.

a. Find the probability distribution for the number of times 1 is rolled.

b. What is the expected number of times 1 is rolled?

22. A delegation of 3 is selected from a city council made up of 5 liberals and 6 conservatives.

a. What is the expected number of liberals in the delegation?

b. What is the expected number of conservatives in the delegation?

23. From a group of 3 women and 5 men, a delegation of 2 is selected. Find the expected number of women in the delegation.

24. In a club with 20 senior and 10 junior members, what is the expected number of junior members on a 4-member committee?

25. If 2 cards are drawn at one time from a deck of 52 cards, what is the expected number of diamonds?

26. Suppose someone offers to pay you \$5 if you draw 2 diamonds in the game in Exercise 25. He says that you should pay 50 cents for the chance to play. Is this a fair game?

27. Your friend missed class the day probability distributions were discussed. How would you explain probability distribution to him?

28. Explain what expected value means in your own words.

29. Four slips of paper numbered 2, 3, 4, and 5 are in a hat. You draw a slip, note the result, and then draw a second slip and note the result (without replacing the first).

a. Find the probability distribution for the sum of the two slips.

b. Draw a histogram for the probability distribution in part a.

c. Find the odds that the sum is even.

d. Find the expected value of the sum.

APPLICATIONS

Business and Economics

30. Complaints A local used-car dealer gets complaints about his cars as shown in the table below. Find the expected number of complaints per day.

Number of Complaints per Day	0	1	2	3	4	5	6
Probability	0.02	0.06	0.16	0.25	0.32	0.13	0.06

31. Payout on Insurance Policies An insurance company has written 100 policies for $100,000, 500 policies for $50,000, and 1000 policies for $10,000 for people of age 20. If experience shows that the probability that a person will die at age 20 is 0.0012, how much can the company expect to pay out during the year the policies were written?

32. Device Failure An insurance policy on an electrical device pays a benefit of $4000 if the device fails during the first year. The amount of the benefit decreases by $1000 each successive year until it reaches 0. If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4. What is the expected benefit under this policy? Choose one of the following. ***Source: Society of Actuaries.***

a. $2234 **b.** $2400 **c.** $2500 **d.** $2667 **e.** $2694

33. Pecans Refer to Exercise 41 in Section 8.4. Suppose that 60% of the pecan "blow outs" are good, and 80% of the "good nuts" are good.

a. If you purchase 50 pecans, what is the expected number of good nuts you will find if you purchase "blow outs"?

b. If you purchase 50 pecans, what is the expected number of bad nuts you will find if you have purchased "good nuts"?

34. Rating Sales Accounts Levi Strauss and Company uses expected value to help its salespeople rate their accounts. For each account, a salesperson estimates potential additional volume and the probability of getting it. The product of these figures gives the expected value of the potential, which is added to the existing volume. The totals are then classified as A, B, or C, as follows: $40,000 or below, class C; from $40,000 up to and including $55,000, class B; above $55,000, class A. Complete the table below for one salesperson. ***Source: James McDonald.***

35. Tour Bus A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists. Each ticket costs $50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay $100 (ticket cost + $50 penalty) to the tourist. What is the expected revenue of the tour operator? Choose one of the following. ***Source: Society of Actuaries.***

a. $935 **b.** $950 **c.** $967 **d.** $976 **e.** $985

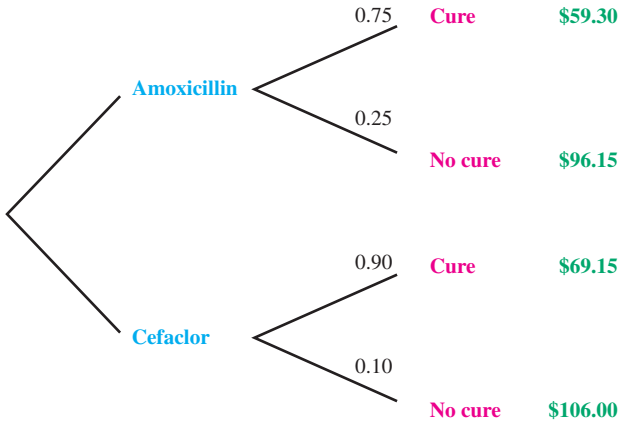
Life Sciences

36. Animal Offspring In a certain animal species, the probability that a healthy adult female will have no offspring in a given year is 0.29, while the probabilities of 1, 2, 3, or 4 offspring are, respectively, 0.23, 0.18, 0.16, and 0.14. Find the expected number of offspring.

37. Ear Infections Otitis media, or middle ear infection, is initially treated with an antibiotic. Researchers have compared two antibiotics, amoxicillin and cefaclor, for their cost effectiveness. Amoxicillin is inexpensive, safe, and effective. Cefaclor is also safe. However, it is considerably more expensive and it is generally more effective. Use the tree diagram below (where the costs are estimated as the total cost of medication, office visit, ear check, and hours of lost work) to answer the following. ***Source: Journal of Pediatric Infectious Disease.***

a. Find the expected cost of using each antibiotic to treat a middle ear infection.

b. To minimize the total expected cost, which antibiotic should be chosen?



Account Number	Existing Volume	Potential Additional Volume	Probability of Getting It	Expected Value of Potential	Existing Volume + Expected Value of Potential	Class
1	$15,000	$10,000	0.25	$2500	$17,500	C
2	$40,000	$0	—	—	$40,000	C
3	$20,000	$10,000	0.20			
4	$50,000	$10,000	0.10			
5	$5000	$50,000	0.50			
6	$0	$100,000	0.60			
7	$30,000	$20,000	0.80			

38. Hospitalization Insurance An insurance policy pays an individual $100 per day for up to 3 days of hospitalization and $25 per day for each day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability function

$$P(X = k) = \begin{cases} \dfrac{6 - k}{15} & \text{for } k = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected payment for hospitalization under this policy. Choose one of the following. ***Source: Society of Actuaries.***

a. $85 **b.** $163 **c.** $168 **d.** $213 **e.** $255

Social Sciences

39. Education Recall from Exercise 65 in Section 8.4 that a study from Cleveland Clinic reported that 74% of very-low-birth-weight babies graduate from high school by age 20. If 250 very-low-birth-weight babies are followed through high school, how many would you expect to graduate from high school? ***Source: The New England Journal of Medicine.***

40. Cheating Recall from Exercise 64 in Section 8.4 that a poll conducted by *U.S. News and World Report* reported that 84% of college students believe they need to cheat to get ahead in the world today. If 500 college students were surveyed, how many would you expect to say that they need to cheat to get ahead in the world today? ***Source: U.S. News and World Report.***

41. Samuel Alito When Supreme Court Justice Samuel Alito was on the U.S. Court of Appeals for the 3rd Circuit, he dissented in the successful appeal of a first-degree murder case. The prosecution used its peremptory challenges to eliminate all African Americans from the jury, as it had in three other first-degree murder trials in the same county that year. According to a majority of the judges, "An amateur with a pocket calculator can calculate the number of blacks that would have served had the State used its strikes in a racially proportionate manner. In the four capital cases there was a total of 82 potential jurors on the venires who were not removed for cause, of whom eight, or 9.76%, were black. If the prosecution had used its peremptory challenges in a manner proportional to the percentage of blacks in the overall venire, then only 3 of the 34 jurors peremptorily struck (8.82%) would have been black and 5 of the 48 actual jurors (10.42%) would have been black. Instead, none of the 48 jurors were black. Admittedly, there was no statistical analysis of these figures presented by either side in the post-conviction proceeding. But is it really necessary to have a sophisticated analysis by a statistician to conclude that there is little chance of randomly selecting four consecutive all white juries?" ***Source: findlaw.com.***

a. Using binomial probability, calculate the probability that no African Americans would be selected out of 48 jurors if the percentage African American is 9.76%.

b. Binomial probability is not entirely accurate in this case, because the jurors were selected without replacement, so the selections were not independent. Recalculate the probability in part a using combinations.

c. In his dissent, Judge Alito wrote, "Statistics can be very revealing—and also terribly misleading in the hands of 'an amateur with a pocket calculator.' . . . Although only about 10% of the population is left-handed, left-handers have won five of the last six presidential elections. Our 'amateur with a calculator' would conclude that 'there is little chance of randomly selecting' left-handers in five out of six presidential elections. But does it follow that the voters cast their ballots based on whether a candidate was right- or left-handed?" Given the figures quoted by Judge Alito, what is the probability that at least 5 out of the last 6 presidents elected would be left-handed?

d. The majority of the judges, in disagreeing with Judge Alito, said, "The dissent has overlooked the obvious fact that there is no provision in the Constitution that protects persons from discrimination based on whether they are right-handed or left handed." Furthermore, according to *Chance News,* only 2 of the last 6 men elected president were left-handed. What is the probability that at least 2 out of the last 6 presidents elected would be left-handed? ***Source: Chance News.***

Physical Sciences

42. Seeding Storms One of the few methods that can be used in an attempt to cut the severity of a hurricane is to *seed* the storm. In this process, silver iodide crystals are dropped into the storm. Unfortunately, silver iodide crystals sometimes cause the storm to *increase* its speed. Wind speeds may also increase or decrease even with no seeding. Use the table below to answer the following. ***Source: Science.***

a. Find the expected amount of damage under each option, "seed" and "do not seed."

b. To minimize total expected damage, what option should be chosen?

	Change in wind speed	Probability	Property damage (millions of dollars)
Seed	+32%	0.038	335.8
	+16%	0.143	191.1
	0	0.392	100.0
	−16%	0.255	46.7
	−34%	0.172	16.3
Do not seed	+32%	0.054	335.8
	+16%	0.206	191.1
	0	0.480	100.0
	−16%	0.206	46.7
	+34%	0.054	16.3

General Interest

43. Cats Kimberly Workman has four cats: Riley, Abby, Beastie, and Sylvester. Each cat has a 30% probability of climbing into

the chair in which Kimberly is sitting, independent of how many cats are already in the chair with Kimberly.

a. Find the probability distribution for the number of cats in the chair with Kimberly.

b. Find the expected number of cats in the chair with Kimberly using the probability distribution in part a.

c. Find the expected number of cats in the chair with Kimberly using the formula for expected value of the binomial distribution.

44. Postal Service Mr. Statistics (a feature in *Fortune* magazine) investigated the claim of the U.S. Postal Service that 83% of first class mail in New York City arrives by the next day. (The figure is 87% nationwide.) He mailed a letter to himself on 10 consecutive days; only 4 were delivered by the next day. ***Source: Fortune.***

a. Find the probability distribution for the number of letters delivered by the next day if the overall probability of next-day delivery is 83%.

b. Using your answer to part a, find the probability that 4 or fewer out of 10 letters would be delivered by the next day.

c. Based on your answer to part b, do you think it is likely that the 83% figure is accurate? Explain.

d. Find the number of letters out of 10 that you would expect to be delivered by the next day if the 83% figure is accurate.

45. Raffle A raffle offers a first prize of $400 and 3 second prizes of $80 each. One ticket costs $2, and 500 tickets are sold. Find the expected payback for a person who buys 1 ticket. Is this a fair game?

46. Raffle A raffle offers a first prize of $1000, 2 second prizes of $300 each, and 20 third prizes of $10 each. If 10,000 tickets are sold at 50¢ each, find the expected payback for a person buying 1 ticket. Is this a fair game?

Find the expected payback for the games of chance described in Exercises 47–52.

47. Lottery A state lottery requires you to choose 4 cards from an ordinary deck: 1 heart, 1 club, 1 diamond, and 1 spade in that order from the 13 cards in each suit. If all four choices are selected by the lottery, you win $5000. It costs $1 to play.

48. Lottery If exactly 3 of the 4 choices in Exercise 47 are selected, the player wins $200. (Ignore the possibility that all 4 choices are selected. It still costs $1 to play.)

49. Roulette In one form of roulette, you bet $1 on "even." If 1 of the 18 even numbers comes up, you get your dollar back, plus another one. If 1 of the 20 noneven (18 odd, 0, and 00) numbers comes up, you lose your dollar.

50. Roulette In another form of roulette, there are only 19 noneven numbers (no 00).

51. Numbers *Numbers* is a game in which you bet $1 on any three-digit number from 000 to 999. If your number comes up, you get $500.

52. Keno In one form of the game *Keno*, the house has a pot containing 80 balls, each marked with a different number from 1 to 80. You buy a ticket for $1 and mark one of the 80 numbers on it. The house then selects 20 numbers at random. If your number is among the 20, you get $3.20 (for a net winning of $2.20).

53. Contests A magazine distributor offers a first prize of $100,000, two second prizes of $40,000 each, and two third prizes of $10,000 each. A total of 2,000,000 entries are received in the contest. Find the expected payback if you submit one entry to the contest. If it would cost you $1 in time, paper, and stamps to enter, would it be worth it?

54. Contests A contest at a fast-food restaurant offered the following cash prizes and probabilities of winning on one visit. Suppose you spend $1 to buy a bus pass that lets you go to 25 different restaurants in the chain and pick up entry forms. Find your expected value.

Prize	Probability
$100,000	1/176,402,500
$25,000	1/39,200,556
$5000	1/17,640,250
$1000	1/1,568,022
$100	1/282,244
$5	1/7056
$1	1/588

55. The Hog Game In the hog game, each player states the number of dice that he or she would like to roll. The player then rolls that many dice. If a 1 comes up on any die, the player's score is 0. Otherwise, the player's score is the sum of the numbers rolled. ***Source: Mathematics Teacher.***

a. Find the expected value of the player's score when the player rolls one die.

b. Find the expected value of the player's score when the player rolls two dice.

c. Verify that the expected nonzero score of a single die is 4, so that if a player rolls n dice that do not result in a score of 0, the expected score is $4n$.

d. Verify that if a player rolls n dice, there are 5^n possible ways to get a nonzero score, and 6^n possible ways to roll the dice. Explain why the expected value, E, of the player's score when the player rolls n dice is then

$$E = \frac{5^n(4n)}{6^n}.$$

56. Football After a team scores a touchdown, it can either attempt to kick an extra point or attempt a two-point conversion. During the 2005 NFL season, two-point conversions were successful 45% of the time and the extra-point kicks were successful 96% of the time. ***Source: NFL.***

a. Calculate the expected value of each strategy.

b. Which strategy, over the long run, will maximize the number of points scored?

c. Using this information, should a team always only use one strategy? Explain.

57. Baseball The 2009 National League batting champion was Hanley Ramirez, with an average of 0.342. This can be interpreted as a probability of 0.342 of getting a hit whenever he bats. Assume that each time at bat is an independent event. Suppose he goes to bat four times in a game. ***Source: baseball-reference.com.***

a. Find the probability distribution for the number of hits.

b. What is the expected number of hits that Hanley Ramirez gets in a game?

YOUR TURN ANSWERS

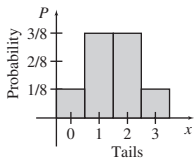
1.

x	0	1	2
$P(x)$	6/11	9/22	1/22

2.

x	0	1	2	3
$P(x)$	1/8	3/8	3/8	1/8

3. −$3.25 **4.** 4.08 **5.** 6



8 CHAPTER REVIEW

SUMMARY

In this chapter we continued our study of probability by introducing some elementary principles of counting. Our primary tool is the multiplication principle:

> If n choices must be made, with m_1 ways to make choice 1, and for each of these ways, m_2 ways to make choice 2, and so on, with m_n ways to make choice n, then there are $m_1 \cdot m_2 \cdot \ldots \cdot m_n$ ways to make the entire sequence of choices.

We learned two counting ideas to efficiently count the number of ways we can select a number of objects without replacement:

- permutations (when order matters), and
- combinations (when order doesn't matter).

We also considered distinguishable permutations, in which some of the objects are indistinguishable. All of these concepts were then used to calculate the numerator and denominator of various probabilities. We next explored binomial probability, in which the following conditions were satisfied:

- the same experiment is repeated a fixed number of times (n),
- there are only two possible outcomes (success and failure), and
- the trials are independent, so the probability of success remains constant (p).

We showed how to quickly calculate an entire set of combinations for binomial probability using Pascal's triangle. Finally, we introduced the following terms regarding probability distributions:

- random variable (a function assigning a real number to each outcome of an experiment),
- probability distribution (the possible values of a random variable, along with the corresponding probabilities),
- histogram (a bar graph displaying a probability distribution), and
- expected value (the average value of a random variable that we would expect in the long run).

In the next chapter, we will see how probability forms the basis of the field known as statistics.

Factorial Notation
$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$
$$0! = 1$$

Permutations
$$P(n,r) = \frac{n!}{(n-r)!}$$

Distinguishable Permutations
If there are n_1 objects of type 1, n_2 of type 2, and so on for r different types, then the number of distinguishable permutations is
$$\frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

Combinations
$$C(n,r) = \frac{n!}{(n-r)!\, r!}$$

Binomial Probability
$$P(x) = C(n,x)\, p^x (1-p)^{n-x}$$

Expected Value
$$E(x) = x_1p_1 + x_2p_2 + x_3p_3 + \cdots + x_np_n$$
For binomial probability, $E(x) = np$.

KEY TERMS

8.1
multiplication principle
factorial notation
permutations
distinguishable permutations

8.2
combinations

8.4
Bernoulli trials
binomial experiment
binomial probability
Pascal's triangle

8.5
random variable
probability distribution
probability function
histogram
expected value
fair game

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. Permutations provide a way of counting possibilities when order matters.

2. Combinations provide a way of counting possibilities when order doesn't matter.

3. The number of distinguishable permutations of n objects, when r are indistinguishable and the remaining $n - r$ are also indistinguishable, is the same as the number of combinations of r objects chosen from n.

4. Calculating the numerator or the denominator of a probability can involve either permutations or combinations.

5. The probability of at least 2 occurrences of an event is equal to the probability of 1 or fewer occurrences.

6. The probability of at least two people in a group having the same birthday is found by subtracting the probability of the complement of the event from 1.

7. The trials in binomial probability must be independent.

8. Binomial probability can be used when each trial has three possible outcomes.

9. A random variable can have negative values.

10. The expected value of a random variable must equal one of the values that the random variable can have.

11. The probabilities in a probability distribution must add up to 1.

12. A fair game can have an expected value that is greater than 0.

PRACTICE AND EXPLORATIONS

13. In how many ways can 6 shuttle vans line up at the airport?

14. How many variations in first-, second-, and third-place finishes are possible in a 100-yd dash with 6 runners?

15. In how many ways can a sample of 3 oranges be taken from a bag of a dozen oranges?

16. In how many ways can a committee of 4 be selected from a club with 10 members?

17. If 2 of the 12 oranges in Exercise 15 are rotten, in how many ways can the sample of 3 include

a. exactly 1 rotten orange?

b. exactly 2 rotten oranges?

c. no rotten oranges?

d. at most 2 rotten oranges?

18. If 6 of the 10 club members in Exercise 16 are males, in how many ways can the sample of 4 include

a. exactly 3 males?

b. no males?

c. at least 2 males?

19. Five different pictures will be arranged in a row on a wall.

a. In how many ways can this be done?

b. In how many ways can this be done if a certain one must be first?

20. In how many ways can the 5 pictures in Exercise 19 be arranged if 2 are landscapes and 3 are puppies and if

a. like types must be kept together?

b. landscapes and puppies must be alternated?

21. In a Chinese restaurant the menu lists 8 items in column A and 6 items in column B.

a. To order a dinner, the diner is told to select 3 items from column A and 2 from column B. How many dinners are possible?

b. How many dinners are possible if the diner can select up to 3 from column A and up to 2 from column B? Assume at least one item must be included from either A or B.

22. A representative is to be selected from each of 3 departments in a small college. There are 7 people in the first department, 5 in the second department, and 4 in the third department.

a. How many different groups of 3 representatives are possible?

b. How many groups are possible if any number (at least 1) up to 3 representatives can form a group? (Each department is still restricted to at most one representative.)

23. Explain under what circumstances a permutation should be used in a probability problem, and under what circumstances a combination should be used.

24. Discuss under what circumstances the binomial probability formula should be used in a probability problem.

A basket contains 4 black, 2 blue, and 7 green balls. A sample of 3 balls is drawn. Find the probabilities that the sample contains the following.

25. All black balls

26. All blue balls

27. 2 black balls and 1 green ball

28. Exactly 2 black balls

29. Exactly 1 blue ball

30. 2 green balls and 1 blue ball

Suppose a family plans 6 children, and the probability that a particular child is a girl is 1/2. Find the probabilities that the 6-child family has the following children.

31. Exactly 3 girls

32. All girls

33. At least 4 girls

34. No more than 2 boys

Suppose 2 cards are drawn without replacement from an ordinary deck of 52. Find the probabilities of the following results.

35. Both cards are red.

36. Both cards are spades.

37. At least 1 card is a spade.

38. One is a face card and the other is not.

39. At least one is a face card.

40. At most one is a queen.

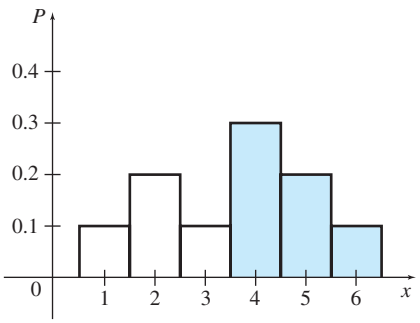
In Exercises 41 and 42, (a) give a probability distribution, (b) sketch its histogram, and (c) find the expected value.

41. A coin is tossed 3 times and the number of heads is recorded.

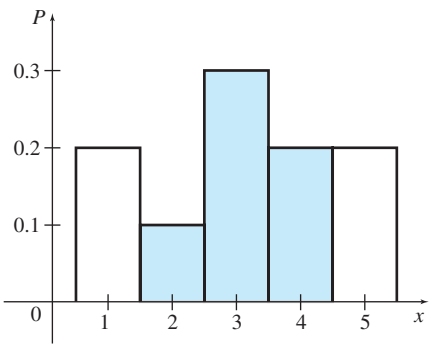
42. A pair of dice is rolled and the sum of the results for each roll is recorded.

In Exercises 43 and 44, give the probability that corresponds to the shaded region of each histogram.

43.



44.



45. You pay \$6 to play in a game where you will roll a die, with payoffs as follows: \$8 for a 6, \$7 for a 5, and \$4 for any other results. What are your expected winnings? Is the game fair?

46. Find the expected number of girls in a family of 5 children.

47. Three cards are drawn from a standard deck of 52 cards.

a. What is the expected number of aces?

b. What is the expected number of clubs?

48. Suppose someone offers to pay you \$100 if you draw 3 cards from a standard deck of 52 cards and all the cards are clubs. What should you pay for the chance to win if it is a fair game?

49. Six students will decide which of them are on a committee by flipping a coin. Each student flips the coin and is on the committee if he or she gets a head. What is the probability that someone is on the committee, but not all 6 students?

50. Find the probability that at most 2 students from Exercise 49 are on the committee.

51. In this exercise we study the connection between sets (from Chapter 7) and combinations (from Chapter 8).

a. Given a set with n elements, what is the number of subsets of size 0? of size 1? of size 2? of size n?

b. Using your answer from part a, give an expression for the total number of subsets of a set with n elements.

c. Using your answer from part b and a result from Chapter 7, explain why the following equation must be true:

$$C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n, n) = 2^n.$$

d. Verify the equation in part c for $n = 4$ and $n = 5$.

e. Explain what the equation in part c tells you about Pascal's triangle.

In the following exercise, find the digit (0 through 9) that belongs in each box. This exercise is from the 1990 University Entrance Center Examination, given in Japan to all applicants for public universities. *Source: Japanese University Entrance Examination Problems in Mathematics.*

52. The numbers 1 through 9 are written individually on nine cards. Choose three cards from the nine, letting x, y, and z denote the numbers of the cards arranged in increasing order.

a. There are $\square\square$ such x, y, and z combinations.

b. The probability of having x, y, and z all even is $\dfrac{\square}{\square\square}$.

c. The probability of having x, y, and z be consecutive numbers is $\dfrac{\square}{\square\square}$.

d. The probability of having $x = 4$ is $\dfrac{\square}{\square\square}$.

e. Possible values of x range from $\square$ to $\square$. If k is an integer such that $\square \le k \le \square$, the probability that $x = k$ is $\dfrac{(\square - k)(\square - k)}{\square\square\square}$. The expected value of x is $\dfrac{\square}{\square}$.

APPLICATIONS

Business and Economics

53. Music Players A popular music player is manufactured in 3 different sizes, 8 different colors, and with or without a camera. How many different varieties of this music player are available?

54. Job Qualifications Of the 12 people applying for an entry level position, 9 are qualified and 3 are not. The personnel manager will hire 4 of the applicants.

a. In how many different ways can she hire the 4 applicants if the jobs are considered the same?

b. In how many ways can she hire 4 applicants that are qualified?

c. In how many ways can she hire at most 1 unqualified applicant?

d. In how many ways can she hire the 4 applicants if the jobs are not the same?

Identity Theft **According to a survey by Javelin Strategy and Research, 1 out of 6 adults in Arizona were victims of identity theft. Suppose that 12 adults are randomly selected from Arizona. Find the probabilities of each of the following results.** ***Source: The New York Times.***

55. None of the adults were victims of identity theft.

56. All of the adults were victims of identity theft.

57. Exactly 10 of the adults were victims of identity theft.

58. Exactly 2 of the adults were victims of identity theft.

59. At least 2 of the adults were victims of identity theft.

60. At most 3 of the adults were victims of identity theft.

61. Find the expected number of victims of identity theft in a sample of 12 adults in Arizona.

62. Land Development A developer can buy a piece of property that will produce a profit of $26,000 with probability 0.7, or a loss of $9000 with probability 0.3. What is the expected profit?

63. Insurance Claims An insurance company determines that N, the number of claims received in a week, is a random variable with $P(N = n) = 1/2^{n+1}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week. Determine the probability that exactly seven claims will be received during a given two-week period. Choose one of the following. ***Source: Society of Actuaries.***

a. $1/256$ **b.** $1/128$ **c.** $7/512$ **d.** $1/64$ **e.** $1/32$

64. Injury Claims The number of injury claims per month is modeled by a random variable N with

$$P(N = n) = \frac{1}{(n + 1)(n + 2)}, \qquad \text{where } n \geq 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month. Choose one of the following. ***Source: Society of Actuaries.***

a. $1/3$ **b.** $2/5$ **c.** $1/2$ **d.** $3/5$ **e.** $5/6$

65. Product Success A company is considering the introduction of a new product that is believed to have probability 0.5 of being successful and probability 0.5 of being unsuccessful. Successful products pass quality control 80% of the time. Unsuccessful products pass quality control 25% of the time. If the product is successful, the net profit to the company will be $40 million; if unsuccessful, the net loss will be $15 million. Determine the expected net profit if the product passes quality control. Choose one of the following. ***Source: Society of Actuaries.***

a. $23 million **b.** $24 million **c.** $25 million

d. $26 million **e.** $27 million

66. Sampling Fruit A merchant buys boxes of fruit from a grower and sells them. Each box of fruit is either Good or Bad. A Good box contains 80% excellent fruit and will earn $200 profit on the retail market. A Bad box contains 30% excellent fruit and will produce a loss of $1000. The a priori probability of receiving a Good box of fruit is 0.9. Before the merchant decides to put the box on the market, he can sample one piece of fruit to test whether it is excellent. Based on that sample, he has the option of rejecting the box without paying for it. Determine the expected value of the right to sample. Choose one of the following. (*Hint:* The a priori probability is the probability before sampling a piece of fruit. If the merchant samples the fruit, what are the probabilities of accepting a Good box, accepting a Bad box, and not accepting the box? What are these probabilities if he does not sample the fruit?) ***Source: Society of Actuaries.***

a. 0 **b.** $16 **c.** $34 **d.** $72 **e.** $80

67. Overbooking Flights The March 1982 issue of *Mathematics Teacher* included "Overbooking Airline Flights," an article by Joe Dan Austin. In this article, Austin developed a model for the expected income for an airline flight. With appropriate assumptions, the probability that exactly x of n people with reservations show up at the airport to buy a ticket is given by the binomial probability formula. Assume the following: 6 reservations have been accepted for 3 seats, $p = 0.6$ is the probability that a person with a reservation will show up, a ticket costs $400, and the airline must pay $400 to anyone with a reservation who does not get a ticket. Complete the following table.

Number Who Show Up (x)	0	1	2	3	4	5	6
Airline's Income							
$P(x)$							

a. Use the table to find $E(I)$, the expected airline income from the 3 seats.

b. Find $E(I)$ for $n = 3$, $n = 4$, and $n = 5$. Compare these answers with $E(I)$ for $n = 6$. For these values of n, how many reservations should the airline book for the 3 seats in order to maximize the expected revenue?

Life Sciences

68. Pharmacology In placebo-controlled trials of Prozac®, a drug that is prescribed to fight depression, 23% of the patients

who were taking the drug experienced nausea, whereas 10% of the patients who were taking the placebo experienced nausea. ***Source: The New England Journal of Medicine.***

a. If 50 patients who are taking Prozac® are selected, what is the probability that 10 or more will experience nausea?

b. Of the 50 patients in part a, what is the expected number of patients who will experience nausea?

c. If a second group of 50 patients receives a placebo, what is the probability that 10 or fewer will experience nausea?

d. If a patient from a study of 1000 people, who are equally divided into two groups (those taking a placebo and those taking Prozac®), is experiencing nausea, what is the probability that he/she is taking Prozac®?

e. Since 0.23 is more than twice as large as 0.10, do you think that people who take Prozac® are more likely to experience nausea than those who take a placebo? Explain.

Social Sciences

69. Education In Exercise 46 of Section 8.3, we saw that a school in Bangkok requires that students take an entrance examination. After the examination, 5 students are randomly drawn from each group of 40 for automatic acceptance into the school regardless of their performance on the examination. The drawing consists of placing 35 red and 5 green pieces of paper into a box. If the lottery is changed so that each student picks a piece of paper from the box and then returns the piece of paper to the box, find the probability that exactly 5 of the 40 students will choose a green piece of paper. ***Source: Mathematics Teacher.***

General Interest

In Exercises 70–73, (a) give a probability distribution, (b) sketch its histogram, and (c) find the expected value.

70. Candy According to officials of Mars, the makers of M&M Plain Chocolate Candies, 20% of the candies in each bag are orange. Four candies are selected from a bag and the number of orange candies is recorded. ***Source: Mars, Inc.***

71. Women Athletes Since 1992, the Big 10 collegiate sports conference has been committed to increasing opportunities for women in sports. In the 2007–2008 season, 48% of the conference athletes were women. Suppose 5 athletes are picked at random from Big 10 universities. The number of women is recorded. ***Source: Big Ten Conference.***

72. Race In the mathematics honors society at a college, 2 of the 8 members are African American. Three members are selected at random to be interviewed by the student newspaper, and the number of African Americans is noted.

73. Homework In a small class of 10 students, 3 did not do their homework. The professor selects half of the class to present solutions to homework problems on the board, and records how many of those selected did not do their homework.

74. Lottery A lottery has a first prize of $5000, two second prizes of $1000 each, and two $100 third prizes. A total of 10,000 tickets is sold, at $1 each. Find the expected payback of a person buying 1 ticket.

75. Contests At one time, game boards for a United Airlines contest could be obtained by sending a self-addressed, stamped envelope to a certain address. The prize was a ticket for any city to which United flies. Assume that the value of the ticket was $2000 (we might as well go first-class), and that the probability that a particular game board would win was 1/8000. If the stamps to enter the contest cost 44¢ and envelopes cost 4¢ each, find the expected payback for a person ordering 1 game board. (Notice that 2 stamps and envelopes were required to enter.)

76. Lottery On June 23, 2003, an interesting thing happened in the Pennsylvania Lottery's Big 4, in which a four-digit number from 0000 to 9999 is chosen twice a day. On this day, the number 3199 was chosen both times. ***Source: Pennsylvania Lottery.***

a. What is the probability of the same number being chosen twice in one day?

b. What is the probability of the number 3199 being chosen twice in one day?

77. Lottery In the Pennsylvania Lottery's Daily Number (Evening) game, a three-digit number between 000 and 999 is chosen each day. The favorite number among players is 000, which on February 13, 2010, was the winning number for the 13th time since 1980. ***Source: Pennsylvania Lottery.***

a. Find the number of times that 000 would be expected to win in 30 years of play. (Assume that the game is played 365 days a year.)

b. In 2003, a Daily Number (Midday) drawing was added. The number 000 has only occurred once. Find the number of times that 000 would be expected to win in 7 years of play. (Assume that the game is played 365 days a year.)

78. Lottery New York has a lottery game called Quick Draw, in which the player can pick anywhere from 1 up to 10 numbers from 1 to 80. The computer then picks 20 numbers, and how much you win is based on how many of your numbers match the computer's. For simplicity, we will only consider the two cases in which you pick 4 or 5 numbers. The payoffs for each dollar that you bet are given in the table below.

	How Many Numbers Match the Computer's Numbers					
	0	**1**	**2**	**3**	**4**	**5**
You Pick 4	0	0	1	5	55	
You Pick 5	0	0	0	2	20	300

a. According to the Quick Draw playing card, the "Overall Chances of Winning" when you pick 4 are "1:3.86," while the chances when you pick 5 are "1:10.34." Verify these figures.

b. Find the expected value when you pick 4 and when you pick 5, betting $1 each time.

c. Based on your results from parts a and b, are you better off picking 4 numbers or picking 5? Explain your reasoning.

79. Murphy's Law Robert Matthews wrote an article about Murphy's Law, which says that if something can go wrong, it will. He considers Murphy's Law of Odd Socks, which says that if an odd sock can be created it will be, in a drawer of 10 loose pairs of socks. ***Source: Sunday Telegraph.***

a. Find the probability of getting a matching pair when the following numbers of socks are selected at random from the drawer.

i. 5 socks **ii.** 6 socks

b. Matthews says that it is necessary to rummage through 30% of the socks to get a matching pair. Using your answers from part a, explain precisely what he means by that.

c. Matthews claims that if you lose 6 socks at random from the drawer, then it is 100 times more likely that you will be left with the worst possible outcome—6 odd socks—than with a drawer free of odd socks. Verify this calculation by finding the probability that you will be left with 6 odd socks and the probability that you will have a drawer free of odd socks.

80. Baseball The number of runs scored in 16,456 half-innings of the 1986 National League Baseball season was analyzed by Hal Stern. Use the table to answer the following questions. ***Source: Chance News.***

Runs	Frequency	Probability
0	12,087	0.7345
1	2451	0.1489
2	1075	0.0653
3	504	0.0306
4	225	0.0137
5	66	0.0040
6	29	0.0018
7	12	0.0007
8	5	0.0003
9	2	0.0001

a. What is the probability that a given team scored 5 or more runs in any given half-inning during the 1986 season?

b. What is the probability that a given team scored fewer than 2 runs in any given half-inning of the 1986 season?

c. What is the expected number of runs that a team scored during any given half-inning of the 1986 season? Interpret this number.

81. St. Petersburg Paradox Suppose you play a gambling game in which you flip a coin until you get a head. If you get a head on the first toss, you win $2. You win $4 if the first head occurs on the second toss, $8 if it occurs on the the third toss, and so forth, with a prize of $2^n if the first head occurs on the nth toss. Show that the expected value of this game is infinite. Explain why this is a paradox.*

82. Pit The card game of Pit was introduced by Parker Brothers in 1904 and is still popular. In the version owned by one of the authors of this book, there are 10 suits of 9 identical cards, plus the Bull and the Bear card, for a total of 92 cards. (Newer versions of the game have only 8 suits of cards.) For this problem, assume that all 92 cards are used, and you are dealt 9 cards.

a. What is the probability that you have one card from each of 9 different suits, but neither the Bull nor the Bear?

b. What is the probability that you have a pair of cards from one suit and one card from each of 7 other suits, but neither the Bull nor the Bear?

c. What is the probability that you have two pairs of cards from two different suits and one card from each of 5 other suits, but neither the Bull nor the Bear?

*Many articles have been written in an attempt to explain this paradox, first posed by the Swiss mathematician Daniel Bernoulli when he lived in St. Petersburg. For example, see Székely, Gábor and Donald St. P. Richards, "The St. Petersburg Paradox and the Crash of High-Tech Stocks in 2000," *The American Statistician*, Vol. 58, No. 3, Aug. 2004, pp. 225–231.

EXTENDED APPLICATION

OPTIMAL INVENTORY FOR A SERVICE TRUCK

For many different items it is difficult or impossible to take the item to a central repair facility when service is required. Washing machines, large television sets, office copiers, and computers are only a few examples of such items. Service for items of this type is commonly performed by sending a repair person to the item, with the person driving to the location in a truck containing various parts that might be required in repairing the item. Ideally, the truck should contain all the parts that might be required. However, most parts would be needed only infrequently, so that inventory costs for the parts would be high.

An optimum policy for deciding on which parts to stock on a truck would require that the probability of not being able to repair an item without a trip back to the warehouse for needed parts be as low as possible, consistent with minimum inventory costs. An analysis similar to the one below was developed at the Xerox Corporation. ***Source: Management Science.***

To set up a mathematical model for deciding on the optimum truck-stocking policy, let us assume that a broken machine might require one of 5 different parts (we could assume any number of different parts—we use 5 to simplify the notation). Suppose also

that the probability that a particular machine requires part 1 is p_1; that it requires part 2 is p_2; and so on. Assume also that failures of different part types are independent, and that at most one part of each type is used on a given job.

Suppose that, on the average, a repair person makes N service calls per time period. If the repair person is unable to make a repair because at least one of the parts is unavailable, there is a penalty cost, L, corresponding to wasted time for the repair person, an extra trip to the parts depot, customer unhappiness, and so on. For each of the parts carried on the truck, an average inventory cost is incurred. Let H_i be the average inventory cost for part i, where $1 \le i \le 5$.

Let M_1 represent a policy of carrying only part 1 on the repair truck, M_{24} represent a policy of carrying only parts 2 and 4, with M_{12345} and M_0 representing policies of carrying all parts and no parts, respectively.

For policy M_{35}, carrying parts 3 and 5 only, the expected cost per time period per repair person, written $C(M_{35})$, is

$$C(M_{35}) = (H_3 + H_5) + NL[1 - (1 - p_1)(1 - p_2)(1 - p_4)].$$

(The expression in brackets represents the probability of needing at least one of the parts not carried, 1, 2, or 4 here.) As further examples,

$$C(M_{125}) = (H_1 + H_2 + H_5) + NL[1 - (1 - p_3)(1 - p_4)],$$

while

$$\begin{aligned} C(M_{12345}) &= (H_1 + H_2 + H_3 + H_4 + H_5) + NL[1 - 1] \\ &= H_1 + H_2 + H_3 + H_4 + H_5, \end{aligned}$$

and

$$C(M_0) = NL[1 - (1 - p_1)(1 - p_2)(1 - p_3)(1 - p_4)(1 - p_5)].$$

To find the best policy, evaluate $C(M_0)$, $C(M_1), \ldots,$ $C(M_{12345})$, and choose the smallest result. (A general solution method is in the *Management Science* paper.)

Example 1

Suppose that for a particular item, only 3 possible parts might need to be replaced. By studying past records of failures of the item, and finding necessary inventory costs, suppose that the following values have been found.

p_1	p_2	p_3
0.09	0.24	0.17

H_1	H_2	H_3
\$15	\$40	\$9

Suppose $N = 3$ and L is \$54. Then, as an example,

$$\begin{aligned} C(M_1) &= H_1 + NL[1 - (1 - p_2)(1 - p_3)] \\ &= 15 + 3(54)[1 - (1 - 0.24)(1 - 0.17)] \\ &= 15 + 3(54)[1 - (0.76)(0.83)] \\ &\approx 15 + 59.81 = 74.81. \end{aligned}$$

Thus, if policy M_1 is followed (carrying only part 1 on the truck), the expected cost per repair person per time period is \$74.81. Also,

$$\begin{aligned} C(M_{23}) &= H_2 + H_3 + NL[1 - (1 - p_1)] \\ &= 40 + 9 + 3(54)(0.09) = 63.58, \end{aligned}$$

so that M_{23} is a better policy than M_1. By finding the expected values for all other possible policies (see the exercises), the optimum policy may be chosen.

EXERCISES

1. Refer to the example and find the following.
   a. $C(M_0)$ **b.** $C(M_2)$ **c.** $C(M_3)$ **d.** $C(M_{12})$
   e. $C(M_{13})$ **f.** $C(M_{123})$
2. Which policy leads to the lowest expected cost?
3. In the example, $p_1 + p_2 + p_3 = 0.09 + 0.24 + 0.17 = 0.50$. Why is it not necessary that the probabilities add up to 1?
4. Suppose an item to be repaired might need one of n different parts. How many different policies would then need to be evaluated?

DIRECTIONS FOR GROUP PROJECT

Suppose you and three others are employed as service repair persons and that you have some disagreement with your supervisor as to the quantity and type of parts to have on hand for your service calls. Use the answers to Exercises 1–4 to prepare a report with a recommendation to your boss on optimal inventory. Make sure that you describe each concept well since your boss is not mathematically minded.

9 Statistics

9.1 Frequency Distributions; Measures of Central Tendency

9.2 Measures of Variation

9.3 The Normal Distribution

9.4 Normal Approximation to the Binomial Distribution

Chapter 9 Review

Extended Application: Statistics in the Law—The *Castaneda* Decision

To understand the economics of large-scale farming, analysts look at historical data on the farming industry. In an exercise in Section 1 you will calculate basic descriptive statistics for U.S. wheat prices and production levels over a recent decade. Later sections in this chapter develop more sophisticated techniques for extracting useful information from this kind of data.

Statistics is a branch of mathematics that deals with the collection and summarization of data. Methods of statistical analysis make it possible for us to draw conclusions about a population based on data from a sample of the population. Statistical models have become increasingly useful in manufacturing, government, agriculture, medicine, the social sciences, and all types of research. In this chapter we give a brief introduction to some of the key topics from statistical theory.

9.1 Frequency Distributions; Measures of Central Tendency

APPLY IT **How can the results of a survey of business executives on the number of college credits in management needed by a business major best be organized to provide useful information?**

Frequency distributions will provide an answer to this question in Example 1.

Often, a researcher wishes to learn something about a characteristic of a population, but because the population is very large or mobile, it is not possible to examine all of its elements. Instead, a limited sample drawn from the population is studied to determine the characteristics of the population. For these inferences to be correct, the sample chosen must be a **random sample**. Random samples are representative of the population because they are chosen so that every element of the population is equally likely to be selected. A hand dealt from a well-shuffled deck of cards is a random sample.

A random sample can be difficult to obtain in real life. For example, suppose you want to take a random sample of voters in your congressional district to see which candidate they prefer in the next election. If you do a telephone survey, you have a random sample of people who are at home to answer the telephone, underrepresenting those who work a lot of hours and are rarely home to answer the phone, or those who have an unlisted number, or those who cannot afford a telephone, or those who only have a cell phone, or those who refuse to answer telephone surveys. Such people may have a different opinion than those you interview.

A famous example of an inaccurate poll was made by the *Literary Digest* in 1936. Their survey indicated that Alfred Landon would win the presidential election; in fact, Franklin Roosevelt won with 62% of the popular vote. The *Digest's* major error was mailing their surveys to a sample of those listed in telephone directories. During the Depression, many poor people did not have telephones, and the poor voted overwhelmingly for Roosevelt. Modern pollsters use sophisticated techniques to ensure that their sample is as random as possible.

Once a sample has been chosen and all data of interest are collected, the data must be organized so that conclusions may be more easily drawn. One method of organization is to group the data into intervals; equal intervals are usually chosen.

EXAMPLE 1 Business Executives

A survey asked a random sample of 30 business executives for their recommendations as to the number of college credits in management that a business major should have.

The results are shown below. Group the data into intervals and find the frequency of each interval.

3	25	22	16	0	9	14	8	34	21
15	12	9	3	8	15	20	12	28	19
17	16	23	19	12	14	29	13	24	18

APPLY IT

SOLUTION The highest number in the list is 34 and the lowest is 0; one convenient way to group the data is in intervals of size 5, starting with 0–4 and ending with 30–34. This gives an interval for each number in the list and results in seven equal intervals of a convenient size. Too many intervals of smaller size would not simplify the data enough, while too few intervals of larger size would conceal information that the data might provide. A rule of thumb is to use from 6 to 15 intervals.

First tally the number of college credits falling into each interval. Then total the tallies in each interval as in the following table. This table is an example of a **grouped frequency distribution**.

Grouped Frequency Distribution

College Credits	Tally	Frequency
0–4	|||	3
5–9	||||	4
10–14	𝍸 |	6
15–19	𝍸 |||	8
20–24	𝍸	5
25–29	|||	3
30–34	|	1
	Total:	30

The frequency distribution in Example 1 shows information about the data that might not have been noticed before. For example, the interval with the largest number of recommended credits is 15–19, and 19 executives (more than half) recommended between 10 and 24 credits, inclusive. Also, the frequency in each interval increases rather evenly (up to 8) and then decreases at about the same pace. However, some information has been lost; for example, we no longer know how many executives recommended 12 credits.

NOTE
In this section, the heights of the histogram bars give the frequencies. The histograms in Chapter 8 were for probability distributions, and so the heights gave the probabilities.

The information in a grouped frequency distribution can be displayed in a histogram similar to the histograms for probability distributions in Chapter 8. The intervals determine the widths of the bars; if equal intervals are used, all the bars have the same width. The heights of the bars are determined by the frequencies.

A **frequency polygon** is another form of graph that illustrates a grouped frequency distribution. The polygon is formed by joining consecutive midpoints of the tops of the histogram bars with straight line segments. The midpoints of the first and last bars are joined to endpoints on the horizontal axis where the next midpoint would appear.

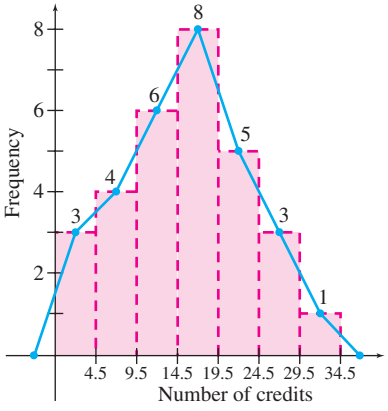


FIGURE 1

EXAMPLE 2 Frequency Distributions

A grouped frequency distribution of college credits was found in Example 1. Draw a histogram and a frequency polygon for this distribution.

SOLUTION First draw a histogram, shown in red in Figure 1. To get a frequency polygon, connect consecutive midpoints of the tops of the bars. The frequency polygon is shown in blue.

TECHNOLOGY NOTE

Many graphing calculators have the capability of drawing a histogram. Figure 2 shows the data of Example 1 drawn on a TI-84 Plus. See the *Graphing Calculator and Excel Spreadsheet Manual* available with this text.

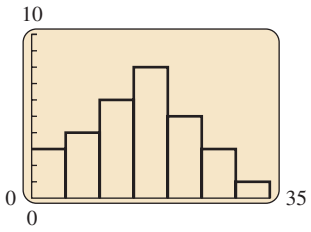


FIGURE 2

Mean

The average value of a probability distribution is the expected value of the distribution. Three measures of central tendency, or "averages," are used with frequency distributions: the mean, the median, and the mode. The most important of these is the mean, which is similar to the expected value of a probability distribution. The **arithmetic mean** (the **mean**) of a set of numbers is the sum of the numbers, divided by the total number of numbers. Recall from Section 1.3 that we can write the sum of n numbers $x_1, x_2, x_3, \ldots, x_n$ in a compact way using *summation notation*:

$$x_1 + x_2 + x_3 + \cdots + x_n = \sum x.$$

The symbol $\bar{x}$ (read x-bar) is used to represent the mean of a sample.

Mean

The **mean** of the n numbers $x_1, x_2, x_3, \ldots, x_n$ is

$$\bar{x} = \frac{\Sigma x}{n}.$$

Bankruptcies	
Year	Petitions Filed
2004	1597
2005	2078
2006	618
2007	851
2008	1118
2009	1474

YOUR TURN 1 Find the mean of the following data: 12, 17, 21, 25, 27, 38, 49.

EXAMPLE 3 Bankruptcy

The table to the left lists the number of bankruptcy petitions (in thousands) filed in the United States in the years 2004–2009. Find the mean number of bankruptcy petitions filed annually during this period. ***Source: American Bankruptcy Institute.***

SOLUTION Let $x_1 = 1597$, $x_2 = 2078$, and so on. Here, $n = 6$, since there are six numbers.

$$\bar{x} = \frac{1597 + 2078 + 618 + 851 + 1118 + 1474}{6}$$

$$\bar{x} = \frac{7736}{6} \approx 1289$$

The mean number of bankruptcy petitions filed during the given years is about 1,289,000. TRY YOUR TURN 1

As another example, the mean response for the number of college credits in management that a business major should have, based on the sample of 30 business executives described in Example 1, is

$$\bar{x} = \frac{(3 + 25 + 22 + \cdots + 18)}{30} = \frac{478}{30} = 15.93.$$

EXAMPLE 4 Mean for Frequency Distributions

Find the mean for the data shown in the following frequency distribution.

Calculations for Frequency Distribution		
Value	**Frequency**	**Value × Frequency**
30	6	$30 \cdot 6 = 180$
32	9	$32 \cdot 9 = 288$
33	7	$33 \cdot 7 = 231$
37	12	$37 \cdot 12 = 444$
42	6	$42 \cdot 6 = 252$
	Total: 40	Total: 1395

SOLUTION The value 30 appears six times, 32 nine times, and so on. To find the mean, first multiply 30 by 6, 32 by 9, and so on.

A new column, "Value × Frequency," has been added to the frequency distribution. Adding the products from this column gives a total of 1395. The total from the frequency column is 40, so $n = 40$. The mean is

$$\bar{x} = \frac{1395}{40} = 34.875.$$

The mean of grouped data is found in a similar way. For grouped data, intervals are used, rather than single values. To calculate the mean, it is assumed that all these values are located at the midpoint of the interval. The letter x is used to represent the midpoints and f represents the frequencies, as shown in the next example.

EXAMPLE 5 Business Executives

Listed below is the grouped frequency distribution for the 30 business executives described in Example 1. Find the mean from the grouped frequency distribution.

Grouped Frequency Distribution for College Credits			
Interval	**Midpoint, x**	**Frequency, f**	**Product, xf**
0–4	2	3	6
5–9	7	4	28
10–14	12	6	72
15–19	17	8	136
20–24	22	5	110
25–29	27	3	81
30–34	32	1	32
		Total: 30	Total: 465

SOLUTION A column for the midpoint of each interval has been added. The numbers in this column are found by adding the endpoints of each interval and dividing by 2. For the interval 0–4, the midpoint is $(0 + 4)/2 = 2$. The numbers in the product column on the right are found by multiplying each frequency by its corresponding midpoint. Finally, we divide the total of the product column by the total of the frequency column to get

$$\bar{x} = \frac{465}{30} = 15.5.$$

Notice that this mean is slightly different from the earlier mean of 15.93. The reason for this difference is that we have acted as if each piece of data is at the midpoint, which is not

YOUR TURN 2 Find the mean of the following grouped frequency.

Interval	Frequency
0–6	2
7–13	4
14–20	7
21–27	10
28–34	3
35–41	1

true here, and is not true in most cases. Information is always lost when the data are grouped. It is more accurate to use the original data, rather than the grouped frequency, when calculating the mean, but the original data might not be available. Furthermore, the mean based upon the grouped data is typically not too far from the mean based upon the original data, and there may be situations in which the extra accuracy is not worth the extra effort. TRY YOUR TURN 2

The formula for the mean of a grouped frequency distribution is given below.

Mean of a Grouped Distribution

The mean of a distribution, where x represents the midpoints, f the frequencies, and $n = \Sigma f$, is

$$\bar{x} = \frac{\Sigma xf}{n}.$$

NOTE

1. The midpoint of the intervals in a grouped frequency distribution may be values that the data cannot take on. For example, if we grouped the data for the 30 business executives into the intervals 0–5, 6–11, 12–17, 18–23, 24–29, and 30–35, the midpoints would be 2.5, 8.5, 14.5, 20.5, 26.5, and 32.5, even though all the data are whole numbers.
2. If we used different intervals in Example 5, the mean would come out to be a slightly different number. Verify that with the intervals 0–5, 6–11, 12–17, 18–23, 24–29, and 30–35, the mean in Example 5 is 16.1.

The mean of a random sample is a random variable, and for this reason it is sometimes called the **sample mean**. The sample mean is a random variable because it assigns a number to the experiment of taking a random sample. If a different random sample were taken, the mean would probably have a different value, with some values more probable than others. If another set of 30 business executives were selected in Example 1, the mean number of college credits in management recommended for a business major might be 13.22 or 17.69. It is unlikely that the mean would be as small as 1.21 or as large as 32.75, although these values are remotely possible.

We saw in Section 8.5 how to calculate the expected value of a random variable when we know its probability distribution. The expected value is sometimes called the **population mean**, denoted by the Greek letter μ. In other words,

$$E(x) = \mu.$$

Furthermore, it can be shown that the expected value of $\bar{x}$ is also equal to μ; that is,

$$E(\bar{x}) = \mu.$$

For instance, consider again the 30 business executives in Example 1. We found that $\bar{x} = 15.93$, but the value of μ, the average for all possible business executives, is unknown. If a good estimate of μ were needed, the best guess (based on this data) is 15.93.

Median

Asked by a reporter to give the average height of the players on his team, a Little League coach lined up his 15 players by increasing height. He picked the player in the middle and pronounced that player to be of average height. This kind of average, called the **median**, is defined as the middle entry in a set of data arranged in either increasing or decreasing order. If there is an even number of entries, the median is defined to be the mean of the two center entries.

Calculating the Median

Odd Number of Entries	Even Number of Entries
1	2
3	3
Median = 4	4
7	7
8	9
	15

Median $= \frac{4 + 7}{2} = 5.5$

EXAMPLE 6 Median

Find the median for each list of numbers.

(a) 11, 12, 17, **20,** 23, 28, 29

SOLUTION The median is the middle number; in this case, 20. (Note that the numbers are already arranged in numerical order.) In this list, three numbers are smaller than 20 and three are larger.

(b) 15, 13, 7, 11, 19, 30, 39, 5, 10

SOLUTION First arrange the numbers in numerical order, from smallest to largest.

5, 7, 10, 11, **13,** 15, 19, 30, 39

The middle number, or median, can now be determined; it is 13.

(c) 47, 59, 32, 81, 74, 153

SOLUTION Write the numbers in numerical order.

32, 47, **59, 74,** 81, 153

There are six numbers here; the median is the mean of the two middle numbers.

$$\text{Median} = \frac{59 + 74}{2} = \frac{133}{2} = 66\frac{1}{2}$$

YOUR TURN 3 Find the median of the data in Your Turn 1.

TRY YOUR TURN 3

Both the mean and the median are examples of a **statistic**, which is simply a number that gives information about a sample. The next example shows a situation in which the median gives a truer representation than the mean.

EXAMPLE 7 Salaries

An office has 10 salespersons, 4 secretaries, the sales manager, and Jeff Weidenaar, who owns the business. Their annual salaries are as follows: secretaries, $35,000 each; salespersons, $45,000 each; manager, $55,000; and owner, $300,000. Calculate the mean and median salaries of those working in the office.

SOLUTION The mean salary is

$$\bar{x} = \frac{(35{,}000)4 + (45{,}000)10 + 55{,}000 + 300{,}000}{16} = \$59{,}062.50$$

However, since 15 people earn less than this mean and only one earns more, the mean does not seem very representative. The median salary is found by ranking the salaries by size: $35,000, $35,000, $35,000, $35,000, $45,000, $45,000, . . . , $300,000. Because there are 16 salaries (an even number) in the list, the mean of the eighth and ninth entries will give the value of the median. The eighth and ninth entries are both $45,000, so the median is $45,000. In this example, the median gives a truer representation than the mean. One salary is much larger than the others and has a pronounced effect on the mean.

Mode

Katie's scores on ten class quizzes include one 7, two 8's, six 9's, and one 10 (out of 10 points possible). She claims that her average grade on quizzes is 9, because most of her scores are 9's. This kind of "average," found by selecting the most frequent entry, is called the **mode**.

EXAMPLE 8 Mode

Find the mode for each list of numbers.

(a) 57, 38, **55, 55,** 80, 87, 98, **55,** 57

SOLUTION The number 55 occurs more often than any other, so it is the mode. It is not necessary to place the numbers in numerical order when looking for the mode.

(b) 182, **185,** 183, **185, 187, 187,** 189

SOLUTION Both 185 and 187 occur twice. This list has *two* modes.

(c) 10,708; 11,519; 10,972; 17,546; 13,905; 12,182

SOLUTION No number occurs more than once. This list has no mode.

The mode has the advantages of being easily found and not being influenced by data that are very large or very small compared to the rest of the data. It is often used in samples where the data to be "averaged" are not numerical. A major disadvantage of the mode is that we cannot always locate exactly one mode for a set of values. There can be more than one mode, in the case of ties, or there can be no mode if all entries occur with the same frequency.

The mean is the most commonly used measure of central tendency. Its advantages are that it is easy to compute, it takes all the data into consideration, and it is reliable—that is, repeated samples are likely to give very similar means. A disadvantage of the mean is that it is influenced by extreme values, as illustrated in the previous salary example.

The median can be easy to compute and is influenced very little by extremes. Like the mode, the median can be found in situations where the data are not numerical. For example, in a taste test, people are asked to rank five soft drinks from the one they like best to the one they like least. The combined rankings then produce an ordered sample, from which the median can be identified. A disadvantage of the median is the need to rank the data in order; this can be difficult when the number of items is large.

EXAMPLE 9 Seed Storage

Seeds that are dried, placed in an airtight container, and stored in a cool, dry place remain ready to be planted for a long time. The table below gives the amount of time that each type of seed can be stored and still remain viable for planting. ***Source: The Handy Science Answer Book.***

	Storage Time		
Vegetable	Years	Vegetable	Years
Beans	3	Cucumbers	5
Cabbage	4	Melons	4
Carrots	1	Peppers	2
Cauliflower	4	Pumpkin	4
Corn	2	Tomatoes	3

Find the mean, median, and mode of the information in the table.

SOLUTION

Method 1 Calculating by Hand

The mean amount of time that the seeds can be stored is

$$\bar{x} = \frac{3 + 4 + 1 + 4 + 2 + 5 + 4 + 2 + 4 + 3}{10} = 3.2 \text{ years.}$$

After the numbers are arranged in order from smallest to largest, the middle number, or median, is found; it is 3.5.

The number 4 occurs more often than any other, so it is the mode.

Method 2 Graphing Calculator

Most scientific calculators have some statistical capability and can calculate the mean of a set of data; graphing calculators can often calculate the median as well. For example, Figure 3 shows the mean and the median for the data above calculated on a TI-84 Plus, where the data was stored in the list L_1. These commands are found under the LIST-MATH menu. This calculator does not include a command for finding the mode.

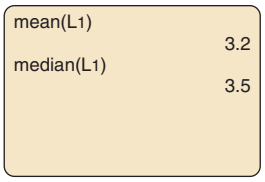


FIGURE 3

Method 3 Spreadsheet

Using Microsoft Excel, place the data in cells A1 through A10. To find the mean of this data, type "=AVERAGE(A1:A10)" in cell A11, or any other unused cell, and then press Enter. The result of 3.2 will appear in cell A11. To find the median of this data, type "=MEDIAN(A1:A10)" in cell A12, or any other unused cell, and press Enter. The result of 3.5 will appear in cell A12. To find the mode of this data, type "=MODE(A1:A10)" in cell A13, or any other unused cell, and press Enter. The result of 4 will appear in cell A13.

9.1 EXERCISES

For Exercises 1–4, do the following:

a. Group the data as indicated.

b. Prepare a frequency distribution with a column for intervals and frequencies.

c. Construct a histogram.

d. Construct a frequency polygon.

1. Use six intervals, starting with 0–24.

7	105	116	73	129	26
29	44	126	82	56	137
43	73	65	141	79	74
121	12	46	37	85	82
2	99	85	95	90	38
86	147	32	84	13	100

2. Use seven intervals, starting with 30–39.

79	71	78	87	69	50	63	51
60	46	65	65	56	88	94	56
74	63	87	62	84	76	82	67
59	66	57	81	93	93	54	88
55	69	78	63	63	48	89	81
98	42	91	66	60	70	64	70
61	75	82	65	68	39	77	81
67	62	73	49	51	76	94	54
83	71	94	45	73	95	72	66
71	77	48	51	54	57	69	87

3. Repeat Exercise 1 using eight intervals, starting with 0–19.

4. Repeat Exercise 2 using six intervals, starting with 39–48.

5. How does a frequency polygon differ from a histogram?

6. Discuss the advantages and disadvantages of the mean as a measure of central tendency.

Find the mean for each list of numbers.

7. 8, 10, 16, 21, 25

8. 67, 89, 75, 86, 100, 93

9. 30,200; 23,700; 33,320; 29,410; 24,600; 27,750; 27,300; 32,680

10. 38,500; 39,720; 42,183; 21,982; 43,250

11. 9.4, 11.3, 10.5, 7.4, 9.1, 8.4, 9.7, 5.2, 1.1, 4.7

12. 15.3, 27.2, 14.8, 16.5, 31.8, 40.1, 18.9, 28.4, 26.3, 35.3

Find the mean for the following.

13.

Value	Frequency
4	6
6	1
9	3
15	2

14.

Value	Frequency
9	3
12	5
15	1
18	1

15. Find the mean for the data in Exercise 1 from the grouped frequency distribution found in each of the following exercises.

a. Exercise 1

b. Exercise 3

16. Find the mean for the data in Exercise 2 from the grouped frequency distribution found in each of the following exercises.

a. Exercise 2

b. Exercise 4

Find the median for each list of numbers.

17. 27, 35, 39, 42, 47, 51, 54

18. 596, 604, 612, 683, 719

19. 100, 114, 125, 135, 150, 172

20. 359, 831, 904, 615, 211, 279, 505

21. 28.4, 9.1, 3.4, 27.6, 59.8, 32.1, 47.6, 29.8

22. 0.2, 1.4, 0.6, 0.2, 2.5, 1.9, 0.8, 1.5

Use a graphing calculator or spreadsheet to calculate the mean and median for the data in the indicated exercises.

23. Exercise 1

24. Exercise 2

Find the mode or modes for each list of numbers.

25. 4, 9, 8, 6, 9, 2, 1, 3

26. 16, 15, 13, 15, 14, 13, 11, 15, 14

27. 55, 62, 62, 71, 62, 55, 73, 55, 71

28. 158, 162, 165, 162, 165, 157, 163

29. 6.8, 6.3, 6.3, 6.9, 6.7, 6.4, 6.1, 6.0

30. 22.35, 14.90, 17.85, 15.46, 14.91, 17.85, 21.35

31. When is the median the most appropriate measure of central tendency?

32. Under what circumstances would the mode be an appropriate measure of central tendency?

For grouped data, the *modal class* is the interval containing the most data values. Find the modal class for each collection of grouped data.

33. Use the distribution in Exercise 1.

34. Use the distribution in Exercise 2.

35. To predict the outcome of the next congressional election, you take a survey of your friends. Is this a random sample of the voters in your congressional district? Explain why or why not.

APPLICATIONS

Business and Economics

Wheat Production **U.S. wheat prices and production figures for a recent decade are given in the following table. *Source: USDA.***

Year	Price ($ per bushel)	Production (millions of bushels)
2000	2.62	2228
2001	2.78	1947
2002	3.56	1606
2003	3.40	2344
2004	3.40	2157
2005	3.42	2103
2006	4.26	1808
2007	6.48	2051
2008	6.78	2499
2009	4.87	2216

Find the mean and median for the following.

36. Price per bushel of wheat

37. Wheat production

38. Salaries The total compensation (in millions of dollars) for the 10 highest paid CEOs in 2009 is given in the following table. ***Source: The Huffington Post.***

Person, Company	Total Compensation
Carol Bartz, Yahoo	47.2
Leslie Moonves, CBS	42.9
Marc Casper, Thermo Fisher Scientific, Inc.	34.1
Phillipe Dauman, Viacom Inc.	33.9
J. Raymond Elliott, Boston Scientific Corp.	33.3
Ray Irani, Occidental Petroleum	31.4
Glen Senk, Urban Outfitters	29.9
Brian Roberts, Comcast	27.2
William Weldon, Johnson & Johnson	25.5
Louis Camilleri, Philip Morris International	24.4

a. Find the mean total compensation for this group of people.

b. Find the median total compensation for this group of people.

39. Household Income The total income for African American households making under $250,000 in 2008 is given in the following table. ***Source: U.S. Census Bureau.***

Income Range	Midpoint Salary	Frequency (in thousands)
Under $20,000	$10,000	4378
$20,000–$39,999	$30,000	3891
$40,000–$59,999	$50,000	2500
$60,000–$79,999	$70,000	1497
$80,000–$99,999	$90,000	866
$100,000–$149,999	$125,000	998
$150,000–$199,999	$175,000	309
$200,000–$249,999	$225,000	79

Use the table to estimate the mean income for African American households in 2008.

40. Household Income The total income for white households making under $250,000 in 2008 is given in the following table. ***Source: U.S. Census Bureau.***

Income Range	Midpoint Salary	Frequency (in thousands)
Under $20,000	$10,000	16,241
$20,000–$39,999	$30,000	20,487
$40,000–$59,999	$50,000	16,065
$60,000–$79,999	$70,000	12,746
$80,000–$99,999	$90,000	9080
$100,000–$149,999	$125,000	12,230
$150,000–$199,999	$175,000	4498
$200,000–$249,999	$225,000	1743

a. Use this table to estimate the mean income for white households in 2008.

b. Compare this estimate with the estimate found in Exercise 39. Discuss whether this provides evidence that white American households have higher earnings than African American households.

41. Airlines The number of consumer complaints against the top U.S. airlines during the first six months of 2010 is given in the following table. ***Source: U.S. Department of Transportation.***

Airline	Complaints	Complaints per 100,000 Passengers Boarding
Delta	1175	2.19
American	660	1.56
United	487	1.84
US Airways	428	1.69
Continental	350	1.64
Southwest	149	0.29
Skywest	77	0.65
American Eagle	68	0.87
Expressjet	56	0.70
Alaska	34	0.44

a. By considering the numbers in the column labeled "Complaints," calculate the mean and median number of complaints per airline.

b. Explain why the averages found in part a are not meaningful.

c. Find the mean and median of the numbers in the column labeled "Complaints per 100,000 Passengers Boarding." Discuss whether these averages are meaningful.

Life Sciences

42. Pandas The size of the home ranges (in square kilometers) of several pandas were surveyed over a year's time, with the following results.

Home Range	Frequency
0.1–0.5	11
0.6–1.0	12
1.1–1.5	7
1.6–2.0	6
2.1–2.5	2
2.6–3.0	1
3.1–3.5	1

a. Sketch a histogram and frequency polygon for the data.

b. Find the mean for the data.

43. Blood Types The number of recognized blood types varies by species, as indicated by the table below. Find the mean, median, and mode of this data. ***Source: The Handy Science Answer Book.***

Animal	Number of Blood Types
Pig	16
Cow	12
Chicken	11
Horse	9
Human	8
Sheep	7
Dog	7
Rhesus monkey	6
Mink	5
Rabbit	5
Mouse	4
Rat	4
Cat	2

General Interest

44. Temperature The following table gives the number of days in June and July of recent years in which the temperature reached 90 degrees or higher in New York's Central Park. ***Source: The New York Times and Accuweather.com.***

Year	Days	Year	Days	Year	Days
1972	11	1985	4	1998	5
1973	8	1986	8	1999	24
1974	11	1987	14	2000	3
1975	3	1988	21	2001	4
1976	8	1989	10	2002	13
1977	11	1990	6	2003	11
1978	5	1991	21	2004	1
1979	7	1992	4	2005	12
1980	12	1993	25	2006	5
1981	12	1994	16	2007	4
1982	11	1995	14	2008	10
1983	20	1996	0	2009	0
1984	7	1997	10	2010	20

a. Prepare a frequency distribution with a column for intervals and frequencies. Use six intervals, starting with 0–4.

b. Sketch a histogram and a frequency polygon, using the intervals in part a.

c. Find the mean for the original data.

d. Find the mean using the grouped data from part a.

e. Explain why your answers to parts c and d are different.

f. Find the median and the mode for the original data.

45. Temperature The table below gives the average monthly temperatures in degrees Fahrenheit for a certain area.

Month	Maximum	Minimum
January	39	16
February	39	18
March	44	21
April	50	26
May	60	32
June	69	37
July	79	43
August	78	42
September	70	37
October	51	31
November	47	24
December	40	20

Find the mean and median for the following.

a. The maximum temperature

b. The minimum temperature

46. Olympics The number of nations participating in the winter Olympic games, from 1968 to 2010, is given below. Find the following measures for the data. ***Source: New York Times 2010 Almanac and International Olympic Committee.***

Year	Nations Participating	Year	Nations Participating
1968	37	1992	64
1972	35	1994	67
1976	37	1998	72
1980	37	2002	77
1984	49	2006	85
1988	57	2010	82

a. Mean **b.** Median **c.** Mode

47. Personal Wealth When Russian billionaire Roman Abramovich became governor of the Russian province Chukotka (in the Bering Straits, opposite Alaska), it instantly became the fourth most prosperous region in Russia, even though its 80,000 other residents are poor. Mr. Abramovich was then worth $5.7 billion. Suppose each of the 80,000 other residents of Chukotka was worth $100. ***Source: National Public Radio.***

a. Calculate the average worth of a citizen of Chukotka.

b. What does this example tell you about the use of the mean to describe an average?

48. Personal Wealth *Washington Post* writer John Schwartz pointed out that if Microsoft Corp. cofounder Bill Gates, who, at the time, was reportedly worth $10 billion, lived in a town with 10,000 totally penniless people, the average personal wealth in the town would make it seem as if everyone were a millionaire. ***Source: The Washington Post.***

a. Verify Schwartz's statement.

b. What would be the median personal wealth in this town?

c. What would be the mode for the personal wealth in this town?

d. In this example, which average is most representative: the mean, the median, or the mode?

49. Baseball Salaries The Major League Baseball team with the highest payroll in 2010 (and most other years) was the New York Yankees. The following table gives the salary of each Yankee in 2010. ***Source: About.com.***

Name	Salary
Alex Rodriguez	$33,000,000
CC Sabathia	$24,285,714
Derek Jeter	$22,600,000
Mark Teixeira	$20,625,000
A.J. Burnett	$16,500,000
Mariano Rivera	$15,000,000
Jorge Posada	$13,100,000
Andy Pettitte	$11,750,000
Javier Vazquez	$11,500,000
Robinson Cano	$9,000,000
Nick Swisher	$6,850,000
Curtis Granderson	$5,500,000
Nick Johnson	$5,500,000
Damaso Marte	$4,000,000
Chan Ho Park	$1,200,000
Randy Winn	$1,100,000
Marcus Thames	$900,000
Sergio Mitre	$850,000
Joba Chamberlain	$487,975
Brett Gardner	$452,500
Phil Hughes	$447,000
Alfredo Aceves	$435,650
David Robertson	$426,650
Ramiro Pena	$412,100
Francisco Cervelli	$410,800

a. Find the mean, median, and mode of the salaries.

b. Which average best describes this data?

c. Why is there such a difference between the mean and the median?

50. SAT I: Reasoning Test Given the following sequence of numbers*

$$1, a, a^2, a^3, \ldots, a^n,$$

*Permission to reprint SAT materials does not constitute review or endorsement by Educational Testing Service or the College Board of this publication as a whole or of any other questions or testing information it may contain. This problem appeared, minus the *additional assumption*, on an SAT in 1996. Colin Rizzio, a high school student at the time, became an instant celebrity when he noticed that the additional assumption was needed to complete the problem. ***Source: The New York Times.***

where n is a positive even integer, with the *additional assumption* that a is a positive number, the median is best described as

a. greater than $a^{n/2}$;

b. smaller than $a^{n/2}$;

c. equal to $a^{n/2}$.

d. The relationship cannot be determined from the information given.

YOUR TURN ANSWERS

1. 27
2. 19.85
3. 25

9.2 Measures of Variation

APPLY IT **How can we tell when a manufacturing process is out of control?**

To answer this question, which we will do in Example 6, we need to understand measures of variation, which tell us how much the numbers in a sample vary from the mean.

The mean gives a measure of central tendency of a list of numbers but tells nothing about the *spread* of the numbers in the list. For example, suppose three inspectors each inspect a sample of five restaurants in their region and record the number of health violations they find in each restaurant. Their results are recorded in the following table.

Three Sets of Data					
I	3	5	6	3	3
II	4	4	4	4	4
III	10	1	0	0	9

Each of these three samples has a mean of 4, and yet they are quite different; the amount of dispersion or variation within the samples is different. Therefore, in addition to a measure of central tendency, another kind of measure is needed that describes how much the numbers vary.

The largest number in sample I is 6, while the smallest is 3, a difference of 3. In sample II this difference is 0; in sample III, it is 10. The difference between the largest and smallest number in a sample is called the **range**, one example of a measure of variation. The range of sample I is 3, of sample II, 0, and of sample III, 10. The range has the advantage of being very easy to compute, and gives a rough estimate of the variation among the data in the sample. It depends only on the two extremes, however, and tells nothing about how the other data are distributed between the extremes.

EXAMPLE 1 Range

Find the range for each list of numbers.

(a) 12, 27, 6, 19, 38, 9, 42, 15

SOLUTION The highest number here is 42; the lowest is 6. The range is the difference between these numbers, or $42 - 6 = 36$.

(b) 74, 112, 59, 88, 200, 73, 92, 175

SOLUTION

$$\text{Range} = 200 - 59 = 141$$

The most useful measure of variation is the *standard deviation.* Before defining it, however, we must find the **deviations from the mean**, the differences found by subtracting the mean from each number in a sample.

EXAMPLE 2 Deviations from the Mean

Find the deviations from the mean for the numbers

$$32, 41, 47, 53, 57.$$

SOLUTION Adding these numbers and dividing by 5 gives a mean of 46. To find the deviations from the mean, subtract 46 from each number in the list. For example, as shown in the table to the left, the first deviation from the mean is $32 - 46 = -14$; the last is $57 - 46 = 11$.

To check your work, find the sum of these deviations. It should always equal 0. (The answer is always 0 because the positive and negative numbers cancel each other.)

Deviations from the Mean

Number	Deviation From Mean
32	−14
41	−5
47	1
53	7
57	11
	0

To find a measure of variation, we might be tempted to use the mean of the deviations. As mentioned above, however, this number is always 0, no matter how widely the data are dispersed. One way to solve this problem is to use absolute value and find the mean of the absolute values of the deviations from the mean. Absolute value is awkward to work with algebraically, and there is an alternative approach that provides better theoretical results. In this method, the way to get a list of positive numbers is to square each deviation and then find the mean. When finding the mean of the squared deviations, most statisticians prefer to divide by $n - 1$, rather than n. We will give the reason later in this section. For the data above, this gives

$$\frac{(-14)^2 + (-5)^2 + 1^2 + 7^2 + 11^2}{5 - 1} = \frac{196 + 25 + 1 + 49 + 121}{4}$$

$$= 98.$$

This number, 98, is called the *variance* of the sample. Since it is found by averaging a list of squares, the variance of a sample is represented by s^2.

For a sample of n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\bar{x}$, the variance is

$$s^2 = \frac{\Sigma(x - \bar{x})^2}{n - 1}.$$

The following shortcut formula for the variance can be derived algebraically from the formula above. You are asked for this derivation in Exercise 23.

Variance

The **variance** of a sample of n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\bar{x}$, is

$$\boldsymbol{s^2 = \frac{\Sigma x^2 - n\bar{x}^2}{n - 1}}.$$

To find the variance, we squared the deviations from the mean, so the variance is in squared units. To return to the same units as the data, we use the *square root* of the variance, called the *standard deviation*.

Standard Deviation

The **standard deviation** of a sample of n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\bar{x}$, is

$$\boldsymbol{s = \sqrt{\frac{\Sigma x^2 - n\bar{x}^2}{n - 1}}}.$$

As its name indicates, the standard deviation is the most commonly used measure of variation. The standard deviation is a measure of the variation from the mean. The size of the standard deviation tells us something about how spread out the data are from the mean.

EXAMPLE 3 Standard Deviation

Find the standard deviation of the numbers

$$7, 9, 18, 22, 27, 29, 32, 40.$$

SOLUTION

Method 1 Calculating by Hand

The mean of the numbers is

$$\frac{7 + 9 + 18 + 22 + 27 + 29 + 32 + 40}{8} = 23.$$

Arrange the work in columns, as shown in the table.

Standard Deviation Calculations	
Number, x	Square of the Number, x^2
7	49
9	81
18	324
22	484
27	729
29	841
32	1024
40	1600
	Total: 5132

The total of the second column gives $\sum x^2 = 5132$. Now, using the formula for variance with $n = 8$, the variance is

$$s^2 = \frac{\sum x^2 - n\bar{x}^2}{n - 1}$$
$$= \frac{5132 - 8(23)^2}{8 - 1}$$
$$\approx 128.6,$$

rounded, and the standard deviation is

$$\sqrt{128.57} \approx 11.3.$$

Method 2 Graphing Calculator

The data are entered into the L_5 list on a TI-84 Plus calculator. Figure 4 shows how the variance and standard deviation are then calculated through the LIST-MATH menu. Figure 5 shows an alternative method, going through the STAT menu, which calculates the mean, the standard deviation using both $n - 1$ (indicated by Sx) and n (indicated by σx) in the denominator, and other statistics.

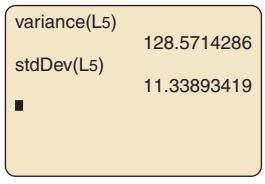


FIGURE 4

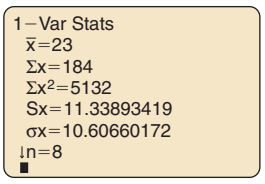


FIGURE 5

Method 3 Spreadsheet

The data are entered in cells A1 through A8. Then, in cell A9, type "=VAR(A1:A8)" and press Enter. The standard deviation can be calculated by either taking the square root of cell A9 or by typing "=STDEV(A1:A8)" in cell A10 and pressing Enter.

YOUR TURN 1 Find the range, variance, and standard deviation for the following list of numbers: 7, 11, 16, 17, 19, 35.

TRY YOUR TURN 1

CAUTION Be careful to divide by $n - 1$, not n, when calculating the standard deviation of a sample. Many calculators are equipped with statistical keys that compute the variance and standard deviation. Some of these calculators use $n - 1$ and others use n for these computations; some may have keys for both. Check your calculator's instruction book before using a statistical calculator for the exercises.

One way to interpret the standard deviation uses the fact that, for many populations, most of the data are within three standard deviations of the mean. (See Section 9.3.) This implies that, in Example 3, most members of the population from which this sample is taken are between

$$\bar{x} - 3s = 23 - 3(11.3) = -10.9$$

and

$$\bar{x} + 3s = 23 + 3(11.3) = 56.9.$$

This has important implications for quality control. If the sample in Example 3 represents measurements of a product that the manufacturer wants to be between 5 and 45, the standard deviation is too large, even though all the numbers in the sample are within these bounds.

We saw in the previous section that the mean of a random sample is a random variable. It should not surprise you, then, to learn that the variance and standard deviation are also random variables. We will refer to the variance and standard deviation of a random sample as the **sample variance** and **sample standard deviation**.

Recall from the previous section that the sample mean $\bar{x}$ is not the same as the population mean μ, which is defined by $\mu = E(x)$, but that $\bar{x}$ gives a good approximation to μ because $E(\bar{x}) = \mu$. Similarly, there is a **population variance**, denoted σ^2, defined by $\sigma^2 = E(x - \mu)^2$, which measures the amount of variation in a population. The **population standard deviation** is simply σ, the square root of the population variance σ^2. (The Greek letter σ is the lowercase version of sigma. You have already seen Σ, the uppercase version.) In more advanced courses in statistics, it is shown that $E(s^2) = \sigma^2$. The reason many statisticians prefer $n - 1$ in the denominator of the standard deviation formula is that it makes $E(s^2) = \sigma^2$ true; this is not true if n is used in the denominator. It may surprise you, then, that $E(s) = \sigma$ is *false,* whether n or $n - 1$ is used. If n is large, the difference between $E(s)$ and σ is slight, so, in practice, the sample standard deviation s gives a good estimate of the population standard deviation σ.

FOR REVIEW

Recall from Section 8.5 that a random variable is a function that assigns a real number to each outcome of an experiment. When the experiment consists of drawing a random sample, the standard deviation and the variance are two real numbers assigned to each outcome. Every time the experiment is performed, the standard deviation and variance will most likely have different values, with some values more probable than others.

For data in a grouped frequency distribution, a slightly different formula for the standard deviation is used.

Standard Deviation for a Grouped Distribution

The standard deviation for a distribution with mean $\bar{x}$, where x is an interval midpoint with frequency f, and $n = \Sigma f$, is

$$s = \sqrt{\frac{\Sigma fx^2 - n\bar{x}^2}{n - 1}}.$$

The formula indicates that the product fx^2 is to be found for each interval. Then these products are summed, n times the square of the mean is subtracted, and the difference is divided by one less than the total frequency; that is, by $n - 1$. The square root of this result is s, the standard deviation.

CAUTION In calculating the standard deviation for either a grouped or ungrouped distribution, using a rounded value for the mean may produce an inaccurate value.

EXAMPLE 4 Standard Deviation for Grouped Data

Find the standard deviation for the grouped data of Example 5, Section 9.1.

SOLUTION Begin by forming columns for x (the midpoint of the interval), x^2, and fx^2. Then sum the f and fx^2 columns. Recall from Example 5 of Section 9.1 that $\bar{x} = 15.5$.

Standard Deviation for Grouped Data

Interval	x	x^2	f	fx^2
0–4	2	4	3	12
5–9	7	49	4	196
10–14	12	144	6	864
15–19	17	289	8	2312
20–24	22	484	5	2420
25–29	27	729	3	2187
30–34	32	1024	1	1024
			Total: 30	Total: 9015

YOUR TURN 2 Find the standard deviation for the following grouped frequency distribution.

Interval	Frequency
0–6	2
7–13	4
14–20	7
21–27	10
28–34	3
35–41	1

The total of the fourth column gives $n = \Sigma f = 30$, and the total of the last column gives $\Sigma fx^2 = 9015$. Use the formula for standard deviation for a grouped distribution to find s.

$$s = \sqrt{\frac{\Sigma fx^2 - n\bar{x}^2}{n - 1}}$$

$$= \sqrt{\frac{9015 - 30(15.5)^2}{30 - 1}}$$

$$\approx 7.89$$

Verify that the standard deviation of the original, ungrouped data in Example 1 of Section 9.1 is 7.92.

TRY YOUR TURN 2

EXAMPLE 5 Nathan's Hot Dog Eating Contest

Since 1916, Nathan's Famous Hot Dogs has held an annual hot dog eating contest, in which each contestant attempts to consume as many hot dogs with buns as possible in a 12-minute period. The following table contains a list of each year's winners since 1997, when the International Federation of Competitive Eating began officiating the contest. ***Source: Wikipedia.*** In what percent of the contests did the number of hot dogs eaten by the winner fall within one standard deviation of the mean number of hot dogs? Within two standard deviations?

Hot Dog Eating Contest Winners

Year	Winner	Hot Dogs Eaten
1997	Hirofumi Nakajima	24.5
1998	Hirofumi Nakajima	19
1999	Steve Keiner	20.25
2000	Kazutoyo Arai	25.125
2001	Takeru Kobayashi	50
2002	Takeru Kobayashi	50.5
2003	Takeru Kobayashi	44.5
2004	Takeru Kobayashi	53.5
2005	Takeru Kobayashi	49
2006	Takeru Kobayashi	53.75
2007	Joey Chestnut	66
2008	Joey Chestnut	59
2009	Joey Chestnut	68
2010	Joey Chestnut	54

SOLUTION First, using the formulas for $\bar{x}$ and s, we calculate the mean and standard deviation:

$$\bar{x} \approx 45.51 \quad \text{and} \quad s \approx 16.57.$$

Subtracting one standard deviation from the mean and then adding one standard deviation to the mean, we find the lower and upper limits:

$$\bar{x} - s = 45.51 - 16.57 = 28.94 \quad \text{(lower)}$$

and

$$\bar{x} + s = 45.51 + 16.57 = 62.08. \quad \text{(upper)}$$

In 8 of the 14 contests, the number of hot dogs eaten by the winner is between 28.94 and 62.08. Therefore, in about 57% $(8/14 \approx 0.57)$ of the recent contests, the number of hot dogs consumed by the winner was within one standard deviation of the mean.

Likewise, subtracting 2 standard deviations from the mean and adding 2 standard deviations to the mean, we get a lower limit of 12.37, and an upper limit of 78.65. All 14 contests fall in this range, so in 100% of the recent contests, the number of hot dogs eaten by the winner was within two standard deviations of the mean.

EXAMPLE 6 Quality Assurance

APPLY IT

Statistical process control is a method of determining when a manufacturing process is out of control, producing defective items. The procedure involves taking samples of a measurement on a product over a production run and calculating the mean and standard deviation of each sample. These results are used to determine when the manufacturing process is out of control. For example, three sample measurements from a manufacturing process on each of four days are given in the table below. The mean $\bar{x}$ and standard deviation s are calculated for each sample.

Samples from a Manufacturing Process

Day	1			2			3			4		
Sample Number	1	2	3	1	2	3	1	2	3	1	2	3
Measurements	−3	0	4	5	−2	4	3	−1	0	4	−2	1
	0	5	3	4	0	3	−2	0	0	3	0	3
	2	2	2	3	1	4	0	1	−2	3	−1	0
$\bar{x}$	−1/3	7/3	3	4	−1/3	11/3	1/3	0	−2/3	10/3	−1	4/3
s	2.5	2.5	1	1	1.5	0.6	2.5	1	1.2	0.6	1	1.5

Next, the mean of the 12 sample means, $\bar{\bar{X}}$, and the mean of the 12 sample standard deviations, $\bar{s}$, are found (using the formula for $\bar{x}$). Here, these measures are

$$\bar{\bar{X}} = 1.3 \quad \text{and} \quad \bar{s} = 1.41.$$

The control limits for the sample means are given by

$$\bar{\bar{X}} \pm k_1 \bar{s},$$

where k_1 is a constant that depends on the sample size, and can be found from a manual. ***Source: Statistical Process Control.*** For samples of size 3, $k_1 = 1.954$, so the control limits for the sample means are

$$1.3 \pm (1.954)(1.41).$$

The upper control limit is 4.06, and the lower control limit is −1.46.

Similarly, the control limits for the sample standard deviations are given by $k_2 \cdot \bar{s}$ and $k_3 \cdot \bar{s}$, where k_2 and k_3 also are values given in the same manual. Here, $k_2 = 2.568$ and $k_3 = 0$, with the upper and lower control limits for the sample standard deviations equal to

$2.568(1.41)$ and $0(1.41)$, or 3.62 and 0. As long as the sample means are between -1.46 and 4.06 and the sample standard deviations are between 0 and 3.62, the process is in control.

9.2 EXERCISES

1. How are the variance and the standard deviation related?

2. Why can't we use the sum of the deviations from the mean as a measure of dispersion of a distribution?

Find the range and standard deviation for each set of numbers.

3. 72, 61, 57, 83, 52, 66, 85

4. 122, 132, 141, 158, 162, 169, 180

5. 241, 248, 251, 257, 252, 287

6. 51, 58, 62, 64, 67, 71, 74, 78, 82, 93

7. 3, 7, 4, 12, 15, 18, 19, 27, 24, 11

8. 17, 57, 48, 13, 26, 3, 36, 21, 9, 40

Use a graphing calculator or spreadsheet to calculate the standard deviation for the data in the indicated exercises.

9. Exercise 1 from Section 9.1

10. Exercise 44 from Section 9.1

Find the standard deviation for the following grouped data.

11. (From Exercise 1, Section 9.1)

Interval	Frequency
0–24	4
25–49	8
50–74	5
75–99	10
100–124	4
125–149	5

12. (From Exercise 2, Section 9.1)

Interval	Frequency
30–39	1
40–49	6
50–59	13
60–69	22
70–79	17
80–89	13
90–99	8

Chebyshev's theorem states that for any set of numbers, the fraction that will lie within k standard deviations of the mean (for $k > 1$) is at least

$$1 - \frac{1}{k^2}.$$

For example, at least $1 - 1/2^2 = 3/4$ of any set of numbers lie within 2 standard deviations of the mean. Similarly, for any probability distribution, the probability that a number will lie within k standard deviations of the mean is at least $1 - 1/k^2$. For example, if the mean is 100 and the standard deviation is 10, the probability that a number will lie within 2 standard deviations of 100, or between 80 and 120, is at least $3/4$. Use Chebyshev's theorem to find the fraction of all the numbers of a data set that must lie within the following numbers of standard deviations from the mean.

13. 3

14. 4

15. 5

16. 6

In a certain distribution of numbers, the mean is 60 with a standard deviation of 8. Use Chebyshev's theorem to tell the probability that a number lies in each interval.

17. Between 36 and 84

18. Between 48 and 72

19. Less than 36 or more than 84

20. Less than 48 or more than 72

21. Discuss what the standard deviation tells us about a distribution.

22. Explain the difference between the sample mean and standard deviation, and the population mean and standard deviation.

23. Derive the shortcut formula for the variance

$$s^2 = \frac{\Sigma x^2 - n\bar{x}^2}{n - 1}$$

from the formula

$$s^2 = \frac{\Sigma (x - \bar{x})^2}{n - 1}$$

and the following summation formulas, in which c is a constant:

$$\Sigma cx = c\Sigma x, \quad \Sigma c = nc, \quad \text{and} \quad \Sigma(x \pm y) = \Sigma x \pm \Sigma y.$$

(*Hint:* Multiply out $(x - \bar{x})^2$.)

24. Consider the set of numbers 9,999,999, 10,000,000, and 10,000,001.

a. Calculate the variance by hand using the formula

$$\frac{\Sigma (x - \bar{x})^2}{n - 1}$$

b. Calculate the variance using your calculator and the shortcut formula.

c. There may be a discrepancy between your answers to parts a and b because of roundoff. Explain this discrepancy, and then discuss advantages and disadvantages of the shortcut formula.

APPLICATIONS

Business and Economics

25. Battery Life Forever Power Company analysts conducted tests on the life of its batteries and those of a competitor (Brand X). They found that their batteries had a mean life (in hours) of 26.2, with a standard deviation of 4.1. Their results for a sample of 10 Brand X batteries were as follows: 15, 18, 19, 23, 25, 25, 28, 30, 34, 38.

a. Find the mean and standard deviation for the sample of Brand X batteries.

b. Which batteries have a more uniform life in hours?

c. Which batteries have the highest average life in hours?

26. Sales Promotion The Quaker Oats Company conducted a survey to determine whether a proposed premium, to be included in boxes of cereal, was appealing enough to generate new sales. Four cities were used as test markets, where the cereal was distributed with the premium, and four cities as control markets, where the cereal was distributed without the premium. The eight cities were chosen on the basis of their similarity in terms of population, per capita income, and total cereal purchase volume. The results were as follows. ***Source: Quaker Oats Company.***

	City	Percent Change in Average Market Share per Month
Test Cities	1	+18
	2	+15
	3	+7
	4	+10
Control Cities	1	+1
	2	−8
	3	−5
	4	0

a. Find the mean of the change in market share for the four test cities.

b. Find the mean of the change in market share for the four control cities.

c. Find the standard deviation of the change in market share for the test cities.

d. Find the standard deviation of the change in market share for the control cities.

e. Find the difference between the means of parts a and b. This difference represents the estimate of the percent change in sales due to the premium.

f. The two standard deviations from parts c and d were used to calculate an "error" of ± 7.95 for the estimate in part e. With this amount of error, what are the smallest and largest estimates of the increase in sales?

On the basis of the interval estimate of part f, the company decided to mass-produce the premium and distribute it nationally.

27. Process Control The following table gives 10 samples of three measurements, made during a production run.

Sample Number									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	3	−2	−3	−1	3	0	−1	2	0
−2	−1	0	1	2	2	1	2	3	0
1	4	1	2	4	2	2	3	2	2

Use the information in Example 6 to find the following.

a. Find the mean $\bar{x}$ for each sample of three measurements.

b. Find the standard deviation s for each sample of three measurements.

c. Find the mean $\bar{X}$ of the sample means.

d. Find the mean $\bar{s}$ of the sample standard deviations.

e. Using $k_1 = 1.954$, find the upper and lower control limits for the sample means.

f. Using $k_2 = 2.568$ and $k_3 = 0$, find the upper and lower control limits for the sample standard deviations.

28. Process Control Given the following measurements from later samples on the process in Exercise 27, decide whether the process is out of control. (*Hint:* Use the results of Exercise 27e and f.)

Sample Number					
1	**2**	**3**	**4**	**5**	**6**
3	−4	2	5	4	0
−5	2	0	1	−1	1
2	1	1	−4	−2	−6

29. Washer Thickness An assembly-line machine turns out washers with the following thicknesses (in millimeters).

1.20	1.01	1.25	2.20	2.58	2.19	1.29	1.15
2.05	1.46	1.90	2.03	2.13	1.86	1.65	2.27
1.64	2.19	2.25	2.08	1.96	1.83	1.17	2.24

Find the mean and standard deviation of these thicknesses.

30. Unemployment The number of unemployed workers in the United States in recent years (in millions) is given below. ***Source: Bureau of Labor Statistics.***

Year	Number Unemployed
2000	5.69
2001	6.80
2002	8.38
2003	8.77
2004	8.15
2005	7.59
2006	7.00
2007	7.08
2008	8.92
2009	14.27

a. Find the mean number unemployed (in millions) in this period. Which years had unemployment closest to the mean?

b. Find the standard deviation for the data.

c. In how many of these years is unemployment within 1 standard deviation of the mean?

d. In how many of these years is unemployment within 3 standard deviations of the mean?

Life Sciences

31. Blood pH A medical laboratory tested 21 samples of human blood for acidity on the pH scale, with the following results.

7.1	7.5	7.3	7.4	7.6	7.2	7.3
7.4	7.5	7.3	7.2	7.4	7.3	7.5
7.5	7.4	7.4	7.1	7.3	7.4	7.4

a. Find the mean and standard deviation.

b. What percentage of the data is within 2 standard deviations of the mean?

32. Blood Types The number of recognized blood types between species is given in the following table. In Exercise 43 of the previous section, the mean was found to be 7.38. ***Source: The Handy Science Answer Book.***

Animal	Number of Blood Types
Pig	16
Cow	12
Chicken	11
Horse	9
Human	8
Sheep	7
Dog	7
Rhesus monkey	6
Mink	5
Rabbit	5
Mouse	4
Rat	4
Cat	2

a. Find the variance and the standard deviation of these data.

b. How many of these animals have blood types that are within 1 standard deviation of the mean?

33. Tumor Growth The amount of time that it takes for various slow-growing tumors to double in size are listed in the following table. ***Source: American Journal of Roentgen.***

Type of Cancer	Doubling Time (days)
Breast cancer	84
Rectal cancer	91
Synovioma	128
Skin cancer	131
Lip cancer	143
Testicular cancer	153
Esophageal cancer	164

a. Find the mean and standard deviation of these data.

b. How many of these cancers have doubling times that are within 2 standard deviations of the mean?

c. If a person had a nonspecified tumor that was doubling every 200 days, discuss whether this particular tumor is growing at a rate that would be expected.

General Interest

34. Box Office Receipts The table below lists the 18 films in which actor Will Smith has starred through 2008, along with the gross domestic box office receipts and the year for each movie. ***Source: The Movie Times.***

Movie	Domestic Box Office Receipts
Hancock, 2008	$227,946,274
Seven Pounds, 2008	$69,951,824
I Am Legend, 2007	$256,386,216
The Pursuit of Happyness, 2006	$162,586,036
Hitch, 2005	$177,575,142
Shark Tale, 2004	$161,412,000
I, Robot, 2004	$144,801,023
Bad Boys II, 2003	$138,540,870
Men in Black 2, 2002	$190,418,803
Ali, 2001	$58,200,000
The Legend of Bagger Vance, 2000	$30,695,000
Wild Wild West, 1999	$113,745,000
Enemy of the State, 1998	$111,544,000
Men in Black, 1997	$250,107,128
Independence Day, 1996	$306,124,000
Bad Boys, 1995	$65,807,000
Made in America, 1993	$44,942,000
Six Degrees of Separation, 1993	$6,410,000

a. Find the mean domestic box office receipts for Will Smith's movies. Which movie has box office receipts closest to the mean?

b. Find the standard deviation for the data.

c. What percent of the movies have box office receipts within 1 standard deviation of the mean? Within 2 standard deviations of the mean?

35. Baseball Salaries The table in Exercise 49 in the previous section lists the salary for each player on the New York Yankees baseball team in 2010.

a. Calculate the standard deviation of these data.

b. What percent of the 2010 New York Yankee players have salaries that are beyond 2 standard deviations from the mean?

c. What does your answer to part b suggest?

36. Cookies Marie Revak and Jihan Williams performed an experiment to determine whether Oreo Double Stuf cookies contain twice as much filling as traditional Oreo cookies. The table on the next page gives the results in grams of the amount of filling inside 49 traditional cookies and 52 Double Stuf cookies. ***Source: The Mathematics Teacher.***

a. Find the mean, maximum, minimum, and standard deviation of the weights for traditional Oreo cookies.

b. Find the mean, maximum, minimum, and standard deviation of the weights for Oreo Double Stuf cookies.

c. What percent of the data of traditional Oreo cookies is within 2 standard deviations of the Double Stuf Oreo mean? (*Hint:* Use the mean and standard deviation for the Double Stuf data.)

d. What percent of the data of traditional Oreo cookies, when multiplied by 2, is within 2 standard deviations of the Double Stuf Oreo mean? (*Hint:* Use the mean and standard deviation for the Double Stuf data.)

e. Is there evidence that Double Stuf Oreos have twice as much filling as the traditional Oreo cookie? Explain.

Traditional	Traditional	Traditional	Double Stuf	Double Stuf	Double Stuf
2.9	2.4	2.7	4.7	6.5	5.8
2.8	2.8	2.8	6.5	6.3	5.9
2.6	3.8	2.6	5.5	4.8	6.2
3.5	3.1	2.6	5.6	3.3	5.9
3.0	2.9	3.0	5.1	6.4	6.5
2.4	3.0	2.8	5.3	5.0	6.5
2.7	2.1	3.5	5.4	5.3	6.1
2.4	3.8	3.3	5.4	5.5	5.8
2.5	3.0	3.3	3.5	5.0	6.0
2.2	3.0	2.8	5.5	6.0	6.2
2.6	2.8	3.1	6.5	5.7	6.2
2.6	2.9	2.6	5.9	6.3	6.0
2.9	2.7	3.5	5.4	6.0	6.8
2.6	3.2	3.5	4.9	6.3	6.2
2.6	2.8	3.1	5.6	6.1	5.4
3.1	3.1	3.1	5.7	6.0	6.6
2.9			5.3	5.8	6.2
			6.9		

YOUR TURN ANSWERS

1. 28, 92.7, 9.628 **2.** 8.52

9.3 The Normal Distribution

APPLY IT

What is the probability that a salesperson drives between 1200 miles and 1600 miles per month?

This question will be answered in Example 4 by using the normal probability distribution introduced in this section.

FOR REVIEW

Empirical probabilities, discussed in Section 7.3, are derived from grouped data by dividing the frequency or amount for each group by the total for all the groups. This gives one example of a probability distribution, discussed further in Sections 7.3 and 8.5.

Suppose a bank is interested in improving its services to customers. The manager decides to begin by finding the amount of time tellers spend on each transaction, rounded to the nearest minute. The times for 75 different transactions are recorded with the results shown in the following table. The frequencies listed in the second column are divided by 75 to find the empirical probabilities.

Teller Transaction Times		
Time	Frequency	Probability
1	3	$3/75 = 0.04$
2	5	$5/75 \approx 0.07$
3	9	$9/75 = 0.12$
4	12	$12/75 = 0.16$
5	15	$15/75 = 0.20$
6	11	$11/75 \approx 0.15$
7	10	$10/75 \approx 0.13$
8	6	$6/75 = 0.08$
9	3	$3/75 = 0.04$
10	1	$1/75 \approx 0.01$

Figure 6(a) shows a histogram and frequency polygon for the data. The heights of the bars are the empirical probabilities rather than the frequencies. The transaction times are given to the nearest minute. Theoretically at least, they could have been timed to the nearest tenth of a minute, or hundredth of a minute, or even more precisely. In each case, a histogram and frequency polygon could be drawn. If the times are measured with smaller and smaller units, there are more bars in the histogram, and the frequency polygon begins to look more and more like the curve in Figure 6(b) instead of a polygon. Actually, it is possible for the transaction times to take on any real number value greater than 0. A distribution in which the outcomes can take any real number value within some interval is a **continuous distribution**. The graph of a continuous distribution is a curve.

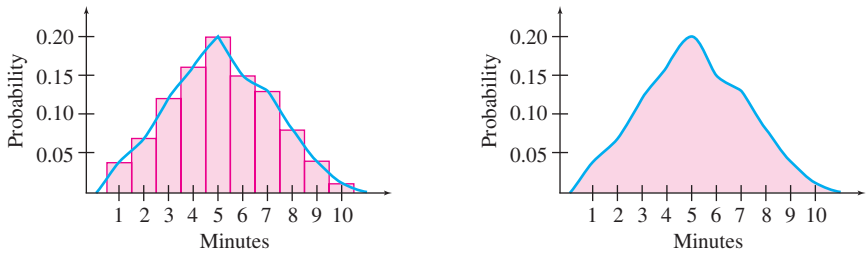


FIGURE 6

The distribution of heights (in inches) of college women is another example of a continuous distribution, since these heights include infinitely many possible measurements, such as 53, 58.5, 66.3, 72.666, . . . , and so on. Figure 7 shows the continuous distribution of heights of college women. Here the most frequent heights occur near the center of the interval shown.

Another continuous curve, which approximates the distribution of yearly incomes in the United States, is given in Figure 8. The graph shows that the most frequent incomes are grouped near the low end of the interval. This kind of distribution, where the peak is not at the center, is called **skewed**.

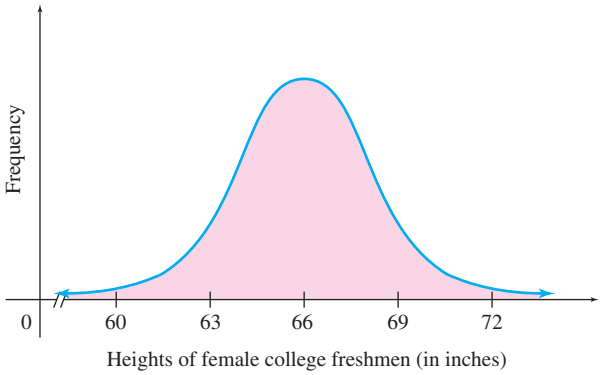


FIGURE 7

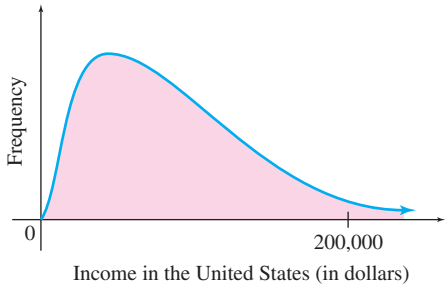
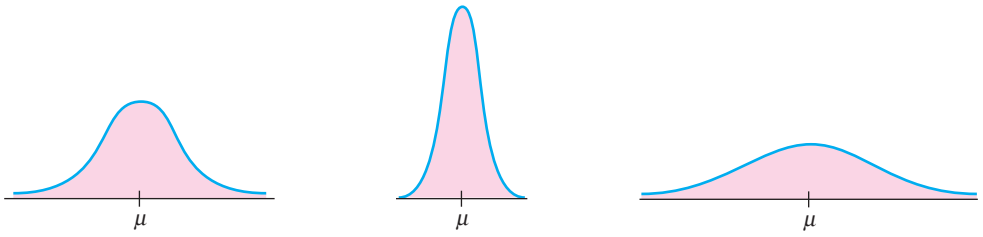


FIGURE 8

Many natural and social phenomena produce continuous probability distributions whose graphs can be approximated very well by bell-shaped curves, such as those shown in Figure 9 on the next page. Such distributions are called **normal distributions** and their graphs are called **normal curves**. Examples of distributions that are approximately normal are the heights of college women and the errors made in filling 1-lb cereal boxes. We use the Greek letters μ (mu) to denote the mean and σ (sigma) to denote the standard deviation of a normal distribution. The definitions of the mean and standard deviation of a continuous distribution require ideas from calculus, but the intuitive ideas are similar to those in the previous section.

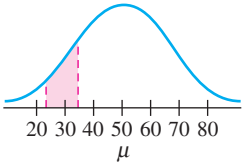


Three normal distributions

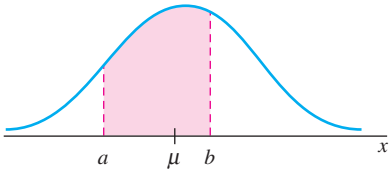
FIGURE 9

There are many normal distributions. Some of the corresponding normal curves are tall and thin and others are short and wide, as shown in Figure 9. But every normal curve has the following properties.

1. Its peak occurs directly above the mean μ.
2. The curve is symmetric about the vertical line through the mean (that is, if you fold the page along this line, the left half of the graph will fit exactly on the right half).
3. The curve never touches the x-axis—it extends indefinitely in both directions.
4. The area under the curve (and above the horizontal axis) is always 1. (This agrees with the fact that the sum of the probabilities in any distribution is 1.)



(a)



(b)

FIGURE 10

It can be shown that a normal distribution is completely determined by its mean μ and standard deviation σ.* A small standard deviation leads to a tall, narrow curve like the one in the center of Figure 9. A large standard deviation produces a flat, wide curve, like the one on the right in Figure 9.

Since the area under a normal curve is 1, parts of this area can be used to determine certain probabilities. For instance, Figure 10(a) is the probability distribution of the annual rainfall in a certain region. Calculus can be used to show that the probability that the annual rainfall will be between 25 in. and 35 in. is the area under the curve from 25 to 35. The general case, shown in Figure 10(b), can be stated as follows.

Normal Probability

The area of the shaded region under a normal curve from a to b is the probability that an observed data value will be between a and b.

To use normal curves effectively, we must be able to calculate areas under portions of these curves. These calculations have already been done for the normal curve with mean $\mu = 0$ and standard deviation $\sigma = 1$ (which is called the **standard normal curve**) and are available in a table in the Appendix. The following examples demonstrate how to use the table or a calculator or spreadsheet to find such areas. Later we shall see how the standard normal curve may be used to find areas under any normal curve.

EXAMPLE 1 Standard Normal Curve

The horizontal axis of the standard normal curve is usually labeled z. Find the following areas under the standard normal curve.

Method 1 Using a Table

(a) The area to the left of $z = 1.25$

SOLUTION Look up 1.25 in the normal curve table. (Find 1.2 in the left-hand column and 0.05 at the top, then locate the intersection of the corresponding row and column.)

*As is shown in more advanced courses, its graph is the graph of the function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)},$$

where $e \approx 2.71828$ is a real number.

The specified area is 0.8944, so the shaded area shown in Figure 11 is 0.8944. This area represents 89.44% of the total area under the normal curve, and so the probability that $z \le 1.25$ is

$$P(z \le 1.25) = 0.8944.$$

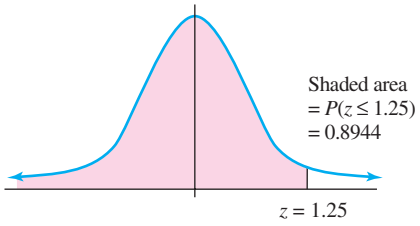


FIGURE 11

(b) The area to the right of $z = 1.25$

SOLUTION From part (a), the area to the left of $z = 1.25$ is 0.8944. The total area under the normal curve is 1, so the area to the right of $z = 1.25$ is

$$1 - 0.8944 = 0.1056.$$

See Figure 12, where the shaded area represents 10.56% of the total area under the normal curve, and the probability that $z \ge 1.25$ is $P(z \ge 1.25) = 0.1056$.

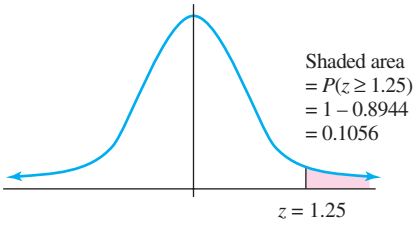


FIGURE 12

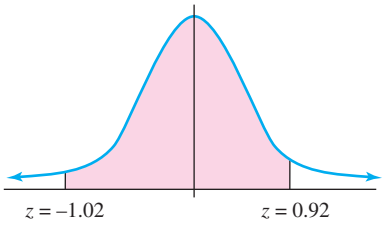


FIGURE 13

(c) The area between $z = -1.02$ and $z = 0.92$

SOLUTION To find this area, which is shaded in Figure 13, start with the area to the left of $z = 0.92$ and subtract the area to the left of $z = -1.02$. See the two shaded regions in Figure 14. The result is

$$P(-1.02 \le z \le 0.92) = 0.8212 - 0.1539 = 0.6673.$$

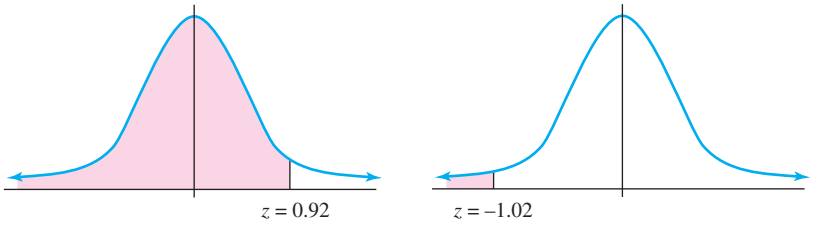


FIGURE 14

Method 2 Graphing Calculator

Because of convenience and accuracy, graphing calculators and computers have made normal curve tables less important. Figure 15 on the next page shows how parts (a) and (b) of this example can be done on a TI-84 Plus using the `normalcdf` command in the DISTR menu. In Figure 15, −1E99 stands for -1×10^{99}. The area between -1×10^{99} and 1.25 is

essentially the same as the area to the left of 1.25. Similarly, the area between 1.25 and 1×10^{99} is essentially the same as the area to the right of 1.25. Verify the results of part (c) with a graphing calculator.

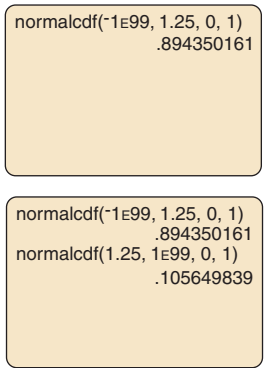


FIGURE 15

Method 3 Spreadsheet

Statistical software packages are widely used today. These packages are set up in a way that is similar to a spreadsheet, and they can be used to generate normal curve values. In addition, most spreadsheets can also perform a wide range of statistical calculations. For example, Microsoft Excel can be used to generate the answers to parts (a), (b), and (c) of this example. In any cell, type "=NORMDIST(1.25,0,1,1)" and press Enter. The value of 0.894350226 is returned. Notice that this value differs slightly from the value returned by a TI-84 Plus. The first three input values represent the z value, mean, and standard deviation. The fourth value is always either a 0 or 1. For applications in this text, we will always place a 1 in this position to indicate that we want the area to the left of the first input value. Similarly, by typing "=1-NORMDIST(1.25,0,1,1)" and pressing Enter, we find that the area to the right of $z = 1.25$ is 0.105649774.

YOUR TURN 1 Find the following areas under the standard normal curve. **(a)** The area to the left of $z = -0.76$. **(b)** The area to the right of $z = -1.36$. **(c)** The area between $z = -1.22$ and $z = 1.33$.

TRY YOUR TURN 1

NOTE Notice in Example 1 that $P(z \le 1.25) = P(z < 1.25)$. The area under the curve is the same, whether we include the endpoint or not. Notice also that $P(z = 1.25) = 0$, because no area is included.

CAUTION When calculating normal probabilities, it is wise to draw a normal curve with the mean and the z-scores every time. This will avoid confusion as to whether you should add or subtract probabilities.

EXAMPLE 2 Normal Probabilities

Find a value of z satisfying the following conditions.

Method 1 Using a Table

(a) 12.1% of the area is to the left of z.

SOLUTION Use the table backwards. Look in the body of the table for an area of 0.1210, and find the corresponding value of z using the left column and the top column of the table. You should find that $z = -1.17$ corresponds to an area of 0.1210.

(b) 20% of the area is to the right of z.

SOLUTION If 20% of the area is to the right, 80% is to the left. Find the value of z corresponding to an area of 0.8000. The closest value is $z = 0.84$.

Method 2 Graphing Calculator

Figure 16 illustrates how a TI-84 Plus can be used to find z values for the particular probabilities given in parts (a) and (b) of this example. The command invNorm is found in the DISTR menu.

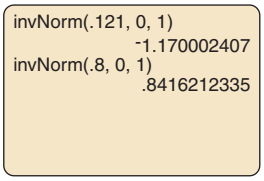


FIGURE 16

Method 3 Spreadsheet

Microsoft Excel can also be used to generate the answers to parts (a) and (b) of this example. In any cell, type "=NORMINV(.121,0,1)" and press Enter. The value of −1.170002408 is returned. Similarly, by typing "=NORMINV(.8,0,1)" and pressing Enter, we find that the corresponding z value is 0.841621234.

YOUR TURN 2 Find a value of z satisfying the following conditions. **(a)** 2.5% of the area is to the left of z. **(b)** 20.9% of the area is to the right of z

TRY YOUR TURN 2

The key to finding areas under *any* normal curve is to express each number x on the horizontal axis in terms of standard deviation above or below the mean. The ***z*-score** for x is the number of standard deviations that x lies from the mean (positive if x is above the mean, negative if x is below the mean).

EXAMPLE 3 *z*-Scores

If a normal distribution has mean 50 and standard deviation 4, find the following z-scores.

(a) The z-score for $x = 46$

SOLUTION Since 46 is 4 units below 50 and the standard deviation is 4, 46 is 1 standard deviation below the mean. So, its z-score is -1.

(b) The z-score for $x = 60$

SOLUTION The z-score is 2.5 because 60 is 10 units above the mean (since $60 - 50 = 10$), and 10 units is 2.5 standard deviations (since $10/4 = 2.5$).

YOUR TURN 3 Find the z-score for $x = 20$ if a normal distribution has a mean 35 and standard deviation 20.

TRY YOUR TURN 3

In Example 3(b), we found the z-score by taking the difference between 60 and the mean and dividing this difference by the standard deviation. The same procedure works in the general case.

z-Score

If a normal distribution has mean μ and standard deviation σ, then the z-score for the number x is

$$z = \frac{x - \mu}{\sigma}.$$

The importance of z-scores lies in the following fact.

Area under a Normal Curve

The area under a normal curve between $x = a$ and $x = b$ is the same as the area under the standard normal curve between the z-score for a and the z-score for b.

Therefore, by converting to z-scores and using the table for the standard normal curve, we can find areas under any normal curve. Since these areas are probabilities, we can now handle a variety of applications.

EXAMPLE 4 Sales

Dixie Office Supplies finds that its sales force drives an average of 1200 miles per month per person, with a standard deviation of 150 miles. Assume that the number of miles driven by a salesperson is closely approximated by a normal distribution.

(a) Find the probability that a salesperson drives between 1200 miles and 1600 miles per month.

APPLY IT

SOLUTION Here $\mu = 1200$ and $\sigma = 150$, and we must find the area under the normal distribution curve between $x_1 = 1200$ and $x_2 = 1600$. We begin by finding the z-score for $x_1 = 1200$.

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{1200 - 1200}{150} = \frac{0}{150} = 0$$

The z-score for $x_2 = 1600$ is

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{1600 - 1200}{150} = \frac{400}{150} \approx 2.67.$$

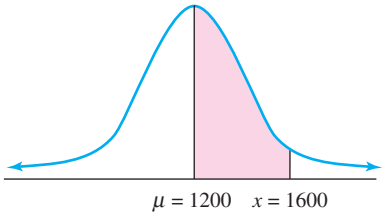


FIGURE 17

From the table, the area to the left of $z_2 = 2.67$ is 0.9962, the area to the left of $z_1 = 0$ is 0.5000, and

$$0.9962 - 0.5000 = 0.4962.$$

Therefore, the probability that a salesperson drives between 1200 miles and 1600 miles per month is 0.4962. See Figure 17.

(b) Find the probability that a salesperson drives between 1000 miles and 1500 miles per month.

SOLUTION As shown in Figure 18, z-scores for both $x_1 = 1000$ and $x_2 = 1500$ are needed.

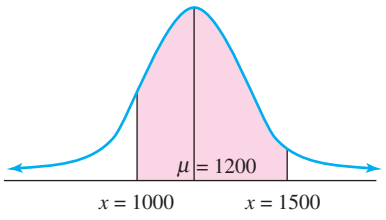


FIGURE 18

$$\begin{array}{ll} \text{For } x_1 = 1000, & \text{For } x_2 = 1500, \\ z_1 = \dfrac{1000 - 1200}{150} & z_2 = \dfrac{1500 - 1200}{150} \\ = \dfrac{-200}{150} & = \dfrac{300}{150} \\ z_1 \approx -1.33. & z_2 = 2.00. \end{array}$$

From the table, $z_1 = -1.33$ leads to an area of 0.0918, while $z_2 = 2.00$ corresponds to 0.9772. A total of $0.9772 - 0.0918 = 0.8854$, or 88.54%, of the drivers travel between 1000 and 1500 miles per month. The probability that a driver travels between 1000 miles and 1500 miles per month is 0.8854.

(c) Find the shortest and longest distances driven by the middle 95% of the sales force.

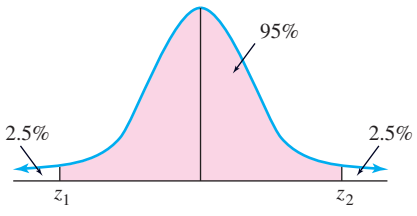


FIGURE 19

SOLUTION First, find the values of z that bound the middle 95% of the data. As Figure 19 illustrates, the lower z value, z_1, has 2.5% of the area to its left, and the higher z value, z_2, has 97.5% of the area to its left. Using the table backwards, we find that $z_1 = -1.96$ and $z_2 = 1.96$.

The shortest distance is, therefore, 1.96 standard deviations *below* the mean, or

$$\text{Shortest} = \mu + z \cdot \sigma = 1200 + (-1.96) \cdot (150) = 906 \text{ miles}.$$

Likewise, the longest distance is 1.96 standard deviations *above* the mean, or

$$\text{Longest} = \mu + z \cdot \sigma = 1200 + (1.96) \cdot (150) = 1494 \text{ miles}.$$

Therefore, the distances driven by the middle 95% of the sales force are between 906 and 1494 miles.

YOUR TURN 4 For Example 4, find the probability that a salesperson drives between 1275 and 1425 miles.

TRY YOUR TURN 4

TECHNOLOGY NOTE

Example 4 can also be done using a graphing calculator or computer, as described before, putting 1200 and 150 in place of 0 and 1 for the mean and standard deviation. The TI-84 Plus command `normalcdf(1000,1500,1200,150)` gives the answer 0.8860386561. This is more accurate than the value of 0.8854 found using the normal curve table, which required rounding the z-scores to two decimal places.

NOTE The answers given to the exercises in this text are found using the normal curve table. If you use a graphing calculator or computer program, your answers may differ slightly.

As mentioned above, z-scores are the number of standard deviations from the mean, so $z = 1$ corresponds to 1 standard deviation above the mean, and so on. Looking up $z = 1.00$ and $z = -1.00$ in the table shows that

$$0.8413 - 0.1587 = 0.6826,$$

or 68.3% of the area under a normal curve lies within 1 standard deviation of the mean. Also, looking up $z = 2.00$ and $z = -2.00$ shows that

$$0.9772 - 0.0228 = 0.9544,$$

or 95.4% of the area lies within 2 standard deviations of the mean. These results, summarized in Figure 20, can be used to get a quick estimate of results when working with normal curves.

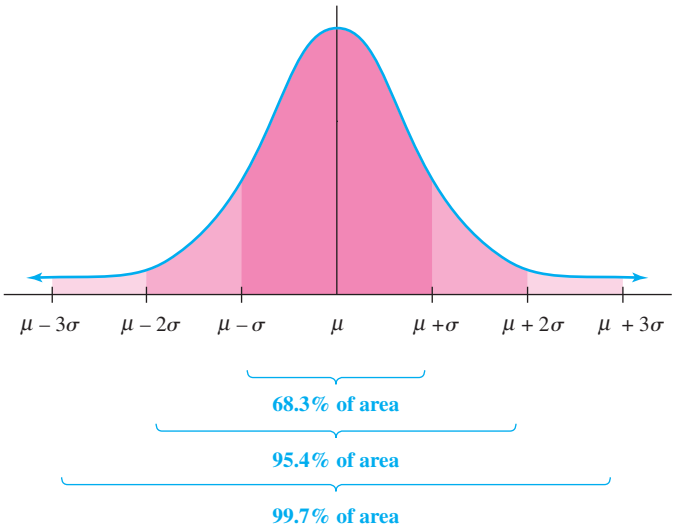


FIGURE 20

9.3 EXERCISES

1. The peak in a normal curve occurs directly above _____.

2. The total area under a normal curve (above the horizontal axis) is _____.

3. How are z-scores found for normal distributions where $\mu \neq 0$ or $\sigma \neq 1$?

4. How is the standard normal curve used to find probabilities for normal distributions?

Find the percent of the area under a normal curve between the mean and the given number of standard deviations from the mean.

5. 1.70

6. 0.93

7. −2.31

8. −1.45

Find the percent of the total area under the standard normal curve between each pair of z-scores.

9. $z = 0.32$ and $z = 3.18$

10. $z = 0.99$ and $z = 2.37$

11. $z = -1.83$ and $z = -0.91$

12. $z = -3.13$ and $z = -2.65$

13. $z = -2.95$ and $z = 2.03$

14. $z = -0.15$ and $z = 0.23$

Find a z-score satisfying the following conditions.

15. 5% of the total area is to the left of z.

16. 1% of the total area is to the left of z.

17. 10% of the total area is to the right of z.

18. 25% of the total area is to the right of z.

19. For any normal distribution, what is the value of $P(x \leq \mu)$? $P(x \geq \mu)$?

20. Compare the probability that a number will lie within 2 standard deviations of the mean of a probability distribution using Chebyshev's theorem and using the normal distribution. (See Exercises 13–20, Section 9.2.) Explain what you observe.

21. Repeat Exercise 20 using 3 standard deviations.

APPLICATIONS

In all of the following applications, assume the distributions are normal. In each case, you should consider whether this is reasonable.

Business and Economics

Life of Light Bulbs **A certain type of light bulb has an average life of 500 hours, with a standard deviation of 100 hours. The length of life of the bulb can be closely approximated by a normal curve. An amusement park buys and installs 10,000 such bulbs. Find the total number that can be expected to last for each period of time.**

22. At least 500 hours

23. Less than 500 hours

24. Between 680 and 780 hours

25. Between 350 and 550 hours

26. Less than 770 hours

27. More than 440 hours

28. Find the shortest and longest lengths of life for the middle 60% of the bulbs.

Quality Control **A box of oatmeal must contain 16 oz. The machine that fills the oatmeal boxes is set so that, on the average, a box contains 16.5 oz. The boxes filled by the machine have weights that can be closely approximated by a normal curve. What fraction of the boxes filled by the machine are underweight if the standard deviation is as follows?**

29. 0.5 oz

30. 0.3 oz

31. 0.2 oz

32. 0.1 oz

Quality Control **The chickens at Colonel Thompson's Ranch have a mean weight of 1850 g, with a standard deviation of 150 g. The weights of the chickens are closely approximated by a normal curve. Find the percent of all chickens having weights in the following ranges.**

33. More than 1700 g

34. Less than 1950 g

35. Between 1750 and 1900 g

36. Between 1600 and 2000 g

37. More than 2100 g or less than 1550 g

38. Find the smallest and largest weights for the middle 95% of the chickens.

39. University Expenditures A 2000 report found that private colleges and universities in the United States had a mean expenditure of \$169 million, with a standard deviation of \$300 million. ***Source: Higher Education.***

a. If the data are normally distributed, what percent of a population is more than one standard deviation below the mean?

b. What percent of private colleges and universities had an expenditure more than one standard deviation below the mean?

c. Based on your answers to parts a and b, discuss whether expenditures at private colleges and universities in the United States are likely to be normally distributed. Explain why this might be expected.

40. Quality Control A machine produces bolts with an average diameter of 0.25 in. and a standard deviation of 0.02 in. What is the probability that a bolt will be produced with a diameter greater than 0.3 in.?

41. Quality Control A machine that fills quart milk cartons is set up to average 32.2 oz per carton, with a standard deviation of 1.2 oz. What is the probability that a filled carton will contain less than 32 oz of milk?

42. Grocery Bills At the Discount Market, the average weekly grocery bill is \$74.50, with a standard deviation of \$24.30. What are the largest and smallest amounts spent by the middle 50% of this market's customers?

43. Grading Eggs To be graded extra large, an egg must weigh at least 2.2 oz. If the average weight for an egg is 1.5 oz, with a standard deviation of 0.4 oz, how many eggs in a sample of five dozen would you expect to grade extra large?

Life Sciences

Vitamin Requirements **In nutrition, the Recommended Daily Allowance of vitamins is a number set by the government as a guide to an individual's daily vitamin intake. Actually, vitamin needs vary drastically from person to person, but the needs are very closely approximated by a normal curve. To calculate the Recommended Daily Allowance, the government first finds the average need for vitamins among people in the population and the standard deviation. The Recommended Daily Allowance is then defined as the mean plus 2.5 times the standard deviation.**

44. What percent of the population will receive adequate amounts of vitamins under this plan?

Find the Recommended Daily Allowance for each vitamin in Exercises 45–47.

45. Mean = 1200 units; standard deviation = 60 units

46. Mean = 159 units; standard deviation = 12 units

47. Mean = 1200 units; standard deviation = 92 units

48. Blood Clotting The mean clotting time of blood is 7.45 seconds, with a standard deviation of 3.6 seconds. What is the probability that an individual's blood clotting time will be less than 7 seconds or greater than 8 seconds?

Social Sciences

Speed Limits **New studies by Federal Highway Administration traffic engineers suggest that speed limits on many thoroughfares are set arbitrarily and often are artificially low. According to traffic engineers, the ideal limit should be the "85th percentile speed." This means the speed at or below which 85 percent of the traffic moves. Assuming speeds are normally distributed, find the 85th percentile speed for roads with the following conditions.** ***Source: Federal Highway Administration.***

49. The mean speed is 52 mph with a standard deviation of 8 mph.

50. The mean speed is 30 mph with a standard deviation of 5 mph.

Education **The grading system known as "grading on the curve" is based on the assumption that grades are often distributed according to the normal curve and that a certain percent of a class should receive each grade, regardless of the performance of the class as a whole. The following is how one professor might grade on the curve.**

Grade	Total Points
A	Greater than $\mu + (3/2)\sigma$
B	$\mu + (1/2)\sigma$ to $\mu + (3/2)\sigma$
C	$\mu - (1/2)\sigma$ to $\mu + (1/2)\sigma$
D	$\mu - (3/2)\sigma$ to $\mu - (1/2)\sigma$
F	Below $\mu - (3/2)\sigma$

What percent of the students receive the following grades?

51. A **52.** B **53.** C

54. Do you think this system would be more likely to be fair in a large freshman class in psychology or in a graduate seminar of five students? Why?

Education **A teacher gives a test to a large group of students. The results are closely approximated by a normal curve. The mean is 76, with a standard deviation of 8. The teacher wishes to give A's to the top 8% of the students and F's to the bottom 8%. A grade of B is given to the next 20%, with D's given similarly. All other students get C's. Find the bottom cutoff (rounded to the nearest whole number) for the following grades.**

55. A **56.** B **57.** C **58.** D

59. Standardized Tests David Rogosa, a professor of educational statistics at Stanford University, has calculated the accuracy of tests used in California to abolish social promotion. Dr. Rogosa has claimed that a fourth grader whose true reading score is exactly at reading level (50th percentile—half of all the students read worse and half read better than this student) has a 58% chance of either scoring above the 55th percentile or below the 45th percentile on any one test. Assume that the results of a given test are normally distributed with mean 0.50 and standard deviation 0.09. ***Source: The New York Times.***

a. Verify that Dr. Rogosa's claim is true.

b. Find the probability that this student will not score between the 40th and 60th percentile.

c. Using the results of parts a and b, discuss problems with the use of standardized testing to prevent social promotion.

General Interest

60. Christopher Columbus Before Christopher Columbus crossed the ocean, he measured the heights of the men on his three ships and found that they were normally distributed with a mean of 69.60 in. and a standard deviation of 3.20 in. What is the probability that a member of his crew had a height less than 66.27 in.? (The answer has another connection with Christopher Columbus!)

61. Lead Poisoning Historians and biographers have collected evidence that suggests that President Andrew Jackson suffered from lead poisoning. Recently, researchers measured the amount of lead in samples of Jackson's hair from 1815. The results of this experiment showed that Jackson had a mean lead level of 130.5 ppm. ***Source: JAMA.***

a. If levels of lead in hair samples from that time period follow a normal distribution with mean 93 and standard deviation 16, find the probability that a randomly selected person from this time period would have a lead level of 130.5 ppm or higher. Does this provide evidence that Jackson suffered from lead poisoning during this time period?* ***Source: Science.***

b. Today's typical lead levels follow a normal distribution with approximate mean 10 ppm and standard deviation 5 ppm. By these standards, calculate the probability that a randomly selected person from today would have a lead level of 130.5 or higher. (*Note:* These standards may not be valid for this experiment.) ***Source: Clinical Chemistry***

c. Discuss whether we can conclude that Andrew Jackson suffered from lead poisoning.

62. Mercury Poisoning Historians and biographers have also collected evidence that suggests that President Andrew Jackson suffered from mercury poisoning. Recently, researchers measured the amount of mercury in samples of Jackson's hair from 1815. The results of this experiment showed that Jackson had a mean mercury level of 6.0 ppm. ***Source: JAMA.***

a. If levels of mercury in hair samples from that time period follow a normal distribution with mean 6.9 and standard deviation 4.6, find the probability that a randomly selected person from that time period would have a mercury level of 6.0 ppm or higher. ***Source: Science of the Total Environment.***

b. Discuss whether this provides evidence that Jackson suffered from mercury poisoning during this time period.

c. Today's accepted normal mercury levels follow a normal distribution with approximate mean 0.6 ppm and standard deviation 0.3 ppm. By present standards, is it likely that a

*Although this provides evidence that Andrew Jackson had elevated lead levels, the authors of the paper concluded that Andrew Jackson did not die from lead poisoning.

randomly selected person from today would have a mercury level of 6.0 ppm or higher?

d. Discuss whether we can conclude that Andrew Jackson suffered from mercury poisoning.

63. Barbie The popularity and voluptuous shape of Barbie dolls have generated much discussion about the influence these dolls may have on young children, particularly with regard to normal body shape. In fact, many people have speculated as to what Barbie's measurements would be if they were scaled to a common human height. Researchers have done this and have compared Barbie's measurements to the average 18- to 35-year-old woman, labeled Reference, and with the average model. The table below illustrates some of the results of their research, where each measurement is in centimeters. Assume that the distributions of measurements for the models and for the reference group follow a normal distribution with the given mean and standard deviation. ***Source: Sex Roles.***

	Models		Reference		
Measurement	**Mean**	**s.d.**	**Mean**	**s.d.**	**Barbie**
Head	50.0	2.4	55.3	2.0	55.0
Neck	31.0	1.0	32.7	1.4	23.9
Chest (bust)	87.4	3.0	90.3	5.5	82.3
Wrist	15.0	0.6	16.1	0.8	10.6
Waist	65.7	3.5	69.8	4.7	40.7

a. Find the probability of Barbie's head size or larger occurring for the reference group and for the models.

b. Find the probability of Barbie's neck size or smaller occurring for the reference group and for the models.

c. Find the probability of Barbie's bust size or larger occurring for the reference group and for the models.

d. Find the probability of Barbie's wrist size or smaller occurring for the reference group and for the models.

e. Find the probability of Barbie's waist size or smaller occurring for the reference group and for the models.

f. Compare the above values and discuss whether Barbie represents either the reference group or models. Any surprises?

64. Ken The same researchers from Exercise 63 wondered how the famous Ken doll measured up to average males and with Australian football players. The table below illustrates some of the results of their research, where each measurement is in centimeters. Assume that the distributions of measurements for the football players and for the reference group follow a normal distribution with the given mean and standard deviation. ***Source: Sex Roles.***

	Football		Reference		
Measurement	**Mean**	**s.d.**	**Mean**	**s.d.**	**Ken**
Head	52.1	2.3	53.7	2.9	53.0
Neck	34.6	1.8	34.2	1.9	32.1
Chest	92.3	3.5	91.2	4.8	75.0
Upper Arm	29.9	1.9	28.8	2.2	27.1
Waist	75.1	3.6	80.9	9.8	56.5

a. Find the probability of Ken's head size or larger occurring for the reference group and for the football players.

b. Find the probability of Ken's neck size or smaller occurring for the reference group and for the football players.

c. Find the probability of Ken's chest size or smaller occurring for the reference group and for the football players.

d. Find the probability of Ken's upper arm size or smaller occurring for the reference group and for the football players.

e. Find the probability of Ken's waist size or smaller occurring for the reference group and for the football players.

f. Compare the above values and discuss whether Ken's measurements are representative of either the reference group or football players. Then compare these results with the results of Exercise 63. Any surprises?

YOUR TURN ANSWERS

1. **(a)** 0.2236 **(b)** 0.9131 **(c)** 0.7970
2. **(a)** −1.96 **(b)** 0.81
3. −0.75 **4.** 0.2417

9.4 Normal Approximation to the Binomial Distribution

APPLY IT **What is the probability that at least 40 out of 100 drivers exceed the speed limit by at least 20 mph in Atlanta?**

This is a binomial probability problem with a large number of trials (100). In this section we will see how the normal curve can be used to approximate the binomial distribution, allowing us to answer this question in Example 2.

As we saw in Section 8.4 on Binomial Probability, many practical experiments have only two possible outcomes, sometimes referred to as success or failure. Such experiments are called Bernoulli trials or Bernoulli processes. Examples of Bernoulli trials include flipping a coin (with heads being a success, for instance, and tails a failure) or testing a computer chip coming off the assembly line to see whether or not it is defective. A binomial experiment consists of repeated independent Bernoulli trials, such as flipping a coin 10 times or taking a random sample of 20 computer chips from the assembly line. In Section 8.5 on Probability Distributions and Expected Value, we found the probability distribution for several binomial experiments, such as sampling five people with bachelor's degrees in education and counting how many are women. The probability distribution for a binomial experiment is known as a **binomial distribution**.

As another example, it was reported in a recent study that 40% of drivers in Atlanta exceed the speed limit by at least 20 mph. ***Source: Laser Atlanta.*** Suppose a state trooper wants to verify this statistic and records the speed of 10 randomly selected drivers. The trooper finds that 5 out of 10, or 50%, exceed the speed limit by at least 20 mph. How likely is this if the 40% figure is accurate? We can answer this question with the binomial probability formula from the previous chapter:

$$C(n, x) \cdot p^x \cdot (1 - p)^{n-x},$$

where n is the size of the sample (10 in this case), x is the number of speeders (5 in this example), and p is the probability that a driver is a speeder (0.40). This gives

$$\begin{aligned} P(x = 5) &= C(10, 5) \cdot 0.40^5 \cdot (1 - 0.40)^5 \\ &= 252(0.01024)(0.07776) \approx 0.2007. \end{aligned}$$

The probability is about 20%, so this result is not unusual.

Suppose that the state trooper takes a larger random sample of 100 drivers. What is the probability that 50 or more drivers speed if the 40% figure is accurate? Calculating $P(x = 50) + P(x = 51) + \cdots + P(x = 100)$ is a formidable task. One solution is provided by graphing calculators or computers. On the TI-84 Plus, for example, we can first calculate the probability that 49 or fewer drivers exceed the speed limit using the DISTR menu command `binomcdf(100,.40,49)`. Subtracting the answer from 1 gives a probability of 0.0271. But this high-tech method fails as n becomes larger; the command `binomcdf(1000000,.40,50000)` gives an error message. On the other hand, there is a low-tech method that works regardless of the size of n. It has further interest because it connects two different distributions: the normal and the binomial. The normal distribution is continuous, since the random variable can take on any real number. The binomial distribution is *discrete*, because the random variable can only take on integer values between 0 and n. Nevertheless, the normal distribution can be used to give a good approximation to binomial probability.

In order to use the normal approximation, we first need to know the mean and standard deviation of the binomial distribution. Recall from Section 8.5 that for the binomial distribution, $E(x) = np$. In Section 9.1, we referred to $E(x)$ as μ, and that notation will be used here. It is shown in more advanced courses in statistics that the standard deviation of the binomial distribution is given by $\sigma = \sqrt{np(1 - p)}$.

FOR REVIEW

Recall from Chapter 8 that the symbol $C(n, r)$ is defined as $\frac{n!}{r!(n - r)!}$. For example,

$$\begin{aligned} C(10, 5) &= \frac{10!}{5!\,5!} \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252. \end{aligned}$$

Mean and Standard Deviation for the Binomial Distribution

For the binomial distribution, the mean and standard deviation are given by

$$\boldsymbol{\mu = np} \qquad \text{and} \qquad \boldsymbol{\sigma = \sqrt{np(1 - p)}},$$

where n is the number of trials and p is the probability of success on a single trial.

EXAMPLE 1 Coin Flip

Suppose a fair coin is flipped 15 times.

(a) Find the mean and standard deviation for the number of heads.

SOLUTION Using $n = 15$ and $p = 1/2$, the mean is

$$\mu = np = 15\left(\frac{1}{2}\right) = 7.5.$$

The standard deviation is

$$\sigma = \sqrt{np(1-p)} = \sqrt{15\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)}$$
$$= \sqrt{15\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \sqrt{3.75} \approx 1.94.$$

We expect, on average, to get 7.5 heads out of 15 tosses. Most of the time, the number of heads will be within 3 standard deviations of the mean, or between $7.5 - 3(1.94) = 1.68$ and $7.5 + 3(1.94) = 13.32$.

(b) Find the probability distribution for the number of heads, and draw a histogram of the probabilities.

SOLUTION The probability distribution is found by putting $n = 15$ and $p = 1/2$ into the formula for binomial probability. For example, the probability of 9 heads is given by

$$P(x = 9) = C(15, 9)\left(\frac{1}{2}\right)^9\left(1-\frac{1}{2}\right)^6 \approx 0.15274.$$

Probabilities for the other values of x between 0 and 15, as well as a histogram of the probabilities, are shown in the table and in Figure 21. TRY YOUR TURN 1

YOUR TURN 1 Suppose a die is rolled 12 times. Find the mean and standard deviation of the number of sixes rolled.

In Figure 21, we have superimposed the normal curve with $\mu = 7.5$ and $\sigma = 1.94$ over the histogram of the distribution. Notice how well the normal distribution fits the binomial distribution. This approximation was first discovered in 1733 by Abraham De Moivre (1667–1754) for the case $p = 1/2$. The result was generalized by the French mathematician

Probability Distribution for Number of Heads	
x	$P(x)$
0	0.00003
1	0.00046
2	0.00320
3	0.01389
4	0.04166
5	0.09164
6	0.15274
7	0.19638
8	0.19638
9	0.15274
10	0.09164
11	0.04166
12	0.01389
13	0.00320
14	0.00046
15	0.00003

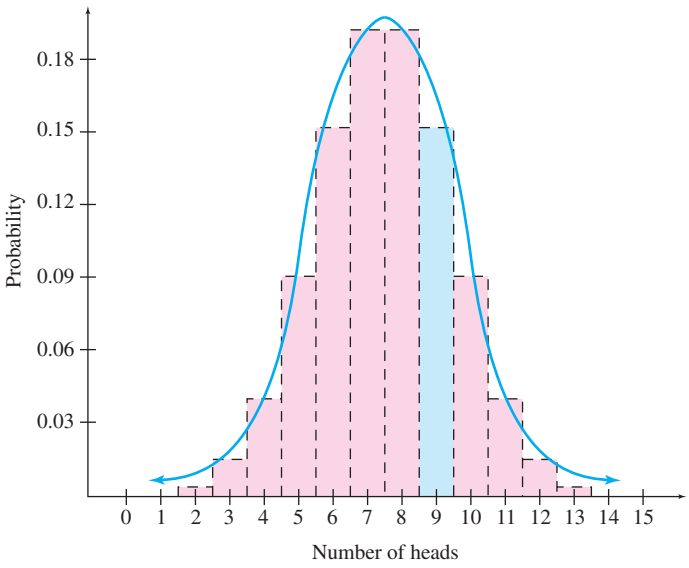


FIGURE 21

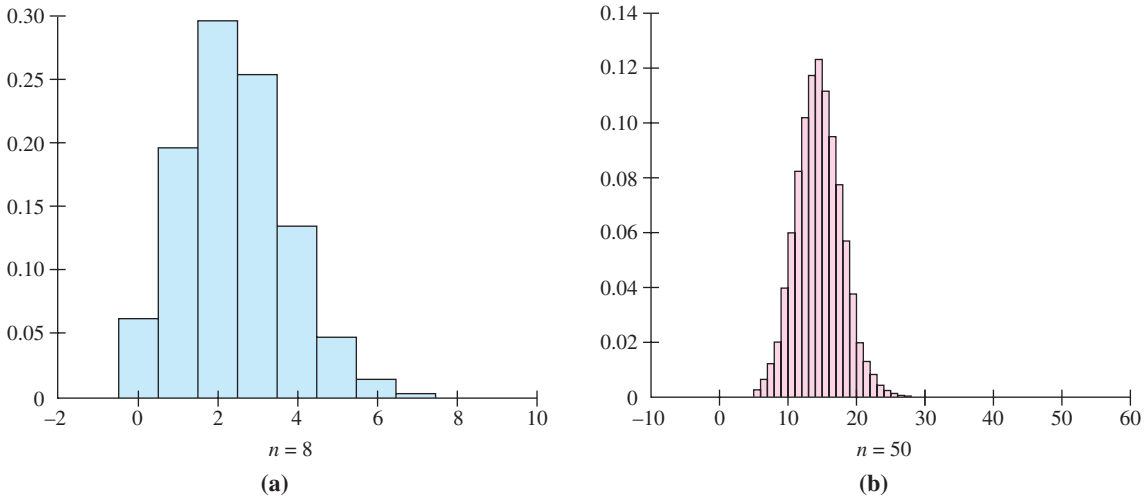


FIGURE 22

Pierre-Simon Laplace (1749–1827) in a book published in 1812.* As n becomes larger and larger, a histogram for the binomial distribution looks more and more like a normal curve. Histograms of the binomial distribution with $p = 0.3$, using $n = 8$ and $n = 50$ are shown in Figures 22(a) and (b), respectively.

The probability of getting exactly 9 heads in 15 tosses, or 0.15274, is the same as the area of the bar in blue in Figure 21. As the graph suggests, the area in blue is approximately equal to the area under the normal curve from $x = 8.5$ to $x = 9.5$. The normal curve is higher than the top of the bar in the left half but lower in the right half.

To find the area under the normal curve from $x = 8.5$ to $x = 9.5$, first find z-scores, as in the previous section. Use the mean and the standard deviation for the distribution, which we have already calculated, to get z-scores for $x_1 = 8.5$ and $x_2 = 9.5$.

$$\begin{aligned} \text{For } x_1 = 8.5, \qquad & \qquad \text{For } x_2 = 9.5, \\ z_1 &= \frac{8.5 - 7.5}{1.94} & z_2 &= \frac{9.5 - 7.5}{1.94} \\ &= \frac{1.00}{1.94} & &= \frac{2.00}{1.94} \\ z_1 &\approx 0.52. & z_2 &\approx 1.03. \end{aligned}$$

From the table in the Appendix, $z_1 = 0.52$ gives an area of 0.6985, and $z_2 = 1.03$ gives 0.8485. The difference between these two numbers is the desired result.

$$P(z \le 1.03) - P(z \le 0.52) = 0.8485 - 0.6985 = 0.1500$$

This answer (0.1500) is not far from the more accurate answer of 0.15274 found in Example 1(b).

CAUTION The normal curve approximation to a binomial distribution is quite accurate *provided that* n is large and p is not close to 0 or 1. As a rule of thumb, the normal curve approximation can be used as long as both np and $n(1 - p)$ are at least 5.

*Laplace's generalization, known as the Central Limit theorem, states that the distribution of the sample mean from *any* distribution approaches the normal distribution as the sample size increases. For more details, see any statistics book, such as *Elementary Statistics* (11th ed.) by Mario F. Triola, Pearson, 2009.

EXAMPLE 2 Speeding

Consider the random sample discussed earlier of 100 drivers in Atlanta, where 40% of the drivers exceed the speed limit by at least 20 mph.

(a) Use the normal distribution to approximate the probability that at least 50 drivers exceed the speed limit.

APPLY IT

SOLUTION First find the mean and the standard deviation using $n = 100$ and $p = 0.40$.

$$\mu = 100(0.40) \qquad \sigma = \sqrt{100(0.40)(1 - 0.40)}$$
$$= 40 \qquad = \sqrt{100(0.40)(0.60)}$$
$$= \sqrt{24} \approx 4.899$$

As the graph in Figure 23 shows, we need to find the area to the right of $x = 49.5$ (since we want 50 or more speeders). The z-score corresponding to $x = 49.5$ is

$$z = \frac{49.5 - 40}{4.899} \approx 1.94.$$

From the table, $z = 1.94$ leads to an area of 0.9738, so

$$P(z > 1.94) = 1 - 0.9738 = 0.0262.$$

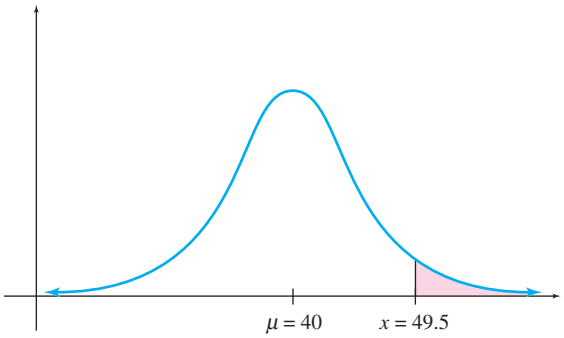


FIGURE 23

This value is close to the value of 0.0271 found earlier with the help of a graphing calculator. Either method tells us there is roughly a 3% chance of finding 50 or more speeders out of a random sample of 100. If the trooper found 50 or more speeders in his sample, he might suspect that either his sample is not truly random, or that the 40% figure for the percent of drivers who speed is too low.

(b) Find the probability of finding between 42 and 48 speeders in a random sample of 100.

SOLUTION As Figure 24 shows, we need to find the area between $x_1 = 41.5$ and $x_2 = 48.5$.

$$\text{If } x_1 = 41.5, \text{ then } z_1 = \frac{41.5 - 40}{4.899} \approx 0.31.$$

$$\text{If } x_2 = 48.5, \text{ then } z_2 = \frac{48.5 - 40}{4.899} \approx 1.74.$$

Use the table to find that $z_1 = 0.31$ gives an area of 0.6217, and $z_2 = 1.74$ yields 0.9591. The final answer is the difference of these numbers, or

$$P(0.31 \le z \le 1.74) = P(z \le 1.74) - P(z \le 0.31)$$
$$= 0.9591 - 0.6217 = 0.3374.$$

The probability of finding between 42 and 48 speeders is about 0.3374. This is close to the value of 0.3352 found using the `binomcdf` command on a TI-84 Plus.

YOUR TURN 2 Suppose that a test consists of 120 multiple choice questions, each with 5 answers from which to choose. Use the normal distribution to find the probability that if you guess at random, you get at least 32 correct.

TRY YOUR TURN 2

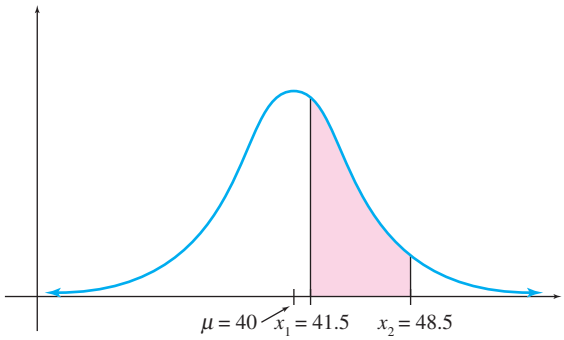


FIGURE 24

9.4 EXERCISES

1. What must be known to find the mean and standard deviation of a binomial distribution?

2. What is the rule of thumb for using the normal distribution to approximate a binomial distribution?

Suppose 16 coins are tossed. Find the probability of getting the following results (a) using the binomial probability formula and (b) using the normal curve approximation.

3. Exactly 4 heads

4. Exactly 10 heads

5. More than 12 tails

6. Fewer than 5 tails

For the remaining exercises in this section, use the normal curve approximation to the binomial distribution.

Suppose 1000 coins are tossed. Find the probability of getting the following results.

7. Exactly 500 heads

8. Exactly 510 heads

9. 475 heads or more

10. Fewer than 490 tails

A die is tossed 120 times. Find the probability of getting the following results.

11. Exactly twenty 5's

12. Exactly twenty-four 6's

13. More than fifteen 3's

14. Fewer than twenty-eight 6's

15. A reader asked Mr. Statistics (a feature in *Fortune* magazine) about the game of 26 once played in the bars of Chicago. The player chooses a number between 1 and 6 and then rolls a cup full of 10 dice 13 times. Out of the 130 numbers rolled, if the number chosen appears at least 26 times, the player wins. Calculate the probability of winning. ***Source: Fortune.***

16. a. Try to use both the binomial probability formula and the normal approximation to the binomial to calculate the probability that exactly half of the coins come up heads if the following number of coins are flipped. You may run into problems using the binomial probability formula for part iii.

i. 10 **ii.** 100 **iii.** 1000

b. If you ran into problems using the binomial probability formula for part iii, tell what happened and explain why it happened.

c. Someone might speculate that with more coins, the probability that exactly half are heads goes up with the number of coin flips. Based on the results from part a, does this happen? Explain the error in the speculation.

APPLICATIONS

Business and Economics

17. Quality Control Two percent of the quartz heaters produced in a certain plant are defective. Suppose the plant produced 10,000 such heaters last month. Find the probabilities that among these heaters, the following numbers were defective.

a. Fewer than 170 **b.** More than 222

18. Quality Control The probability that a certain machine turns out a defective item is 0.05. Find the probabilities that in a run of 75 items, the following results are obtained.

a. Exactly 5 defectives

b. No defectives

c. At least 1 defective

19. Survey Results A company is taking a survey to find out if people like its product. Their last survey indicated that 70% of the population like the product. Based on that, of a sample of 58 people, find the probabilities of the following.

a. All 58 like the product.

b. From 28 to 30 (inclusive) like the product.

20. Minimum Wage A recent study of minimum wage earners found that 50.4% of them are 16 to 24 years old. Suppose a random sample of 600 minimum wage earners is selected. What is the probability that more than 330 of them are 16 to 24 years old? ***Source: Bureau of Labor Statistics.***

Life Sciences

21. Nest Predation For certain bird species, with appropriate assumptions, the number of nests escaping predation has a binomial distribution. Suppose the probability of success (that is, a nest escaping predation) is 0.3. Find the probability that at least half of 24 nests escape predation. ***Source: The American Naturalist.***

22. Food Consumption Under certain appropriate assumptions, the probability of a competing young animal eating x units of food is binomially distributed, with n equal to the maximum number of food units the animal can acquire and p equal to the probability per time unit that an animal eats a unit of food. Suppose $n = 120$ and $p = 0.6$. ***Source: The American Naturalist.***

a. Find the probability that an animal consumes exactly 80 units of food.

b. Suppose the animal must consume at least 70 units of food to survive. What is the probability that this happens?

23. Coconuts A 4-year review of trauma admissions to the Provincial Hospital, Alotau, Milne Bay Providence, reveals that 2.5% of such admissions were due to being struck by falling coconuts. ***Source: The Journal of Trauma.***

a. Suppose 20 patients are admitted to the hospital during a certain time period. What is the probability that no more than 1 of these patients are there because they were struck by falling coconuts? Do not use the normal distribution here.

b. Suppose 2000 patients are admitted to the hospital during a longer time period. What is the approximate probability that no more than 70 of these patients are there because they were struck by falling coconuts?

24. Drug Effectiveness A new drug cures 80% of the patients to whom it is administered. It is given to 25 patients. Find the probabilities that among these patients, the following results occur.

a. Exactly 20 are cured.

b. All are cured.

c. No one is cured.

d. Twelve or fewer are cured.

25. Flu Inoculations A flu vaccine has a probability of 80% of preventing a person who is inoculated from getting the flu. A county health office inoculates 134 people. Find the probabilities of the following.

a. Exactly 10 of the people inoculated get the flu.

b. No more than 10 of the people inoculated get the flu.

c. None of the people inoculated get the flu.

26. Blood Types The blood types B− and AB− are the rarest of the eight human blood types, representing 1.5% and 0.6% of the population, respectively. ***Source: The Handy Science Answer Book.***

a. If the blood types of a random sample of 1000 blood donors are recorded, what is the probability that 10 or more of the samples are AB−?

b. If the blood types of a random sample of 1000 blood donors are recorded, what is the probability that 20 to 40 inclusive of the samples are B−?

c. If a particular city had a blood drive in which 500 people gave blood and 3% of the donations were B−, would we have reason to believe that this town has a higher than normal number of donors who are B−? (*Hint:* Calculate the probability of 15 or more donors being B− for a random sample of 500 and then discuss the probability obtained.)

27. Motorcycles According to a recent report, 24.1% of non-fatal injuries suffered by motorcycle riders occur between 3 P.M. and 6 P.M. If 200 injured motorcyclists are surveyed, what is the probability that at most 50 were injured between 3 P.M. and 6 P.M.? ***Source: Insurance Information Institute.***

Social Sciences

28. Straw Votes In one state, 55% of the voters expect to vote for Vale Leist. Suppose 1400 people are asked the name of the person for whom they expect to vote. Find the probability that at least 750 people will say that they expect to vote for Leist.

29. Smoking A recent study found that 46.3% of all ninth grade students in the United States have tried cigarette smoking, even if only one or two puffs. If 500 ninth grade students are surveyed, what is the probability that at most half have ever tried cigarettes? ***Source: Centers for Disease Control and Prevention.***

30. Weapons and Youth A recent study found that 17.5% of all high school students in the United States have carried a weapon, including a gun, knife, or club. If 1200 high school students are surveyed, what is the probability that more than 200 students, but fewer than 250, have carried a weapon? ***Source: Centers for Disease Control and Prevention.***

31. Election 2000 The Florida recount in the 2000 presidential election gave George W. Bush 2,912,790 votes and Al Gore 2,912,253 votes. What is the likelihood of the vote being so close, even if the electorate is evenly divided? Assume that the number of votes for Bush is binomially distributed with $n = 5{,}825{,}043$ (the sum of the votes for the two candidates) and $p = 0.5$. ***Source: historycentral.com.***

a. Using the binomial probability feature on a graphing calculator, try to calculate $P(2{,}912{,}253 \le X \le 2{,}912{,}790)$. What happens?

b. Use the normal approximation to calculate the probability in part a.

General Interest

32. Homework Only 1 out of 12 American parents requires that children do their homework before watching TV. If your neighborhood is typical, what is the probability that out of 51 parents, 5 or fewer require their children to do homework before watching TV? ***Source: Harper's.***

33. True-False Test A professor gives a test with 100 true-false questions. If 60 or more correct is necessary to pass, what is the probability that a student will pass by random guessing?

34. Hole in One In the 1989 U.S. Open, four golfers each made a hole in one on the same par-3 hole on the same day. *Sports Illustrated* writer R. Reilly stated the probability of a hole in one for a given golf pro on a given par-3 hole to be 1/3709. ***Source: Sports Illustrated.***

a. For a specific par-3 hole, use the binomial distribution to find the probability that 4 or more of the 156 golf pros in the tournament field shoot a hole in one. ***Source: School Science and Mathematics.***

b. For a specific par-3 hole, use the normal approximation to the binomial distribution to find the probability that 4 or

more of the 156 golf pros in the tournament field shoot a hole in one. Why must we be very cautious when using this approximation for this application?

c. If the probability of a hole in one remains constant and is 1/3709 for any par-3 hole, find the probability that in 20,000 attempts by golf pros, there will be 4 or more hole in ones. Discuss whether this assumption is reasonable.

YOUR TURN ANSWERS

1. 2, 1.291 **2.** 0.0436

9 CHAPTER REVIEW

SUMMARY

In this chapter we introduced the field of statistics. Measures of central tendency, such as mean, median, and mode, were defined and illustrated by examples. We determined how much the numbers in a sample vary from the mean of a distribution by calculating the variance and standard deviation. The normal distribution, perhaps the most important and widely used probability distribution, was defined and used to study a wide range of problems. The normal approximation to the binomial distribution was then developed, as were several important applications.

Mean The mean of the n numbers $x_1, x_2, x_3, \ldots, x_n$ is

$$\bar{x} = \frac{\Sigma x}{n}.$$

Mean of a Grouped Distribution The mean of a distribution, where x represents the midpoints, f the frequencies, and $n = \Sigma f$, is

$$\bar{x} = \frac{\Sigma xf}{n}.$$

Variance The variance of a sample of n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\bar{x}$, is

$$s^2 = \frac{\Sigma x^2 - n\bar{x}^2}{n - 1}.$$

Standard Deviation The standard deviation of a sample of n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\bar{x}$, is

$$s = \sqrt{\frac{\Sigma x^2 - n\bar{x}^2}{n - 1}}.$$

Standard Deviation for a Grouped Distribution The standard deviation for a distribution with mean $\bar{x}$, where x is an interval midpoint with frequency f, and $n = \Sigma f$, is

$$s = \sqrt{\frac{\Sigma fx^2 - n\bar{x}^2}{n - 1}}.$$

Normal Distribution The area of the shaded region under a normal curve from a to b is the probability that an observed data value will be between a and b.

z-scores If a normal distribution has mean μ and standard deviation σ, then the z-score for the number x is

$$z = \frac{x - \mu}{\sigma}.$$

Area Under a Normal Curve The area under a normal curve between $x = a$ and $x = b$ is the same as the area under the standard normal curve between the z-score for a and the z-score for b.

Mean and Standard Deviation for the Binomial Distribution For the binomial distribution, the mean and standard deviation are given by

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{np(1 - p)},$$

where n is the number of trials and p is the probability of success on a single trial.

KEY TERMS

9.1
random sample
grouped frequency distribution
frequency polygon
(arithmetic) mean
sample mean
population mean
median
statistic
mode

9.2
range
deviations from the mean
variance
standard deviation
sample variance
sample standard deviation
population variance
population standard deviation
Chebyshev's theorem

9.3
continuous distribution
skewed distribution
normal distribution
normal curve
standard normal curve
z-score

9.4
binomial distribution

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The mean, median, and mode of a normal distribution are all equal.

2. If the mean, median, and mode of a distribution are all equal, then the distribution must be a normal distribution.

3. If the means of two distributions are equal, then the variances must also be equal.

4. The sample mean $\bar{x}$ is not the same as the population mean μ.

5. A large variance indicates that the data are grouped closely together.

6. The mode of a distribution is the middle element of the distribution.

7. For a random variable X that is normally distributed, we know that $P(X \geq 2) = P(X \leq -2)$.

8. For a random variable X that is normally distributed with $\mu = 5$, we know that $P(X > 10) = P(X < 0)$.

9. The normal curve approximation to the binomial distribution should not be used on an experiment where $n = 30$ and $p = 0.1$.

10. The expected value of a sample mean is the population mean.

11. The expected value of a sample standard deviation is the population standard deviation.

12. For a standard normal random variable Z, $P(-1.5 < Z < 0) = 0.50 - P(Z > 1.5)$.

PRACTICE AND EXPLORATIONS

13. Discuss some reasons for organizing data into a grouped frequency distribution.

14. What is the rule of thumb for an appropriate interval in a grouped frequency distribution?

In Exercises 15 and 16, (a) write a frequency distribution, (b) draw a histogram, and (c) draw a frequency polygon.

15. The following numbers give the sales (in dollars) for the lunch hour at a local hamburger stand for the last 20 Fridays. Use intervals 450–474, 475–499, and so on.

480 451 501 478 512 473 509 515 458 566
516 535 492 558 488 547 461 475 492 471

16. The number of credits carried in one semester by students in a business mathematics class was as follows. Use intervals 9–10, 11–12, 13–14, 15–16.

10 9 16 12 13 15 13 16 15 11 13
12 12 15 12 14 10 12 14 15 15 13

Find the mean for the following.

17. 30, 24, 34, 30, 29, 28, 30, 29

18. 105, 108, 110, 115, 106, 110, 104, 113, 117

19.

Interval	Frequency
10–19	6
20–29	12
30–39	14
40–49	10
50–59	8

20.

Interval	Frequency
40–44	3
45–49	6
50–54	7
55–59	14
60–64	3
65–69	2

21. What do the mean, median, and mode of a distribution have in common? How do they differ? Describe each in a sentence or two.

Find the median and the mode (or modes) for each list of numbers.

22. 12, 17, 21, 23, 27, 27, 34

23. 38, 36, 42, 44, 38, 36, 48, 35

Find the modal class for the indicated distributions.

24. Exercise 20 **25.** Exercise 19

26. What is meant by the range of a distribution?

27. How are the variance and the standard deviation of a distribution related? What is measured by the standard deviation?

Find the range and standard deviation for each distribution.

28. 22, 27, 31, 35, 41

29. 26, 43, 51, 29, 37, 56, 29, 82, 74, 93

Find the standard deviation for the following.

30. Exercise 20 **31.** Exercise 19

32. Describe the characteristics of a normal distribution.

33. What is meant by a skewed distribution?

Find the following areas under the standard normal curve.

34. Between $z = 0$ and $z = 2.17$

35. To the left of $z = 0.84$

36. Between $z = -2.13$ and $z = 1.11$

37. Between $z = 1.53$ and $z = 2.82$

38. Find a z-score such that 7% of the area under the curve is to the right of z.

39. Why is the normal distribution not a good approximation of a binomial distribution that has a value of p close to 0 or 1?

40. Suppose a card is drawn at random from an ordinary deck 1,000,000 times with replacement.

a. What is the probability that between 249,500 and 251,000 hearts (inclusive) are drawn?

b. Why must the normal approximation to the binomial distribution be used to solve part a?

41. Suppose four coins are flipped and the number of heads counted. This experiment is repeated 20 times. The data might look something like the following. (You may wish to try this yourself and use your own results rather than these.)

Number of Heads	Frequency
0	1
1	5
2	7
3	5
4	2

a. Calculate the sample mean $\bar{x}$ and sample standard deviation s.

b. Calculate the population mean μ and population standard deviation σ for this binomial population.

c. Compare your answer to parts a and b. What do you expect to happen?

42. Much of our work in Chapters 8 and 9 is interrelated. Note the similarities in the following parallel treatments of a frequency distribution and a probability distribution.

Frequency Distribution

Complete the table below for the following data. (Recall that x is the midpoint of the interval.)

14, 7, 1, 11, 2, 3, 11, 6, 10, 13, 11, 11, 16, 12, 9, 11, 9, 10, 7, 12, 9, 6, 4, 5, 9, 16, 12, 12, 11, 10, 14, 9, 13, 10, 15, 11, 11, 1, 12, 12, 6, 7, 8, 2, 9, 12, 10, 15, 9, 3

Interval	x	Tally	f	$x \cdot f$
1–3	2	𝍸 |	6	12
4–6				
7–9				
10–12				
13–15				
16–18				

Probability Distribution

A binomial distribution has $n = 10$ and $p = 0.5$. Complete the following table.

x	$P(x)$	$x \cdot P(x)$
0	0.001	
1	0.010	
2	0.044	
3	0.117	
4		
5		
6		
7		
8		
9		
10		

a. Find the mean (or expected value) for each distribution.

b. Find the standard deviation for each distribution.

c. Use the normal approximation of the binomial probability distribution to find an interval centered on the mean that contains 95.44% of that distribution.

d. Why can't we use the normal distribution to answer probability questions about the frequency distribution?

APPLICATIONS

Business and Economics

43. **Stock Returns** The annual returns of two stocks for 3 years are given below.

Stock	2009	2010	2011
Stock I	11%	−1%	14%
Stock II	9%	5%	10%

a. Find the mean and standard deviation for each stock over the 3-year period.

b. If you are looking for security (hence, less variability) with an average 8% return, which of these stocks should you choose?

44. Quality Control A machine that fills quart orange juice cartons is set to fill them with 32.1 oz. If the actual contents of the cartons vary normally, with a standard deviation of 0.1 oz, what percentage of the cartons contain less than a quart (32 oz)?

45. Quality Control About 4% of the frankfurters produced by a certain machine are overstuffed and thus defective. For a sample of 500 frankfurters, find the following probabilities—first by using the binomial probability formula, and then by using the normal approximation.

a. Twenty-five or fewer are overstuffed.

b. Exactly 25 are overstuffed.

c. At least 30 are overstuffed.

46. Bankruptcy The probability that a small business will go bankrupt in its first year is 0.21. For 50 such small businesses, find the following probabilities—first by using the binomial probability formula, and then by using the normal approximation.

a. Exactly 8 go bankrupt.

b. No more than 2 go bankrupt.

Life Sciences

47. Rat Diets The weight gains of 2 groups of 10 rats fed different experimental diets were as follows.

Diet	Weight Gains									
A	1	0	3	7	1	1	5	4	1	4
B	2	1	1	2	3	2	1	0	1	0

Compute the mean and standard deviation for each group.

a. Which diet produced the greatest mean gain?

b. Which diet produced the most consistent gain?

Chemical Effectiveness **White flies are devastating California crops. An area infested with white flies is to be sprayed with a chemical that is known to be 98% effective for each application. Assume a sample of 1000 flies is checked.**

48. Use the normal distribution to find the approximate probability that exactly 980 of the flies are killed in one application.

49. Use the normal distribution to find the approximate probability that no more than 986 of the flies are killed in one application.

50. Use the normal distribution to find the approximate probability that at least 975 of the flies are killed in one application.

51. Use the normal distribution to find the approximate probability that between 973 and 993 (inclusive) of the flies are killed in one application.

Social Sciences

Commuting Times **The average resident of a certain East Coast suburb spends 42 minutes per day commuting, with a standard deviation of 12 minutes. Assume a normal distribution. Find the percent of all residents of this suburb who have the following commuting times.**

52. At least 50 minutes per day

53. No more than 40 minutes per day

54. Between 32 and 40 minutes per day

55. Between 38 and 60 minutes per day

56. I.Q. Scores On standard IQ tests, the mean is 100, with a standard deviation of 15. The results are very close to fitting a normal curve. Suppose an IQ test is given to a very large group of people. Find the percent of those people whose IQ scores are as follows.

a. More than 130

b. Less than 85

c. Between 85 and 115

General Interest

57. Olympics The number of countries participating in the Summer Olympics since 1960 is given in the following table. *Source: The New York Times 2010 Almanac.*

Olympic City	Year	Number of Countries
Rome	1960	83
Tokyo	1964	93
Mexico City	1968	112
Munich	1972	121
Montreal	1976	92
Moscow	1980	80
Los Angeles	1984	140
Seoul	1988	159
Barcelona	1992	169
Atlanta	1996	197
Sydney	2000	199
Athens	2004	202
Beijing	2008	205

a. Find the mean, median, and mode of the data.

b. Find the standard deviation of the data.

c. What percent of the data is within 1 standard deviation of the mean?

d. What percent of the data is within 2 standard deviations of the mean?

58. Broadway A survey was given to 313 performers appearing in 23 Broadway companies. The percentage of performers injured during practice or a performance was 55.5%. If a random sample of 500 Broadway performers is taken, use the normal approximation to the binomial distribution to find the approximate probability that more than 300 performers have been injured. *Source: American Journal of Public Health.*

59. Broadway In the survey described in Exercise 58, the demographics of the Broadway performers were recorded as shown below. Assume that all of these demographics follow a normal distribution, an assumption that always must be verified prior to using it in real situations. ***Source: American Journal of Public Health.***

	Mean	Standard Deviation
Dancer's Age (female)	28.0	5.5
Dancer's Age (male)	32.2	8.4
Height (in m) (female)	1.64	0.08
Duration as Professional in yr (female)	11.0	8.9
Total No. of Injuries as Performer (female)	3.0	2.2

a. Find the probability that a female dancer is 35 years old or older.

b. Find the probability that a male dancer is 35 years old or older.

c. Compare your answers to parts a and b.

d. Find the probability that a female performer is 1.4 m tall or taller.

e. Find the probability that a female performer has a career duration that is more than 1.5 standard deviations from the mean.

f. Would a female who has more than 6 injuries during her career be considered a rare event? Explain.

EXTENDED APPLICATION

STATISTICS IN THE LAW—THE *CASTANEDA* DECISION

Statistical evidence is now routinely presented in both criminal and civil cases. In this application we'll look at a famous case that established use of the binomial distribution and measurement by standard deviation as an accepted procedure.*

Defendants who are convicted in criminal cases sometimes appeal their conviction on the grounds that the jury that indicted or convicted them was drawn from a pool of jurors that does not represent the population of the district in which they live. These appeals almost always cite the Supreme Court's decision in *Castaneda v. Partida* [430 U.S. 482], a case that dealt with the selection of grand juries in the state of Texas. The decision summarizes the facts this way:

> *After respondent, a Mexican-American, had been convicted of a crime in a Texas District Court and had exhausted his state remedies on his claim of discrimination in the selection of the grand jury that had indicted him, he filed a habeas corpus petition in the Federal District Court, alleging a denial of due process and equal protection under the Fourteenth Amendment, because of gross underrepresentation of Mexican-Americans on the county grand juries.*

The case went to the Appeals Court, which noted that "the county population was 79% Mexican-American, but, over an 11-year period, only 39% of those summoned for grand jury service were Mexican-American," and concluded that together with other testimony about the selection process, "the proof offered by respondent was sufficient to demonstrate a prima facie case of intentional discrimination in grand jury selection. . . ."

The state appealed to the Supreme Court, and the Supreme Court needed to decide whether the underrepresentation of Mexican-Americans on grand juries was indeed too extreme to be an effect of chance. To do so, they invoked the binomial distribution. Here is the argument:

> *Given that 79.1% of the population is Mexican-American, the expected number of Mexican-Americans among the 870 persons summoned to serve as grand jurors over the 11-year period is approximately 688. The observed number is 339. Of course, in any given drawing, some fluctuation from the expected number is predicted. The important point, however, is that the statistical model shows that the results of a random drawing are likely to fall in the vicinity of the expected value. . . .*

*The *Castaneda* case and many other interesting applications of statistics in law are discussed in Finkelstein and Levin, *Statistics for Lawyers,* New York, Springer-Verlag, 1990. U.S. Supreme Court decisions are online at http://www.findlaw.com/casecode/supreme.html. In addition, most states now have important state court decisions online.

The measure of the predicted fluctuations from the expected value is the standard deviation, defined for the binomial distribution as the square root of the product of the total number in the sample (here 870) times the probability of selecting a Mexican-American (0.791) times the probability of selecting a non-Mexican-American (0.209) . . . . Thus, in this case the standard deviation is approximately 12. As a general rule for such large samples, if the difference between the expected value and the observed number is greater than two or three standard deviations, then the hypothesis that the jury drawing was random would be suspect to a social scientist. The 11-year data here reflect a difference between the expected and observed number of Mexican-Americans of approximately 29 standard deviations. A detailed calculation reveals that the likelihood that such a substantial departure from the expected value would occur by chance is less than 1 in 10^{140}.

The Court decided that the statistical evidence supported the conclusion that jurors were not randomly selected, and that it was up to the state to show that its selection process did not discriminate against Mexican-Americans. The Court concluded:

The proof offered by respondent was sufficient to demonstrate a prima facie case of discrimination in grand jury selection. Since the State failed to rebut the presumption of purposeful discrimination by competent testimony, despite two opportunities to do so, we affirm the Court of Appeals' holding of a denial of equal protection of the law in the grand jury selection process in respondent's case.

EXERCISES

1. Check the Court's calculation of 29 standard deviations as the difference between the expected number of Mexican-Americans and the number actually chosen.

2. Where do you think the Court's figure of 1 in 10^{140} came from?

3. The *Castaneda* decision also presents data from a $2\frac{1}{2}$-year period during which the State District Judge supervised the selection process. During this period, 220 persons were called to serve as grand jurors, and only 100 of these were Mexican-American.

a. Considering the 220 jurors as a random selection from a large population, what is the expected number of Mexican-Americans, using the 79.1% population figure?

b. If we model the drawing of jurors as a sequence of 220 independent Bernoulli trials, what is the standard deviation of the number of Mexican-Americans?

c. About how many standard deviations is the actual number of Mexican-Americans drawn (100) from the expected number that you calculated in part a?

d. What does the normal distribution table at the back of the book tell you about this result?

4. The following information is from an appeal brought by Hy-Vee stores before the Iowa Supreme Court, appealing a ruling by the Iowa Civil Rights Commission in favor of a female employee of one of their grocery stores.

In 1985, there were 112 managerial positions in the ten Hy-Vee stores located in Cedar Rapids. Only 6 of these managers were women. During that same year there were 294 employees; 206 were men and 88 were women.

a. How far from the expected number of women in management was the actual number, assuming that gender had nothing to do with promotion? Measure the difference in standard deviations.

b. Does this look like evidence of purposeful discrimination?

5. Go to the website WolframAlpha.com and enter "normal probability." Use the results to calculate the probability of being 12.3 standard deviations below the mean, as calculated in Exercise 3c. Compare this result with the result using a graphing calculator and using Excel.

DIRECTIONS FOR GROUP PROJECT

Suppose that you and three other students are serving as interns at a prestigious law firm. One of the partners is interested in the use of probability in court cases and would like the four of you to prepare a brief on the Castaneda *decision. She insists that you describe the case and highlight the mathematics used in your brief. Be sure to use the results from the case along with the results of Exercises 1–3 in preparing your brief. Also, make recommendations of other types of cases where probability may be used in law. Presentation software, such as Microsoft PowerPoint, should be used to present your brief to the partners of the firm.*

10 Nonlinear Functions

10.1 Properties of Functions

10.2 Quadratic Functions; Translation and Reflection

10.3 Polynomial and Rational Functions

10.4 Exponential Functions

10.5 Logarithmic Functions

10.6 Applications: Growth and Decay; Mathematics of Finance

Chapter 10 Review

Extended Application: Power Functions

There are fourteen mountain peaks over 8000 meters on the Earth's surface. At these altitudes climbers face the challenge of "thin air," since atmospheric pressure is about one third of the pressure at sea level. An exercise in Section 4 of this chapter shows how the change in atmospheric pressure with altitude can be modeled with an exponential function.

Figure 1 below shows the average price of gold for each year in the last few decades. *Source: finfacts.ie.* The graph is not a straight line and illustrates a function that, unlike those studied in Chapter 1, is *nonlinear*. Linear functions are simple to study, and they can be used to approximate many functions over short intervals. But most functions exhibit behavior that, in the long run, does not follow a straight line. In this chapter we will study some of the most common nonlinear functions.

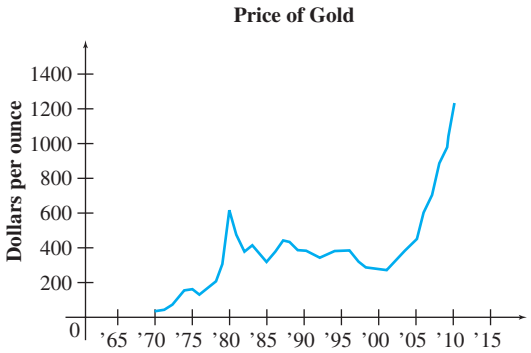


FIGURE 1

10.1 Properties of Functions

APPLY IT **How has the world's use of different energy sources changed over time?**
We will analyze this question in Exercise 78 in this section, after developing the concept of nonlinear functions.

As we saw in Chapter 1, the linear cost function $C(x) = 12x + 5000$ for video games is related to the number of items produced. The number of games produced is the independent variable and the total cost is the dependent variable because it depends on the number produced. When a specific number of games (say 1000) is substituted for x, the cost $C(x)$ has one specific value $(12 \cdot 1000 + 5000)$. Because of this, the variable $C(x)$ is said to be a *function* of x.

Function

A **function** is a rule that assigns to each element from one set exactly one element from another set.

In most cases in this book, the "rule" mentioned in the box is expressed as an equation, such as $C(x) = 12x + 5000$, and each set mentioned in the definition will ordinarily be the real numbers or some subset of the reals. When an equation is given for a function, we say that the equation *defines* the function. Whenever x and y are used in this book to define a function, x represents the independent variable and y the dependent variable. Of course, letters other than x and y could be used and are often more meaningful. For example, if the independent variable represents the number of items sold for \$4 each and the dependent variable represents revenue, we might write $R = 4s$.

The independent variable in a function can take on any value within a specified set of values called the *domain*.

Domain and Range

The set of all possible values of the independent variable in a function is called the **domain** of the function, and the resulting set of possible values of the dependent variable is called the **range**.

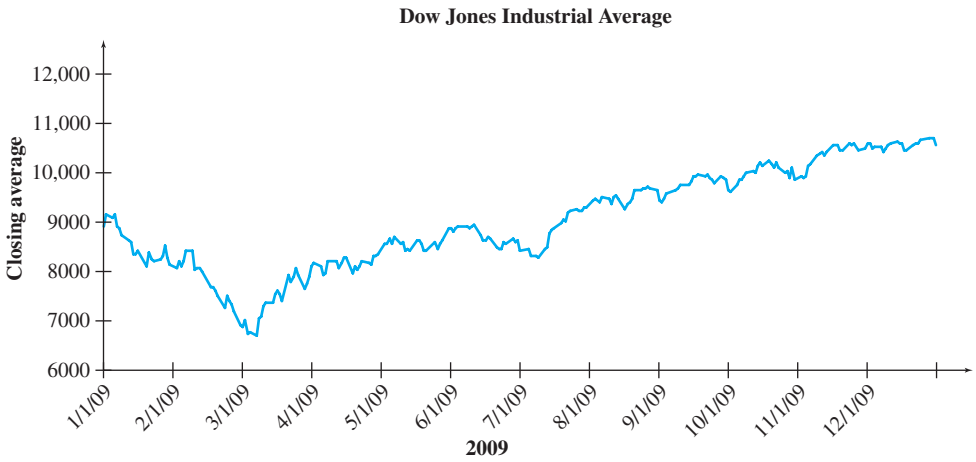


FIGURE 2

An important function to investors around the world is the Dow Jones industrial average, a performance measure of the stock market. Figure 2 shows how this average varied over the year 2009. ***Source: Yahoo! Finance.*** Let us label this function $y = f(x)$, where y is the Dow Jones industrial average and x is the time in days from the beginning of 2009. Notice that the function increases and decreases during the year, so it is not linear, although a linear function could be used as a very rough approximation. Such a function, whose graph is not a straight line, is called a *nonlinear function*.

The concepts you learned in the section on linear functions apply to this and other nonlinear functions as well. The independent variable here is x, the time in days; the dependent variable is y, the average at any time. The domain is $\{x | 0 \le x \le 365\}$, or $[0, 365]$; $x = 0$ corresponds to the beginning of the day on January 1, and $x = 365$ corresponds to the end of the day on December 31. By looking for the lowest and highest values of the function, we estimate the range to be approximately $\{y | 6500 \le y \le 10{,}500\}$, or $[6500, 10{,}500]$. As with linear functions, the domain is mapped along the horizontal axis and the range along the vertical axis.

Dow Jones Industrial Average	
Day (x)	**Close (y)**
0	8772.25
1	8772.25
2	9034.69
3	9034.69
4	9034.69
5	8952.89
6	9015.10
7	8769.70

We do not have a formula for $f(x)$. (If we had possessed such a formula at the beginning of 2009, we could have made a lot of money!) Instead, we can use the graph to estimate values of the function. To estimate $f(10)$, for example, we draw a vertical line from January 10, as shown in Figure 3(a). The y-coordinate seems to be roughly 8600, so we estimate $f(10) \approx 8600$. Similarly, if we wanted to solve the equation $f(x) = 7000$, we would look for points on the graph that have a y-coordinate of 7000. As Figure 3(b) shows, this occurs at two points. The first time is around February 27 (the 58th day of the year), and the last time is around March 11 (the 70th day of the year). Thus $f(x) = 7000$ when $x = 58$ and $x = 70$.

This function can also be given as a table. The table in the margin shows the value of the function for several values of x.

Notice from the table that $f(0) = f(1) = 8772.25$. The stock market was closed for the first day of 2009, so the Dow Jones average did not change. This illustrates an important

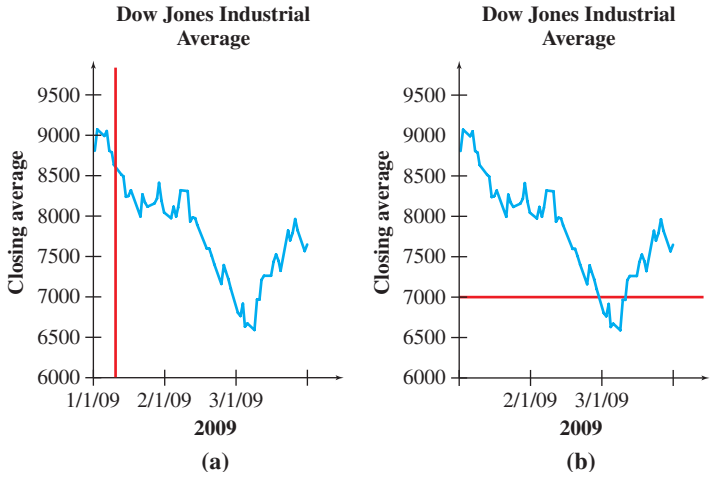


FIGURE 3

property of functions: Several different values of the independent variable can have the same value for the dependent variable. On the other hand, we cannot have several different y-values corresponding to the same value of x; if we did, this would not be a function.

What is $f(5.5)$? We do not know. When the stock market closed on January 5, the Dow Jones industrial average was 8952.89. The closing value the following day was 9015.10. We do not know what happened in between, although this information is recorded by the New York Stock Exchange.

Functions arise in numerous applications, and an understanding of them is critical for understanding calculus. The following example shows some of the ways functions can be represented and will help you in determining whether a relationship between two variables is a function or not.

EXAMPLE 1 Functions

Which of the following are functions?

(a)

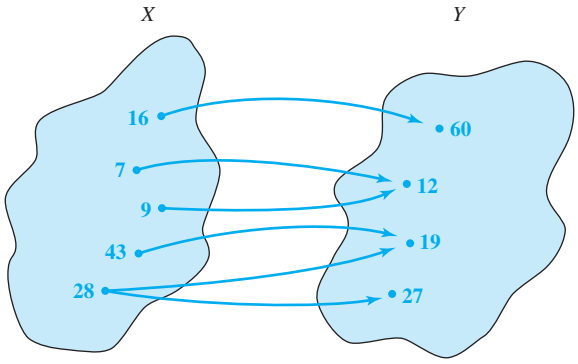


FIGURE 4

SOLUTION Figure 4 shows that an x-value of 28 corresponds to *two* y-values, 19 and 27. In a function, each x must correspond to exactly one y, so this correspondence is not a function.

(b) The x^2 key on a calculator

SOLUTION This correspondence between input and output is a function because the calculator produces just one x^2 (one y-value) for each x-value entered. Notice also that

two x-values, such as 3 and -3, produce the same y-value of 9, but this does not violate the definition of a function.

(c)

x	1	1	2	2	3	3
y	3	-3	5	-5	8	-8

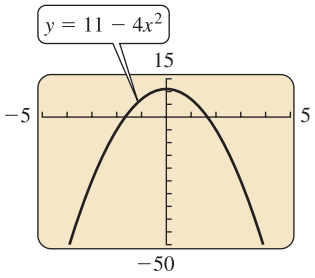
SOLUTION Since at least one x-value corresponds to more than one y-value, this table does not define a function.

(d) The set of ordered pairs with first elements mothers and second elements their children

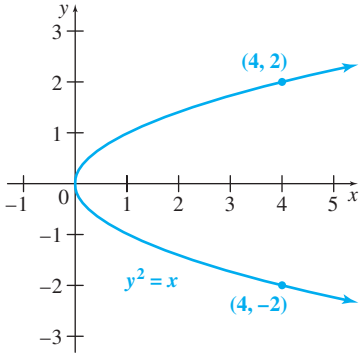
SOLUTION Here the mother is the independent variable and the child is the dependent variable. For a given mother, there may be several children, so this correspondence is not a function.

(e) The set of ordered pairs with first elements children and second elements their birth mothers

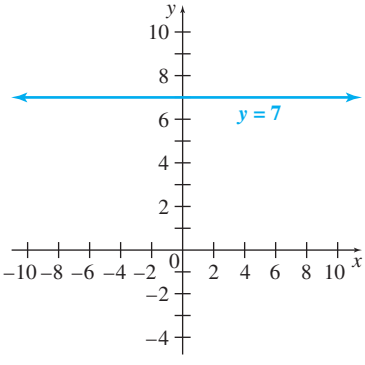
SOLUTION In this case the child is the independent variable and the mother is the dependent variable. Since each child has only one birth mother, this is a function.



(a)



(b)



(c)

FIGURE 5

EXAMPLE 2 Functions

Decide whether each equation or graph represents a function. (Assume that x represents the independent variable here, an assumption we shall make throughout this book.) Give the domain and range of any functions.

(a) $y = 11 - 4x^2$

SOLUTION For a given value of x, calculating $11 - 4x^2$ produces exactly one value of y. (For example, if $x = -7$, then $y = 11 - 4(-7)^2 = -185$, so $f(-7) = -185$.) Since one value of the independent variable leads to exactly one value of the dependent variable, $y = 11 - 4x^2$ meets the definition of a function.

Because x can take on any real-number value, the domain of this function is the set of all real numbers. Finding the range is more difficult. One way to find it would be to ask what possible values of y could come out of this function. Notice that the value of y is 11 minus a quantity that is always 0 or positive, since $4x^2$ can never be negative. There is no limit to how large $4x^2$ can be, so the range is $(-\infty, 11]$.

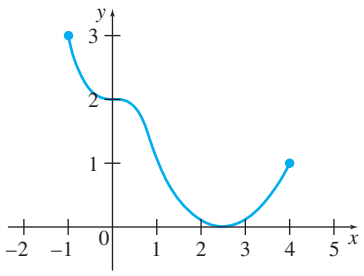
Another way to find the range would be to examine the graph. Figure 5(a) shows a graphing calculator view of this function, and we can see that the function takes on y-values of 11 or less. The calculator cannot tell us, however, whether the function continues to go down past the viewing window, or turns back up. To find out, we need to study this type of function more carefully, as we will do in the next section.

(b) $y^2 = x$

SOLUTION Suppose $x = 4$. Then $y^2 = x$ becomes $y^2 = 4$, from which $y = 2$ or $y = -2$, as illustrated in Figure 5(b). Since one value of the independent variable can lead to two values of the dependent variable, $y^2 = x$ does not represent a function.

(c) $y = 7$

SOLUTION No matter what the value of x, the value of y is always 7. This is indeed a function; it assigns exactly one element, 7, to each value of x. Such a function is known as a **constant function.** The domain is the set of all real numbers, and the range is the set $\{7\}$. Its graph is the horizontal line that intersects the y-axis at $y = 7$, as shown in Figure 5(c). Every constant function has a horizontal line for its graph.



(d)

FIGURE 5

(d) The graph in Figure 5(d).

SOLUTION For each value of x, there is only one value of y. For example, the point $(-1, 3)$ on the graph shows that $f(-1) = 3$. Therefore, the graph represents a function. From the graph, we see that the values of x go from -1 to 4, so the domain is $[-1, 4]$. By looking at the values of y, we see that the range is $[0, 3]$.

The following agreement on domains is customary.

Agreement on Domains

Unless otherwise stated, assume that the domain of all functions defined by an equation is the largest set of real numbers that are meaningful replacements for the independent variable.

For example, suppose

$$y = \frac{-4x}{2x - 3}.$$

Any real number can be used for x except $x = 3/2$, which makes the denominator equal 0. By the agreement on domains, the domain of this function is the set of all real numbers except $3/2$, which we denote $\{x | x \neq 3/2\}$, $\{x \neq 3/2\}$, or $(-\infty, 3/2) \cup (3/2, \infty)$.*

CAUTION When finding the domain of a function, there are two operations to avoid: (1) dividing by zero; and (2) taking the square root (or any even root) of a negative number. Later sections will present other functions, such as logarithms, which require further restrictions on the domain. For now, just remember these two restrictions on the domain.

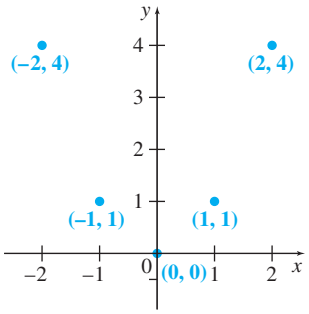


FIGURE 6

FOR REVIEW

Section R.5 demonstrates the method for solving a quadratic inequality. To solve $2x^2 + 5x - 12 \geq 0$, factor the quadratic to get $(2x - 3)(x + 4) \geq 0$. Setting each factor equal to 0 gives $x = 3/2$ or $x = -4$, leading to the intervals $(-\infty, -4]$, $[-4, 3/2]$, and $[3/2, \infty)$. Testing a number from each interval shows that the solution is $(-\infty, -4] \cup [3/2, \infty)$.

EXAMPLE 3 Domain and Range

Find the domain and range for each function defined as follows.

(a) $f(x) = x^2$

SOLUTION Any number may be squared, so the domain is the set of all real numbers, written $(-\infty, \infty)$. Since $x^2 \geq 0$ for every value of x, the range is $[0, \infty)$.

(b) $y = x^2$, with the domain specified as $\{-2, -1, 0, 1, 2\}$.

SOLUTION With the domain specified, the range is the set of values found by applying the function to the domain. Since $f(0) = 0$, $f(-1) = f(1) = 1$, and $f(-2) = f(2) = 4$, the range is $\{0, 1, 4\}$. The graph of the set of ordered pairs is shown in Figure 6.

(c) $y = \sqrt{6 - x}$

SOLUTION For y to be a real number, $6 - x$ must be nonnegative. This happens only when $6 - x \geq 0$, or $6 \geq x$, making the domain $(-\infty, 6]$. The range is $[0, \infty)$ because $\sqrt{6 - x}$ is always nonnegative.

(d) $y = \sqrt{2x^2 + 5x - 12}$

SOLUTION The domain includes only those values of x satisfying $2x^2 + 5x - 12 \geq 0$. Using the methods for solving a quadratic inequality produces the domain

$$(-\infty, -4] \cup [3/2, \infty).$$

As in part (c), the range is $[0, \infty)$.

*The *union* of sets A and B, written $A \cup B$, is defined as the set of all elements in A or B or both.

(e) $y = \dfrac{2}{x^2 - 9}$

SOLUTION Since the denominator cannot be zero, $x \neq 3$ and $x \neq -3$. The domain is

$$(-\infty, -3) \cup (-3, 3) \cup (3, \infty).$$

Because the numerator can never be zero, $y \neq 0$. The denominator can take on any real number except for 0, allowing y to take on any value except for 0, so the range is $(-\infty, 0) \cup (0, \infty)$.

YOUR TURN 1 Find the domain and range for the function $y = \dfrac{1}{\sqrt{x^2 - 4}}$.

TRY YOUR TURN 1

To understand how a function works, think of a function f as a machine—for example, a calculator or computer—that takes an input x from the domain and uses it to produce an output $f(x)$ (which represents the y-value), as shown in Figure 7. In the Dow Jones example, when we put 4 into the machine, we get out 9034.69, since $f(4) = 9034.69$.

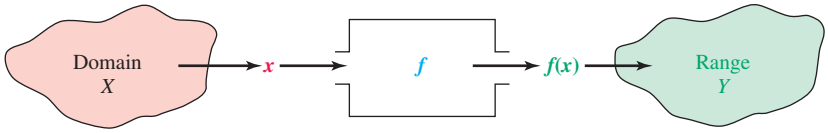


FIGURE 7

EXAMPLE 4 Evaluating Functions

Let $g(x) = -x^2 + 4x - 5$. Find the following.

(a) $g(3)$

SOLUTION Replace x with 3.

$$g(3) = -3^2 + 4 \cdot 3 - 5 = -9 + 12 - 5 = -2$$

(b) $g(a)$

SOLUTION Replace x with a to get $g(a) = -a^2 + 4a - 5$.

This replacement of one variable with another is important in later chapters.

(c) $g(x + h)$

SOLUTION Replace x with the expression $x + h$ and simplify.

$$\begin{aligned} g(x + h) &= -(x + h)^2 + 4(x + h) - 5 \\ &= -(x^2 + 2xh + h^2) + 4(x + h) - 5 \\ &= -x^2 - 2xh - h^2 + 4x + 4h - 5 \end{aligned}$$

(d) $g\left(\dfrac{2}{r}\right)$

SOLUTION Replace x with $2/r$ and simplify.

$$g\left(\frac{2}{r}\right) = -\left(\frac{2}{r}\right)^2 + 4\left(\frac{2}{r}\right) - 5 = -\frac{4}{r^2} + \frac{8}{r} - 5$$

(e) Find all values of x such that $g(x) = -12$.

SOLUTION Set $g(x)$ equal to -12, and then add 12 to both sides to make one side equal to 0.

$$\begin{aligned} -x^2 + 4x - 5 &= -12 \\ -x^2 + 4x + 7 &= 0 \end{aligned}$$

This equation does not factor, but can be solved with the quadratic formula, which says that if $ax^2 + bx + c = 0$, where $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this case, with $a = -1$, $b = 4$, and $c = 7$, we have

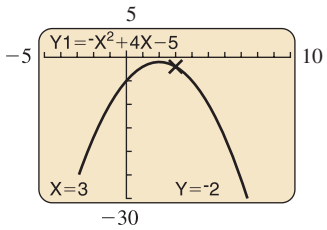
$$\begin{aligned} x &= \frac{-4 \pm \sqrt{16 - 4(-1)7}}{2(-1)} \\ &= \frac{-4 \pm \sqrt{44}}{-2} \\ &= 2 \pm \sqrt{11} \\ &\approx -1.317 \quad \text{or} \quad 5.317. \end{aligned}$$

YOUR TURN 2 Given the function $f(x) = 2x^2 - 3x - 4$, find each of the following.
(a) $f(x + h)$ **(b)** All values of x such that $f(x) = -5$.

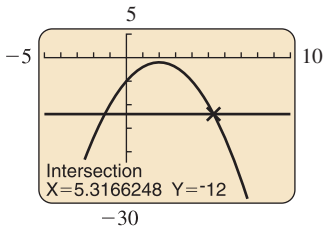
TRY YOUR TURN 2

TECHNOLOGY NOTE

We can verify the results of parts (a) and (e) of the previous example using a graphing calculator. In Figure 8(a), after graphing $f(x) = -x^2 + 4x - 5$, we have used the "value" feature on the TI-84 Plus to support our answer from part (a). In Figure 8(b) we have used the "intersect" feature to find the intersection of $y = g(x)$ and $y = -12$. The result is $x = 5.3166248$, which is one of our two answers to part (e). The graph clearly shows that there is another answer on the opposite side of the y-axis.



(a)



(b)

FIGURE 8

CAUTION Notice from Example 4(c) that $g(x + h)$ is *not* the same as $g(x) + h$, which equals $-x^2 + 4x - 5 + h$. There is a significant difference between applying a function to the quantity $x + h$ and applying a function to x and adding h afterward.

If you tend to get confused when replacing x with $x + h$, as in Example 4(c), you might try replacing the x in the original function with a box, like this:

$$g(\square) = -(\square)^2 + 4(\square) - 5$$

Then, to compute $g(x + h)$, just enter $x + h$ into the box:

$$g(\boxed{x+h}) = -(\boxed{x+h})^2 + 4(\boxed{x+h}) - 5$$

and proceed as in Example 4(c).

Notice in the Dow Jones example that to find the value of the function for a given value of x, we drew a vertical line from the value of x and found where it intersected the graph. If a graph is to represent a function, each value of x from the domain must lead to exactly one value of y. In the graph in Figure 9, the domain value x_1 leads to *two* y-values, y_1 and y_2. Since the given x-value corresponds to two different y-values, this is not the graph of a function. This example suggests the **vertical line test** for the graph of a function.

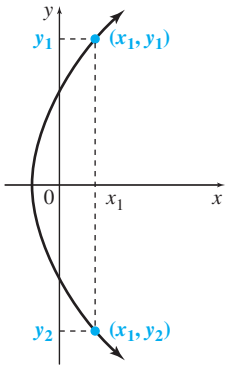


FIGURE 9

Vertical Line Test

If a vertical line intersects a graph in more than one point, the graph is not the graph of a function.

A graph represents a function if and only if every vertical line intersects the graph in no more than one point.

EXAMPLE 5 Vertical Line Test

Use the vertical line test to determine which of the graphs in Example 2 represent functions.

SOLUTION Every vertical line intersects the graphs in Figure 5(a), (c), and (d) in at most one point, so these are the graphs of functions. It is possible for a vertical line to intersect the graph in Figure 5(b) twice, so this is not a function.

A function f is called an **even function** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis, so the left side is a mirror image of the right side. The function f is called an **odd function** if $f(-x) = -f(x)$. This means that the graph is symmetric about the origin, so the left side of the graph can be found by rotating the right side by 180° about the origin.

EXAMPLE 6 Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

(a) $f(x) = x^4 - x^2$

SOLUTION Calculate $f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x)$, so the function is even. Its graph, shown in Figure 10(a), is symmetric about the y-axis.

FOR REVIEW

Recall from Sec. R.6 that $(-a)^n = a^n$ if n is an even integer, and $(-a)^n = -a^n$ if n is an odd integer.

(b) $f(x) = \dfrac{x}{x^2 + 1}$

SOLUTION Calculate $f(-x) = \dfrac{(-x)}{(-x)^2 + 1} = \dfrac{-x}{x^2 + 1} = -f(x)$, so the function is odd. Its graph, shown in Figure 10(b), is symmetric about the origin. You can see this by turning the book upside down and observing that the graph looks the same.

(c) $f(x) = x^4 - 4x^3$

SOLUTION Calculate $f(-x) = (-x)^4 - 4(-x)^3 = x^4 + 4x^3$, which is equal neither to $f(x)$ nor to $-f(x)$. So the function is neither even nor odd. Its graph, shown in Figure 10(c), has no symmetry.

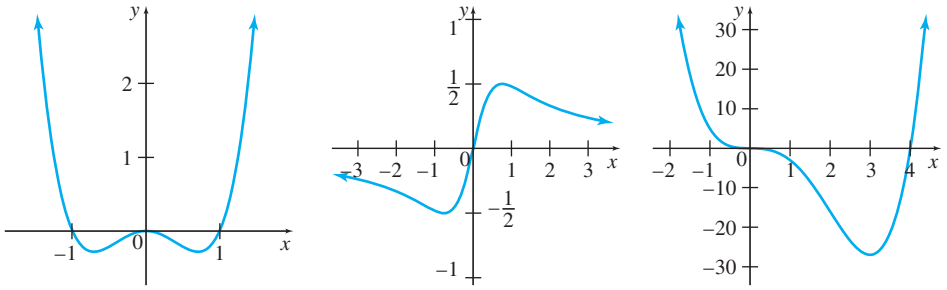


FIGURE 10

EXAMPLE 7 Delivery Charges

An overnight delivery service charges \$25 for a package weighing up to 2 lb. For each additional pound, or portion thereof, there is an additional charge of \$3. Let $D(w)$ represent the cost to send a package weighing w lb. Graph $D(w)$ for w in the interval $(0, 6]$.

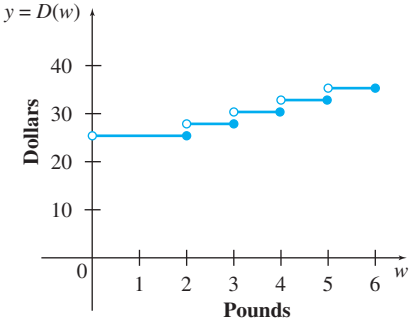


FIGURE 11

SOLUTION For w in the interval $(0, 2]$, the shipping cost is $y = 25$. For w in $(2, 3]$, the shipping cost is $y = 25 + 3 = 28$. For w in $(3, 4]$, the shipping cost is $y = 28 + 3 = 31$, and so on. The graph is shown in Figure 11.

The function discussed in Example 7 is called a **step function**. Many real-life situations are best modeled by step functions. Additional examples are given in the exercises.

In Chapter 1 you saw several examples of linear models. In Example 8, we use a quadratic equation to model the area of a lot.

EXAMPLE 8 Area

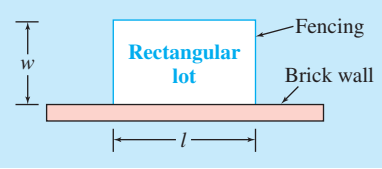


FIGURE 12

A fence is to be built against a brick wall to form a rectangular lot, as shown in Figure 12. Only three sides of the fence need to be built, because the wall forms the fourth side. The contractor will use 200 m of fencing. Let the length of the wall be l and the width w, as shown in Figure 12.

(a) Find the area of the lot as a function of the length l.

SOLUTION The area formula for a rectangle is area = length × width, or

$$A = lw.$$

We want the area as a function of the length only, so we must eliminate the width. We use the fact that the total amount of fencing is the sum of the three sections, one length and two widths, so $200 = l + 2w$. Solve this for w:

$$\begin{aligned} 200 &= l + 2w \\ 200 - l &= 2w && \text{Subtract } l \text{ from both sides.} \\ 100 - l/2 &= w. && \text{Divide both sides by 2.} \end{aligned}$$

Substituting this into the formula for area gives

$$A = l(100 - l/2).$$

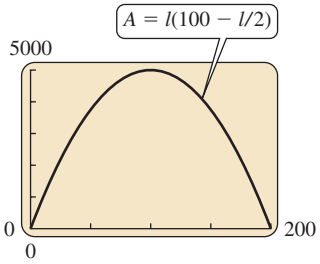


FIGURE 13

(b) Find the domain of the function in part (a).

SOLUTION The length cannot be negative, so $l \geq 0$. Similarly, the width cannot be negative, so $100 - l/2 \geq 0$, from which we find $l \leq 200$. Therefore, the domain is $[0, 200]$.

(c) Sketch a graph of the function in part (a).

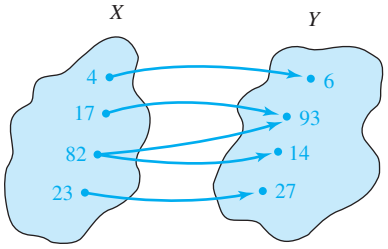
SOLUTION The result from a graphing calculator is shown in Figure 13. Notice that at the endpoints of the domain, when $l = 0$ and $l = 200$, the area is 0. This makes sense: If the length or width is 0, the area will be 0 as well. In between, as the length increases from 0 to 100 m, the area gets larger, and seems to reach a peak of 5000 m^2 when $l = 100$ m. After that, the area gets smaller as the length continues to increase because the width is becoming smaller.

In the next section, we will study this type of function in more detail and determine exactly where the maximum occurs.

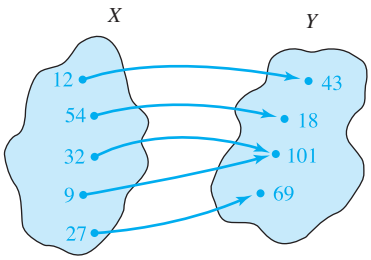
10.1 EXERCISES

Which of the following rules define y as a function of x?

1.



2.



3.

x	y
3	9
2	4
1	1
0	0
−1	1
−2	4
−3	9

4.

x	y
9	3
4	2
1	1
0	0
1	−1
4	−2
9	−3

5. $y = x^3 + 2$

6. $y = \sqrt{x}$

7. $x = |y|$

8. $x = y^2 + 4$

List the ordered pairs obtained from each equation, given {−2, −1, 0, 1, 2, 3} as the domain. Graph each set of ordered pairs. Give the range.

9. $y = 2x + 3$

10. $y = -3x + 9$

11. $2y - x = 5$

12. $6x - y = -1$

13. $y = x(x + 2)$

14. $y = (x - 2)(x + 2)$

15. $y = x^2$

16. $y = -4x^2$

Give the domain of each function defined as follows.

17. $f(x) = 2x$

18. $f(x) = 2x + 3$

19. $f(x) = x^4$

20. $f(x) = (x + 3)^2$

21. $f(x) = \sqrt{4 - x^2}$

22. $f(x) = |3x - 6|$

23. $f(x) = (x - 3)^{1/2}$

24. $f(x) = (3x + 5)^{1/2}$

25. $f(x) = \dfrac{2}{1 - x^2}$

26. $f(x) = \dfrac{-8}{x^2 - 36}$

27. $f(x) = -\sqrt{\dfrac{2}{x^2 - 16}}$

28. $f(x) = -\sqrt{\dfrac{5}{x^2 + 36}}$

29. $f(x) = \sqrt{x^2 - 4x - 5}$

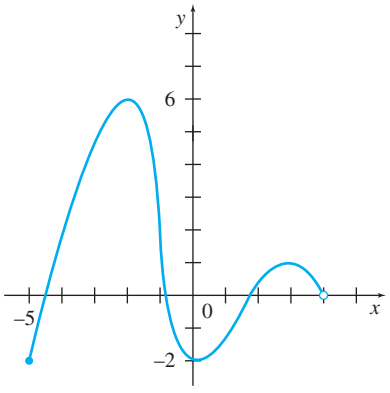
30. $f(x) = \sqrt{15x^2 + x - 2}$

31. $f(x) = \dfrac{1}{\sqrt{3x^2 + 2x - 1}}$

32. $f(x) = \sqrt{\dfrac{x^2}{3 - x}}$

Give the domain and the range of each function. Where arrows are drawn, assume the function continues in the indicated direction.

33.



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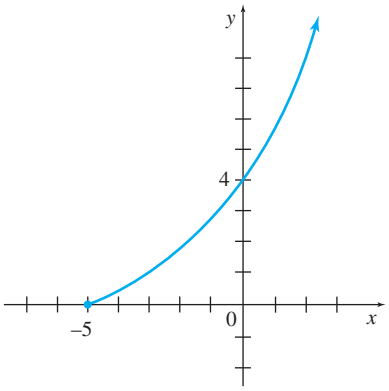

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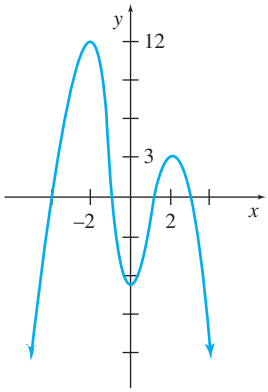

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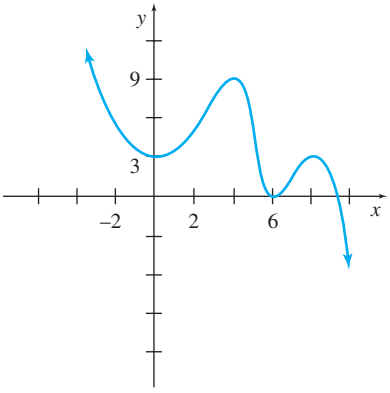
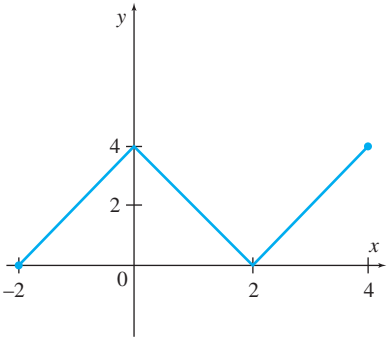


In Exercises 37–40, give the domain and range. Then, use each graph to find (a) $f(-2)$, (b) $f(0)$, (c) $f(1/2)$, and (d) any values of x such that $f(x) = 1$.

37.



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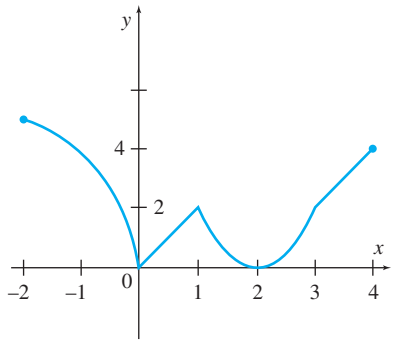

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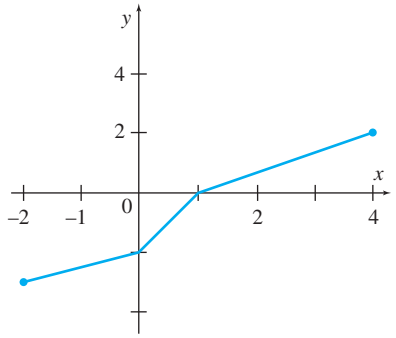

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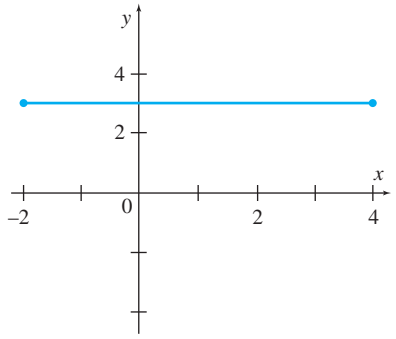


For each function, find (a) $f(4)$, (b) $f(-1/2)$, (c) $f(a)$, (d) $f(2/m)$, and (e) any values of x such that $f(x) = 1$.

41. $f(x) = 3x^2 - 4x + 1$

42. $f(x) = (x + 3)(x - 4)$

43. $f(x) = \begin{cases} \dfrac{2x+1}{x-4} & \text{if } x \neq 4 \\ 7 & \text{if } x = 4 \end{cases}$

44. $f(x) = \begin{cases} \dfrac{x-4}{2x+1} & \text{if } x \neq -\dfrac{1}{2} \\ 10 & \text{if } x = -\dfrac{1}{2} \end{cases}$

Let $f(x) = 6x^2 - 2$ and $g(x) = x^2 - 2x + 5$ to find the following values.

45. $f(t + 1)$

46. $f(2 - r)$

47. $g(r + h)$

48. $g(z - p)$

49. $g\left(\dfrac{3}{q}\right)$

50. $g\left(-\dfrac{5}{z}\right)$

For each function defined as follows, find (a) $f(x + h)$, (b) $f(x + h) - f(x)$, and (c) $[f(x + h) - f(x)]/h$.

51. $f(x) = 2x + 1$

52. $f(x) = x^2 - 3$

53. $f(x) = 2x^2 - 4x - 5$

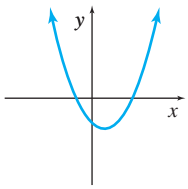
54. $f(x) = -4x^2 + 3x + 2$

55. $f(x) = \dfrac{1}{x}$

56. $f(x) = -\dfrac{1}{x^2}$

Decide whether each graph represents a function.

57.



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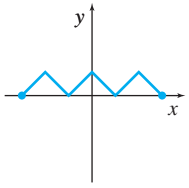

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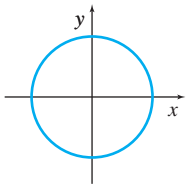

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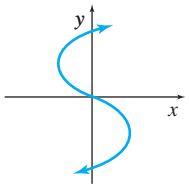

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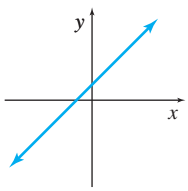

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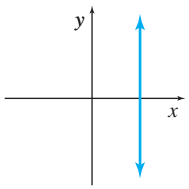


Classify each of the functions in Exercises 63–70 as even, odd, or neither.

63. $f(x) = 3x$

64. $f(x) = 5x$

65. $f(x) = 2x^2$

66. $f(x) = x^2 - 3$

67. $f(x) = \dfrac{1}{x^2 + 4}$

68. $f(x) = x^3 + x$

69. $f(x) = \dfrac{x}{x^2 - 9}$

70. $f(x) = |x - 2|$

APPLICATIONS

Business and Economics

71. Saw Rental A chain-saw rental firm charges \$28 per day or fraction of a day to rent a saw, plus a fixed fee of \$8 for resharpening the blade. Let $S(x)$ represent the cost of renting a saw for x days. Find the following.

a. $S\left(\dfrac{1}{2}\right)$ **b.** $S(1)$ **c.** $S\left(1\dfrac{1}{4}\right)$

d. $S\left(3\dfrac{1}{2}\right)$ **e.** $S(4)$ **f.** $S\left(4\dfrac{1}{10}\right)$

g. What does it cost to rent a saw for $4\frac{9}{10}$ days?

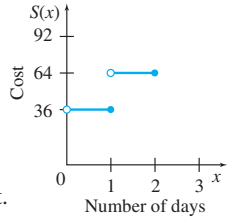
h. A portion of the graph of $y = S(x)$ is shown here. Explain how the graph could be continued.

i. What is the independent variable?

j. What is the dependent variable?

k. Is S a linear function? Explain.

l. Write a sentence or two explaining what part f and its answer represent.



m. We have left $x = 0$ out of the graph. Discuss why it should or shouldn't be included. If it were included, how would you define $S(0)$?

72. Rental Car Cost The cost to rent a mid-size car is \$54 per day or fraction of a day. If the car is picked up in Pittsburgh and dropped off in Cleveland, there is a fixed \$44 drop-off charge. Let $C(x)$ represent the cost of renting the car for x days, taking it from Pittsburgh to Cleveland. Find the following.

a. $C(3/4)$ **b.** $C(9/10)$ **c.** $C(1)$ **d.** $C\left(1\frac{5}{8}\right)$

e. Find the cost of renting the car for 2.4 days.

f. Graph $y = C(x)$.

g. Is C a function? Explain.

h. Is C a linear function? Explain.

i. What is the independent variable?

j. What is the dependent variable?

73. Attorney Fees According to Massachusetts state law, the maximum amount of a jury award that attorneys can receive is:

40% of the first \$150,000,

33.3% of the next \$150,000,

30% of the next \$200,000, and

24% of anything over \$500,000.

Let $f(x)$ represent the maximum amount of money that an attorney in Massachusetts can receive for a jury award of size x. Find each of the following, and describe in a sentence what the answer tells you. ***Source: The New Yorker.***

a. $f(250{,}000)$ **b.** $f(350{,}000)$ **c.** $f(550{,}000)$

d. Sketch a graph of $f(x)$.

74. Tax Rates In New York state in 2010, the income tax rates for a single person were as follows:

4% of the first \$8000 earned,

4.5% of the next \$3000 earned,

5.25% of the next \$2000 earned,

5.9% of the next \$7000 earned,

6.85% of the next \$180,000 earned,

7.85% of the next \$300,000 earned, and

8.97% of any amount earned over \$500,000.

Let $f(x)$ represent the amount of tax owed on an income of x dollars. Find each of the following, and explain in a sentence what the answer tells you. ***Source: New York State.***

a. $f(10{,}000)$

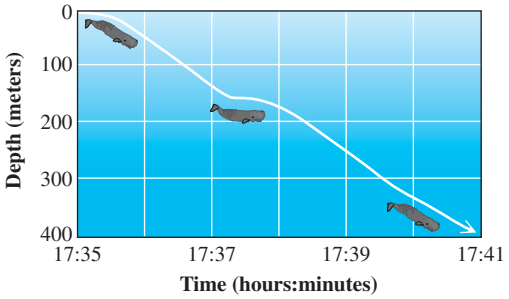
b. $f(12{,}000)$

c. $f(18{,}000)$

d. Sketch a graph of $f(x)$.

Life Sciences

75. Whales Diving The figure in the next column shows the depth of a diving sperm whale as a function of time, as recorded by researchers at the Woods Hole Oceanographic Institution in Massachusetts. ***Source: Peter Tyack, Woods Hole Oceanographic Institution.***



Find the depth of the whale at the following times.

a. 17 hours and 37 minutes

b. 17 hours and 39 minutes

76. Metabolic Rate The basal metabolic rate (in kcal/day) for large anteaters is given by

$$y = f(x) = 19.7x^{0.753},$$

where x is the anteater's weight in kilograms.* ***Source: Wildlife Feeding and Nutrition.***

a. Find the basal metabolic rate for anteaters with the following weights.

i. 5 kg **ii.** 25 kg

b. Suppose the anteater's weight is given in pounds rather than kilograms. Given that 1 lb = 0.454 kg, find a function $x = g(z)$ giving the anteater's weight in kilograms if z is the animal's weight in pounds.

c. Write the basal metabolic rate as a function of the weight in pounds in the form $y = az^b$ by calculating $f(g(z))$.

77. Swimming Energy The energy expenditure (in kcal/km) for animals swimming at the surface of the water is given by

$$y = f(x) = 0.01x^{0.88},$$

where x is the animal's weight in grams. ***Source: Wildlife Feeding and Nutrition.***

a. Find the energy for the following animals swimming at the surface of the water.

i. A muskrat weighing 800 g

ii. A sea otter weighing 20,000 g

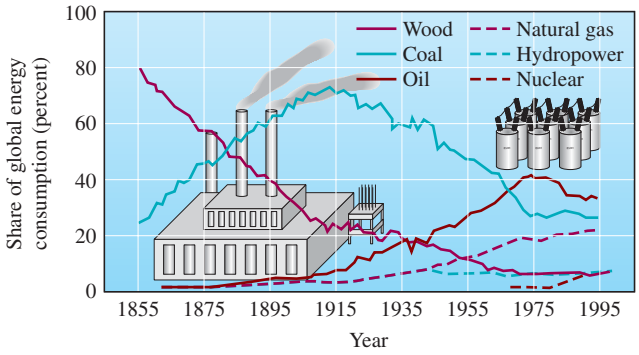
b. Suppose the animal's weight is given in kilograms rather than grams. Given that 1 kg = 1000 g, find a function $x = g(z)$ giving the animal's weight in grams if z is the animal's weight in kilograms.

c. Write the energy expenditure as a function of the weight in kilograms in the form $y = az^b$ by calculating $f(g(z))$.

*Technically, kilograms are a measure of mass, not weight. Weight is a measure of the force of gravity, which varies with the distance from the center of Earth. For objects on the surface of Earth, weight and mass are often used interchangeably, and we will do so in this text.

GENERAL INTEREST

78. APPLY IT Energy Consumption Over the last century, the world has shifted from using high-carbon sources of energy such as wood to lower carbon fuels such as oil and natural gas, as shown in the figure. ***Source: The New York Times.*** The rise in carbon emissions during this time has caused concern because of its suspected contribution to global warming.



a. In what year were the percent of wood and coal use equal? What was the percent of each used in that year?

b. In what year were the percent of oil and coal use equal? What was the percent of each used in that year?

79. Perimeter A rectangular field is to have an area of 500 m^2.

a. Write the perimeter, P, of the field as a function of the width, w.

b. Find the domain of the function in part a.

c. Use a graphing calculator to sketch the graph of the function in part a.

d. Describe what the graph found in part c tells you about how the perimeter of the field varies with the width.

80. Area A rectangular field is to have a perimeter of 6000 ft.

a. Write the area, A, of the field as a function of the width, w.

b. Find the domain of the function in part a.

c. Use a graphing calculator to sketch the graph of the function in part a.

d. Describe what the graph found in part c tells you about how the area of the field varies with the width.

YOUR TURN ANSWERS

1. $(-\infty, -2) \cup (2, \infty), (0, \infty)$

2. (a) $2x^2 + 4xh + 2h^2 - 3x - 3h - 4$ (b) 1 and 1/2

10.2 Quadratic Functions; Translation and Reflection

APPLY IT

How much should a company charge for its seminars? When Power and Money, Inc., charges $600 for a seminar on management techniques, it attracts 1000 people. For each $20 decrease in the fee, an additional 100 people will attend the seminar. The managers wonder how much to charge for the seminar to maximize their revenue.

In Example 6 in this section we will see how knowledge of quadratic functions *will help provide an answer to the question above.*

FOR REVIEW

In this section you will need to know how to solve a quadratic equation by factoring and by the quadratic formula, which are covered in Sections R.2 and R.4. Factoring is usually easiest; when a polynomial is set equal to zero and factored, then a solution is found by setting any one factor equal to zero. But factoring is not always possible. The quadratic formula will provide the solution to *any* quadratic equation.

A linear function is defined by

$$f(x) = ax + b,$$

for real numbers a and b. In a *quadratic function* the independent variable is squared. A quadratic function is an especially good model for many situations with a maximum or a minimum function value. Quadratic functions also may be used to describe supply and demand curves; cost, revenue, and profit; as well as other quantities. Next to linear functions, they are the simplest type of function, and well worth studying thoroughly.

Quadratic Function

A **quadratic function** is defined by

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers, with $a \neq 0$.

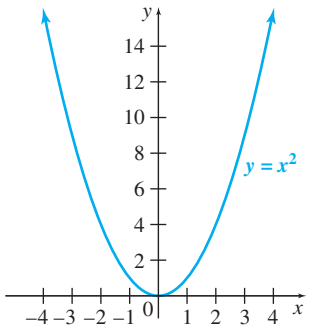


FIGURE 14

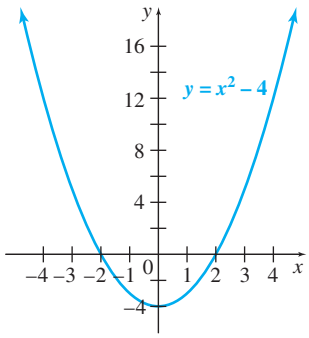


FIGURE 15

The simplest quadratic function has $f(x) = x^2$, with $a = 1$, $b = 0$, and $c = 0$. This function describes situations where the dependent variable y is proportional to the *square* of the independent variable x. The graph of this function is shown in Figure 14. This graph is called a **parabola**. Every quadratic function has a parabola as its graph. The lowest (or highest) point on a parabola is the **vertex** of the parabola. The vertex of the parabola in Figure 14 is $(0, 0)$.

If the graph in Figure 14 were folded in half along the y-axis, the two halves of the parabola would match exactly. This means that the graph of a quadratic function is *symmetric* with respect to a vertical line through the vertex; this line is the **axis** of the parabola.

There are many real-world instances of parabolas. For example, cross sections of spotlight reflectors or radar dishes form parabolas. Also, a projectile thrown in the air follows a parabolic path. For such applications, we need to study more complicated quadratic functions than $y = x^2$, as in the next several examples.

EXAMPLE 1 Graphing a Quadratic Function

Graph $y = x^2 - 4$.

SOLUTION Each value of y will be 4 less than the corresponding value of y in $y = x^2$. The graph of $y = x^2 - 4$ has the same shape as that of $y = x^2$ but is 4 units lower. See Figure 15. The vertex of the parabola (on this parabola, the *lowest* point) is at $(0, -4)$. The x-intercepts can be found by letting $y = 0$ to get

$$0 = x^2 - 4,$$

from which $x = 2$ and $x = -2$ are the x-intercepts. The axis of the parabola is the vertical line $x = 0$.

Example 1 suggests that the effect of c in $ax^2 + bx + c$ is to lower the graph if c is negative and to raise the graph if c is positive. This is true for any function; the movement up or down is referred to as a **vertical translation** of the function.

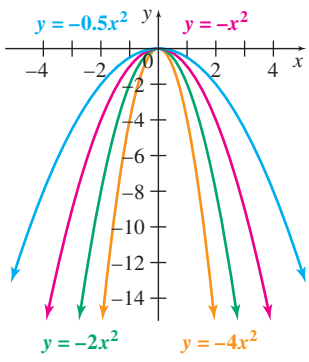


FIGURE 16

EXAMPLE 2 Graphing Quadratic Functions

Graph $y = ax^2$ with $a = -0.5$, $a = -1$, $a = -2$, and $a = -4$.

SOLUTION Figure 16 shows all four functions plotted on the same axes. We see that since a is negative, the graph opens downward. When a is between -1 and 1 (that is, when $a = -0.5$), the graph is wider than the original graph, because the values of y are smaller in magnitude. On the other hand, when a is greater than 1 or less than -1, the graph is steeper.

Example 2 shows that the sign of a in $ax^2 + bx + c$ determines whether the parabola opens upward or downward. Multiplying $f(x)$ by a negative number flips the graph of f upside down. This is called a **vertical reflection** of the graph. The magnitude of a determines how steeply the graph increases or decreases.

EXAMPLE 3 Graphing Quadratic Functions

Graph $y = (x - h)^2$ for $h = 3$, 0, and -4.

SOLUTION Figure 17 shows all three functions on the same axes. Notice that since the number is subtracted *before* the squaring occurs, the graph does not move up or down but instead moves left or right. Evaluating $f(x) = (x - 3)^2$ at $x = 3$ gives the same result as evaluating $f(x) = x^2$ at $x = 0$. Therefore, when we subtract the positive number 3 from x, the graph shifts 3 units to the right, so the vertex is at $(3, 0)$. Similarly, when we subtract

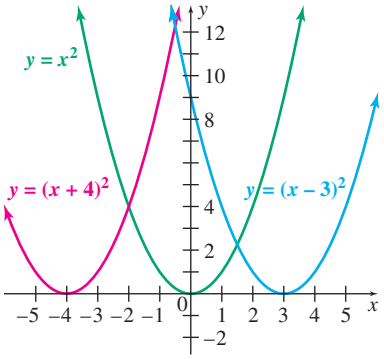


FIGURE 17

the negative number -4 from x—in other words, when the function becomes $f(x) = (x + 4)^2$ — the graph shifts to the left 4 units.

The left or right shift of the graph illustrated in Figure 17 is called a **horizontal translation** of the function.

If a quadratic equation is given in the form $ax^2 + bx + c$, we can identify the translations and any vertical reflection by rewriting it in the form

$$y = a(x - h)^2 + k.$$

In this form, we can identify the vertex as (h, k). A quadratic equation not given in this form can be converted by a process called **completing the square.** The next example illustrates the process.

EXAMPLE 4 Graphing a Quadratic Function

Graph $y = -3x^2 - 2x + 1$.

Method 1
Completing the Square

SOLUTION To begin, factor -3 from the x-terms so the coefficient of x^2 is 1:

$$y = -3\left(x^2 + \frac{2}{3}x\right) + 1.$$

Next, we make the expression inside the parentheses a perfect square by adding the square of one-half of the coefficient of x, which is $(\frac{1}{2} \cdot \frac{2}{3})^2 = \frac{1}{9}$. Since there is a factor of -3 outside the parentheses, we are actually adding $-3 \cdot (\frac{1}{9})$. To make sure that the value of the function is not changed, we must also add $3 \cdot (\frac{1}{9})$ to the function. Actually, we are simply adding $-3 \cdot (\frac{1}{9}) + 3 \cdot (\frac{1}{9}) = 0$, and not changing the function. To summarize our steps,

$$\begin{aligned} y &= -3\left(x^2 + \frac{2}{3}x\right) + 1 && \text{Factor out } -3. \\ &= -3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + 1 + 3\left(\frac{1}{9}\right) && \text{Add and subtract } -3 \text{ times } (\tfrac{1}{2} \text{ the coefficient of } x)^2. \\ &= -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}. && \text{Factor and combine terms.} \end{aligned}$$

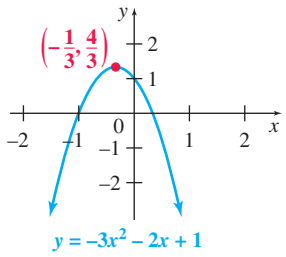


FIGURE 18

The function is now in the form $y = a(x - h)^2 + k$. Since $h = -1/3$ and $k = 4/3$, the graph is the graph of the parabola $y = x^2$ translated $1/3$ unit to the left and $4/3$ units upward. This puts the vertex at $(-1/3, 4/3)$. Since $a = -3$ is negative, the graph will be flipped upside down. The 3 will cause the parabola to be stretched vertically by a factor of 3. These results are shown in Figure 18.

Method 2
The Quadratic Formula

Instead of completing the square to find the vertex of the graph of a quadratic function given in the form $y = ax^2 + bx + c$, we can develop a formula for the vertex. By the quadratic formula, if $ax^2 + bx + c = 0$, where $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice that this is the same as

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm Q,$$

YOUR TURN 1 For the function $y = 2x^2 - 6x - 1$, **(a)** complete the square, **(b)** find the y-intercept, **(c)** find the x-intercepts, **(d)** find the vertex, and **(e)** sketch the graph.

where $Q = \sqrt{b^2 - 4ac}/(2a)$. Since a parabola is symmetric with respect to its axis, the vertex is halfway between its two roots. Halfway between $x = -b/(2a) + Q$ and $x = -b/(2a) - Q$ is $x = -b/(2a)$. Once we have the x-coordinate of the vertex, we can easily find the y-coordinate by substituting the x-coordinate into the original equation. For the function in this example, use the quadratic formula to verify that the x-intercepts are at $x = -1$ and $x = 1/3$, and the vertex is at $(-1/3, 4/3)$. The y-intercept (where $x = 0$) is 1. The graph is in Figure 18.

TRY YOUR TURN 1

Graph of the Quadratic Function

The graph of the quadratic function $f(x) = ax^2 + bx + c$ has its vertex at

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right).$$

The graph opens upward if $a > 0$ and downward if $a < 0$.

Another situation that may arise is the absence of any x-intercepts, as in the next example.

EXAMPLE 5 Graphing a Quadratic Function

Graph $y = x^2 + 4x + 6$.

SOLUTION This does not appear to factor, so we'll try the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 1, b = 4, c = 6$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)} = \frac{-4 \pm \sqrt{-8}}{2}$$

As soon as we see the negative value under the square root sign, we know the solutions are complex numbers. Therefore, there are no x-intercepts. Nevertheless, the vertex is still at

$$x = \frac{-b}{2a} = \frac{-4}{2} = -2.$$

Substituting this into the equation gives

$$y = (-2)^2 + 4(-2) + 6 = 2.$$

The y-intercept is at $(0, 6)$, which is 2 units to the right of the parabola's axis $x = -2$. Using the symmetry of the figure, we can also plot the mirror image of this point on the opposite side of the parabola's axis: at $x = -4$ (2 units to the left of the axis), y is also equal to 6. Plotting the vertex, the y-intercept, and the point $(-4, 6)$ gives the graph in Figure 19.

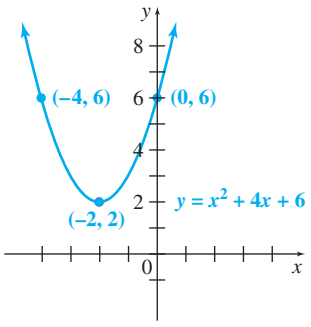


FIGURE 19

We now return to the question with which we started this section.

EXAMPLE 6 Management Science

When Power and Money, Inc., charges $600 for a seminar on management techniques, it attracts 1000 people. For each $20 decrease in the fee, an additional 100 people will attend the seminar. The managers are wondering how much to charge for the seminar to maximize their revenue.

APPLY IT

SOLUTION Let x be the number of $20 decreases in the price. Then the price charged per person will be

$$\text{Price per person} = 600 - 20x,$$

and the number of people in the seminar will be

$$\text{Number of people} = 1000 + 100x.$$

The total revenue, $R(x)$, is given by the product of the price and the number of people attending, or

$$\begin{aligned} R(x) &= (600 - 20x)(1000 + 100x) \\ &= 600{,}000 + 40{,}000x - 2000x^2. \end{aligned}$$

We see by the negative in the x^2-term that this defines a parabola opening downward, so the maximum revenue is at the vertex. The x-coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-40{,}000}{2(-2000)} = 10.$$

YOUR TURN 2 Solve Example 6 with the following changes: a \$1650 price attracts 900 people, and each \$40 decrease in the price attracts an additional 80 people.

The y-coordinate is then

$$\begin{aligned} y &= 600{,}000 + 40{,}000(10) - 2000(10^2) \\ &= 800{,}000. \end{aligned}$$

Therefore, the maximum revenue is \$800,000, which is achieved by charging $600 - 20x = 600 - 20(10) = \$400$ per person. TRY YOUR TURN 2

Notice in this last example that the maximum revenue was achieved by charging less than the current price of \$600, which was more than made up for by the increase in sales. This is typical of many applications. Mathematics is a powerful tool for solving such problems, since the answer is not always what one might have guessed intuitively.

To solve problems such as Example 6, notice the following:

1. The key step after reading and understanding the problem is identifying a useful variable.
2. Revenue is always price times the number sold.
3. The expressions for the price and for the number of people are both linear functions of x.
4. We know the constant term in each linear function because we know what happens when $x = 0$.
5. We know how much both the price and the number of people change each time x increases by 1, which gives us the slope of each linear function.
6. The maximum or minimum of a quadratic function occurs at its vertex.

The concept of maximizing or minimizing a function is important in calculus, as we shall see in future chapters.

In the next example, we show how the calculation of profit can involve a quadratic function.

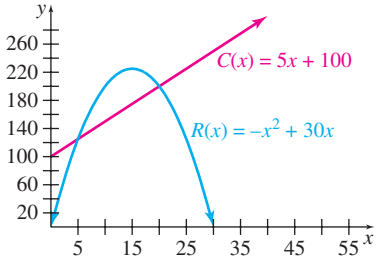


FIGURE 20

EXAMPLE 7 Profit

A deli owner has found that his revenue from producing x pounds of vegetable cream cheese is given by $R(x) = -x^2 + 30x$, while the cost in dollars is given by $C(x) = 5x + 100$.

(a) Find the minimum break-even quantity.

SOLUTION Notice from the graph in Figure 20 that the revenue function is a parabola opening downward and the cost function is a linear function that crosses the revenue function at two points. To find the minimum break-even quantity, we find where the two functions are equal.

$$\begin{aligned} R(x) &= C(x) \\ -x^2 + 30x &= 5x + 100 \\ 0 &= x^2 - 25x + 100 && \text{Subtract } -x^2 + 30x \text{ from both sides.} \\ &= (x - 5)(x - 20) && \text{Factor.} \end{aligned}$$

The two graphs cross when $x = 5$ and $x = 20$. The minimum break-even point is at $x = 5$. The deli owner must sell at least 5 lb of cream cheese to break even.

(b) Find the maximum revenue.

SOLUTION By factoring the revenue function, $R(x) = -x^2 + 30x = x(-x + 30)$, we can see that it has two roots, $x = 0$ and $x = 30$. The maximum is at the vertex, which has a value of x halfway between the two roots, or $x = 15$. (Alternatively, we could use the formula $x = -b/(2a) = -30/(-2) = 15$.) The maximum revenue is $R(15) = -15^2 + 30(15) = 225$, or \$225.

(c) Find the maximum profit.

SOLUTION The profit is the difference between the revenue and the cost, or

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-x^2 + 30x) - (5x + 100) \\ &= -x^2 + 25x - 100. \end{aligned}$$

This is just the negative of the expression factored in part (a), where we found the roots to be $x = 5$ and $x = 20$. The value of x at the vertex is halfway between these two roots, or $x = (5 + 20)/2 = 12.5$. (Alternatively, we could use the formula $x = -b/(2a) = -25/(-2) = 12.5$.) The value of the function here is $P(12.5) = -12.5^2 + 25(12.5) - 100 = 56.25$. It is clear that this is a maximum, not only from Figure 20, but also because the profit function is a quadratic with a negative x^2-term. A maximum profit of \$56.25 is achieved by selling 12.5 lb of cream cheese. TRY YOUR TURN 3

YOUR TURN 3 Suppose the revenue in dollars is given by $R(x) = -x^2 + 40x$ and the cost is given by $C(x) = 8x + 192$. Find **(a)** the minimum break-even quantity, **(b)** the maximum revenue, and **(c)** the maximum profit.

Below and on the next page, we provide guidelines for sketching graphs that involve translations and reflections.

Translations and Reflections of Functions

Let f be any function, and let h and k be positive constants (Figure 21).

The graph of $y = f(x) + k$ is the graph of $y = f(x)$ translated upward by an amount k (Figure 22).

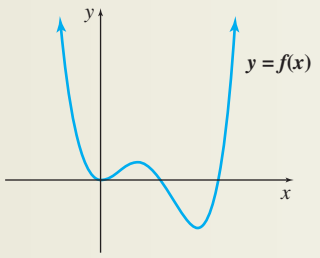


FIGURE 21

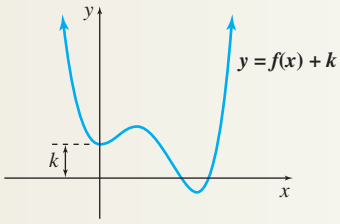


FIGURE 22

The graph of $y = f(x) - k$ is the graph of $y = f(x)$ translated downward by an amount k (Figure 23).

The graph of $y = f(x - h)$ is the graph of $y = f(x)$ translated to the right by an amount h (Figure 24).

The graph of $y = f(x + h)$ is the graph of $y = f(x)$ translated to the left by an amount h (Figure 25).

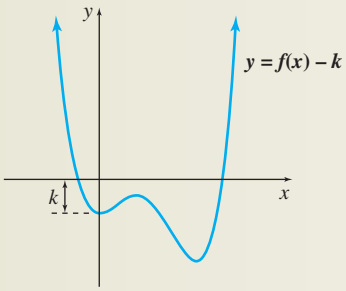


FIGURE 23

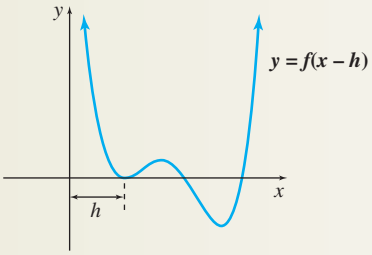


FIGURE 24

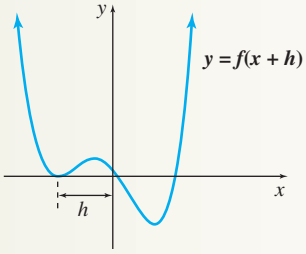


FIGURE 25

Translations and Reflections of Functions

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected vertically across the x-axis, that is, turned upside down (Figure 26).
The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected horizontally across the y-axis, that is, its mirror image (Figure 27).

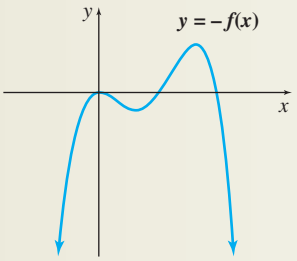


FIGURE 26

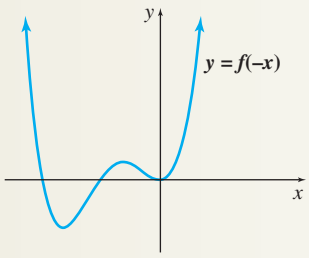


FIGURE 27

Notice in Figure 27 another type of reflection, known as a **horizontal reflection**. Multiplying x or $f(x)$ by a constant a, to get $y = f(ax)$ or $y = a \cdot f(x)$ does not change the general appearance of the graph, except to compress or stretch it. When a is negative, it also causes a reflection, as shown in the last two figures in the summary for $a = -1$. Also see Exercises 39–42 in this section.

EXAMPLE 8 Translations and Reflections of a Graph

Graph $f(x) = -\sqrt{4 - x} + 3$.

SOLUTION Begin with the simplest possible function, then add each variation in turn. Start with the graph of $f(x) = \sqrt{x}$. As Figure 28 reveals, this is just one-half of the graph of $f(x) = x^2$ lying on its side.

Now add another component of the original function, the negative in front of the x, giving $f(x) = \sqrt{-x}$. This is a horizontal reflection of the $f(x) = \sqrt{x}$ graph, as shown in Figure 29. Next, include the 4 under the square root sign. To get $4 - x$ into the form $f(x - h)$ or $f(x + h)$, we need to factor out the negative: $\sqrt{4 - x} = \sqrt{-(x - 4)}$. Now the 4 is subtracted, so this function is a translation to the right of the function $f(x) = \sqrt{-x}$ by 4 units, as Figure 30 indicates.

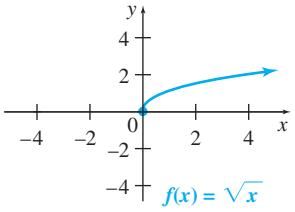


FIGURE 28

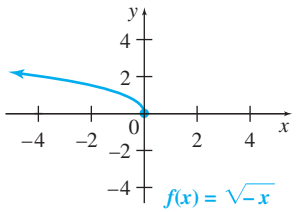


FIGURE 29

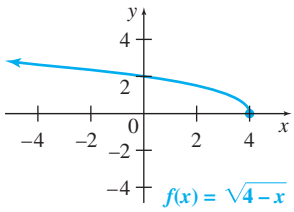


FIGURE 30

The effect of the negative in front of the radical is a vertical reflection, as in Figure 31, which shows the graph of $f(x) = -\sqrt{4 - x}$. Finally, adding the constant 3 raises the entire graph by 3 units, giving the graph of $f(x) = -\sqrt{4 - x} + 3$ in Figure 32(a).

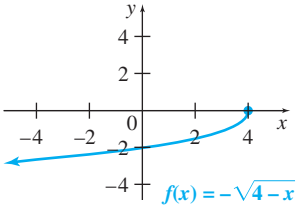
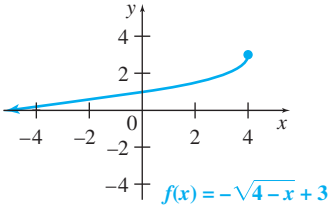
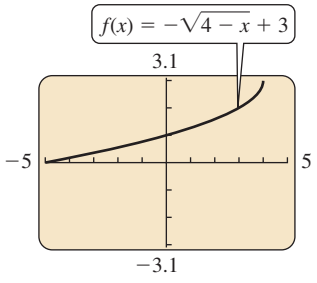


FIGURE 31



(a)



(b)

FIGURE 32

TECHNOLOGY NOTE

If you viewed a graphing calculator image such as Figure 32(b), you might think the function continues to go up and to the right. By realizing that $(4, 3)$ is the vertex of the sideways parabola, we see that this is the rightmost point on the graph. Another approach is to find the domain of f by setting $4 - x \geq 0$, from which we conclude that $x \leq 4$. This demonstrates the importance of knowing the algebraic techniques in order to interpret a graphing calculator image correctly.

10.2 EXERCISES

1. How does the value of a affect the graph of $y = ax^2$? Discuss the case for $a \geq 1$ and for $0 \leq a \leq 1$.

2. How does the value of a affect the graph of $y = ax^2$ if $a \leq 0$?

In Exercises 3–8, match the correct graph A–F to the function without using your calculator. Then, if you have a graphing calculator, use it to check your answers. Each graph in this group shows x and y in $[-10, 10]$.

3. $y = x^2 - 3$

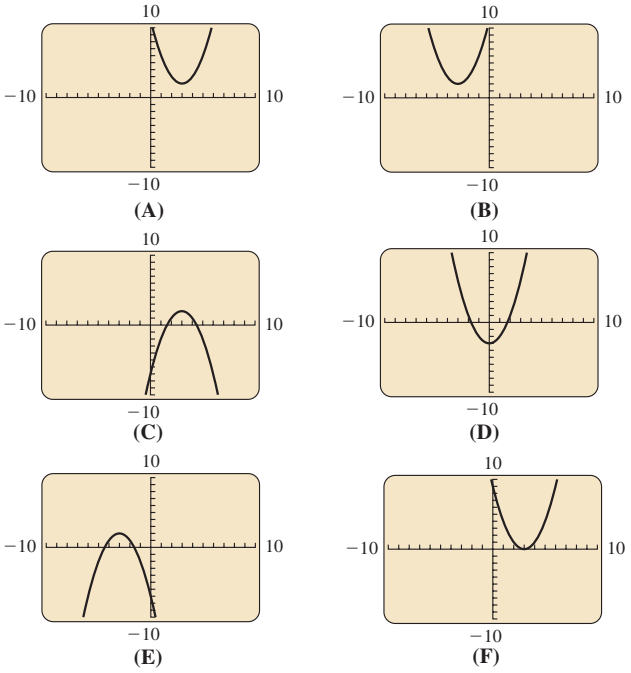
4. $y = (x - 3)^2$

5. $y = (x - 3)^2 + 2$

6. $y = (x + 3)^2 + 2$

7. $y = -(3 - x)^2 + 2$

8. $y = -(x + 3)^2 + 2$



Complete the square and determine the vertex for each of the following.

9. $y = 3x^2 + 9x + 5$

10. $y = 4x^2 - 20x - 7$

11. $y = -2x^2 + 8x - 9$

12. $y = -5x^2 - 8x + 3$

In Exercises 13–24, graph each parabola and give its vertex, axis, x-intercepts, and y-intercept.

13. $y = x^2 + 5x + 6$

14. $y = x^2 + 4x - 5$

15. $y = -2x^2 - 12x - 16$

16. $y = -3x^2 - 6x + 4$

17. $f(x) = 2x^2 + 8x - 8$

18. $f(x) = -x^2 + 6x - 6$

19. $f(x) = 2x^2 - 4x + 5$

20. $f(x) = \frac{1}{2}x^2 + 6x + 24$

21. $f(x) = -2x^2 + 16x - 21$

22. $f(x) = \frac{3}{2}x^2 - x - 4$

23. $f(x) = \frac{1}{3}x^2 - \frac{8}{3}x + \frac{1}{3}$

24. $f(x) = -\frac{1}{2}x^2 - x - \frac{7}{2}$

In Exercises 25–30, follow the directions for Exercises 3–8.

25. $y = \sqrt{x + 2} - 4$

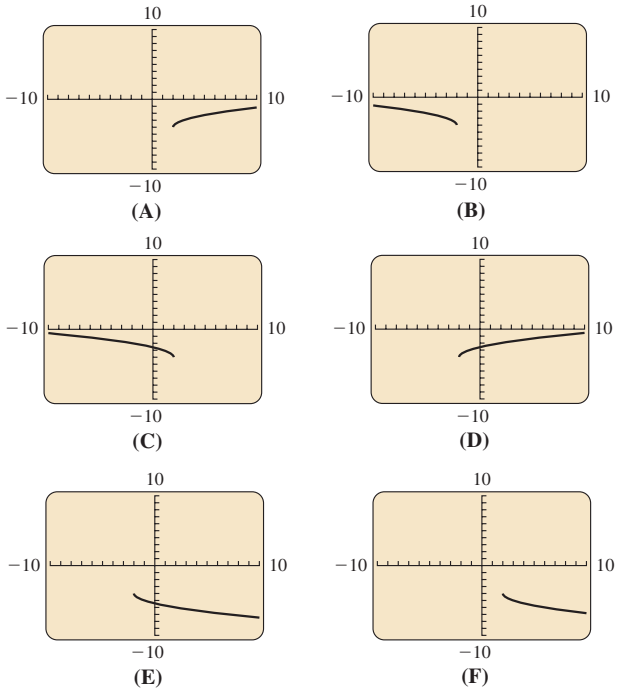
26. $y = \sqrt{x - 2} - 4$

27. $y = \sqrt{-x + 2} - 4$

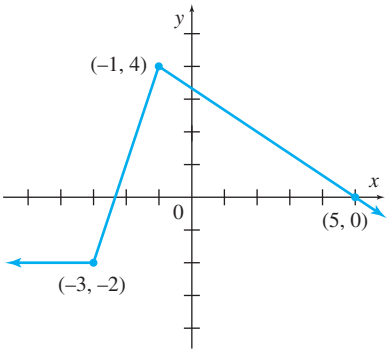
28. $y = \sqrt{-x - 2} - 4$

29. $y = -\sqrt{x + 2} - 4$

30. $y = -\sqrt{x - 2} - 4$



Given the following graph, sketch by hand the graph of the function described, giving the new coordinates for the three points labeled on the original graph.



31. $y = -f(x)$

32. $y = f(x - 2) + 2$

33. $y = f(-x)$

34. $y = f(2 - x) + 2$

Use the ideas in this section to graph each function without a calculator.

35. $f(x) = \sqrt{x - 2} + 2$ **36.** $f(x) = \sqrt{x + 2} - 3$

37. $f(x) = -\sqrt{2 - x} - 2$ **38.** $f(x) = -\sqrt{2 - x} + 2$

Using the graph of $f(x)$ in Figure 21, show the graph of $f(ax)$ where a satisfies the given condition.

39. $0 < a < 1$ **40.** $1 < a$

41. $-1 < a < 0$ **42.** $a < -1$

Using the graph of $f(x)$ in Figure 21, show the graph of $af(x)$ where a satisfies the given condition.

43. $0 < a < 1$ **44.** $1 < a$

45. $-1 < a < 0$ **46.** $a < -1$

47. If r is an x-intercept of the graph of $y = f(x)$, what is an x-intercept of the graph of each of the following?

a. $y = -f(x)$ **b.** $y = f(-x)$

c. $y = -f(-x)$

48. If b is the y-intercept of the graph of $y = f(x)$, what is the y-intercept of the graph of each of the following?

a. $y = -f(x)$ **b.** $y = f(-x)$

c. $y = -f(-x)$

APPLICATIONS

Business and Economics

Profit In Exercises 49–52, let $C(x)$ be the cost to produce x batches of widgets, and let $R(x)$ be the revenue in thousands of dollars. For each exercise, (a) graph both functions, (b) find the minimum break-even quantity, (c) find the maximum revenue, and (d) find the maximum profit.

49. $R(x) = -x^2 + 8x, \quad C(x) = 2x + 5$

50. $R(x) = -\dfrac{x^2}{2} + 5x, \quad C(x) = \dfrac{3}{2}x + 3$

51. $R(x) = -\dfrac{4}{5}x^2 + 10x, \quad C(x) = 2x + 15$

52. $R(x) = -4x^2 + 36x, \quad C(x) = 16x + 24$

53. Maximizing Revenue The revenue of a charter bus company depends on the number of unsold seats. If the revenue $R(x)$ is given by

$$R(x) = 8000 + 70x - x^2,$$

where x is the number of unsold seats, find the maximum revenue and the number of unsold seats that corresponds to maximum revenue.

54. Maximizing Revenue A charter flight charges a fare of \$200 per person plus \$4 per person for each unsold seat on the plane. The plane holds 100 passengers. Let x represent the number of unsold seats.

a. Find an expression for the total revenue received for the flight $R(x)$. (*Hint:* Multiply the number of people flying, $100 - x$, by the price per ticket.)

b. Graph the expression from part a.

c. Find the number of unsold seats that will produce the maximum revenue.

d. What is the maximum revenue?

e. Some managers might be concerned about the empty seats, arguing that it doesn't make economic sense to leave any seats empty. Write a few sentences explaining why this is not necessarily so.

55. Maximizing Revenue The demand for a certain type of cosmetic is given by

$$p = 500 - x,$$

where p is the price in dollars when x units are demanded.

a. Find the revenue $R(x)$ that would be obtained at a price p. (*Hint:* Revenue = Demand × Price)

b. Graph the revenue function $R(x)$.

c. Find the price that will produce maximum revenue.

d. What is the maximum revenue?

56. Revenue The manager of a peach orchard is trying to decide when to arrange for picking the peaches. If they are picked now, the average yield per tree will be 100 lb, which can be sold for 80¢ per pound. Past experience shows that the yield per tree will increase about 5 lb per week, while the price will decrease about 4¢ per pound per week.

a. Let x represent the number of weeks that the manager should wait. Find the income per pound.

b. Find the number of pounds per tree.

c. Find the total revenue from a tree.

d. When should the peaches be picked in order to produce maximum revenue?

e. What is the maximum revenue?

57. Income The manager of an 80-unit apartment complex is trying to decide what rent to charge. Experience has shown that at a rent of \$800, all the units will be full. On the average, one additional unit will remain vacant for each \$25 increase in rent.

a. Let x represent the number of \$25 increases. Find an expression for the rent for each apartment.

b. Find an expression for the number of apartments rented.

c. Find an expression for the total revenue from all rented apartments.

d. What value of x leads to maximum revenue?

e. What is the maximum revenue?

58. Advertising A study done by an advertising agency reveals that when x thousands of dollars are spent on advertising, it results in a sales increase in thousands of dollars given by the function

$$S(x) = -\frac{1}{4}(x - 10)^2 + 40, \quad \text{for } 0 \le x \le 10.$$

a. Find the increase in sales when no money is spent on advertising.

b. Find the increase in sales when \$10,000 is spent on advertising.

c. Sketch the graph of $S(x)$ without a calculator.

Life Sciences

59. Length of Life According to recent data from the Teachers Insurance and Annuity Association (TIAA), the survival function for life after 65 is approximately given by

$$S(x) = 1 - 0.058x - 0.076x^2,$$

where x is measured in decades. This function gives the probability that an individual who reaches the age of 65 will live at least x decades ($10x$ years) longer. ***Source: Ralph DeMarr.***

a. Find the median length of life for people who reach 65, that is, the age for which the survival rate is 0.50.

b. Find the age beyond which virtually nobody lives. (There are, of course, exceptions.)

60. Tooth Length The length (in mm) of the mesiodistal crown of the first molar for human fetuses can be approximated by

$$L(t) = -0.01t^2 + 0.788t - 7.048,$$

where t is the number of weeks since conception. ***Source: American Journal of Physical Anthropology.***

a. What does this formula predict for the length at 14 weeks? 24 weeks?

b. What does this formula predict for the maximum length, and when does that occur? Explain why the formula does not make sense past that time.

61. Splenic Artery Resistance Blood flow to the fetal spleen is of research interest because several diseases are associated with increased resistance in the splenic artery (the artery that goes to the spleen). Researchers have found that the index of splenic artery resistance in the fetus can be described by the function

$$y = 0.057x - 0.001x^2,$$

where x is the number of weeks of gestation. ***Source: American Journal of Obstetrics and Gynecology.***

a. At how many weeks is the splenic artery resistance a maximum?

b. What is the maximum splenic artery resistance?

c. At how many weeks is the splenic artery resistance equal to 0, according to this formula? Is your answer reasonable for this function? Explain.

62. Cancer From 1975 to 2007, the age-adjusted incidence rate of invasive lung and bronchial cancer among women can be closely approximated by

$$f(t) = -0.040194t^2 + 2.1493t + 23.921,$$

where t is the number of years since 1975. ***Source: National Cancer Institute.*** Based on this model, in what year did the incidence rate reach a maximum? On what years was the rate increasing? Decreasing?

Social Sciences

63. Age of Marriage The following table gives the median age at their first marriage of women in the United States for some selected years. ***Source: U.S. Census Bureau.***

Year	Age
1940	21.5
1950	20.3
1960	20.3
1970	20.8
1980	22.0
1990	23.9
2000	25.1

a. Plot the data using $x = 40$ for 1940, and so on.

b. Would a linear or quadratic function best model this data? Explain.

c. If your graphing calculator has a regression feature, find the quadratic function that best fits the data. Graph this function on the same calculator window as the data. (On a TI-84 Plus calculator, press the STAT key, and then select the CALC menu. QuadReg is item 5. The command QuadReg L_1, L_2, Y_1 finds the quadratic regression equation for the data in L_1 and L_2 and stores the function in Y_1.)

d. Find a quadratic function defined by $f(x) = a(x - h)^2 + k$ that models the data using (60, 20.3) as the vertex and then choosing (100, 25.1) as a second point to determine the value of a.

e. Graph the function from part d on the same calculator window as the data and function from part c. Do the graphs of the two functions differ by much?

64. Gender Ratio The number of males per 100 females, age 65 or over, in the United States for some recent years is shown in the following table. ***Source: The New York Times 2010 Almanac.***

Year	Males per 100 Females
1960	82.8
1970	72.1
1980	67.6
1990	67.2
2000	70.0
2007	72.9

a. Plot the data, letting x be the years since 1900.

b. Would a linear or quadratic function best model this data? Explain.

c. If your graphing calculator has a quadratic regression feature, find the quadratic function that best fits the data. Graph this function on the same calculator window as the data. (See Exercise 63(c).)

d. Choose the lowest point in the table above as the vertex and (60, 82.8) as a second point to find a quadratic function defined by $f(x) = a(x - h)^2 + k$ that models the data.

e. Graph the function from part d on the same calculator window as the data and function from part c. Do the graphs of the two functions differ by much?

f. Predict the number of males per 100 females in 2004 using the two functions from parts c and d, and compare with the actual figure of 71.7.

65. Accident Rate According to data from the National Highway Traffic Safety Administration, the accident rate as a function of the age of the driver in years x can be approximated by the function

$$f(x) = 60.0 - 2.28x + 0.0232x^2$$

for $16 \le x \le 85$. Find the age at which the accident rate is a minimum and the minimum rate. ***Source: Ralph DeMarr.***

Physical Sciences

66. Maximizing the Height of an Object If an object is thrown upward with an initial velocity of 32 ft/second, then its height after t seconds is given by

$$h = 32t - 16t^2.$$

a. Find the maximum height attained by the object.

b. Find the number of seconds it takes the object to hit the ground.

67. Stopping Distance According to data from the National Traffic Safety Institute, the stopping distance y in feet of a car traveling x mph can be described by the equation $y = 0.056057x^2 + 1.06657x$. ***Source: National Traffic Safety Institute.***

a. Find the stopping distance for a car traveling 25 mph.

b. How fast can you drive if you need to be certain of stopping within 150 ft?

General Interest

68. Maximizing Area Glenview Community College wants to construct a rectangular parking lot on land bordered on one side by a highway. It has 380 ft of fencing to use along the other three sides. What should be the dimensions of the lot if the enclosed area is to be a maximum? (*Hint:* Let x represent the width of the lot, and let $380 - 2x$ represent the length.)

69. Maximizing Area What would be the maximum area that could be enclosed by the college's 380 ft of fencing if it decided to close the entrance by enclosing all four sides of the lot? (See Exercise 68.)

In Exercises 70 and 71, draw a sketch of the arch or culvert on coordinate axes, with the horizontal and vertical axes through the vertex of the parabola. Use the given information to label points on the parabola. Then give the equation of the parabola and answer the question.

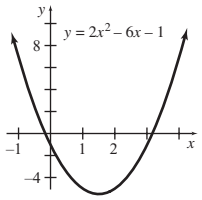
70. Parabolic Arch An arch is shaped like a parabola. It is 30 m wide at the base and 15 m high. How wide is the arch 10 m from the ground?

71. Parabolic Culvert A culvert is shaped like a parabola, 18 ft across the top and 12 ft deep. How wide is the culvert 8 ft from the top?

YOUR TURN ANSWERS

1. **(a)** $y = 2(x - 3/2)^2 - 11/2$ **(b)** -1 **(c)** $(3 \pm \sqrt{11})/2$ **(d)** $(3/2, -11/2)$

(e)



2. Charge \$1050 for a maximum revenue of \$2,205,000.

3. **(a)** 8 **(b)** \$400 **(c)** \$64

10.3 Polynomial and Rational Functions

APPLY IT **How does the revenue collected by the government vary with the tax rate?**

In Exercises 48–50 in this section, we will explore this question using polynomial *and* rational functions.

Polynomial Functions

Earlier, we discussed linear and quadratic functions and their graphs. Both of these functions are special types of *polynomial functions*.

Polynomial Function

A **polynomial function** of **degree** n, where n is a nonnegative integer, is defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1$, and a_0 are real numbers, called **coefficients**, with $a_n \neq 0$. The number a_n is called the **leading coefficient**.

For $n = 1$, a polynomial function takes the form

$$f(x) = a_1 x + a_0,$$

a linear function. A linear function, therefore, is a polynomial function of degree 1. (Note, however, that a linear function of the form $f(x) = a_0$ for a real number a_0 is a polynomial

function of degree 0, the constant function.) A polynomial function of degree 2 is a quadratic function.

Accurate graphs of polynomial functions of degree 3 or higher require methods of calculus to be discussed later. Meanwhile, a graphing calculator is useful for obtaining such graphs, but care must be taken in choosing a viewing window that captures the significant behavior of the function.

The simplest polynomial functions of higher degree are those of the form $f(x) = x^n$. Such a function is known as a **power function**. Figure 33 below shows the graphs of $f(x) = x^3$ and $f(x) = x^5$, as well as tables of their values. These functions are simple enough that they can be drawn by hand by plotting a few points and connecting them with a smooth curve. An important property of all polynomials is that their graphs are smooth curves.

The graphs of $f(x) = x^4$ and $f(x) = x^6$, shown in Figure 34 along with tables of their values, can be sketched in a similar manner. These graphs have symmetry about the y-axis, as does the graph of $f(x) = ax^2$ for a nonzero real number a. As with the graph of $f(x) = ax^2$, the value of a in $f(x) = ax^n$ affects the direction of the graph. When $a > 0$, the graph has the same general appearance as the graph of $f(x) = x^n$. However, if $a < 0$, the graph is reflected vertically. Notice that $f(x) = x^3$ and $f(x) = x^5$ are odd functions, while $f(x) = x^4$ and $f(x) = x^6$ are even functions.

$f(x) = x^3$		$f(x) = x^5$	
x	$f(x)$	x	$f(x)$
-2	-8	-1.5	-7.6
-1	-1	-1	-1
0	0	0	0
1	1	1	1
2	8	1.5	7.6

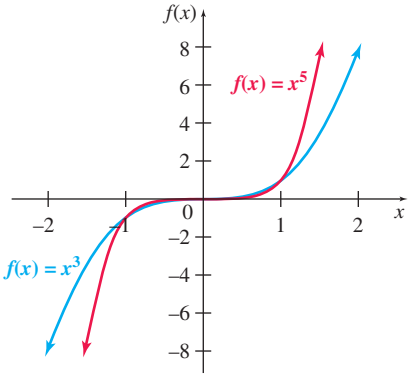


FIGURE 33

$f(x) = x^4$		$f(x) = x^6$	
x	$f(x)$	x	$f(x)$
-2	16	-1.5	11.4
-1	1	-1	1
0	0	0	0
1	1	1	1
2	16	1.5	11.4

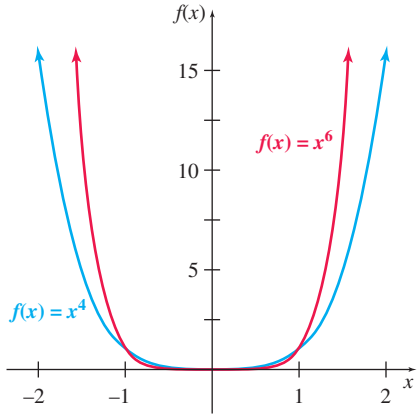


FIGURE 34

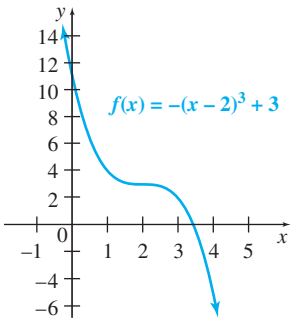


FIGURE 35

YOUR TURN 1 Graph $f(x) = 64 - x^6$.

EXAMPLE 1 Translations and Reflections

Graph $f(x) = -(x - 2)^3 + 3$.

SOLUTION Using the principles of translation and reflection from the previous section, we recognize that this is similar to the graph of $y = x^3$, but reflected vertically (because of the negative in front of $(x - 2)^3$), and with its center moved 2 units to the right and 3 units up. The result is shown in Figure 35.

TRY YOUR TURN 1

A polynomial of degree 3, such as that in the previous example and in the next, is known as a **cubic polynomial.** A polynomial of degree 4, such as that in Example 3, is known as a **quartic polynomial.**

TECHNOLOGY

EXAMPLE 2 Graphing a Polynomial

Graph $f(x) = 8x^3 - 12x^2 + 2x + 1$.

SOLUTION Figure 36 shows the function graphed on the x- and y-intervals $[-0.5, 0.6]$ and $[-2, 2]$. In this view, it appears similar to a parabola opening downward. Zooming out to $[-1, 2]$ by $[-8, 8]$, we see in Figure 37 that the graph goes upward as x gets large. There are also two **turning points** near $x = 0$ and $x = 1$. (In a later chapter, we will introduce another term for such turning points: *relative extrema*.) By zooming in with the graphing calculator, we can find these turning points to be at approximately $(0.09175, 1.08866)$ and $(0.90825, -1.08866)$.

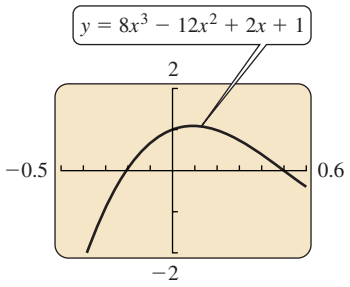


FIGURE 36

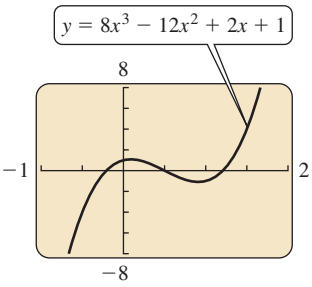


FIGURE 37

Zooming out still further, we see the function on $[-10, 10]$ by $[-300, 300]$ in Figure 38. From this viewpoint, we don't see the turning points at all, and the graph seems similar in shape to that of $y = x^3$. This is an important point: when x is large in magnitude, either positive or negative, $8x^3 - 12x^2 + 2x + 1$ behaves a lot like $8x^3$, because the other terms are small in comparison with the cubic term. So this viewpoint tells us something useful about the function, but it is less useful than the previous graph for determining the turning points.

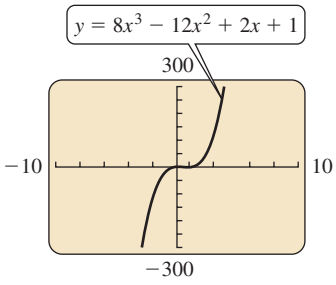


FIGURE 38

After the previous example, you may wonder how to be sure you have the viewing window that exhibits all the important properties of a function. We will find an answer to this question in later chapters using the techniques of calculus. Meanwhile, let us consider one more example to get a better idea of what polynomials look like.

TECHNOLOGY

EXAMPLE 3 Graphing a Polynomial

Graph $f(x) = -3x^4 + 14x^3 - 54x + 3$.

SOLUTION Figure 39 shows a graphing calculator view on $[-3, 5]$ by $[-50, 50]$. If you have a graphing calculator, we recommend that you experiment with various viewpoints and verify for yourself that this viewpoint captures the important behavior of the function. Notice that it has three turning points. Notice also that as $|x|$ gets large, the graph turns downward. This is because as $|x|$ becomes large, the x^4-term dominates the other terms, which are small in comparison, and the x^4-term has a negative coefficient.

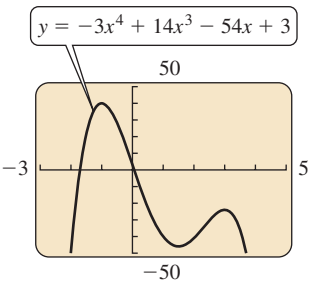


FIGURE 39

As suggested by the graphs above, the domain of a polynomial function is the set of all real numbers. The range of a polynomial function of odd degree is also the set of all real numbers. Some typical graphs of polynomial functions of odd and even degree are shown in Figure 40 on the next page. The first two graphs suggest that for every polynomial function f of odd degree, there is at least one real value of x for which $f(x) = 0$. Such a value of x is called a **real zero** of f; these values are also the x-intercepts of the graph.

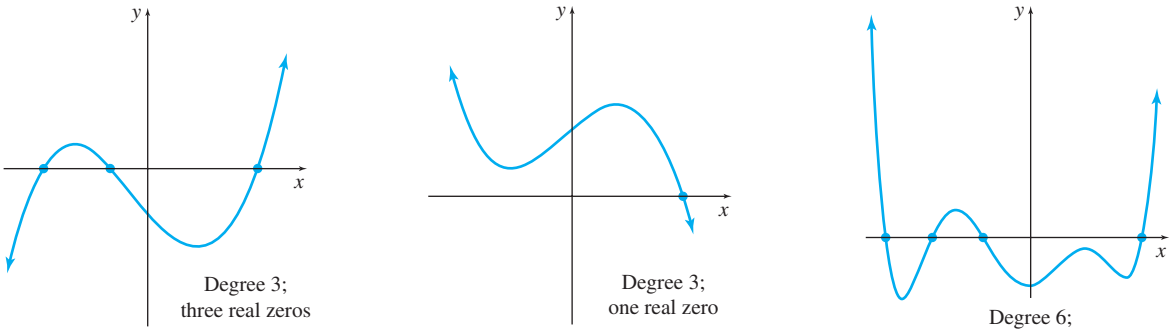


FIGURE 40

EXAMPLE 4 Identifying the Degree of a Polynomial

Identify the degree of the polynomial in each of the figures, and give the sign (+ or −) for the leading coefficient.

(a) Figure 41(a)

SOLUTION Notice that the polynomial has a range $[k, \infty)$. This must be a polynomial of even degree, because if the highest power of x is an odd power, the polynomial can take on all real numbers, positive and negative. Notice also that the polynomial becomes a large positive number as x gets large in magnitude, either positive or negative, so the leading coefficient must be positive. Finally, notice that it has three turning points. Observe from the previous examples that a polynomial of degree n has at most $n - 1$ turning points. In a later chapter, we will use calculus to see why this is true. So the polynomial graphed in Figure 41(a) might be degree 4, although it could also be of degree 6, 8, etc. We can't be sure from the graph alone.

(b) Figure 41(b)

SOLUTION Because the range is $(-\infty, \infty)$, this must be a polynomial of odd degree. Notice also that the polynomial becomes a large negative number as x becomes a large positive number, so the leading coefficient must be negative. Finally, notice that it has four turning points, so it might be degree 5, although it could also be of degree 7, 9, etc.

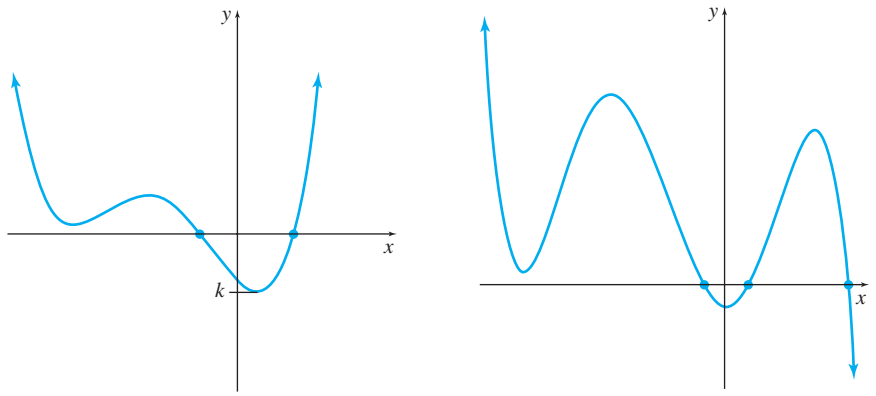


FIGURE 41

Properties of Polynomial Functions

1. A polynomial function of degree n can have at most $n - 1$ turning points. Conversely, if the graph of a polynomial function has n turning points, it must have degree at least $n + 1$.
2. In the graph of a polynomial function of even degree, both ends go up or both ends go down. For a polynomial function of odd degree, one end goes up and one end goes down.
3. If the graph goes up as x becomes a large positive number, the leading coefficient must be positive. If the graph goes down as x becomes a large positive number, the leading coefficient is negative.

Rational Functions

Many situations require mathematical models that are quotients. A common model for such situations is a *rational function*.

Rational Function

A **rational function** is defined by

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Since any values of x such that $q(x) = 0$ are excluded from the domain, a rational function often has a graph with one or more breaks.

EXAMPLE 5 Graphing a Rational Function

Graph $y = \frac{1}{x}$.

SOLUTION This function is undefined for $x = 0$, since 0 is not allowed as the denominator of a fraction. For this reason, the graph of this function will not intersect the vertical line $x = 0$, which is the y-axis. Since x can take on any value except 0, the values of x can approach 0 as closely as desired from either side of 0.

Values of $1/x$ for Small x

x	-0.5	-0.2	-0.1	-0.01	0.01	0.1	0.2	0.5
				(x approaches 0.) ↓				
$y = \frac{1}{x}$	-2	-5	-10	-100	100	10	5	2
					↑ ($\lvert y \rvert$ gets larger and larger.)			

The table above suggests that as x gets closer and closer to 0, $|y|$ gets larger and larger. This is true in general: as the denominator gets smaller, the fraction gets larger. Thus, the graph of the function approaches the vertical line $x = 0$ (the y-axis) without ever touching it.

As $|x|$ gets larger and larger, $y = 1/x$ gets closer and closer to 0, as shown in the table below. This is also true in general: as the denominator gets larger, the fraction gets smaller.

Values of $1/x$ for Large $|x|$

x	-100	-10	-4	-1	1	4	10	100
$y = \frac{1}{x}$	-0.01	-0.1	-0.25	-1	1	0.25	0.1	0.01

The graph of the function approaches the horizontal line $y = 0$ (the x-axis). The information from both tables supports the graph in Figure 42.

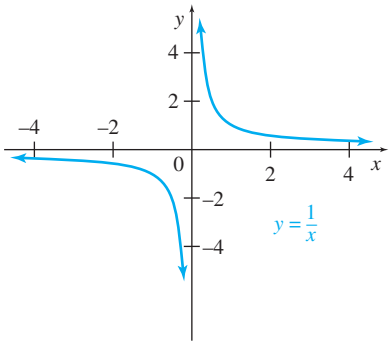


FIGURE 42

In Example 5, the vertical line $x = 0$ and the horizontal line $y = 0$ are *asymptotes*, defined as follows.

Asymptotes

If a function gets larger and larger in magnitude without bound as x approaches the number k, then the line $x = k$ is a **vertical asymptote**.

If the values of y approach a number k as $|x|$ gets larger and larger, the line $y = k$ is a **horizontal asymptote**.

There is an easy way to find any vertical asymptotes of a rational function. First, find the roots of the denominator. If a number k makes the denominator 0 but does not make the numerator 0, then the line $x = k$ is a vertical asymptote. If, however, a number k makes both the denominator and the numerator 0, then further investigation will be necessary, as we will see in the next example. In the next chapter we will show another way to find asymptotes using the concept of a *limit*.

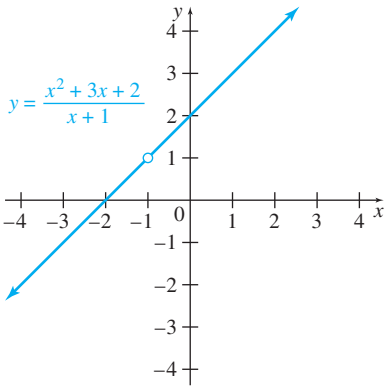


FIGURE 43

EXAMPLE 6 Graphing a Rational Function

Graph the following rational functions:

(a) $y = \dfrac{x^2 + 3x + 2}{x + 1}$.

SOLUTION The value $x = -1$ makes the denominator 0, and so -1 is not in the domain of this function. Note that the value $x = -1$ also makes the numerator 0. In fact, if we factor the numerator and simplify the function, we get

$$y = \frac{x^2 + 3x + 2}{x + 1} = \frac{(x + 2)(x + 1)}{(x + 1)} = x + 2 \quad \text{for} \quad x \neq -1.$$

The graph of this function, therefore, is the graph of $y = x + 2$ with a hole at $x = -1$, as shown in Figure 43.

(b) $y = \dfrac{3x + 2}{2x + 4}$.

SOLUTION The value $x = -2$ makes the denominator 0, but not the numerator, so the line $x = -2$ is a vertical asymptote. To find a horizontal asymptote, let x get larger and larger, so that $3x + 2 \approx 3x$ because the 2 is very small compared with $3x$. Similarly, for x very large, $2x + 4 \approx 2x$. Therefore, $y = (3x + 2)/(2x + 4) \approx (3x)/(2x) = 3/2$. This means that the line $y = 3/2$ is a horizontal asymptote. (A more precise way of approaching this idea will be seen in the next chapter when limits at infinity are discussed.)

The intercepts should also be noted. When $x = 0$, the y-intercept is $y = 2/4 = 1/2$. To make a fraction 0, the numerator must be 0; so to make $y = 0$, it is necessary that $3x + 2 = 0$. Solve this for x to get $x = -2/3$ (the x-intercept). We can also use these values to determine where the function is positive and where it is negative. Using the techniques described in Chapter R, verify that the function is negative on $(-2, -2/3)$ and positive on $(-\infty, -2) \cup (-2/3, \infty)$. With this information, the two asymptotes to guide us, and the fact that there are only two intercepts, we suspect the graph is as shown in Figure 44. A graphing calculator can support this. TRY YOUR TURN 2

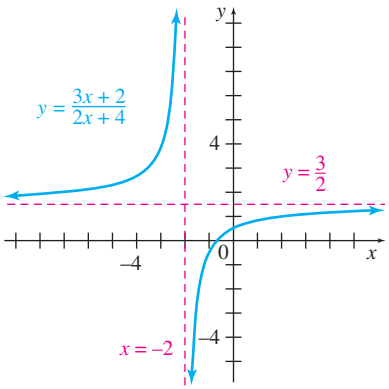


FIGURE 44

YOUR TURN 2

Graph $y = \dfrac{4x - 6}{x - 3}$.

Rational functions occur often in practical applications. In many situations involving environmental pollution, much of the pollutant can be removed from the air or water at a fairly reasonable cost, but the last small part of the pollutant can be very expensive to remove. Cost as a function of the percentage of pollutant removed from the environment can be calculated for various percentages of removal, with a curve fitted through the resulting data points. This curve then leads to a mathematical model of the situation. Rational functions are often a good choice for these **cost-benefit models** because they rise rapidly as they approach a vertical asymptote.

EXAMPLE 7 Cost-Benefit Analysis

Suppose a cost-benefit model is given by

$$y = \frac{18x}{106 - x},$$

where y is the cost (in thousands of dollars) of removing x percent of a certain pollutant. The domain of x is the set of all numbers from 0 to 100 inclusive; any amount of pollutant from 0% to 100% can be removed. Find the cost to remove the following amounts of the pollutant: 100%, 95%, 90%, and 80%. Graph the function.

SOLUTION Removal of 100% of the pollutant would cost

$$y = \frac{18(100)}{106 - 100} = 300,$$

or \$300,000. Check that 95% of the pollutant can be removed for \$155,000, 90% for \$101,000, and 80% for \$55,000. Using these points, as well as others obtained from the function, gives the graph shown in Figure 45.

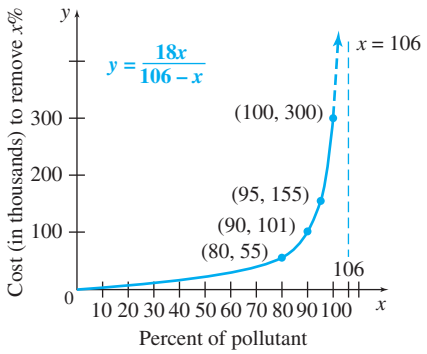


FIGURE 45

If a cost function has the form $C(x) = mx + b$, where x is the number of items produced, m is the marginal cost per item and b is the fixed cost, then the **average cost** per item is given by

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{mx + b}{x}.$$

Notice that this is a rational function with a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = m$. The vertical asymptote reflects the fact that, as the number of items produced approaches 0, the average cost per item becomes infinitely large, because the fixed costs are spread over fewer and fewer items. The horizontal asymptote shows that, as the number of items becomes large, the fixed costs are spread over more and more items, so most of the average cost per item is the marginal cost to produce each item. This is another example of how asymptotes give important information in real applications.

10.3 EXERCISES

1. Explain how translations and reflections can be used to graph $y = -(x - 1)^4 + 2$.

2. Describe an asymptote, and explain when a rational function will have (a) a vertical asymptote and (b) a horizontal asymptote.

Use the principles of the previous section with the graphs of this section to sketch a graph of the given function.

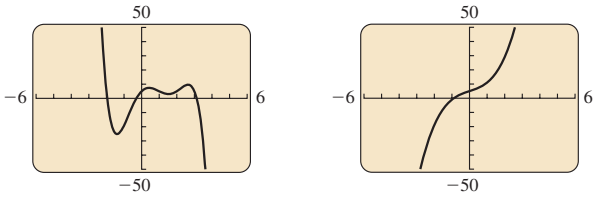
3. $f(x) = (x - 2)^3 + 3$

4. $f(x) = (x + 1)^3 - 2$

5. $f(x) = -(x + 3)^4 + 1$

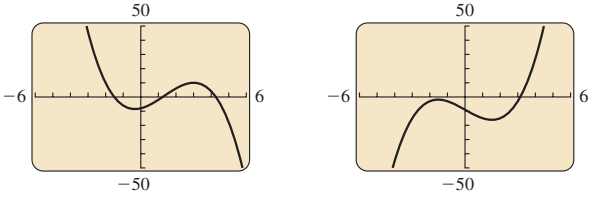
6. $f(x) = -(x - 1)^4 + 2$

In Exercises 7–15, match the correct graph A–I to the function without using your calculator. Then, after you have answered all of them, if you have a graphing calculator, use your calculator to check your answers. Each graph is plotted on $[-6, 6]$ by $[-50, 50]$.



(A)

(B)



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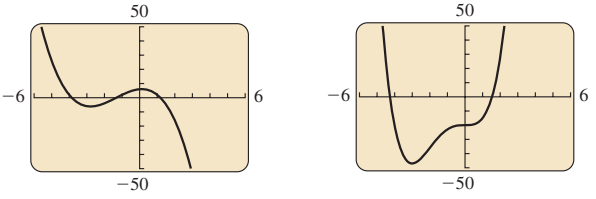
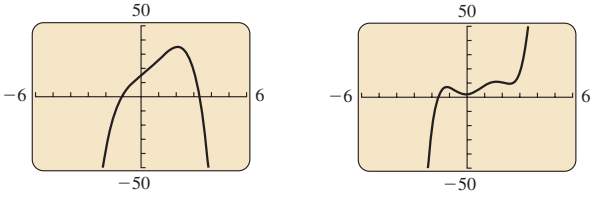


(E)

(F)



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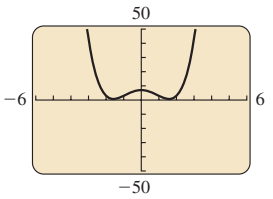


(I)

7. $y = x^3 - 7x - 9$

8. $y = -x^3 + 4x^2 + 3x - 8$

9. $y = -x^3 - 4x^2 + x + 6$

10. $y = 2x^3 + 4x + 5$

11. $y = x^4 - 5x^2 + 7$

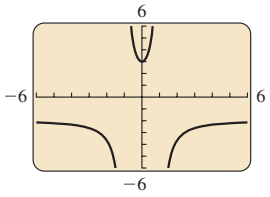
12. $y = x^4 + 4x^3 - 20$

13. $y = -x^4 + 2x^3 + 10x + 15$

14. $y = 0.7x^5 - 2.5x^4 - x^3 + 8x^2 + x + 2$

15. $y = -x^5 + 4x^4 + x^3 - 16x^2 + 12x + 5$

In Exercises 16–20, match the correct graph A–E to the function without using your calculator. Then, after you have answered all of them, if you have a graphing calculator, use your calculator to check your answers. Each graph in this group is plotted on $[-6, 6]$ by $[-6, 6]$. Hint: Consider the asymptotes. (If you try graphing these graphs in Connected mode rather than Dot mode, you will see some lines that are not part of the graph, but the result of the calculator connecting disconnected parts of the graph.)



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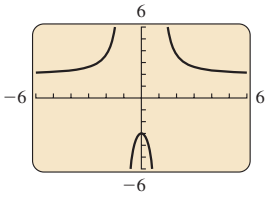

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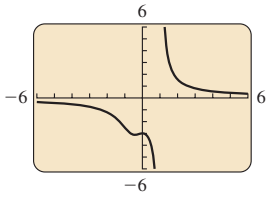

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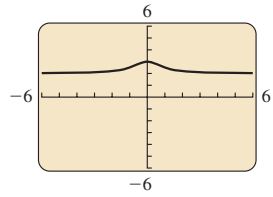

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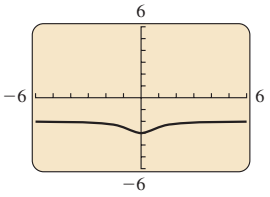


(E)

16. $y = \dfrac{2x^2 + 3}{x^2 - 1}$

17. $y = \dfrac{2x^2 + 3}{x^2 + 1}$

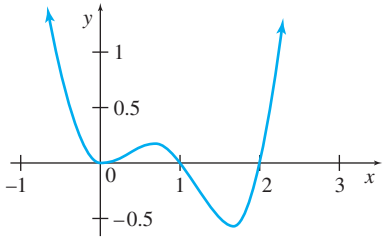
18. $y = \dfrac{-2x^2 - 3}{x^2 - 1}$

19. $y = \dfrac{-2x^2 - 3}{x^2 + 1}$

20. $y = \dfrac{2x^2 + 3}{x^3 - 1}$

Each of the following is the graph of a polynomial function. Give the possible values for the degree of the polynomial, and give the sign (+ or −) for the leading coefficient.

21.



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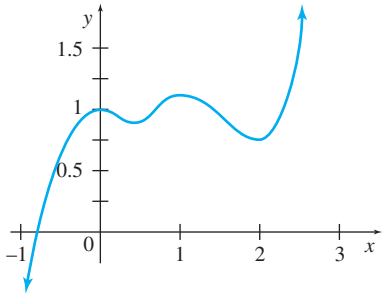

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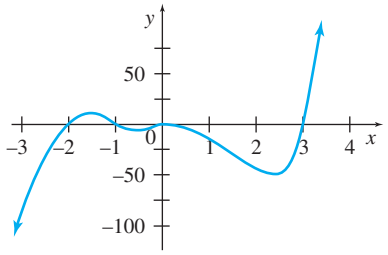

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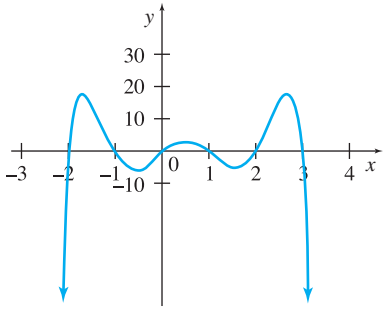

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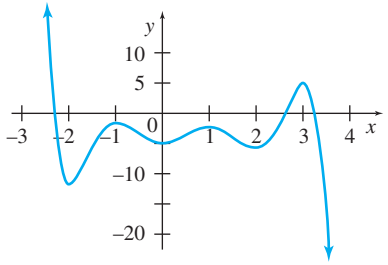

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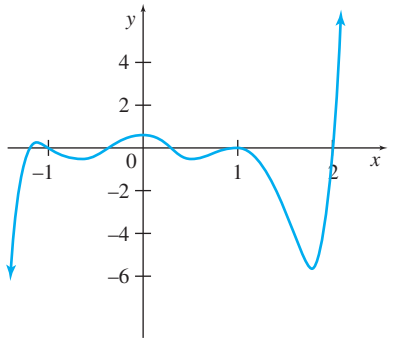


Find any horizontal and vertical asymptotes and any holes that may exist for each rational function. Draw the graph of each function, including any *x*- and *y*-intercepts.

27. $y = \dfrac{-4}{x + 2}$

28. $y = \dfrac{-1}{x + 3}$

29. $y = \dfrac{2}{3 + 2x}$

30. $y = \dfrac{8}{5 - 3x}$

31. $y = \dfrac{2x}{x - 3}$

32. $y = \dfrac{4x}{3 - 2x}$

33. $y = \dfrac{x + 1}{x - 4}$

34. $y = \dfrac{x - 4}{x + 1}$

35. $y = \dfrac{3 - 2x}{4x + 20}$

36. $y = \dfrac{6 - 3x}{4x + 12}$

37. $y = \dfrac{-x - 4}{3x + 6}$

38. $y = \dfrac{-2x + 5}{x + 3}$

39. $y = \dfrac{x^2 + 7x + 12}{x + 4}$

40. $y = \dfrac{9 - 6x + x^2}{3 - x}$

41. Write an equation that defines a rational function with a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.

42. Write an equation that defines a rational function with a vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 0$.

43. Consider the polynomial functions defined by $f(x) = (x - 1)(x - 2)(x + 3)$, $g(x) = x^3 + 2x^2 - x - 2$, and $h(x) = 3x^3 + 6x^2 - 3x - 6$.

a. What is the value of $f(1)$?

b. For what values, other than 1, is $f(x) = 0$?

c. Verify that $g(-1) = g(1) = g(-2) = 0$.

d. Based on your answer from part c, what do you think is the factored form of $g(x)$? Verify your answer by multiplying it out and comparing with $g(x)$.

e. Using your answer from part d, what is the factored form of $h(x)$?

f. Based on what you have learned in this exercise, fill in the blank: If f is a polynomial and $f(a) = 0$ for some number a, then one factor of the polynomial is _____.

44. Consider the function defined by

$$f(x) = \frac{x^7 - 4x^5 - 3x^4 + 4x^3 + 12x^2 - 12}{x^7}.$$

Source: The Mathematics Teacher.

a. Graph the function on $[-6, 6]$ by $[-6, 6]$. From your graph, estimate how many x-intercepts the function has and what their values are.

b. Now graph the function on $[-1.5, -1.4]$ by $[-10^{-4}, 10^{-4}]$ and on $[1.4, 1.5]$ by $[-10^{-5}, 10^{-5}]$. From your graphs, estimate how many x-intercepts the function has and what their values are.

c. From your results in parts a and b, what advice would you give a friend on using a graphing calculator to find x-intercepts?

45. Consider the function defined by

$$f(x) = \frac{1}{x^5 - 2x^3 - 3x^2 + 6}.$$

Source: The Mathematics Teacher.

a. Graph the function on $[-3.4, 3.4]$ by $[-3, 3]$. From your graph, estimate how many vertical asymptotes the function has and where they are located.

b. Now graph the function on $[-1.5, -1.4]$ by $[-10, 10]$ and on $[1.4, 1.5]$ by $[-1000, 1000]$. From your graphs, estimate how many vertical asymptotes the function has and where they are located.

c. From your results in parts a and b, what advice would you give a friend on using a graphing calculator to find vertical asymptotes?

APPLICATIONS

Business and Economics

46. Average Cost Suppose the average cost per unit $\overline{C}(x)$, in dollars, to produce x units of yogurt is given by

$$\overline{C}(x) = \frac{600}{x + 20}.$$

a. Find $\overline{C}(10)$, $\overline{C}(20)$, $\overline{C}(50)$, $\overline{C}(75)$, and $\overline{C}(100)$.

b. Which of the intervals $(0, \infty)$ and $[0, \infty)$ would be a more reasonable domain for $\overline{C}$? Why?

c. Give the equations of any asymptotes. Find any intercepts.

d. Graph $y = \overline{C}(x)$.

47. Cost Analysis In a recent year, the cost per ton, y, to build an oil tanker of x thousand deadweight tons was approximated by

$$\overline{C}(x) = \frac{220{,}000}{x + 475}$$

for $x > 0$.

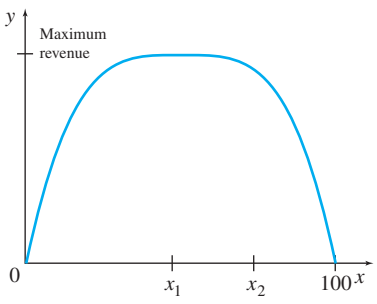
a. Find $\overline{C}(25)$, $\overline{C}(50)$, $\overline{C}(100)$, $\overline{C}(200)$, $\overline{C}(300)$, and $\overline{C}(400)$.

b. Find any asymptotes.

c. Find any intercepts.

d. Graph $y = \overline{C}(x)$.

APPLY IT Tax Rates **Exercises 48–50 refer to the *Laffer curve*, originated by the economist Arthur Laffer. An idealized version of this curve is shown here. According to this curve, decreasing a tax rate, say from x_2 percent to x_1 percent on the graph, can actually lead to an increase in government revenue. The theory is that people will work harder and earn more money if they are taxed at a lower rate, so the government ends up with more revenue than it would at a higher tax rate. All economists agree on the endpoints—0 revenue at tax rates of both 0% and 100%—but there is much disagreement on the location of the tax rate x_1 that produces the maximum revenue.**



Tax rate (percent)

48. A function that might describe the entire Laffer curve is

$$y = x(100 - x)(x^2 + 500),$$

where y is government revenue in hundreds of thousands of dollars from a tax rate of x percent, with the function valid for $0 \le x \le 100$. Find the revenue from the following tax rates.

a. 10% **b.** 40% **c.** 50% **d.** 80%

e. Graph the function.

49. Find the equations of two quadratic functions that could describe the Laffer curve by having zeros at $x = 0$ and $x = 100$. Give the first a maximum of 100 and the second a maximum of 250, then multiply them together to get a new Laffer curve with a maximum of 25,000. Plot the resulting function.

50. An economist might argue that the models in the two previous exercises are unrealistic because they predict that a tax rate of 50% gives the maximum revenue, while the actual value is probably less than 50%. Consider the function

$$y = \frac{300x - 3x^2}{5x + 100},$$

where y is government revenue in millions of dollars from a tax rate of x percent, where $0 \le x \le 100$. ***Source: Dana Lee Ling.***

a. Graph the function, and discuss whether the shape of the graph is appropriate.

b. Use a graphing calculator to find the tax rate that produces the maximum revenue. What is the maximum revenue?

51. Cost-Benefit Model Suppose a cost-benefit model is given by

$$y = \frac{6.7x}{100 - x},$$

where y is the cost in thousands of dollars of removing x percent of a given pollutant.

a. Find the cost of removing each percent of pollutants: 50%; 70%; 80%; 90%; 95%; 98%; 99%.

b. Is it possible, according to this function, to remove *all* the pollutant?

c. Graph the function.

52. Cost-Benefit Model Suppose a cost-benefit model is given by

$$y = \frac{6.5x}{102 - x},$$

where y is the cost in thousands of dollars of removing x percent of a certain pollutant.

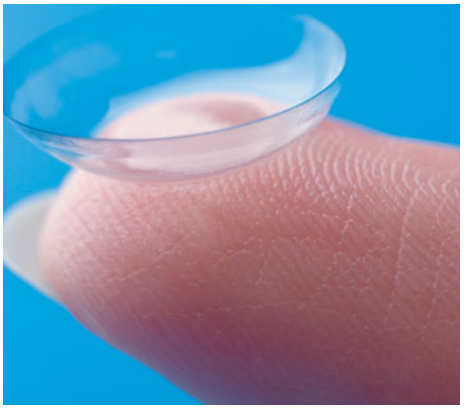
a. Find the cost of removing each percent of pollutants: 0%; 50%; 80%; 90%; 95%; 99%; 100%.

b. Graph the function.

Life Sciences

53. Contact Lenses The strength of a contact lens is given in units known as diopters, as well as in mm of arc. The following is taken from a chart used by optometrists to convert diopters to mm of arc. ***Source: Bausch & Lomb.***

Diopters	mm of Arc
36.000	9.37
36.125	9.34
36.250	9.31
36.375	9.27
36.500	9.24
36.625	9.21
36.750	9.18
36.875	9.15
37.000	9.12



a. Notice that as the diopters increase, the mm of arc decrease. Find a value of k so the function $a = f(d) = k/d$ gives a, the mm of arc, as a function of d, the strength in diopters. (Round k to the nearest integer. For a more accurate answer, average all the values of k given by each pair of data.)

b. An optometrist wants to order 40.50 diopter lenses for a patient. The manufacturer needs to know the strength in mm of arc. What is the strength in mm of arc?

54. Cardiac Output A technique for measuring cardiac output depends on the concentration of a dye after a known amount is injected into a vein near the heart. In a normal heart, the concentration of the dye at time x (in seconds) is given by the function

$$g(x) = -0.006x^4 + 0.140x^3 - 0.053x^2 + 1.79x.$$

a. Graph $g(x)$ on $[0, 6]$ by $[0, 20]$.

b. In your graph from part a, notice that the function initially increases. Considering the form of $g(x)$, do you think it can keep increasing forever? Explain.

c. Write a short paragraph about the extent to which the concentration of dye might be described by the function $g(x)$.

55. Alcohol Concentration The polynomial function

$$A(x) = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$$

gives the approximate blood alcohol concentration in a 170-lb woman x hours after drinking 2 oz of alcohol on an empty stomach, for x in the interval $[0, 5]$. ***Source: Medical Aspects of Alcohol Determination in Biological Specimens.***

a. Graph $A(x)$ on $0 \le x \le 5$.

b. Using the graph from part a, estimate the time of maximum alcohol concentration.

c. In many states, a person is legally drunk if the blood alcohol concentration exceeds 0.08%. Use the graph from part a to estimate the period in which this 170-lb woman is legally drunk.

56. Medical School For the years 1998 to 2009, the number of applicants to U.S. medical schools can be closely approximated by

$$A(t) = -6.7615t^4 + 114.7t^3 - 240.1t^2 - 2129t + 40{,}966,$$

where t is the number of years since 1998. ***Source: Association of American Medical Colleges.***

a. Graph the number of applicants on $0 \le t \le 11$.

b. Based on the graph in part a, during what years did the number of medical school applicants increase?

57. Population Biology The function

$$f(x) = \frac{\lambda x}{1 + (ax)^b}$$

is used in population models to give the size of the next generation $(f(x))$ in terms of the current generation (x). ***Source: Models in Ecology.***

a. What is a reasonable domain for this function, considering what x represents?

b. Graph this function for $\lambda = a = b = 1$.

c. Graph this function for $\lambda = a = 1$ and $b = 2$.

d. What is the effect of making b larger?

58. Growth Model The function

$$f(x) = \frac{Kx}{A + x}$$

is used in biology to give the growth rate of a population in the presence of a quantity x of food. This is called Michaelis-Menten kinetics. ***Source: Mathematical Models in Biology.***

a. What is a reasonable domain for this function, considering what x represents?

b. Graph this function for $K = 5$ and $A = 2$.

c. Show that $y = K$ is a horizontal asymptote.

d. What do you think K represents?

e. Show that A represents the quantity of food for which the growth rate is half of its maximum.

59. Brain Mass The mass (in grams) of the human brain during the last trimester of gestation and the first two years after birth can be approximated by the function

$$m(c) = \frac{c^3}{100} - \frac{1500}{c},$$

where c is the circumference of the head in centimeters. ***Source: Early Human Development.***

a. Find the approximate mass of brains with a head circumference of 30, 40, or 50 cm.

b. Clearly the formula is invalid for any values of c yielding negative values of w. For what values of c is this true?

c. Use a graphing calculator to sketch this graph on the interval $20 \le c \le 50$.

d. Suppose an infant brain has mass of 700 g. Use features on a graphing calculator to find what the circumference of the head is expected to be.

Social Sciences

60. Head Start The enrollment in Head Start for some recent years is included in the table. ***Source: Administration for Children & Families.***

Year	Enrollment
1966	733,000
1970	477,400
1980	376,300
1990	540,930
1995	750,696
2000	857,664
2005	906,993

a. Plot the points from the table using 0 for 1960, and so on.

b. Use the quadratic regression feature of a graphing calculator to get a quadratic function that approximates the data. Graph the function on the same window as the scatterplot.

c. Use cubic regression to get a cubic function that approximates the data. Graph the function on the same window as the scatterplot.

d. Which of the two functions in part b and c appears to be a better fit for the data? Explain your reasoning.

Physical Sciences

61. Length of a Pendulum A simple pendulum swings back and forth in regular time intervals. Grandfather clocks use pendulums to keep accurate time. The relationship between the length of a pendulum L and the period (time) T for one complete oscillation can be expressed by the function $L = kT^n$, where k is a constant and n is a positive integer to be determined. The data below were taken for different lengths of pendulums.* ***Source: Gary Rockswold.***

T (sec)	L (ft)
1.11	1.0
1.36	1.5
1.57	2.0
1.76	2.5
1.92	3.0
2.08	3.5
2.22	4.0

a. Find the value of k for $n = 1$, 2, and 3, using the data for the 4-ft pendulum.

b. Use a graphing calculator to plot the data in the table and to graph the function $L = kT^n$ for the three values of k (and their corresponding values of n) found in part a. Which function best fits the data?

*See Exercise 23, Section 1.3.

c. Use the best-fitting function from part a to predict the period of a pendulum having a length of 5 ft.

d. If the length of pendulum doubles, what happens to the period?

e. If you have a graphing calculator or computer program with a quadratic regression feature, use it to find a quadratic function that approximately fits the data. How does this answer compare with the answer to part b?

62. Coal Consumption The table gives U.S. coal consumption for selected years. ***Source: U.S. Department of Energy.***

Year	Millions of Short Tons
1950	494.1
1960	398.1
1970	523.2
1980	702.7
1985	818.0
1990	902.9
1995	962.1
2000	1084.1
2005	1128.3

a. Draw a scatterplot, letting $x = 0$ represent 1950.

b. Use the quadratic regression feature of a graphing calculator to get a quadratic function that approximates the data.

c. Graph the function from part b on the same window as the scatterplot.

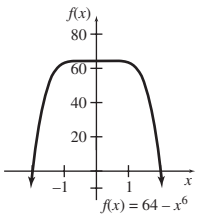
d. Use cubic regression to get a cubic function that approximates the data.

e. Graph the cubic function from part d on the same window as the scatterplot.

f. Which of the two functions in parts b and d appears to be a better fit for the data? Explain your reasoning.

YOUR TURN ANSWERS

1.



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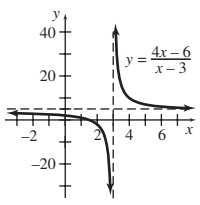


10.4 Exponential Functions

APPLY IT **How much interest will an investment earn? What is the oxygen consumption of yearling salmon?**

Later in this section, in Examples 5 and 6, we will see that the answers to these questions depend on exponential functions.

In earlier sections we discussed functions involving expressions such as x^2, $(2x + 1)^3$, or x^{-1}, where the variable or variable expression is the base of an exponential expression, and the exponent is a constant. In an exponential function, the variable is in the exponent and the base is a constant.

Exponential Function

An **exponential function** with base a is defined as

$$f(x) = a^x, \quad \text{where } a > 0 \text{ and } a \neq 1.$$

(If $a = 1$, the function is the constant function $f(x) = 1$.)

Exponential functions may be the single most important type of functions used in practical applications. They are used to describe growth and decay, which are important ideas in management, social science, and biology.

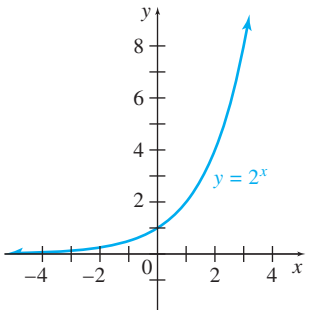


FIGURE 46

Figure 46 shows a graph of the exponential function defined by $f(x) = 2^x$. You could plot such a curve by hand by noting that $2^{-2} = 1/4$, $2^{-1} = 1/2$, $2^0 = 1$, $2^1 = 2$, and $2^2 = 4$, and then drawing a smooth curve through the points $(-2, 1/4)$, $(-1, 1/2)$, $(0, 1)$, $(1, 2)$, and $(2, 4)$. This graph is typical of the graphs of exponential functions of the form $y = a^x$, where $a > 1$. The y-intercept is $(0, 1)$. Notice that as x gets larger and larger, the function also gets larger. As x gets more and more negative, the function becomes smaller and smaller, approaching but never reaching 0. Therefore, the x-axis is a horizontal asymptote, but the function only approaches the left side of the asymptote. In contrast, rational functions approach both the left and right sides of the asymptote. The graph suggests that the domain is the set of all real numbers and the range is the set of all positive numbers.

FOR REVIEW

To review the properties of exponents used in this section, see Section R.6.

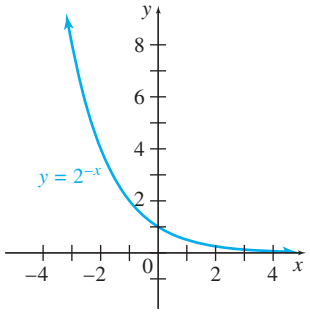


FIGURE 47

EXAMPLE 1 Graphing an Exponential Function

Graph $f(x) = 2^{-x}$.

SOLUTION The graph, shown in Figure 47, is the horizontal reflection of the graph of $f(x) = 2^x$ given in Figure 46. Since $2^{-x} = 1/2^x = (1/2)^x$, this graph is typical of the graphs of exponential functions of the form $y = a^x$ where $0 < a < 1$. The domain includes all real numbers and the range includes all positive numbers. The y-intercept is $(0, 1)$. Notice that this function, with $f(x) = 2^{-x} = (1/2)^x$, is decreasing over its domain.

In the definition of an exponential function, notice that the base a is restricted to positive values, with negative or zero bases not allowed. For example, the function $y = (-4)^x$ could not include such numbers as $x = 1/2$ or $x = 1/4$ in the domain because the y-values would not be real numbers. The resulting graph would be at best a series of separate points having little practical use.

FOR REVIEW

Recall from Section 10.2 that the graph of $f(-x)$ is the reflection of the graph of $f(x)$ about the y-axis.

EXAMPLE 2 Graphing an Exponential Function

Graph $f(x) = -2^x + 3$.

SOLUTION The graph of $y = -2^x$ is the vertical reflection of the graph of $y = 2^x$, so this is a decreasing function. (Notice that -2^x is not the same as $(-2)^x$. In -2^x, we raise 2 to the x power and then take the negative.) The 3 indicates that the graph should be translated

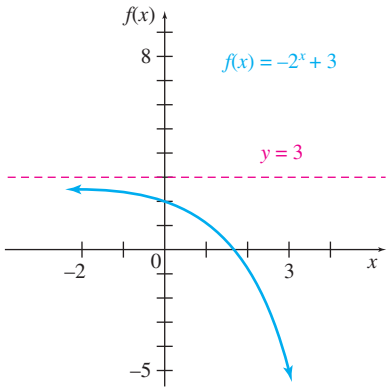


FIGURE 48

vertically 3 units, as compared to the graph of $y = -2^x$. Since $y = -2^x$ would have y-intercept $(0, -1)$, this function has y-intercept $(0, 2)$, which is up 3 units. For negative values of x, the graph approaches the line $y = 3$, which is a horizontal asymptote. The graph is shown in Figure 48.

Exponential Equations

In Figures 46 and 47, which are typical graphs of exponential functions, a given value of x leads to exactly one value of a^x. Because of this, an equation with a variable in the exponent, called an **exponential equation**, often can be solved using the following property.

If $a > 0$, $a \neq 1$, and $a^x = a^y$, then $x = y$.

The value $a = 1$ is excluded, since $1^2 = 1^3$, for example, even though $2 \neq 3$. To solve $2^{3x} = 2^7$ using this property, work as follows.

$$\begin{aligned} 2^{3x} &= 2^7 \\ 3x &= 7 \\ x &= \frac{7}{3} \end{aligned}$$

EXAMPLE 3 Solving Exponential Equations

(a) Solve $9^x = 27$.

FOR REVIEW

Recall from Section R.6 that $(a^m)^n = a^{mn}$.

SOLUTION First rewrite both sides of the equation so the bases are the same. Since $9 = 3^2$ and $27 = 3^3$,

$$\begin{aligned} 9^x &= 27 \\ (3^2)^x &= 3^3 \\ 3^{2x} &= 3^3 \qquad \text{Multiply exponents.} \\ 2x &= 3 \\ x &= \frac{3}{2}. \end{aligned}$$

(b) Solve $32^{2x-1} = 128^{x+3}$.

SOLUTION Since the bases must be the same, write 32 as 2^5 and 128 as 2^7, giving

$$\begin{aligned} 32^{2x-1} &= 128^{x+3} \\ (2^5)^{2x-1} &= (2^7)^{x+3} \\ 2^{10x-5} &= 2^{7x+21}. \qquad \text{Multiply exponents.} \end{aligned}$$

Now use the property from above to get

$$\begin{aligned} 10x - 5 &= 7x + 21 \\ 3x &= 26 \\ x &= \frac{26}{3}. \end{aligned}$$

Verify this solution in the original equation. TRY YOUR TURN 1

YOUR TURN 1 Solve $25^{x/2} = 125^{x+3}$.

Compound Interest

The calculation of compound interest is an important application of exponential functions. The cost of borrowing money or the return on an investment is called **interest**. The amount borrowed or invested is the **principal**, P. The **rate of interest** r is given as a percent per year, and t is the **time,** measured in years.

Simple Interest

The product of the principal P, rate r, and time t gives **simple interest**, I:

$$I = Prt.$$

With **compound interest**, interest is charged (or paid) on interest as well as on the principal. To find a formula for compound interest, first suppose that P dollars, the principal, is deposited at a rate of interest r per year. The interest earned during the first year is found using the formula for simple interest.

$$\text{First-year interest} = P \cdot r \cdot 1 = Pr.$$

At the end of one year, the amount on deposit will be the sum of the original principal and the interest earned, or

$$P + Pr = P(1 + r). \tag{1}$$

If the deposit earns compound interest, the interest earned during the second year is found from the total amount on deposit at the end of the first year. Thus, the interest earned during the second year (again found by the formula for simple interest), is

$$[P(1 + r)](r)(1) = P(1 + r)r, \tag{2}$$

so the total amount on deposit at the end of the second year is the sum of amounts from (1) and (2) above, or

$$P(1 + r) + P(1 + r)r = P(1 + r)(1 + r) = P(1 + r)^2.$$

In the same way, the total amount on deposit at the end of three years is

$$P(1 + r)^3.$$

After t years, the total amount on deposit, called the *compound amount,* is $P(1 + r)^t$.

When interest is compounded more than once a year, the compound interest formula is adjusted. For example, if interest is to be paid quarterly (four times a year), $1/4$ of the interest rate is used each time interest is calculated, so the rate becomes $r/4$, and the number of compounding periods in t years becomes $4t$. Generalizing from this idea gives the following formula.

Compound Amount

If P dollars is invested at a yearly rate of interest r per year, compounded m times per year for t years, the **compound amount** is

$$A = P\left(1 + \frac{r}{m}\right)^{tm} \text{ dollars.}$$

EXAMPLE 4 Compound Interest

Inga Moffitt invests a bonus of $9000 at 6% annual interest compounded semiannually for 4 years. How much interest will she earn?

SOLUTION Use the formula for compound interest with $P = 9000$, $r = 0.06$, $m = 2$, and $t = 4$.

$$\begin{aligned} A &= P\left(1 + \frac{r}{m}\right)^{tm} \\ &= 9000\left(1 + \frac{0.06}{2}\right)^{4(2)} \\ &= 9000(1.03)^8 \\ &\approx 11{,}400.93 \end{aligned}$$

Use a calculator.

The investment plus the interest is $11,400.93. The interest amounts to $11,400.93 − $9000 = $2400.93.

TRY YOUR TURN 2

YOUR TURN 2 Find the interest earned on $4400 at 3.25% interest compounded quarterly for 5 years.

NOTE When using a calculator to compute the compound interest, store each partial result in the calculator and avoid rounding off until the final answer.

The Number *e*

Perhaps the single most useful base for an exponential function is the number ***e***, an irrational number that occurs often in practical applications. The famous Swiss mathematician Leonhard Euler (pronounced "oiler") (1707–1783) was the first person known to have referred to this number as e, and the notation has continued to this day. To see how the number e occurs in an application, begin with the formula for compound interest,

$$P\left(1+\frac{r}{m}\right)^{tm}.$$

Suppose that a lucky investment produces annual interest of 100%, so that $r = 1.00 = 1$. Suppose also that you can deposit only \$1 at this rate, and for only one year. Then $P = 1$ and $t = 1$. Substituting these values into the formula for compound interest gives

$$P\left(1+\frac{r}{m}\right)^{t(m)} = 1\left(1+\frac{1}{m}\right)^{1(m)} = \left(1+\frac{1}{m}\right)^{m}.$$

X	Y1
1	2
8	2.5658
50	2.6916
100	2.7048
1000	2.7169
10000	2.7181
100000	2.7183

X=100000

FIGURE 49

As interest is compounded more and more often, m gets larger and the value of this expression will increase. For example, if $m = 1$ (interest is compounded annually),

$$\left(1+\frac{1}{m}\right)^{m} = \left(1+\frac{1}{1}\right)^{1} = 2^1 = 2,$$

so that your \$1 becomes \$2 in one year. Using a graphing calculator, we produced Figure 49 (where m is represented by X and $(1 + 1/m)^m$ by Y_1) to see what happens as m becomes larger and larger. A spreadsheet can also be used to produce this table.

The table suggests that as m increases, the value of $(1 + 1/m)^m$ gets closer and closer to a fixed number, called e. As we shall see in the next chapter, this is an example of a limit.

Definition of e

As m becomes larger and larger, $\left(1+\frac{1}{m}\right)^{m}$ becomes closer and closer to the number e, whose approximate value is 2.718281828.

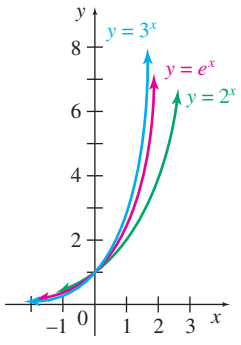


FIGURE 50

The value of e is approximated here to 9 decimal places. Euler approximated e to 18 decimal places. Many calculators give values of e^x, usually with a key labeled e^x. Some require two keys, either INV LN or 2nd LN. (We will define ln x in the next section.) In Figure 50, the functions $y = 2^x$, $y = e^x$, and $y = 3^x$ are graphed for comparison. Notice that e^x is between 2^x and 3^x, because e is between 2 and 3. For $x > 0$, the graphs show that $3^x > e^x > 2^x$. All three functions have y-intercept $(0, 1)$. It is difficult to see from the graph, but $3^x < e^x < 2^x$ when $x < 0$.

The number e is often used as the base in an exponential equation because it provides a good model for many natural, as well as economic, phenomena. In the exercises for this section, we will look at several examples of such applications.

Continuous Compounding

In economics, the formula for **continuous compounding** is a good example of an exponential growth function. Recall the formula for compound amount

$$A = P\left(1+\frac{r}{m}\right)^{tm},$$

where m is the number of times annually that interest is compounded. As m becomes larger and larger, the compound amount also becomes larger but not without bound. Recall that as m becomes larger and larger, $(1 + 1/m)^m$ becomes closer and closer to e. Similarly,

$$\left(1 + \frac{1}{(m/r)}\right)^{m/r}$$

becomes closer and closer to e. Let us rearrange the formula for compound amount to take advantage of this fact.

$$\begin{aligned} A &= P\left(1 + \frac{r}{m}\right)^{tm} \\ &= P\left(1 + \frac{1}{(m/r)}\right)^{tm} \\ &= P\left[\left(1 + \frac{1}{(m/r)}\right)^{m/r}\right]^{rt} \qquad \frac{m}{r} \cdot rt = tm \end{aligned}$$

This last expression becomes closer and closer to Pe^{rt} as m becomes larger and larger, which describes what happens when interest is compounded continuously. Essentially, the number of times annually that interest is compounded becomes infinitely large. We thus have the following formula for the compound amount when interest is compounded continuously.

Continuous Compounding

If a deposit of P dollars is invested at a rate of interest r compounded continuously for t years, the compound amount is

$$A = Pe^{rt} \text{ dollars.}$$

EXAMPLE 5 Continuous Compound Interest

APPLY IT If $600 is invested in an account earning 2.75% compounded continuously, how much would be in the account after 5 years?

SOLUTION In the formula for continuous compounding, let $P = 600$, $t = 5$, and $r = 0.0275$ to get

$$A = 600e^{5(0.0275)} \approx 688.44,$$

or $688.44.

YOUR TURN 3 Find the amount after 4 years if $800 is invested in an account earning 3.15% compounded continuously.

TRY YOUR TURN 3

In situations that involve growth or decay of a population, the size of the population at a given time t often is determined by an exponential function of t. The next example illustrates a typical application of this kind.

EXAMPLE 6 Oxygen Consumption

APPLY IT Biologists studying salmon have found that the oxygen consumption of yearling salmon (in appropriate units) increases exponentially with the speed of swimming according to the function defined by

$$f(x) = 100e^{0.6x},$$

where x is the speed in feet per second. Find the following.

(a) The oxygen consumption when the fish are still

FOR REVIEW

Refer to the discussion on linear regression in Section 1.3. A similar process is used to fit data points to other types of functions. Many of the functions in this chapter's applications were determined in this way, including that given in Example 6.

SOLUTION When the fish are still, their speed is 0. Substitute 0 for x:

$$f(0) = 100e^{(0.6)(0)} = 100e^0$$
$$= 100 \cdot 1 = 100. \qquad e^0 = 1$$

When the fish are still, their oxygen consumption is 100 units.

(b) The oxygen consumption at a speed of 2 ft per second

SOLUTION Find $f(2)$ as follows.

$$f(2) = 100e^{(0.6)(2)} = 100e^{1.2} \approx 332$$

At a speed of 2 ft per second, oxygen consumption is about 332 units rounded to the nearest integer. Because the function is only an approximation of the real situation, further accuracy is not realistic.

EXAMPLE 7 Food Surplus

A magazine article argued that the cause of the obesity epidemic in the United States is the decreasing cost of food (in real terms) due to the increasing surplus of food. ***Source: The New York Times Magazine.*** As one piece of evidence, the following table was provided, which we have updated, showing U.S. corn production (in billions of bushels) for selected years.

Year	Production (billions of bushels)
1930	1.757
1940	2.207
1950	2.764
1960	3.907
1970	4.152
1980	6.639
1990	7.934
2000	9.968
2005	11.112

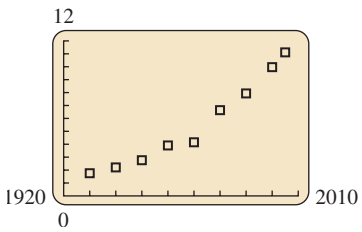


FIGURE 51

(a) Plot the data. Does the production appear to grow linearly or exponentially?

SOLUTION Figure 51 shows a graphing calculator plot of the data, which suggests that corn production is growing exponentially.

(b) Find an exponential function in the form of $p(x) = p_0a^{x-1930}$ that models this data, where x is the year and $p(x)$ is the production of corn. Use the data for 1930 and 2005.

SOLUTION Since $p(1930) = p_0a^0 = p_0$, we have $p_0 = 1.757$. Using $x = 2005$, we have

$$p(2005) = 1.757a^{2005-1930} = 1.757a^{75} = 11.112$$
$$a^{75} = \frac{11.112}{1.757} \qquad \text{Divide by 1.757.}$$
$$a = \left(\frac{11.112}{1.757}\right)^{1/75} \qquad \text{Take the 75th root.}$$
$$\approx 1.0249.$$

Thus $p(x) = 1.757(1.0249)^{x-1930}$. Figure 52 shows that this function fits the data well.

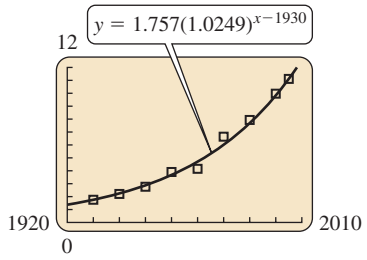


FIGURE 52

(c) Determine the expected annual percentage increase in corn production during this time period.

SOLUTION Since a is 1.0249, the production of corn each year is 1.0249 times its value the previous year, for a rate of increase of $0.0249 = 2.49\%$ per year.

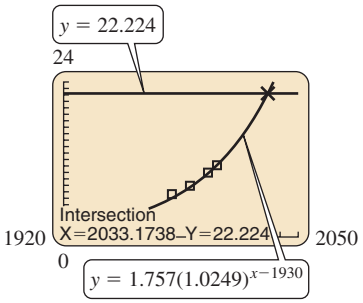


FIGURE 53

(d) Graph p and estimate the year when corn production will be double what it was in 2005.

SOLUTION Figure 53 shows the graphs of $p(x)$ and $y = 2 \cdot 11.112 = 22.224$ on the same coordinate axes. (Note that the scale in Figure 53 is different than the scale in Figures 51 and 52 so that larger values of x and $p(x)$ are visible.) Their graphs intersect at approximately 2033, which is thus the year when corn production will be double its 2005 level. In the next section, we will see another way to solve such problems that does not require the use of a graphing calculator.

Another way to check whether an exponential function fits the data is to see if points whose x-coordinates are equally spaced have y-coordinates with a constant ratio. This must be true for an exponential function because if $f(x) = a \cdot b^x$, then $f(x_1) = a \cdot b^{x_1}$ and $f(x_2) = a \cdot b^{x_2}$, so

$$\frac{f(x_2)}{f(x_1)} = \frac{a \cdot b^{x_2}}{a \cdot b^{x_1}} = b^{x_2 - x_1}.$$

This last expression is constant if $x_2 - x_1$ is constant, that is, if the x-coordinates are equally spaced.

In the previous example, all data points but the last have x-coordinates 10 years apart, so we can compare the ratios of corn production for any of these first pairs of years. Here are the ratios for 1930–1940 and for 1990–2000:

$$\frac{2.207}{1.757} = 1.256$$

$$\frac{9.968}{7.934} = 1.256$$

These ratios are identical to 3 decimal places, so an exponential function fits the data very well. Not all ratios are this close; using the values at 1970 and 1980, we have $6.639/4.152 = 1.599$. From Figure 52, we can see that this is because the 1970 value is below the exponential curve and the 1980 value is above the curve.

TECHNOLOGY NOTE

Another way to find an exponential function that fits a set of data is to use a graphing calculator or computer program with an exponential regression feature. This fits an exponential function through a set of points using the least squares method, introduced in Section 1.3 for fitting a line through a set of points. On a TI-84 Plus, for example, enter the year into the list L_1 and the corn production into L_2. For simplicity, subtract 1930 from each year, so that 1930 corresponds to $x = 0$. Selecting ExpReg from the STAT CALC menu yields $y = 1.728(1.0254)^x$, which is close to the function we found in Example 7(b).

10.4 EXERCISES

A ream of 20-lb paper contains 500 sheets and is about 2 in. high. Suppose you take one sheet, fold it in half, then fold it in half again, continuing in this way as long as possible. *Source: The AMATYC Review*.

1. Complete the table.

Number of Folds	1	2	3	4	5	…	10	…	50
Layers of Paper									

2. After folding 50 times (if this were possible), what would be the height (in miles) of the folded paper?

For Exercises 3–11, match the correct graph A–F to the function without using your calculator. Notice that there are more functions than graphs; some of the functions are equivalent. After you have answered all of them, use a graphing calculator to check your answers. Each graph in this group is plotted on the window $[-2, 2]$ by $[-4, 4]$.

3. $y = 3^x$

4. $y = 3^{-x}$

5. $y = \left(\frac{1}{3}\right)^{1-x}$

6. $y = 3^{x+1}$

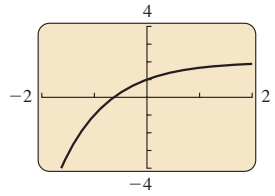
7. $y = 3(3)^x$

8. $y = \left(\frac{1}{3}\right)^x$

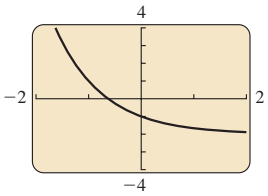
9. $y = 2 - 3^{-x}$

10. $y = -2 + 3^{-x}$

11. $y = 3^{x-1}$



(A)



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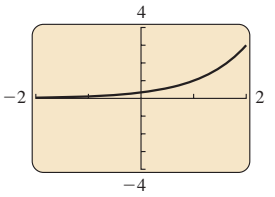

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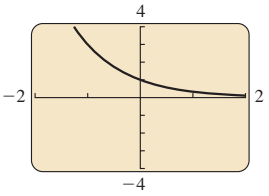

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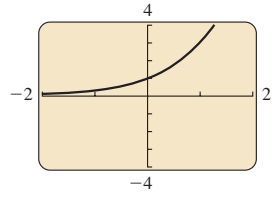
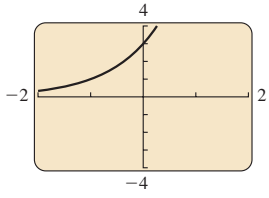


(E)



(F)

12. In Exercises 3–11, there were more formulas for functions than there were graphs. Explain how this is possible.

Solve each equation.

13. $2^x = 32$

14. $4^x = 64$

15. $3^x = \frac{1}{81}$

16. $e^x = \frac{1}{e^5}$

17. $4^x = 8^{x+1}$

18. $25^x = 125^{x+2}$

19. $16^{x+3} = 64^{2x-5}$

20. $(e^3)^{-2x} = e^{-x+5}$

21. $e^{-x} = (e^4)^{x+3}$

22. $2^{|x|} = 8$

23. $5^{-|x|} = \frac{1}{25}$

24. $2^{x^2-4x} = \left(\frac{1}{16}\right)^{x-4}$

25. $5^{x^2+x} = 1$

26. $8^{x^2} = 2^{x+4}$

27. $27^x = 9^{x^2+x}$

28. $e^{x^2+5x+6} = 1$

Graph each of the following.

29. $y = 5e^x + 2$

30. $y = -2e^x - 3$

31. $y = -3e^{-2x} + 2$

32. $y = 4e^{-x/2} - 1$

33. In our definition of exponential function, we ruled out negative values of a. The author of a textbook on mathematical economics, however, obtained a "graph" of $y = (-2)^x$ by plotting the following points and drawing a smooth curve through them.

x	-4	-3	-2	-1	0	1	2	3
y	$1/16$	$-1/8$	$1/4$	$-1/2$	1	-2	4	-8

The graph oscillates very neatly from positive to negative values of y. Comment on this approach. (This exercise shows the dangers of point plotting when drawing graphs.)

34. Explain why the exponential equation $4^x = 6$ cannot be solved using the method described in Example 3.

35. Explain why $3^x > e^x > 2^x$ when $x > 0$, but $3^x < e^x < 2^x$ when $x < 0$.

36. A friend claims that as x becomes large, the expression $1 + 1/x$ gets closer and closer to 1, and 1 raised to any power is still 1. Therefore, $f(x) = (1 + 1/x)^x$ gets closer and closer to 1 as x gets larger. Use a graphing calculator to graph f on $0.1 \le x \le 50$. How might you use this graph to explain to the friend why $f(x)$ does not approach 1 as x becomes large? What does it approach?

APPLICATIONS

Business and Economics

37. Interest Find the interest earned on $10,000 invested for 5 years at 4% interest compounded as follows.

a. Annually **b.** Semiannually (twice a year)

c. Quarterly **d.** Monthly **e.** Continuously

38. Interest Suppose $26,000 is borrowed for 4 years at 6% interest. Find the interest paid over this period if the interest is compounded as follows.

a. Annually **b.** Semiannually

c. Quarterly **d.** Monthly **e.** Continuously

39. Interest Ron Hampton needs to choose between two investments: One pays 6% compounded annually, and the other pays 5.9% compounded monthly. If he plans to invest $18,000 for 2 years, which investment should he choose? How much extra interest will he earn by making the better choice?

40. Interest Find the interest rate required for an investment of $5000 to grow to $7500 in 5 years if interest is compounded as follows.

a. Annually **b.** Quarterly

41. Inflation Assuming continuous compounding, what will it cost to buy a $10 item in 3 years at the following inflation rates?

a. 3% **b.** 4% **c.** 5%

42. Interest Ali Williams invests a $25,000 inheritance in a fund paying 5.5% per year compounded continuously. What will be the amount on deposit after each time period?

a. 1 year **b.** 5 years **c.** 10 years

43. Interest Leigh Jacks plans to invest $500 into a money market account. Find the interest rate that is needed for the money to grow to $1200 in 14 years if the interest is compounded quarterly.

44. Interest Kristi Perez puts $10,500 into an account to save money to buy a car in 12 years. She expects the car of her dreams to cost $30,000 by then. Find the interest rate that is necessary if the interest is computed using the following methods.

a. Compounded quarterly **b.** Compounded continuously

45. Inflation If money loses value at the rate of 8% per year, the value of $1 in t years is given by

$$y = (1 - 0.08)^t = (0.92)^t.$$

a. Use a calculator to help complete the following table.

t	0	1	2	3	4	5	6	7	8	9	10
y	1					0.66					0.43

b. Graph $y = (0.92)^t$.

c. Suppose a house costs $165,000 today. Use the results of part a to estimate the cost of a similar house in 10 years.

d. Find the cost of a $50 textbook in 8 years.

46. Interest On January 1, 2000, Jack deposited $1000 into Bank X to earn interest at the rate of j per annum compounded semiannually. On January 1, 2005, he transferred his account to Bank Y to earn interest at the rate of k per annum compounded quarterly. On January 1, 2008, the balance at Bank Y was $1990.76. If Jack could have earned interest at the rate of k per annum compounded quarterly from January 1, 2000, through January 1, 2008, his balance would have been $2203.76. Which of the following represents the ratio k/j? ***Source: Society of Actuaries.***

a. 1.25 **b.** 1.30 **c.** 1.35 **d.** 1.40 **e.** 1.45

Life Sciences

47. Population Growth Since 1960, the growth in world population (in millions) closely fits the exponential function defined by

$$A(t) = 3100e^{0.0166t},$$

where t is the number of years since 1960. ***Source: United Nations.***

a. World population was about 3686 million in 1970. How closely does the function approximate this value?

b. Use the function to approximate world population in 2000. (The actual 2000 population was about 6115 million.)

c. Estimate world population in the year 2015.

48. Growth of Bacteria Salmonella bacteria, found on almost all chicken and eggs, grow rapidly in a nice warm place. If just a few hundred bacteria are left on the cutting board when a chicken is cut up, and they get into the potato salad, the population begins compounding. Suppose the number present in the potato salad after x hours is given by

$$f(x) = 500 \cdot 2^{3x}.$$

a. If the potato salad is left out on the table, how many bacteria are present 1 hour later?

b. How many were present initially?

c. How often do the bacteria double?

d. How quickly will the number of bacteria increase to 32,000?

49. Minority Population According to the U.S. Census Bureau, the United States is becoming more diverse. Based on U.S. Census population projections for 2000 to 2050, the projected Hispanic population (in millions) can be modeled by the exponential function

$$h(t) = 37.79(1.021)^t,$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. ***Source: U.S. Census Bureau.***

a. Find the projected Hispanic population for 2005. Compare this to the actual value of 42.69 million.

b. The U.S. Asian population is also growing exponentially, and the projected Asian population (in millions) can be modeled by the exponential function

$$a(t) = 11.14(1.023)^t,$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. Find the projected Asian population for 2005, and compare this to the actual value of 12.69 million.

c. Determine the expected annual percentage increase for Hispanics and for Asians. Which minority population, Hispanic or Asian, is growing at a faster rate?

d. The U.S. black population is growing at a linear rate, and the projected black population (in millions) can be modeled by the linear function

$$b(t) = 0.5116t + 35.43,$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. Find the projected black population for 2005 and compare this projection to the actual value of 37.91 million.

e. Graph the projected population function for Hispanics and estimate when the Hispanic population will be double its actual value for 2005. Then do the same for the Asian and black populations. Comment on the accuracy of these numbers.

50. Physician Demand The demand for physicians is expected to increase in the future, as shown in the table on the following page. ***Source: Association of American Medical Colleges.***

Year	Demand for Physicians (in thousands)
2006	680.5
2015	758.6
2020	805.8
2025	859.3

a. Plot the data, letting $t = 0$ correspond to 2000. Does fitting an exponential curve to the data seem reasonable?

b. Use the data for 2006 and 2015 to find a function of the form $f(x) = Ce^{kt}$ that goes through these two points.

c. Use your function from part c to predict the demand for physicians in 2020 and 2025. How well do these predictions fit the data?

d. If you have a graphing calculator or computer program with an exponential regression feature, use it to find an exponential function that approximately fits the data. How does this answer compare with the answer to part b?

Physical Sciences

51. Carbon Dioxide The table gives the estimated global carbon dioxide (CO_2) emissions from fossil-fuel burning, cement production, and gas flaring over the last century. The CO_2 estimates are expressed in millions of metric tons. ***Source: U.S. Department of Energy***.

Year	CO_2 Emissions (millions of metric tons)
1900	534
1910	819
1920	932
1930	1053
1940	1299
1950	1630
1960	2577
1970	4076
1980	5330
1990	6143
2000	6672

a. Plot the data, letting $x = 0$ correspond to 1900. Do the emissions appear to grow linearly or exponentially?

b. Find an exponential function in the form of $f(x) = f_0a^x$ that fits this data at 1900 and 2000, where x is the number of years since 1900 and $f(x)$ is the CO_2 emissions.

c. Approximate the average annual percentage increase in CO_2 emissions during this time period.

d. Graph $f(x)$ and estimate the first year when emissions will be at least double what they were in 2000.

52. Radioactive Decay Suppose the quantity (in grams) of a radioactive substance present at time t is

$$Q(t) = 1000(5^{-0.3t}),$$

where t is measured in months.

a. How much will be present in 6 months?

b. How long will it take to reduce the substance to 8 g?

53. Atmospheric Pressure The atmospheric pressure (in millibars) at a given altitude (in meters) is listed in the table. ***Source: Elements of Meteorology.***

Altitude	Pressure
0	1013
1000	899
2000	795
3000	701
4000	617
5000	541
6000	472
7000	411
8000	357
9000	308
10,000	265

a. Find functions of the form $P = ae^{kx}$, $P = mx + b$, and $P = 1/(ax + b)$ that fit the data at $x = 0$ and $x = 10{,}000$, where P is the pressure and x is the altitude.

b. Plot the data in the table and graph the three functions found in part a. Which function best fits the data?

c. Use the best-fitting function from part b to predict pressure at 1500 m and 11,000 m. Compare your answers to the true values of 846 millibars and 227 millibars, respectively.

d. If you have a graphing calculator or computer program with an exponential regression feature, use it to find an exponential function that approximately fits the data. How does this answer compare with the answer to part b?

54. Computer Chips The power of personal computers has increased dramatically as a result of the ability to place an increasing number of transistors on a single processor chip. The following table lists the number of transistors on some popular computer chips made by Intel. ***Source: Intel.***

Year	Chip	Transistors (in millions)
1985	386	0.275
1989	486	1.2
1993	Pentium	3.1
1997	Pentium II	7.5
1999	Pentium III	9.5
2000	Pentium 4	42
2005	Pentium D	291
2007	Penryn	820
2009	Nehalem	1900

a. Let t be the year, where $t = 0$ corresponds to 1985, and y be the number of transistors (in millions). Find functions of the form $y = mt + b$, $y = at^2 + b$, and $y = ab^t$ that fit the data at 1985 and 2009.

b. Use a graphing calculator to plot the data in the table and to graph the three functions found in part a. Which function best fits the data?

c. Use the best-fitting function from part b to predict the number of transistors on a chip in the year 2015.

d. If you have a graphing calculator or computer program with an exponential regression feature, use it to find an exponential function that approximately fits the data. How does this answer compare with the answer to part b?

e. In 1965 Gordon Moore wrote a paper predicting how the power of computer chips would grow in the future. Moore's law says that the number of transistors that can be put on a chip doubles roughly every 18 months. Discuss the extent to which the data in this exercise confirms or refutes Moore's law.

55. Wind Energy The following table gives the total world wind energy capacity (in megawatts) in recent years. ***Source: World Wind Energy Association.***

Year	Capacity (MW)
2001	24,322
2002	31,181
2003	39,295
2004	47,693
2005	58,024
2006	74,122
2007	93,930
2008	120,903
2009	159,213

a. Let t be the number of years since 2000, and C the capacity (in MW). Find functions of the form $C = mt + b$, $C = at^2 + b$, and $C = ab^t$ that fit the data at 2001 and 2009.

b. Use a graphing calculator to plot the data in the table and to graph the three functions found in part a. Which function best fits the data?

c. If you have a graphing calculator or computer program with an exponential regression feature, use it to find an exponential function that approximately fits the data in the table. How does this answer compare with the answer to part b?

d. Using the three functions from part b and the function from part c, predict the total world wind capacity in 2010. Compare these with the World Wind Energy Association's prediction of 203,500.

YOUR TURN ANSWERS

1. $-9/2$ **2.** \$772.97
3. \$907.43

10.5 Logarithmic Functions

APPLY IT **With an inflation rate averaging 5% per year, how long will it take for prices to double?**

The number of years it will take for prices to double under given conditions is called the **doubling time**. For \$1 to double (become \$2) in t years, assuming 5% annual compounding, means that

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

becomes

$$2 = 1\left(1 + \frac{0.05}{1}\right)^{1(t)}$$

or

$$2 = (1.05)^t.$$

This equation would be easier to solve if the variable were not in the exponent. **Logarithms** are defined for just this purpose. In Example 8, we will use logarithms to answer the question posed above.

Logarithm

For $a > 0$, $a \neq 1$, and $x > 0$,

$$y = \log_a x \quad \text{means} \quad a^y = x.$$

(Read $y = \log_a x$ as "y is the logarithm of x to the base a.") For example, the exponential statement $2^4 = 16$ can be translated into the logarithmic statement $4 = \log_2 16$. Also, in the problem discussed above, $(1.05)^t = 2$ can be rewritten with this definition as $t = \log_{1.05} 2$. A logarithm is an exponent: **$\log_a x$ is the exponent used with the base a to get x.**

EXAMPLE 1 Equivalent Expressions

This example shows the same statements written in both exponential and logarithmic forms.

Exponential Form	*Logarithmic Form*
(a) $3^2 = 9$	$\log_3 9 = 2$
(b) $(1/5)^{-2} = 25$	$\log_{1/5} 25 = -2$
(c) $10^5 = 100{,}000$	$\log_{10} 100{,}000 = 5$
(d) $4^{-3} = 1/64$	$\log_4(1/64) = -3$
(e) $2^{-4} = 1/16$	$\log_2(1/16) = -4$
(f) $e^0 = 1$	$\log_e 1 = 0$

TRY YOUR TURN 1

YOUR TURN 1 Write the equation $5^{-2} = 1/25$ in logarithmic form.

EXAMPLE 2 Evaluating Logarithms

Evaluate each of the following logarithms.

(a) $\log_4 64$

SOLUTION We seek a number x such that $4^x = 64$. Since $4^3 = 64$, we conclude that $\log_4 64 = 3$.

(b) $\log_2(-8)$

SOLUTION We seek a number x such that $2^x = -8$. Since 2^x is positive for all real numbers x, we conclude that $\log_2(-8)$ is undefined. (Actually, $\log_2(-8)$ can be defined if we use complex numbers, but in this textbook, we restrict ourselves to real numbers.)

(c) $\log_5 80$

SOLUTION We know that $5^2 = 25$ and $5^3 = 125$, so $\log_5 25 = 2$ and $\log_5 125 = 3$. Therefore, $\log_5 80$ must be somewhere between 2 and 3. We will find a more accurate answer in Example 4.

TRY YOUR TURN 2

YOUR TURN 2 Evaluate $\log_3(1/81)$.

Logarithmic Functions

For a given positive value of x, the definition of logarithm leads to exactly one value of y, so $y = \log_a x$ defines the *logarithmic function* of base a (the base a must be positive, with $a \neq 1$).

Logarithmic Function

If $a > 0$ and $a \neq 1$, then the **logarithmic function** of base a is defined by

$$f(x) = \log_a x$$

for $x > 0$.

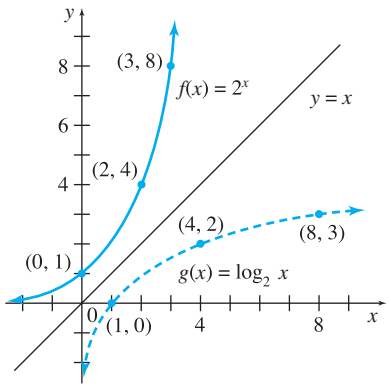


FIGURE 54

The graphs of the exponential function with $f(x) = 2^x$ and the logarithmic function with $g(x) = \log_2 x$ are shown in Figure 54. The graphs show that $f(3) = 2^3 = 8$, while $g(8) = \log_2 8 = 3$. Thus, $f(3) = 8$ and $g(8) = 3$. Also, $f(2) = 4$ and $g(4) = 2$. In fact, for any number m, if $f(m) = p$, then $g(p) = m$. Functions related in this way are called **inverse functions** of each other. The graphs also show that the domain of the exponential function (the set of real numbers) is the range of the logarithmic function. Also, the range of the exponential function (the set of positive real numbers) is the domain of the logarithmic function. Every logarithmic function is the inverse of some exponential function. This means that we can graph logarithmic functions by rewriting them as exponential functions using the definition of logarithm. The graphs in Figure 54 show a characteristic of a pair of inverse functions: their graphs are mirror images about the line $y = x$. Therefore, since exponential functions go through the point (0, 1), logarithmic functions go through the point (1, 0). Notice that because the exponential function has the x-axis as a horizontal asymptote, the logarithmic function has the y-axis as a vertical asymptote. A more complete discussion of inverse functions is given in most standard intermediate algebra and college algebra books.

The graph of $\log_2 x$ is typical of logarithms with bases $a > 1$. When $0 < a < 1$, the graph is the vertical reflection of the logarithm graph in Figure 54. Because logarithms with bases less than 1 are rarely used, we will not explore them here.

CAUTION The domain of $\log_a x$ consists of all $x > 0$. In other words, you cannot take the logarithm of zero or a negative number. This also means that in a function such as $g(x) = \log_a(x - 2)$, the domain is given by $x - 2 > 0$, or $x > 2$.

Properties of Logarithms

The usefulness of logarithmic functions depends in large part on the following **properties of logarithms**.

Properties of Logarithms

Let x and y be any positive real numbers and r be any real number. Let a be a positive real number, $a \neq 1$. Then

a. $\log_a xy = \log_a x + \log_a y$

b. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

c. $\log_a x^r = r \log_a x$

d. $\log_a a = 1$

e. $\log_a 1 = 0$

f. $\log_a a^r = r$.

To prove property (a), let $m = \log_a x$ and $n = \log_a y$. Then, by the definition of logarithm,

$$a^m = x \qquad \text{and} \qquad a^n = y.$$

Hence,

$$a^m a^n = xy.$$

By a property of exponents, $a^m a^n = a^{m+n}$, so

$$a^{m+n} = xy.$$

Now use the definition of logarithm to write

$$\log_a xy = m + n.$$

Since $m = \log_a x$ and $n = \log_a y$,

$$\log_a xy = \log_a x + \log_a y.$$

Proofs of properties (b) and (c) are left for the exercises. Properties (d) and (e) depend on the definition of a logarithm. Property (f) follows from properties (c) and (d).

EXAMPLE 3 Properties of Logarithms

If all the following variable expressions represent positive numbers, then for $a > 0$, $a \neq 1$, the statements in (a)–(c) are true.

(a) $\log_a x + \log_a(x - 1) = \log_a x(x - 1)$

(b) $\log_a \dfrac{x^2 - 4x}{x + 6} = \log_a(x^2 - 4x) - \log_a(x + 6)$

(c) $\log_a(9x^5) = \log_a 9 + \log_a(x^5) = \log_a 9 + 5 \cdot \log_a x$

TRY YOUR TURN 3

YOUR TURN 3 Write the expression $\log_a (x^2/y^3)$ as a sum, difference, or product of simpler logarithms.

Evaluating Logarithms

The invention of logarithms is credited to John Napier (1550–1617), who first called logarithms "artificial numbers." Later he joined the Greek words *logos* (ratio) and *arithmos* (number) to form the word used today. The development of logarithms was motivated by a need for faster computation. Tables of logarithms and slide rule devices were developed by Napier, Henry Briggs (1561–1631), Edmund Gunter (1581–1626), and others.

For many years logarithms were used primarily to assist in involved calculations. Current technology has made this use of logarithms obsolete, but logarithmic functions play an important role in many applications of mathematics. Since our number system has base 10, logarithms to base 10 were most convenient for numerical calculations and so base 10 logarithms were called **common logarithms.** Common logarithms are still useful in other applications. For simplicity,

$$\textbf{log}_{10}\, \boldsymbol{x} \text{ is abbreviated } \textbf{log}\, \boldsymbol{x}.$$

Most practical applications of logarithms use the number e as base. (Recall that to 7 decimal places, $e = 2.7182818$.) Logarithms to base e are called **natural logarithms**, and

$$\textbf{log}_{e}\, \boldsymbol{x} \text{ is abbreviated } \textbf{ln}\, \boldsymbol{x}$$

(read "el-en x"). A graph of $f(x) = \ln x$ is given in Figure 55.

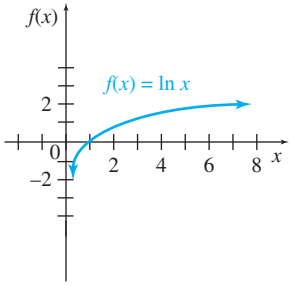


FIGURE 55

NOTE Keep in mind that $\ln x$ is a logarithmic function. Therefore, all of the properties of logarithms given previously are valid when a is replaced with e and $\log_e$ is replaced with ln.

Although common logarithms may seem more "natural" than logarithms to base e, there are several good reasons for using natural logarithms instead. The most important reason is discussed later, in the section on Derivatives of Logarithmic Functions.

A calculator can be used to find both common and natural logarithms. For example, using a calculator and 4 decimal places, we get the following values.

$$\log 2.34 = 0.3692, \quad \log 594 = 2.7738, \quad \text{and} \quad \log 0.0028 = -2.5528.$$
$$\ln 2.34 = 0.8502, \quad \ln 594 = 6.3869, \quad \text{and} \quad \ln 0.0028 = -5.8781.$$

Notice that logarithms of numbers less than 1 are negative when the base is greater than 1. A look at the graph of $y = \log_2 x$ or $y = \ln x$ will show why.

Sometimes it is convenient to use logarithms to bases other than 10 or e. For example, some computer science applications use base 2. In such cases, the following theorem is useful for converting from one base to another.

Change-of-Base Theorem for Logarithms

If x is any positive number and if a and b are positive real numbers, $a \neq 1$, $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

To prove this result, use the definition of logarithm to write $y = \log_a x$ as $x = a^y$ or $x = a^{\log_a x}$ (for positive x and positive a, $a \neq 1$). Now take base b logarithms of both sides of this last equation.

$$\log_b x = \log_b a^{\log_a x}$$
$$\log_b x = (\log_a x)(\log_b a), \qquad \log_a x^r = r \log_a x$$
$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{Solve for } \log_a x.$$

If the base b is equal to e, then by the change-of-base theorem,

$$\log_a x = \frac{\log_e x}{\log_e a}.$$

Using $\ln x$ for $\log_e x$ gives the special case of the theorem using natural logarithms.

For any positive numbers a and x, $a \neq 1$,

$$\log_a x = \frac{\ln x}{\ln a}.$$

The change-of-base theorem for logarithms is useful when graphing $y = \log_a x$ on a graphing calculator for a base a other than e or 10. For example, to graph $y = \log_2 x$, let $y = \ln x / \ln 2$. The change-of-base theorem is also needed when using a calculator to evaluate a logarithm with a base a other than e or 10.

EXAMPLE 4 Evaluating Logarithms

Evaluate $\log_5 80$.

SOLUTION As we saw in Example 2, this number is between 2 and 3. Using the second form of the change-of-base theorem for logarithms with $x = 80$ and $a = 5$ gives

$$\log_5 80 = \frac{\ln 80}{\ln 5} \approx \frac{4.3820}{1.6094} \approx 2.723.$$

To check, use a calculator to verify that $5^{2.723} \approx 80$.

YOUR TURN 4
Evaluate $\log_3 50$.

TRY YOUR TURN 4

CAUTION As mentioned earlier, when using a calculator, do not round off intermediate results. Keep all numbers in the calculator until you have the final answer. In Example 4, we showed the rounded intermediate values of ln 80 and ln 5, but we used the unrounded quantities when doing the division.

Logarithmic Equations

Equations involving logarithms are often solved by using the fact that exponential functions and logarithmic functions are inverses, so a logarithmic equation can be rewritten (with the definition of logarithm) as an exponential equation. In other cases, the properties of logarithms may be useful in simplifying a **logarithmic equation**.

EXAMPLE 5 Solving Logarithmic Equations

Solve each equation.

(a) $\log_x \frac{8}{27} = 3$

SOLUTION Using the definition of logarithm, write the expression in exponential form. To solve for x, take the cube root on both sides.

$$x^3 = \frac{8}{27}$$
$$x = \frac{2}{3}$$

(b) $\log_4 x = \frac{5}{2}$

SOLUTION In exponential form, the given statement becomes

$$4^{5/2} = x$$
$$(4^{1/2})^5 = x$$
$$2^5 = x$$
$$32 = x.$$

(c) $\log_2 x - \log_2(x-1) = 1$

SOLUTION By a property of logarithms,

$$\log_2 x - \log_2(x-1) = \log_2 \frac{x}{x-1},$$

so the original equation becomes

$$\log_2 \frac{x}{x-1} = 1.$$

Now write this equation in exponential form, and solve.

$$\frac{x}{x-1} = 2^1 = 2$$

Solve this equation.

$$\frac{x}{x-1}(x-1) = 2(x-1) \qquad \text{Multiply both sides by } x-1.$$
$$x = 2(x-1)$$
$$x = 2x - 2$$
$$-x = -2$$
$$x = 2$$

(d) $\log x + \log(x-3) = 1$

SOLUTION Similar to part (c), we have

$$\log x + \log(x-3) = \log[x(x-3)] = 1.$$

Since the logarithm base is 10, this means that

$$x(x-3) = 10 \qquad 10^1 = 10$$
$$x^2 - 3x - 10 = 0 \qquad \text{Subtract 10 from both sides.}$$
$$(x-5)(x+2) = 0. \qquad \text{Factor.}$$

This leads to two solutions: $x = 5$ and $x = -2$. But notice that -2 is not a valid value for x in the original equation, since the logarithm of a negative number is undefined. The only solution is, therefore, $x = 5$.

YOUR TURN 5 Solve for x: $\log_2 x + \log_2(x+2) = 3$.

TRY YOUR TURN 5

CAUTION It is important to check solutions when solving equations involving logarithms because $\log_a u$, where u is an expression in x, has domain given by $u > 0$.

Exponential Equations

In the previous section exponential equations like $(1/3)^x = 81$ were solved by writing each side of the equation as a power of 3. That method cannot be used to solve an equation such as $3^x = 5$, however, since 5 cannot easily be written as a power of 3. Such equations can be solved approximately with a graphing calculator, but an algebraic method is also useful, particularly when the equation involves variables such as a and b rather than just numbers such as 3 and 5. A general method for solving these equations is shown in the following example.

EXAMPLE 6 Solving Exponential Equations

Solve each equation.

(a) $3^x = 5$

SOLUTION Taking natural logarithms (logarithms to any base could be used) on both sides gives

$$\begin{aligned} \ln 3^x &= \ln 5 \\ x \ln 3 &= \ln 5 && \ln u^r = r \ln u \\ x &= \frac{\ln 5}{\ln 3} \approx 1.465 \end{aligned}$$

(b) $3^{2x} = 4^{x+1}$

SOLUTION Taking natural logarithms on both sides gives

$$\begin{aligned} \ln 3^{2x} &= \ln 4^{x+1} \\ 2x \ln 3 &= (x + 1) \ln 4 && \ln u^r = r \ln u \\ (2 \ln 3)x &= (\ln 4)x + \ln 4 \\ (2 \ln 3)x - (\ln 4)x &= \ln 4 && \text{Subtract } (\ln 4)x \text{ from both sides.} \\ (2 \ln 3 - \ln 4)x &= \ln 4 && \text{Factor } x. \\ x &= \frac{\ln 4}{2 \ln 3 - \ln 4}. && \text{Divide both sides by } 2 \ln 3 - \ln 4. \end{aligned}$$

Use a calculator to evaluate the logarithms, then divide, to get

$$x \approx \frac{1.3863}{2(1.0986) - 1.3863} \approx 1.710.$$

(c) $5e^{0.01x} = 9$

SOLUTION

$$\begin{aligned} e^{0.01x} &= \frac{9}{5} = 1.8 && \text{Divide both sides by 5.} \\ \ln e^{0.01x} &= \ln 1.8 && \text{Take natural logarithms on both sides.} \\ 0.01x &= \ln 1.8 && \ln e^u = u \\ x &= \frac{\ln 1.8}{0.01} \approx 58.779 \end{aligned}$$

TRY YOUR TURN 6

YOUR TURN 6 Solve for x: $2^{x+1} = 3^x$.

Just as $\log_a x$ can be written as a base e logarithm, any exponential function $y = a^x$ can be written as an exponential function with base e. For example, there exists a real number k such that

$$2 = e^k.$$

Raising both sides to the power x gives

$$2^x = e^{kx},$$

so that powers of 2 can be found by evaluating appropriate powers of e. To find the necessary number k, solve the equation $2 = e^k$ for k by first taking logarithms on both sides.

$$\begin{aligned} 2 &= e^k \\ \ln 2 &= \ln e^k \\ \ln 2 &= k \ln e \\ \ln 2 &= k \qquad \ln e = 1 \end{aligned}$$

Thus, $k = \ln 2$. In the section on Derivatives of Exponential Functions, we will see why this change of base is useful. A general statement can be drawn from this example.

Change-of-Base Theorem for Exponentials

For every positive real number a,

$$a^x = e^{(\ln a)x}.$$

Another way to see why the change-of-base theorem for exponentials is true is to first observe that $e^{\ln a} = a$. Combining this with the fact that $e^{ab} = (e^a)^b$, we have $e^{(\ln a)x} = (e^{\ln a})^x = a^x$.

EXAMPLE 7 Change-of-Base-Theorem

(a) Write 7^x using base e rather than base 7.

SOLUTION According to the change-of-base theorem,

$$7^x = e^{(\ln 7)x}.$$

Using a calculator to evaluate ln 7, we could also approximate this as $e^{1.9459x}$.

(b) Approximate the function $f(x) = e^{2x}$ as $f(x) = a^x$ for some base a.

SOLUTION We do not need the change-of-base theorem here. Just use the fact that

$$e^{2x} = (e^2)^x \approx 7.389^x,$$

where we have used a calculator to approximate e^2.

TRY YOUR TURN 7

YOUR TURN 7 Approximate $e^{0.025x}$ in the form a^x.

EXAMPLE 8 Doubling Time

APPLY IT

Complete the solution of the problem posed at the beginning of this section.

SOLUTION Recall that if prices will double after t years at an inflation rate of 5%, compounded annually, t is given by the equation

$$2 = (1.05)^t.$$

We solve this equation by first taking natural logarithms on both sides.

$$\begin{aligned} \ln 2 &= \ln(1.05)^t \\ \ln 2 &= t \ln 1.05 \qquad \ln x^r = r \ln x \\ t &= \frac{\ln 2}{\ln 1.05} \approx 14.2 \end{aligned}$$

It will take about 14 years for prices to double.

The problem solved in Example 8 can be generalized for the compound interest equation

$$A = P(1 + r)^t.$$

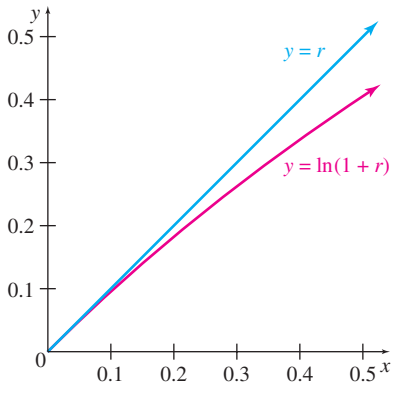


FIGURE 56

Solving for t as in Example 8 (with $A = 2$ and $P = 1$) gives the doubling time in years as

$$t = \frac{\ln 2}{\ln(1 + r)}.$$

If r is small, $\ln(1 + r) \approx r$, as Figure 56 shows, so that

$$t = \frac{\ln 2}{\ln(1 + r)} \approx \frac{\ln 2}{r} \approx \frac{0.693}{r}.$$

Notice from Figure 56 that the actual value of r is larger than $\ln(1 + r)$, causing the above formula to give a doubling time that is too small. By changing 0.693 to 0.70, the formula becomes reasonably accurate for $0.001 \le r < 0.05$. By increasing the numerator further to 0.72, the formula becomes reasonably accurate for $0.05 \le r < 0.12$. This leads to two useful rules for estimating the doubling time. The **rule of 70** says that for $0.001 \le r < 0.05$, the value of $70/100r$ gives a good approximation of t. The **rule of 72** says that for $0.05 \le r \le 0.12$, the value of $72/100r$ approximates t quite well.

Figure 57 shows the three functions $y = \ln 2/\ln(1 + r)$, $y = 70/100r$, and $y = 72/100r$ graphed on the same axes. Observe how close to each other they are. In an exercise we will ask you to explore the relationship between these functions further.

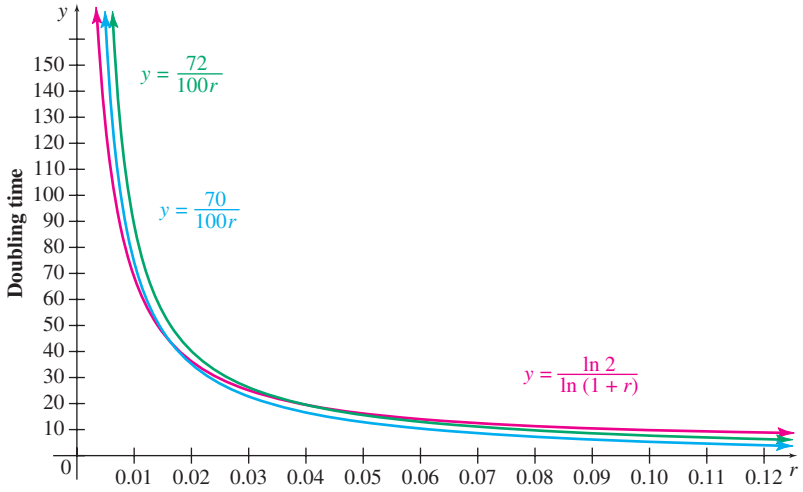


FIGURE 57

EXAMPLE 9 Rules of 70 and 72

Approximate the years to double at an interest rate of 6% using first the rule of 70, then the rule of 72.

SOLUTION By the rule of 70, money will double at 6% interest after

$$\frac{70}{100r} = \frac{70}{100(0.06)} = \frac{70}{6} = 11.67\left(\text{or } 11\frac{2}{3}\right)$$

years.

Using the rule of 72 gives

$$\frac{72}{100r} = \frac{72}{6} = 12$$

years doubling time. Since a more precise answer is given by

$$\frac{\ln 2}{\ln(1 + r)} = \frac{\ln 2}{\ln(1.06)} \approx \frac{0.693}{0.058} \approx 11.9,$$

the rule of 72 gives a better approximation than the rule of 70. This agrees with the statement that the rule of 72 works well for values of r where $0.05 \le r \le 0.12$, since $r = 0.06$ falls into this category.

EXAMPLE 10 Index of Diversity

One measure of the diversity of the species in an ecological community is given by the **index of diversity** H, where

$$H = -[P_1 \ln P_1 + P_2 \ln P_2 + \cdots + P_n \ln P_n],$$

and $P_1, P_2, \ldots, P_n$ are the proportions of a sample belonging to each of n species found in the sample. ***Source: Statistical Ecology.*** For example, in a community with two species, where there are 90 of one species and 10 of the other, $P_1 = 90/100 = 0.9$ and $P_2 = 10/100 = 0.1$, with

$$H = -[0.9 \ln 0.9 + 0.1 \ln 0.1] \approx 0.325.$$

Verify that $H \approx 0.673$ if there are 60 of one species and 40 of the other. As the proportions of n species get closer to $1/n$ each, the index of diversity increases to a maximum of $\ln n$.

10.5 EXERCISES

Write each exponential equation in logarithmic form.

1. $5^3 = 125$ **2.** $7^2 = 49$

3. $3^4 = 81$ **4.** $2^7 = 128$

5. $3^{-2} = \frac{1}{9}$ **6.** $\left(\frac{5}{4}\right)^{-2} = \frac{16}{25}$

Write each logarithmic equation in exponential form.

7. $\log_2 32 = 5$ **8.** $\log_3 81 = 4$

9. $\ln \frac{1}{e} = -1$ **10.** $\log_2 \frac{1}{8} = -3$

11. $\log 100{,}000 = 5$ **12.** $\log 0.001 = -3$

Evaluate each logarithm without using a calculator.

13. $\log_8 64$ **14.** $\log_9 81$

15. $\log_4 64$ **16.** $\log_3 27$

17. $\log_2 \frac{1}{16}$ **18.** $\log_3 \frac{1}{81}$

19. $\log_2 \sqrt[3]{\frac{1}{4}}$ **20.** $\log_8 \sqrt[4]{\frac{1}{2}}$

21. $\ln e$ **22.** $\ln e^3$

23. $\ln e^{5/3}$ **24.** $\ln 1$

25. Is the "logarithm to the base 3 of 4" written as $\log_4 3$ or $\log_3 4$?

26. Write a few sentences describing the relationship between e^x and $\ln x$.

Use the properties of logarithms to write each expression as a sum, difference, or product of simpler logarithms. For example, $\log_2(\sqrt{3}x) = \frac{1}{2}\log_2 3 + \log_2 x$.

27. $\log_5(3k)$ **28.** $\log_9(4m)$

29. $\log_3 \frac{3p}{5k}$ **30.** $\log_7 \frac{15p}{7y}$

31. $\ln \frac{3\sqrt{5}}{\sqrt[3]{6}}$ **32.** $\ln \frac{9\sqrt[3]{5}}{\sqrt[4]{3}}$

Suppose $\log_b 2 = a$ **and** $\log_b 3 = c$. **Use the properties of logarithms to find the following.**

33. $\log_b 32$ **34.** $\log_b 18$

35. $\log_b(72b)$ **36.** $\log_b(9b^2)$

Use natural logarithms to evaluate each logarithm to the nearest thousandth.

37. $\log_5 30$ **38.** $\log_{12} 210$

39. $\log_{1.2} 0.95$ **40.** $\log_{2.8} 0.12$

Solve each equation in Exercises 41–64. Round decimal answers to four decimal places.

41. $\log_x 36 = -2$ **42.** $\log_9 27 = m$

43. $\log_8 16 = z$ **44.** $\log_y 8 = \frac{3}{4}$

45. $\log_r 5 = \frac{1}{2}$ **46.** $\log_4(5x + 1) = 2$

47. $\log_5(9x - 4) = 1$

48. $\log_4 x - \log_4(x + 3) = -1$

49. $\log_9 m - \log_9(m - 4) = -2$

50. $\log(x + 5) + \log(x + 2) = 1$

51. $\log_3(x - 2) + \log_3(x + 6) = 2$

52. $\log_3(x^2 + 17) - \log_3(x + 5) = 1$

53. $\log_2(x^2 - 1) - \log_2(x + 1) = 2$

54. $\ln(5x + 4) = 2$

55. $\ln x + \ln 3x = -1$ **56.** $\ln(x + 1) - \ln x = 1$

57. $2^x = 6$ **58.** $5^x = 12$

59. $e^{k-1} = 6$ **60.** $e^{2y} = 15$

61. $3^{x+1} = 5^x$ **62.** $2^{x+1} = 6^{x-1}$

63. $5(0.10)^x = 4(0.12)^x$ **64.** $1.5(1.05)^x = 2(1.01)^x$

Write each expression using base e rather than base 10.

65. 10^{x+1} **66.** 10^{x^2}

Approximate each expression in the form a^x without using e.

67. e^{3x} **68.** e^{-4x}

Find the domain of each function.

69. $f(x) = \log(5 - x)$ **70.** $f(x) = \ln(x^2 - 9)$

71. Lucky Larry was faced with solving

$$\log(2x + 1) - \log(3x - 1) = 0.$$

Larry just dropped the logs and proceeded:

$$(2x + 1) - (3x - 1) = 0$$
$$-x + 2 = 0$$
$$x = 2.$$

Although Lucky Larry is wrong in dropping the logs, his procedure will always give the correct answer to an equation of the form

$$\log A - \log B = 0,$$

where A and B are any two expressions in x. Prove that this last equation leads to the equation $A - B = 0$, which is what you get when you drop the logs. ***Source: The AMATYC Review.***

72. Find all errors in the following calculation.

$$(\log(x + 2))^2 = 2\log(x + 2)$$
$$= 2(\log x + \log 2)$$
$$= 2(\log x + 100)$$

73. Prove: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

74. Prove: $\log_a x^r = r \log_a x$.

APPLICATIONS

Business and Economics

75. Inflation Assuming annual compounding, find the time it would take for the general level of prices in the economy to double at the following annual inflation rates.

a. 3% **b.** 6% **c.** 8%

d. Check your answers using either the rule of 70 or the rule of 72, whichever applies.

76. Interest Mary Klingman invests $15,000 in an account paying 7% per year compounded annually.

a. How many years are required for the compound amount to at least double? (Note that interest is only paid at the end of each year.)

b. In how many years will the amount at least triple?

c. Check your answer to part a using either the rule of 70 or the rule of 72, whichever applies.

77. Interest Leigh Jacks plans to invest $500 into a money market account. Find the interest rate that is needed for the money to grow to $1200 in 14 years if the interest is compounded continuously. (Compare with Exercise 43 in the previous section.)

78. Rule of 72 Complete the following table, and use the results to discuss when the rule of 70 gives a better approximation for the doubling time, and when the rule of 72 gives a better approximation.

r	0.001	0.02	0.05	0.08	0.12
$(\ln 2)/\ln(1 + r)$					
$70/100r$					
$72/100r$					

79. Pay Increases You are offered two jobs starting July 1, 2013. Humongous Enterprises offers you $45,000 a year to start, with a raise of 4% every July 1. At Crabapple Inc. you start at $30,000, with an annual increase of 6% every July 1. On July 1 of what year would the job at Crabapple Inc. pay more than the job at Humongous Enterprises? Use the algebra of logarithms to solve this problem, and support your answer by using a graphing calculator to see where the two salary functions intersect.

Life Sciences

80. Insect Species An article in *Science* stated that the number of insect species of a given mass is proportional to $m^{-0.6}$, where m is the mass in grams. ***Source: Science.*** A graph accompanying the article shows the common logarithm of the mass on the horizontal axis and the common logarithm of the number of species on the vertical axis. Explain why the graph is a straight line. What is the slope of the line?

Index of Diversity **For Exercises 81–83, refer to Example 10.**

81. Suppose a sample of a small community shows two species with 50 individuals each.

a. Find the index of diversity H.

b. What is the maximum value of the index of diversity for two species?

c. Does your answer for part a equal ln 2? Explain why.

82. A virgin forest in northwestern Pennsylvania has 4 species of large trees with the following proportions of each: hemlock, 0.521; beech, 0.324; birch, 0.081; maple, 0.074. Find the index of diversity H.

83. Find the value of the index of diversity for populations with n species and $1/n$ of each if

a. $n = 3$; **b.** $n = 4$.

c. Verify that your answers for parts a and b equal ln 3 and ln 4, respectively.

84. Allometric Growth The allometric formula is used to describe a wide variety of growth patterns. It says that $y = nx^m$, where x and y are variables, and n and m are constants. For example, the famous biologist J. S. Huxley used this formula to relate the weight of the large claw of the fiddler crab to the weight of the body without the claw. ***Source: Problems of Relative Growth.*** Show that if x and y are given by the allometric formula, then

$X = \log_b x$, $Y = \log_b y$, and $N = \log_b n$ are related by the linear equation

$$Y = mX + N.$$

85. Drug Concentration When a pharmaceutical drug is injected into the bloodstream, its concentration at time t can be approximated by $C(t) = C_0 e^{-kt}$, where C_0 is the concentration at $t = 0$. Suppose the drug is ineffective below a concentration C_1 and harmful above a concentration C_2. Then it can be shown that the drug should be given at intervals of time T, where

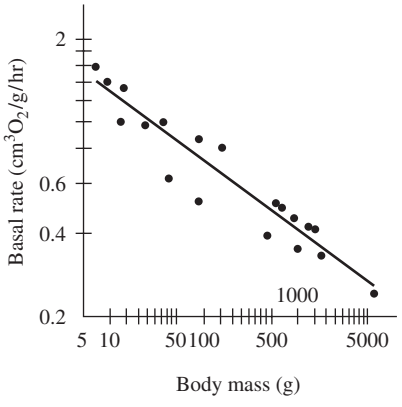
$$T = \frac{1}{k} \ln \frac{C_2}{C_1}.$$

Source: Applications of Calculus to Medicine.

A certain drug is harmful at a concentration five times the concentration below which it is ineffective. At noon an injection of the drug results in a concentration of 2 mg per liter of blood. Three hours later the concentration is down to 1 mg per liter. How often should the drug be given?

The graph for Exercise 86 is plotted on a logarithmic scale where differences between successive measurements are not always the same. Data that do not plot in a linear pattern on the usual Cartesian axes often form a linear pattern when plotted on a logarithmic scale. Notice that on the horizontal scale, the distance from 5 to 10 is not the same as the distance from 10 to 15, and so on. This is characteristic of a graph drawn on logarithmic scales.

86. Metabolism Rate The accompanying graph shows the basal metabolism rate (in cm^3 of oxygen per gram per hour) for marsupial carnivores, which include the Tasmanian devil. This rate is inversely proportional to body mass raised to the power 0.25. ***Source: The Quarterly Review of Biology.***



a. Estimate the metabolism rate for a marsupial carnivore with body mass of 10 g. Do the same for one with body mass of 1000 g.

b. Verify that if the relationship between x and y is of the form $y = ax^b$, then there will be a linear relationship between $\ln x$ and $\ln y$. (*Hint:* Apply ln to both sides of $y = ax^b$.)

c. If a function of the form $y = ax^b$ contains the points (x_1, y_1) and (x_2, y_2), then values for a and b can be found by dividing $y_1 = ax_1^b$ by $y_2 = ax_2^b$, solving the resulting equation for b, and putting the result back into either equation to solve for a. Use this procedure and the results from part a to find an equation of the form $y = ax^b$ that gives the basal metabolism rate as a function of body mass.

d. Use the result of part c to predict the basal metabolism rate of a marsupial carnivore whose body mass is 100 g.

87. Minority Population The U.S. Census Bureau has reported that the United States is becoming more diverse. In Exercise 49 of the previous section, the projected Hispanic population (in millions) was modeled by the exponential function

$$h(t) = 37.79\,(1.021)^t$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. ***Source: U.S. Census Bureau.***

a. Estimate in what year the Hispanic population will double the 2005 population of 42.69 million. Use the algebra of logarithms to solve this problem.

b. The projected U.S. Asian population (in millions) was modeled by the exponential function

$$h\,(t) = 11.14\,(1.023)^t,$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. Estimate in what year the Asian population will double the 2005 population of 12.69 million.

Social Sciences

88. Evolution of Languages The number of years $N(r)$ since two independently evolving languages split off from a common ancestral language is approximated by

$$N(r) = -5000\ \ln r,$$

where r is the proportion of the words from the ancestral language that are common to both languages now. Find the following.

a. $N\,(0.9)$ **b.** $N\,(0.5)$ **c.** $N\,(0.3)$

d. How many years have elapsed since the split if 70% of the words of the ancestral language are common to both languages today?

e. If two languages split off from a common ancestral language about 1000 years ago, find r.

Physical Sciences

89. Communications Channel According to the Shannon-Hartley theorem, the capacity of a communications channel in bits per second is given by

$$C = B \log_2\left(\frac{s}{n} + 1\right),$$

where B is the frequency bandwidth of the channel in hertz and s/n is its signal-to-noise ratio. ***Source: Scientific American.*** It is physically impossible to exceed this limit. Solve the equation for the signal-to-noise ratio s/n.

For Exercises 90–93, recall that log x represents the common (base 10) logarithm of x.

90. Intensity of Sound The loudness of sounds is measured in a unit called a *decibel*. To do this, a very faint sound, called the *threshold sound*, is assigned an intensity I_0. If a particular sound has intensity I, then the decibel rating of this louder sound is

$$10 \log \frac{I}{I_0}.$$

Find the decibel ratings of the following sounds having intensities as given. Round answers to the nearest whole number.

a. Whisper, $115I_0$

b. Busy street, $9{,}500{,}000I_0$

c. Heavy truck, 20 m away, $1{,}200{,}000{,}000I_0$

d. Rock music concert, $895{,}000{,}000{,}000I_0$

e. Jetliner at takeoff, $109{,}000{,}000{,}000{,}000I_0$

f. In a noise ordinance instituted in Stamford, Connecticut, the threshold sound I_0 was defined as 0.0002 microbars. ***Source: The New York Times.*** Use this definition to express the sound levels in parts c and d in microbars.

91. Intensity of Sound A story on the National Public Radio program *All Things Considered*, discussed a proposal to lower the noise limit in Austin, Texas, from 85 decibels to 75 decibels. A manager for a restaurant was quoted as saying, "If you cut from 85 to 75, . . . you're basically cutting the sound down in half." Is this correct? If not, to what fraction of its original level is the sound being cut? ***Source: National Public Radio.***

92. Earthquake Intensity The magnitude of an earthquake, measured on the Richter scale, is given by

$$R(I) = \log \frac{I}{I_0},$$

where I is the amplitude registered on a seismograph located 100 km from the epicenter of the earthquake, and I_0 is the amplitude of a certain small size earthquake. Find the Richter scale ratings of earthquakes with the following amplitudes.

a. $1{,}000{,}000I_0$ **b.** $100{,}000{,}000I_0$

c. On June 15, 1999, the city of Puebla in central Mexico was shaken by an earthquake that measured 6.7 on the Richter scale. Express this reading in terms of I_0. ***Source: Exploring Colonial Mexico.***

d. On September 19, 1985, Mexico's largest recent earthquake, measuring 8.1 on the Richter scale, killed about 10,000 people. Express the magnitude of an 8.1 reading in terms of I_0. ***Source: History.com.***

e. Compare your answers to parts c and d. How much greater was the force of the 1985 earthquake than the 1999 earthquake?

f. The relationship between the energy E of an earthquake and the magnitude on the Richter scale is given by

$$R(E) = \frac{2}{3} \log\left(\frac{E}{E_0}\right),$$

where E_0 is the energy of a certain small earthquake. Compare the energies of the 1999 and 1985 earthquakes.

g. According to a newspaper article, "Scientists say such an earthquake of magnitude 7.5 could release 15 times as much energy as the magnitude 6.7 trembler that struck the Northridge section of Los Angeles" in 1994. ***Source: The New York Times.*** Using the formula from part f, verify this quote by computing the magnitude of an earthquake with 15 times the energy of a magnitude 6.7 earthquake.

93. Acidity of a Solution A common measure for the acidity of a solution is its pH. It is defined by $\text{pH} = -\log[H^+]$, where H^+ measures the concentration of hydrogen ions in the solution. The pH of pure water is 7. Solutions that are more acidic than pure water have a lower pH, while solutions that are less acidic (referred to as basic solutions) have a higher pH.

a. Acid rain sometimes has a pH as low as 4. How much greater is the concentration of hydrogen ions in such rain than in pure water?

b. A typical mixture of laundry soap and water for washing clothes has a pH of about 11, while black coffee has a pH of about 5. How much greater is the concentration of hydrogen ions in black coffee than in the laundry mixture?

94. Music Theory A music theorist associates the fundamental frequency of a pitch f with a real number defined by

$$p = 69 + 12 \log_2 (f/440).$$

Source: Science.

a. Standard concert pitch for an A is 440 cycles per second. Find the associated value of p.

b. An A one octave higher than standard concert pitch is 880 cycles per second. Find the associated value of p.

YOUR TURN ANSWERS

1. $\log_5 (1/25) = -2$ **2.** -4

3. $2 \log_a x - 3 \log_a y$ **4.** 3.561

5. 2 **6.** $(\ln 2)/\ln(3/2) \approx 1.7095$.

7. 1.0253^x

10.6 Applications: Growth and Decay; Mathematics of Finance

APPLY IT **What interest rate will cause $5000 to grow to $7250 in 6 years if money is compounded continuously?**

This question, which will be answered in Example 7, is one of many situations that occur in biology, economics, and the social sciences, in which a quantity changes at a rate proportional to the amount of the quantity present.

In cases such as continuous compounding described above, the amount present at time t is a function of t, called the **exponential growth and decay function**. (The derivation of this equation is presented in a later section on Differential Equations.)

Exponential Growth and Decay Function

Let y_0 be the amount or number of some quantity present at time $t = 0$. The quantity is said to grow or decay exponentially if for some constant k, the amount present at time t is given by

$$y = y_0 e^{kt}.$$

If $k > 0$, then k is called the **growth constant**; if $k < 0$, then k is called the **decay constant**. A common example is the growth of bacteria in a culture. The more bacteria present, the faster the population increases.

EXAMPLE 1 Yeast Production

Yeast in a sugar solution is growing at a rate such that 1 g becomes 1.5 g after 20 hours. Find the growth function, assuming exponential growth.

SOLUTION The values of y_0 and k in the exponential growth function $y = y_0 e^{kt}$ must be found. Since y_0 is the amount present at time $t = 0$, $y_0 = 1$. To find k, substitute $y = 1.5$, $t = 20$, and $y_0 = 1$ into the equation.

$$y = y_0 e^{kt}$$

$$1.5 = 1e^{k(20)}$$

Now take natural logarithms on both sides and use the power rule for logarithms and the fact that $\ln e = 1$.

$$1.5 = e^{20k}$$

$$\ln 1.5 = \ln e^{20k} \qquad \text{Take ln of both sides.}$$

$$\ln 1.5 = 20k \qquad \ln e^x = x$$

$$\frac{\ln 1.5}{20} = k \qquad \text{Divide both sides by 20.}$$

$$k \approx 0.02 \text{ (to the nearest hundredth)}$$

The exponential growth function is $y = e^{0.02t}$, where y is the number of grams of yeast present after t hours.

TRY YOUR TURN 1

YOUR TURN 1 Find the growth function if 5 g grows exponentially to 18 g after 16 hours.

The decline of a population or decay of a substance may also be described by the exponential growth function. In this case the decay constant k is negative, since an increase in time leads to a decrease in the quantity present. Radioactive substances provide a good example of exponential decay. By definition, the **half-life** of a radioactive substance is the time it takes for exactly half of the initial quantity to decay.

EXAMPLE 2 Carbon Dating

Carbon-14 is a radioactive form of carbon that is found in all living plants and animals. After a plant or animal dies, the carbon-14 disintegrates. Scientists determine the age of the remains by comparing its carbon-14 with the amount found in living plants and animals. The amount of carbon-14 present after t years is given by the exponential equation

$$A(t) = A_0 e^{kt},$$

with $k = -[(\ln 2)/5600]$.

(a) Find the half-life of carbon-14.

SOLUTION Let $A(t) = (1/2)A_0$ and $k = -[(\ln 2)/5600]$.

$$\frac{1}{2}A_0 = A_0 e^{-[(\ln 2)/5600]t}$$

$$\frac{1}{2} = e^{-[(\ln 2)/5600]t} \qquad \text{Divide by } A_0.$$

$$\ln \frac{1}{2} = \ln e^{-[(\ln 2)/5600]t} \qquad \text{Take ln of both sides.}$$

$$\ln \frac{1}{2} = -\frac{\ln 2}{5600}t \qquad \ln e^x = x$$

$$-\frac{5600}{\ln 2}\ln \frac{1}{2} = t \qquad \text{Multiply by } -\frac{5600}{\ln 2}.$$

$$-\frac{5600}{\ln 2}(\ln 1 - \ln 2) = t \qquad \ln \tfrac{x}{y} = \ln x - \ln y$$

$$-\frac{5600}{\ln 2}(-\ln 2) = t \qquad \ln 1 = 0$$

$$5600 = t$$

The half-life is 5600 years.

(b) Charcoal from an ancient fire pit on Java had $1/4$ the amount of carbon-14 found in a living sample of wood of the same size. Estimate the age of the charcoal.

SOLUTION Let $A(t) = (1/4)A_0$ and $k = -[(\ln 2)/5600]$.

$$\frac{1}{4}A_0 = A_0 e^{-[(\ln 2)/5600]t}$$

$$\frac{1}{4} = e^{-[(\ln 2)/5600]t}$$

$$\ln \frac{1}{4} = \ln e^{-[(\ln 2)/5600]t}$$

$$\ln \frac{1}{4} = -\frac{\ln 2}{5600}t$$

$$-\frac{5600}{\ln 2}\ln \frac{1}{4} = t$$

$$t = 11{,}200$$

The charcoal is about 11,200 years old.

YOUR TURN 2 Estimate the age of a sample with 1/10 the amount of carbon-14 as a live sample.

TRY YOUR TURN 2

By following the steps in Example 2, we get the general equation giving the half-life T in terms of the decay constant k as

$$T = -\frac{\ln 2}{k}.$$

For example, the decay constant for potassium-40, where t is in billions of years, is approximately -0.5545 so its half-life is

$$T = -\frac{\ln 2}{(-0.5545)} \approx 1.25 \text{ billion years.}$$

We can rewrite the growth and decay function as

$$y = y_0 e^{kt} = y_0 (e^k)^t = y_0 a^t,$$

where $a = e^k$. This is sometimes a helpful way to look at an exponential growth or decay function.

EXAMPLE 3 Radioactive Decay

Rewrite the function for radioactive decay of carbon-14 in the form $A(t) = A_0 a^{f(t)}$.

SOLUTION From the previous example, we have

$$\begin{aligned} A(t) = A_0 e^{kt} &= A_0 e^{-[(\ln 2)/5600]t} \\ &= A_0 (e^{\ln 2})^{-t/5600} \\ &= A_0 2^{-t/5600} = A_0 (2^{-1})^{t/5600} = A_0 \left(\frac{1}{2}\right)^{t/5600}. \end{aligned}$$

This last expression shows clearly that every time t increases by 5600 years, the amount of carbon-14 decreases by a factor of $1/2$.

Effective Rate

We could use a calculator to see that \$1 at 8% interest (per year) compounded semiannually is $1(1.04)^2 = 1.0816$ or \$1.0816. The actual increase of \$0.0816 is 8.16% rather than the 8% that would be earned with interest compounded annually. To distinguish between these two amounts, 8% (the annual interest rate) is called the **nominal** or **stated** interest rate, and 8.16% is called the **effective** interest rate. We will continue to use r to designate the stated rate and we will use r_E for the effective rate.

Effective Rate for Compound Interest

If r is the annual stated rate of interest and m is the number of compounding periods per year, the effective rate of interest is

$$r_E = \left(1 + \frac{r}{m}\right)^m - 1.$$

Effective rate is sometimes called *annual yield*.

With continuous compounding, \$1 at 8% for 1 year becomes $(1)e^{1(0.08)} = e^{0.08} \approx 1.0833$.

The increase is 8.33% rather than 8%, so a stated interest rate of 8% produces an effective rate of 8.33%.

Effective Rate for Continuous Compounding

If interest is compounded continuously at an annual stated rate of r, the effective rate of interest is

$$r_E = e^r - 1.$$

EXAMPLE 4 Effective Rate

Find the effective rate corresponding to each stated rate.

(a) 6% compounded quarterly

SOLUTION Using the formula, we get

$$\left(1 + \frac{0.06}{4}\right)^4 - 1 = (1.015)^4 - 1 \approx 0.0614.$$

The effective rate is 6.14%.

(b) 6% compounded continuously

SOLUTION The formula for continuous compounding gives

$$e^{0.06} - 1 \approx 0.0618,$$

so the effective rate is 6.18%. TRY YOUR TURN 3

YOUR TURN 3 Find the effective rate corresponding to each stated rate. **(a)** 4.25% compounded monthly **(b)** 3.75% compounded continuously.

The formula for interest compounded m times a year, $A = P(1 + r/m)^{tm}$, has five variables: A, P, r, m, and t. If the values of any four are known, then the value of the fifth can be found.

EXAMPLE 5 Interest

Meghan Moreau has received a bonus of $25,000. She invests it in an account earning 7.2% compounded quarterly. Find how long it will take for her $25,000 investment to grow to $40,000.

SOLUTION Here $P = \$25{,}000$, $r = 0.072$, and $m = 4$. We also know the amount she wishes to end up with, $A = \$40{,}000$. Substitute these values into the compound interest formula and solve for time, t.

$$\begin{aligned}
40{,}000 &= 25{,}000\left(1 + \frac{0.072}{4}\right)^{4t} \\
40{,}000 &= 25{,}000(1.018)^{4t} \\
1.6 &= 1.018^{4t} && \text{Divide both sides by 25,000.} \\
\ln 1.6 &= \ln(1.018)^{4t} && \text{Take ln of both sides.} \\
\ln 1.6 &= 4t \cdot \ln 1.018 \\
t &= \frac{\ln 1.6}{4 \ln 1.018} \approx 6.586 && \text{Divide both sides by 4 ln 1.018.}
\end{aligned}$$

Note that the interest is calculated quarterly and is added only at the *end* of each quarter. Therefore, we need to round up to the nearest quarter. She will have $40,000 in 6.75 years. TRY YOUR TURN 4

YOUR TURN 4 Find the time needed for $30,000 to grow to $50,000 when invested in an account that pays 3.15% compounded quarterly.

CAUTION When calculating the time it takes for an investment to grow, take into account that interest is added only at the *end* of each compounding period. In Example 5, interest is added quarterly. At the end of the second quarter of the sixth year ($t = 6.5$), she will have only $39,754.13, but at the end of the third quarter of that year ($t = 6.75$), she will have $40,469.70.

If A, the amount of money we wish to end up with, is given as well as r, m, and t, then P can be found using the formula for compounded interest. Here P is the amount that should be deposited today to produce A dollars in t years. The amount P is called the **present value** of A dollars.

EXAMPLE 6 Present Value

Tom Shaffer has a balloon payment of $100,000 due in 3 years. What is the present value of that amount if the money earns interest at 4% annually?

SOLUTION Here P in the compound interest formula is unknown, with $A = 100{,}000$, $r = 0.04$, $t = 3$, and $m = 1$. Substitute the known values into the formula to get $100{,}000 = P(1.04)^3$. Solve for P, using a calculator to find $(1.04)^3$.

$$P = \frac{100{,}000}{(1.04)^3} \approx 88{,}889.64$$

The present value of $100,000 in 3 years at 4% per year is $88,889.64.

In general, to find the present value for an interest rate r compounded m times per year for t years, solve the equation

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

for the variable P. To find the present value for an interest rate r compounded continuously for t years, solve the equation

$$A = Pe^{rt}$$

for the variable P.

EXAMPLE 7 Continuous Compound Interest

Find the interest rate that will cause $5000 to grow to $7250 in 6 years if the money is compounded continuously.

APPLY IT

SOLUTION Use the formula for continuous compounding, $A = Pe^{rt}$, with $A = 7250$, $P = 5000$, and $t = 6$. Solve first for e^{rt}, then for r.

$$\begin{aligned} A &= Pe^{rt} \\ 7250 &= 5000e^{6r} \\ 1.45 &= e^{6r} && \text{Divide by 5000.} \\ \ln 1.45 &= \ln e^{6r} && \text{Take ln of both sides.} \\ \ln 1.45 &= 6r && \ln e^x = x \\ r &= \frac{\ln 1.45}{6} \\ r &\approx 0.0619 \end{aligned}$$

The required interest rate is 6.19%.

TRY YOUR TURN 5

YOUR TURN 5 Find the interest rate that will cause $3200 to grow to $4500 in 7 years if the money is compounded continuously.

Limited Growth Functions

The exponential growth functions discussed so far all continued to grow without bound. More realistically, many populations grow exponentially for a while, but then the growth is slowed by some external constraint that eventually limits the growth. For example, an animal population may grow to the point where its habitat can no longer support the population and the growth rate begins to dwindle until a stable population size is reached. Models that reflect this pattern are called **limited growth functions.** The next example discusses a function of this type that occurs in industry.

EXAMPLE 8 Employee Turnover

Assembly-line operations tend to have a high turnover of employees, forcing companies to spend much time and effort in training new workers. It has been found that a worker new to a task on the line will produce items according to the function defined by

$$P(x) = 25 - 25e^{-0.3x},$$

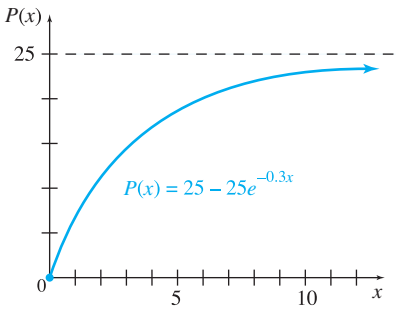


FIGURE 58

where $P(x)$ items are produced by the worker on day x.

(a) What happens to the number of items a worker can produce as x gets larger and larger?

SOLUTION As x gets larger, $e^{-0.3x}$ becomes closer to 0, so $P(x)$ approaches 25. This represents the limit on the number of items a worker can produce in a day. Note that this limit represents a horizontal asymptote on the graph of P, shown in Figure 58.

(b) How many days will it take for a new worker to produce at least 20 items in a day?

SOLUTION Let $P(x) = 20$ and solve for x.

$$\begin{aligned} P(x) &= 25 - 25e^{-0.3x} \\ 20 &= 25 - 25e^{-0.3x} \\ -5 &= -25e^{-0.3x} \\ 0.2 &= e^{-0.3x} \end{aligned}$$

Now take natural logarithms of both sides and use properties of logarithms.

$$\begin{aligned} \ln 0.2 &= \ln e^{-0.3x} \\ \ln 0.2 &= -0.3x \qquad \ln e^u = u \\ x &= \frac{\ln 0.2}{-0.3} \approx 5.4 \end{aligned}$$

This means that 5 days are not quite enough; on the fifth day, a new worker produces $P(5) = 25 - 25e^{-0.3(5)} \approx 19.4$ items. It takes 6 days, and on the sixth day, a new worker produces $P(6) = 25 - 25e^{-0.3(5)} \approx 20.9$ items.

Graphs such as the one in Figure 58 are called **learning curves**. According to such a graph, a new worker tends to learn quickly at first; then learning tapers off and approaches some upper limit. This is characteristic of the learning of certain types of skills involving the repetitive performance of the same task.

10.6 EXERCISES

1. What is the difference between stated interest rate and effective interest rate?
2. In the exponential growth or decay function $y = y_0e^{kt}$, what does y_0 represent? What does k represent?
3. In the exponential growth or decay function, explain the circumstances that cause k to be positive or negative.
4. What is meant by the half-life of a quantity?
5. Show that if a radioactive substance has a half-life of T, then the corresponding constant k in the exponential decay function is given by $k = -(\ln 2)/T$.
6. Show that if a radioactive substance has a half-life of T, then the corresponding exponential decay function can be written as $y = y_0(1/2)^{(t/T)}$.

APPLICATIONS

Business and Economics

Effective Rate **Find the effective rate corresponding to each nominal rate of interest.**

7. 4% compounded quarterly
8. 6% compounded monthly
9. 8% compounded continuously
10. 5% compounded continuously

Present Value **Find the present value of each amount.**

11. $10,000 if interest is 6% compounded quarterly for 8 years
12. $45,678.93 if interest is 7.2% compounded monthly for 11 months

13. $7300 if interest is 5% compounded continuously for 3 years

14. $25,000 if interest is 4.6% compounded continuously for 8 years

15. **Effective Rate** Tami Dreyfus bought a television set with money borrowed from the bank at 9% interest compounded semiannually. What effective interest rate did she pay?

16. **Effective Rate** A firm deposits some funds in a special account at 6.2% compounded quarterly. What effective rate will they earn?

17. **Effective Rate** Robin Kim deposits $7500 of lottery winnings in an account paying 6% interest compounded monthly. What effective rate does the account earn?

18. **Present Value** Frank Steed must make a balloon payment of $20,000 in 4 years. Find the present value of the payment if it includes annual interest of 6.5% compounded monthly.

19. **Present Value** A company must pay a $307,000 settlement in 3 years.

    a. What amount must be deposited now at 6% compounded semiannually to have enough money for the settlement?

    b. How much interest will be earned?

    c. Suppose the company can deposit only $200,000 now. How much more will be needed in 3 years?

    d. Suppose the company can deposit $200,000 now in an account that pays interest continuously. What interest rate would they need to accumulate the entire $307,000 in 3 years?

20. **Present Value** A couple wants to have $40,000 in 5 years for a down payment on a new house.

    a. How much should they deposit today, at 6.4% compounded quarterly, to have the required amount in 5 years?

    b. How much interest will be earned?

    c. If they can deposit only $20,000 now, how much more will they need to complete the $40,000 after 5 years?

    d. Suppose they can deposit $20,000 now in an account that pays interest continuously. What interest rate would they need to accumulate the entire $40,000 in 5 years?

21. **Interest** Christine O'Brien, who is self-employed, wants to invest $60,000 in a pension plan. One investment offers 8% compounded quarterly. Another offers 7.75% compounded continuously.

    a. Which investment will earn the most interest in 5 years?

    b. How much more will the better plan earn?

    c. What is the effective rate in each case?

    d. If Ms. O'Brien chooses the plan with continuous compounding, how long will it take for her $60,000 to grow to $80,000?

    e. How long will it take for her $60,000 to grow to at least $80,000 if she chooses the plan with quarterly compounding? (Be careful; interest is added to the account only every quarter. See Example 5.)

22. **Interest** Greg Tobin wishes to invest a $5000 bonus check into a savings account that pays 6.3% interest. Find how many years it will take for the $5000 to grow to at least $11,000 if interest is compounded

    a. quarterly. (Be careful; interest is added to the account only every quarter. See Example 5.)

    b. continuously.

23. **Sales** Sales of a new model of compact disc player are approximated by the function $S(x) = 1000 - 800e^{-x}$, where $S(x)$ is in appropriate units and x represents the number of years the disc player has been on the market.

    a. Find the sales during year 0.

    b. In how many years will sales reach 500 units?

    c. Will sales ever reach 1000 units?

    d. Is there a limit on sales for this product? If so, what is it?

24. **Sales** Sales of a new model of digital camera are approximated by

$$S(x) = 5000 - 4000e^{-x},$$

where x represents the number of years that the digital camera has been on the market, and $S(x)$ represents sales in thousands of dollars.

    a. Find the sales in year 0.

    b. When will sales reach $4,500,000?

    c. Find the limit on sales.

Life Sciences

25. **Population Growth** The population of the world in the year 1650 was about 500 million, and in the year 2010 was 6756 million. ***Source: U.S. Census Bureau.***

    a. Assuming that the population of the world grows exponentially, find the equation for the population $P(t)$ in millions in the year t.

    b. Use your answer from part a to find the population of the world in the year 1.

    c. Is your answer to part b reasonable? What does this tell you about how the population of the world grows?

26. **Giardia** When a person swallows giardia cysts, stomach acids and pancreatic enzymes cause the cysts to release trophozoites, which divide every 12 hours. ***Source: The New York Times.***

    a. Suppose the number of trophozoites at time $t = 0$ is y_0. Write a function in the form $y = y_0e^{kt}$ giving the number after t hours.

    b. Write the function from part a in the form $y = y_0 2^{f(t)}$.

    c. The article cited above said that a single trophozoite can multiply to a million in just 10 days and a billion in 15 days. Verify this fact.

27. **Growth of Bacteria** A culture contains 25,000 bacteria, with the population increasing exponentially. The culture contains 40,000 bacteria after 10 hours.

    a. Write a function in the form $y = y_0e^{kt}$ giving the number of bacteria after t hours.

    b. Write the function from part a in the form $y = y_0a^t$.

    c. How long will it be until there are 60,000 bacteria?

28. **Decrease in Bacteria** When an antibiotic is introduced into a culture of 50,000 bacteria, the number of bacteria decreases exponentially. After 9 hours, there are only 20,000 bacteria.

a. Write an exponential equation to express the growth function y in terms of time t in hours.

b. In how many hours will half the number of bacteria remain?

29. Growth of Bacteria The growth of bacteria in food products makes it necessary to time-date some products (such as milk) so that they will be sold and consumed before the bacteria count is too high. Suppose for a certain product that the number of bacteria present is given by

$$f(t) = 500e^{0.1t},$$

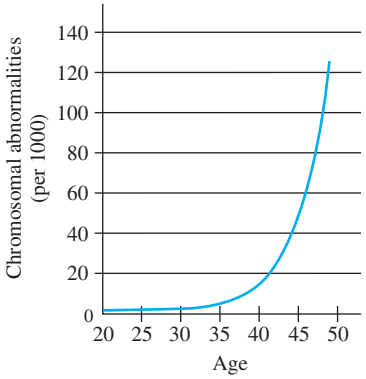
under certain storage conditions, where t is time in days after packing of the product and the value of $f(t)$ is in millions.

a. If the product cannot be safely eaten after the bacteria count reaches 3000 million, how long will this take?

b. If $t = 0$ corresponds to January 1, what date should be placed on the product?

30. Cancer Research An article on cancer treatment contains the following statement: A 37% 5-year survival rate for women with ovarian cancer yields an estimated annual mortality rate of 0.1989. The authors of this article assume that the number of survivors is described by the exponential decay function given at the beginning of this section, where y is the number of survivors and k is the mortality rate. Verify that the given survival rate leads to the given mortality rate. ***Source: American Journal of Obstetrics and Gynecology.***

31. Chromosomal Abnormality The graph below shows how the risk of chromosomal abnormality in a child rises with the age of the mother. ***Source: downsyndrome.about.com.***



a. Read from the graph the risk of chromosomal abnormality (per 1000) at ages 20, 35, 42, and 49.

b. Assuming the graph to be of the form $y = Ce^{kt}$, find k using $t = 20$ and $t = 35$.

c. Still assuming the graph to be of the form $y = Ce^{kt}$, find k using $t = 42$ and $t = 49$.

d. Based on your results from parts a–c, is it reasonable to assume the graph is of the form $y = Ce^{kt}$? Explain.

e. In situations such as parts a–c, where an exponential function does not fit because different data points give different values for the growth constant k, it is often appropriate to describe the data using an equation of the form $y = Ce^{kt^n}$. Parts b and c show that $n = 1$ results in a smaller constant using the interval $[20, 35]$ than using the interval $[42, 49]$. Repeat parts b and c using $n = 2, 3$, etc., until the interval $[20, 35]$ yields a larger value of k than the interval $[42, 49]$, and then estimate what n should be.

Physical Sciences

32. Carbon Dating Refer to Example 2. A sample from a refuse deposit near the Strait of Magellan had 60% of the carbon-14 found in a contemporary living sample. How old was the sample?

Half-Life Find the half-life of each radioactive substance. See Example 2.

33. Plutonium-241; $A(t) = A_0e^{-0.053t}$

34. Radium-226; $A(t) = A_0e^{-0.00043t}$

35. Half-Life The half-life of plutonium-241 is approximately 13 years.

a. How much of a sample weighing 4 g will remain after 100 years?

b. How much time is necessary for a sample weighing 4 g to decay to 0.1 g?

36. Half-Life The half-life of radium-226 is approximately 1620 years.

a. How much of a sample weighing 4 g will remain after 100 years?

b. How much time is necessary for a sample weighing 4 g to decay to 0.1 g?

37. Radioactive Decay 500 g of iodine-131 is decaying exponentially. After 3 days 386 g of iodine-131 is left.

a. Write a function in the form $y = y_0e^{kt}$ giving the number of grams of iodine-131 after t days.

b. Write the function from part a in the form $y = y_0(386/500)^{f(t)}$.

c. Use your answer from part a to find the half-life of iodine-131.

38. Radioactive Decay 25 g of polonium-210 is decaying exponentially. After 50 days 19.5 g of polonium-210 is left.

a. Write a function in the form $y = y_0e^{kt}$ giving the number of grams of polonium-210 after t days.

b. Write the function from part a in the form $y = y_0a^{t/50}$.

c. Use your answer from part a to find the half-life of polonium-210.

39. Nuclear Energy Nuclear energy derived from radioactive isotopes can be used to supply power to space vehicles. The output of the radioactive power supply for a certain satellite is given by the function $y = 40e^{-0.004t}$, where y is in watts and t is the time in days.

a. How much power will be available at the end of 180 days?

b. How long will it take for the amount of power to be half of its original strength?

c. Will the power ever be completely gone? Explain.

40. Botany A group of Tasmanian botanists have claimed that a King's holly shrub, the only one of its species in the world, is also the oldest living plant. Using carbon-14 dating of charcoal found along with fossilized leaf fragments, they arrived at an age of 43,000 years for the plant, whose exact location in southwest Tasmania is being kept a secret. What percent of the original carbon-14 in the charcoal was present? ***Source: Science.***

41. Decay of Radioactivity A large cloud of radioactive debris from a nuclear explosion has floated over the Pacific Northwest, contaminating much of the hay supply. Consequently, farmers in the area are concerned that the cows who eat this hay will give contaminated milk. (The tolerance level for radioactive iodine in milk is 0.) The percent of the initial amount of radioactive iodine still present in the hay after t days is approximated by $P(t)$, which is given by the mathematical model

$$P(t) = 100e^{-0.1t}.$$

a. Find the percent remaining after 4 days.

b. Find the percent remaining after 10 days.

c. Some scientists feel that the hay is safe after the percent of radioactive iodine has declined to 10% of the original amount. Solve the equation $10 = 100e^{-0.1t}$ to find the number of days before the hay may be used.

d. Other scientists believe that the hay is not safe until the level of radioactive iodine has declined to only 1% of the original level. Find the number of days that this would take.

42. Chemical Dissolution The amount of chemical that will dissolve in a solution increases exponentially as the temperature is increased. At 0°C, 10 g of the chemical dissolves, and at 10°C, 11 g dissolves.

a. Write an equation to express the amount of chemical dissolved, y, in terms of temperature, t, in degrees Celsius.

b. At what temperature will 15 g dissolve?

Newton's Law of Cooling **Newton's law of cooling says that the rate at which a body cools is proportional to the difference in temperature between the body and an environment into which it is introduced. This leads to an equation where the temperature $f(t)$ of the body at time t after being introduced into an environment having constant temperature T_0 is**

$$f(t) = T_0 + Ce^{-kt},$$

where C and k are constants. Use this result in Exercises 43–45.

43. Find the temperature of an object when $t = 9$ if $T_0 = 18$, $C = 5$, and $k = 0.6$.

44. If $C = 100$, $k = 0.1$, and t is time in minutes, how long will it take a hot cup of coffee to cool to a temperature of 25°C in a room at 20°C?

45. If $C = -14.6$ and $k = 0.6$ and t is time in hours, how long will it take a frozen pizza to thaw to 10°C in a room at 18°C?

YOUR TURN ANSWERS

1. $y = 5e^{0.08t}$ **2.** 18,600 years old

3. **(a)** 4.33% **(b)** 3.82% **4.** 16.5 years

5. 4.87%

10 CHAPTER REVIEW

SUMMARY

In this chapter we defined functions and studied some of their properties. In particular, we studied several families of functions including quadratic, polynomial, rational, exponential, and logarithmic functions. By knowing the properties of a family of functions, we can immediately apply that knowledge to any member of the family we encounter, giving us valuable information about the domain and the behavior of the function. Furthermore, this knowledge can help us to choose an appropriate function for an application. Exponential functions have so many important applications that we highlighted some of them in the last section of the chapter. In the next chapters, we see how calculus gives us even more information about the behavior of functions.

Function A function is a rule that assigns to each element from one set exactly one element from another set.

Domain and Range The set of all possible values of the independent variable in a function is called the domain of the function, and the resulting set of possible values of the dependent variable is called the range.

Vertical Line Test A graph represents a function if and only if every vertical line intersects the graph in no more than one point.

Quadratic Function A quadratic function is defined by

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers, with $a \neq 0$.

Graph of a Quadratic Function The graph of the quadratic function $f(x) = ax^2 + bx + c$ is a parabola with its vertex at

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right).$$

The graph opens upward if $a > 0$ and downward if $a < 0$.

Polynomial Function

A polynomial function of degree n, where n is a nonnegative integer, is defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1$ and a_0 are real numbers, called coefficients, with $a_n \neq 0$. The number a_n is called the leading coefficient.

Properties of Polynomial Functions

1. A polynomial function of degree n can have at most $n - 1$ turning points. Conversely, if the graph of a polynomial function has n turning points, it must have degree at least $n + 1$.
2. In the graph of a polynomial function of even degree, both ends go up or both ends go down. For a polynomial function of odd degree, one end goes up and one end goes down.
3. If the graph goes up as x becomes large, the leading coefficient must be positive. If the graph goes down as x becomes large, the leading coefficient is negative.

Rational Function

A rational function is defined by

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ *and* $q(x)$ are polynomial functions and $q(x) \neq 0$.

Asymptotes

If a function gets larger and larger in magnitude without bound as x approaches the number k, then the line $x = k$ is a vertical asymptote.

If the values of y approach a number k as $|x|$ gets larger and larger, the line $y = k$ is a horizontal asymptote.

Exponential Function

An exponential function with base a is defined as

$$f(x) = a^x, \text{ where } a > 0 \text{ and } a \neq 1.$$

Simple Interest

If P dollars is invested at a yearly simple interest rate r per year for time t (in years), the interest I is given by

$$I = Prt.$$

Math of Finance Formulas

If P is the principal or present value, r is the annual interest rate, t is time in years, and m is the number of compounding periods per year:

	Compounded m Times per Year	Compounded Continuously
Compound amount	$A = P\left(1 + \frac{r}{m}\right)^{tm}$	$A = Pe^{rt}$
Effective rate	$r_E = \left(1 + \frac{r}{m}\right)^m - 1$	$r_E = e^r - 1$

Definition of e

As m becomes larger and larger, $\left(1 + \frac{1}{m}\right)^m$ becomes closer and closer to the number e, whose approximate value is 2.718281828.

Logarithm

For $a > 0$, $a \neq 1$, and $x > 0$,

$$y = \log_a x \text{ means } a^y = x.$$

Logarithmic Function

If $a > 0$ and $a \neq 1$, then the logarithmic function of base a is defined by

$$f(x) = \log_a x,$$

for $x > 0$.

Properties of Logarithms

Let x and y be any positive real numbers and r be any real number. Let a be a positive real number, $a \neq 1$. Then

a. $\log_a xy = \log_a x + \log_a y$

b. $\log_a \frac{x}{y} = \log_a x - \log_a y$

c. $\log_a x^r = r \log_a x$

d. $\log_a a = 1$

e. $\log_a 1 = 0$

f. $\log_a a^r = r$.

Change-of-Base Theorem for Logarithms

If x is any positive number and if a and b are positive real numbers, $a \neq 1$, $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}.$$

Change-of-Base Theorem for Exponentials

For every positive real number a

$$a^x = e^{(\ln a)x}.$$

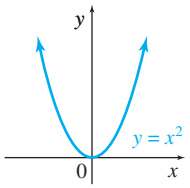
Exponential Growth and Decay Function

Let y_0 be the amount or number of some quantity present at time $t = 0$. The quantity is said to grow or decay exponentially if, for some constant k, the amount present at any time t is given by

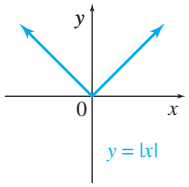
$$y = y_0 e^{kt}.$$

Graphs of Basic Functions

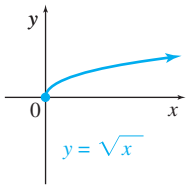
Quadratic



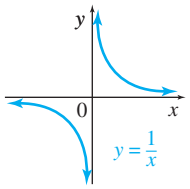
Absolute Value



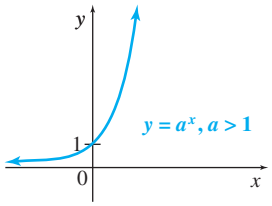
Square Root



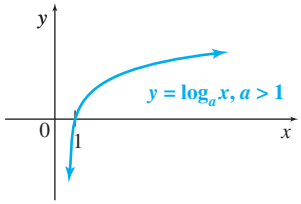
Rational



Exponential



Logarithmic



KEY TERMS

To understand the concepts presented in this chapter, you should know the meaning and use of the following terms. For easy reference, the section in the chapter where a word (or expression) was first used is provided.

10.1
function
domain
range
constant function
vertical line test
even function
odd function
step function

10.2
quadratic function
parabola
vertex
axis
vertical translation
vertical reflection
horizontal translation
completing the square
horizontal reflection

10.3
polynomial function
degree
coefficient
leading coefficient
power function
cubic polynomial
quartic polynomial
turning point
real zero
rational function
vertical asymptote
horizontal asymptote
cost–benefit model
average cost

10.4
exponential function
exponential equation
interest
principal
rate of interest
time
simple interest
compound interest
compound amount
e
continuous compounding

10.5
doubling time
logarithm
logarithmic function
inverse function
properties of logarithms
common logarithms
natural logarithms
change-of-base theorem for logarithms
logarithmic equation
change-of-base theorem for exponentials
rule of 70
rule of 72
index of diversity

10.6
exponential growth and decay function
growth constant
decay constant
half-life
nominal (stated) rate
effective rate
present value
limited growth function
learning curve

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. A linear function is an example of a polynomial function.

2. A rational function is an example of an exponential function.

3. The function $f(x) = 3x^2 - 75x + 2$ is a quadratic function.

4. The function $f(x) = x^2 - 6x + 4$ has a vertex at $x = 3$.

5. The function $g(x) = x^{\pi}$ is an exponential function.

6. The function $f(x) = \dfrac{1}{x - 6}$ has a vertical asymptote at $y = 6$.

7. Since $3^{-2} = \dfrac{1}{9}$ we can conclude that $\log_3 \dfrac{1}{9} = -2$.

8. The domain of the function $f(x) = \dfrac{1}{x^2 - 4}$ includes all real numbers except $x = 2$.

9. The amount of money after two years if \$2000 is invested in an account that is compounded monthly with an annual rate of 4% is $A = 2000\left(1 + \dfrac{4}{12}\right)^{24}$ dollars.

10. $\log_1 1 = 0$

11. $\ln(5 + 7) = \ln 5 + \ln 7$

12. $(\ln 3)^4 = 4 \ln 3$

13. $\log_{10} 0 = 1$

14. $e^{\ln 2} = 2$

15. $e^{\ln(-2)} = -2$

16. $\dfrac{\ln 4}{\ln 8} = \ln 4 - \ln 8$

17. The function $g(x) = e^x$ grows faster than the function $f(x) = \ln x$.

18. The half-life of a radioactive substance is the time required for half of the initial quantity to decay.

PRACTICE AND EXPLORATIONS

19. What is a function? A linear function? A quadratic function? A rational function?

20. How do you find a vertical asymptote? A horizontal asymptote?

21. What can you tell about the graph of a polynomial function of degree n before you plot any points?

22. Describe in words what a logarithm is.

List the ordered pairs obtained from the following if the domain of x for each exercise is $\{-3, -2, -1, 0, 1, 2, 3\}$. Graph each set of ordered pairs. Give the range.

23. $y = (2x - 1)(x + 1)$

24. $y = \dfrac{x}{x^2 + 1}$

25. Let $f(x) = 5x^2 - 3$ and $g(x) = -x^2 + 4x + 1$. Find the following.

a. $f(-2)$ **b.** $g(3)$ **c.** $f(-k)$ **d.** $g(3m)$

e. $f(x + h)$ **f.** $g(x + h)$ **g.** $\dfrac{f(x + h) - f(x)}{h}$

h. $\dfrac{g(x + h) - g(x)}{h}$

26. Let $f(x) = 2x^2 + 5$ and $g(x) = 3x^2 + 4x - 1$. Find the following.

a. $f(-3)$ **b.** $g(2)$ **c.** $f(3m)$ **d.** $g(-k)$

e. $f(x + h)$ **f.** $g(x + h)$ **g.** $\dfrac{f(x + h) - f(x)}{h}$

h. $\dfrac{g(x + h) - g(x)}{h}$

Find the domain of each function defined as follows.

27. $y = \dfrac{3x - 4}{x}$

28. $y = \dfrac{\sqrt{x - 2}}{2x + 3}$

29. $y = \ln(x + 7)$

30. $y = \ln(x^2 - 16)$

Graph the following by hand.

31. $y = 2x^2 + 3x - 1$

32. $y = -\dfrac{1}{4}x^2 + x + 2$

33. $y = -x^2 + 4x + 2$

34. $y = 3x^2 - 9x + 2$

35. $f(x) = x^3 - 3$

36. $f(x) = 1 - x^4$

37. $y = -(x - 1)^4 + 4$

38. $y = -(x + 2)^3 - 2$

39. $f(x) = \dfrac{8}{x}$

40. $f(x) = \dfrac{2}{3x - 6}$

41. $f(x) = \dfrac{4x - 2}{3x + 1}$

42. $f(x) = \dfrac{6x}{x + 2}$

43. $y = 4^x$

44. $y = 4^{-x} + 3$

45. $y = \left(\dfrac{1}{5}\right)^{2x-3}$

46. $y = \left(\dfrac{1}{2}\right)^{x-1}$

47. $y = \log_2(x - 1)$

48. $y = 1 + \log_3 x$

49. $y = -\ln(x + 3)$

50. $y = 2 - \ln x^2$

Solve each equation.

51. $2^{x+2} = \dfrac{1}{8}$

52. $\left(\dfrac{9}{16}\right)^x = \dfrac{3}{4}$

53. $9^{2y+3} = 27^y$

54. $\dfrac{1}{2} = \left(\dfrac{b}{4}\right)^{1/4}$

Write each equation using logarithms.

55. $3^5 = 243$

56. $5^{1/2} = \sqrt{5}$

57. $e^{0.8} = 2.22554$

58. $10^{1.07918} = 12$

Write each equation using exponents.

59. $\log_2 32 = 5$

60. $\log_9 3 = \dfrac{1}{2}$

61. $\ln 82.9 = 4.41763$

62. $\log 3.21 = 0.50651$

Evaluate each expression without using a calculator. Then support your work using a calculator and the change-of-base theorem for logarithms.

63. $\log_3 81$ **64.** $\log_{32} 16$

65. $\log_4 8$ **66.** $\log_{100} 1000$

Simplify each expression using the properties of logarithms.

67. $\log_5 3k + \log_5 7k^3$ **68.** $\log_3 2y^3 - \log_3 8y^2$

69. $4 \log_3 y - 2 \log_3 x$ **70.** $3 \log_4 r^2 - 2 \log_4 r$

Solve each equation. If necessary, round each answer to the nearest thousandth.

71. $6^p = 17$ **72.** $3^{z-2} = 11$

73. $2^{1-m} = 7$ **74.** $12^{-k} = 9$

75. $e^{-5-2x} = 5$ **76.** $e^{3x-1} = 14$

77. $\left(1 + \frac{m}{3}\right)^5 = 15$ **78.** $\left(1 + \frac{2p}{5}\right)^2 = 3$

79. $\log_k 64 = 6$ **80.** $\log_3(2x + 5) = 5$

81. $\log(4p + 1) + \log p = \log 3$

82. $\log_2(5m - 2) - \log_2(m + 3) = 2$

83. Give the following properties of the exponential function $f(x) = a^x; a > 0, a \neq 1$.

a. Domain **b.** Range **c.** y-intercept

d. Asymptote(s)

e. Increasing if a is _____

f. Decreasing if a is _____

84. Give the following properties of the logarithmic function $f(x) = \log_a x; a > 0, a \neq 1$.

a. Domain **b.** Range **c.** x-intercept

d. Asymptote(s)

e. Increasing if a is _____

f. Decreasing if a is _____

85. Compare your answers for Exercises 83 and 84. What similarities do you notice? What differences?

APPLICATIONS

Business and Economics

86. Car Rental To rent a mid-size car from one agency costs \$60 per day or fraction of a day. If you pick up the car in Boston and drop it off in Utica, there is a fixed \$40 charge. Let $C(x)$ represent the cost of renting the car for x days and taking it from Boston to Utica. Find the following.

a. $C\left(\frac{3}{4}\right)$ **b.** $C\left(\frac{9}{10}\right)$ **c.** $C(1)$

d. $C\left(1\frac{5}{8}\right)$ **e.** $C\left(2\frac{1}{9}\right)$

f. Graph the function defined by $y = C(x)$ for $0 < x \leq 5$.

g. What is the independent variable?

h. What is the dependent variable?

87. Pollution The cost to remove x percent of a pollutant is

$$y = \frac{7x}{100 - x},$$

in thousands of dollars. Find the cost of removing the following percents of the pollutant.

a. 80% **b.** 50% **c.** 90%

d. Graph the function.

e. Can all of the pollutant be removed?

Interest Find the amount of interest earned by each deposit.

88. \$6902 if interest is 6% compounded semiannually for 8 years

89. \$2781.36 if interest is 4.8% compounded quarterly for 6 years

Interest Find the compound amount if \$12,104 is invested at 6.2% compounded continuously for each period.

90. 2 years **91.** 4 years

Interest Find the compound amounts for the following deposits if interest is compounded continuously.

92. \$1500 at 6% for 9 years

93. \$12,000 at 5% for 8 years

94. How long will it take for \$1000 deposited at 6% compounded semiannually to double? To triple?

95. How long will it take for \$2100 deposited at 4% compounded quarterly to double? To triple?

Effective Rate Find the effective rate to the nearest hundredth for each nominal interest rate.

96. 7% compounded quarterly

97. 6% compounded monthly

98. 5% compounded continuously

Present Value Find the present value of each amount.

99. \$2000 if interest is 6% compounded annually for 5 years

100. \$10,000 if interest is 8% compounded semiannually for 6 years

101. Interest To help pay for college expenses, Julie Davis borrowed \$10,000 at 7% interest compounded semiannually for 8 years. How much will she owe at the end of the 8-year period?

102. Inflation How long will it take for \$1 to triple at an annual inflation rate of 8% compounded continuously?

103. Interest Find the interest rate needed for \$6000 to grow to \$8000 in 3 years with continuous compounding.

104. Present Value Frank Steed wants to open a camera shop. How much must he deposit now at 6% interest compounded monthly to have \$25,000 at the end of 3 years?

105. Revenue A concert promoter finds she can sell 1000 tickets at \$50 each. She will not sell the tickets for less than \$50, but she finds that for every \$1 increase in the ticket price above \$50, she will sell 10 fewer tickets.

a. Express n, the number of tickets sold, as a function of p, the price.

b. Express R, the revenue, as a function of p, the price.

c. Find the domain of the function found in part b.

d. Express R, the revenue, as a function of n, the number sold.

e. Find the domain of the function found in part d.

f. Find the price that produces the maximum revenue.

g. Find the number of tickets sold that produces the maximum revenue.

h. Find the maximum revenue.

i. Sketch the graph of the function found in part b.

j. Describe what the graph found in part i tells you about how the revenue varies with price.

106. Cost Suppose the cost in dollars to produce x posters is given by

$$C(x) = \frac{5x + 3}{x + 1}.$$

a. Sketch a graph of $C(x)$.

b. Find a formula for $C(x + 1) - C(x)$, the cost to produce an additional poster when x posters are already produced.

c. Find a formula for $A(x)$, the average cost per poster.

d. Find a formula for $A(x + 1) - A(x)$, the change in the average cost per poster when one additional poster is produced. (This quantity is approximately equal to the marginal average cost, which will be discussed in the chapter on the derivative.)

107. Cost Suppose the cost in dollars to produce x hundreds of nails is given by

$$C(x) = x^2 + 4x + 7.$$

a. Sketch a graph of $C(x)$.

b. Find a formula for $C(x + 1) - C(x)$, the cost to produce an additional hundred nails when x hundred are already produced. (This quantity is approximately equal to the marginal cost.)

c. Find a formula for $A(x)$, the average cost per hundred nails.

d. Find a formula for $A(x + 1) - A(x)$, the change in the average cost per nail when one additional batch of 100 nails is produced. (This quantity is approximately equal to the marginal average cost, which will be discussed in the chapter on the derivative.)

108. Consumer Price Index The U.S. consumer price index (CPI, or cost of living index) has risen over the years, as shown in the table in the next column, using an index in which the average over the years 1982 to 1984 is set to 100. ***Source: Bureau of Labor Statistics.***

Year	CPI
1960	29.6
1970	38.8
1980	82.4
1990	130.7
1995	152.4
2000	172.2
2005	195.3

a. Letting t be the years since 1960, write an exponential function in the form $y = a^t$ that fits the data at 1960 and 2005.

b. If your calculator has an exponential regression feature, find the best fitting exponential function for the data.

c. Use a graphing calculator to plot the answers to parts a and b on the same axes as the data. Are the answers to parts a and b close to each other?

d. If your calculator has a quadratic and cubic regression feature, find the best-fitting quadratic and cubic functions for the data.

e. Use a graphing calculator to plot the answers to parts b and d on the same window as the data. Discuss the extent to which any one of these functions models the data better than the others.

Life Sciences

109. Fever A certain viral infection causes a fever that typically lasts 6 days. A model of the fever (in °F) on day x, $1 \le x \le 6$, is

$$F(x) = -\frac{2}{3}x^2 + \frac{14}{3}x + 96.$$

According to the model, on what day should the maximum fever occur? What is the maximum fever?

110. Sunscreen An article in a medical journal says that a sunscreen with a sun protection factor (SPF) of 2 provides 50% protection against ultraviolet B (UVB) radiation, an SPF of 4 provides 75% protection, and an SPF of 8 provides 87.5% protection (which the article rounds to 87%). ***Source: Family Practice.***

a. 87.5% protection means that 87.5% of the UVB radiation is screened out. Write as a fraction the amount of radiation that is let in, and then describe how this fraction, in general, relates to the SPF rating.

b. Plot UVB percent protection (y) against x, where $x = 1/\text{SPF}$.

c. Based on your graph from part b, give an equation relating UVB protection to SPF rating.

d. An SPF of 8 has double the chemical concentration of an SPF 4. Find the increase in the percent protection.

e. An SPF of 30 has double the chemical concentration of an SPF 15. Find the increase in the percent protection.

f. Based on your answers from parts d and e, what happens to the increase in the percent protection as the SPF continues to double?

111. HIV in Infants The following table lists the reported number of cases of infants born in the United States with HIV in recent years because their mother was infected.* ***Source: Centers for Disease Control and Prevention.***

Year	Cases
1995	295
1997	166
1999	109
2001	115
2003	94
2005	107
2007	79

a. Plot the data on a graphing calculator, letting $t = 0$ correspond to the year 1995.

b. Using the regression feature on your calculator, find a quadratic, a cubic, and an exponential function that models this data.

c. Plot the three functions with the data on the same coordinate axes. Which function or functions best capture the behavior of the data over the years plotted?

d. Find the number of cases predicted by all three functions for 2015. Which of these are realistic? Explain.

112. Respiratory Rate Researchers have found that the 95th percentile (the value at which 95% of the data is at or below) for respiratory rates (in breaths per minute) during the first 3 years of infancy are given by

$$y = 10^{1.82411 - 0.0125995x + 0.00013401x^2}$$

for awake infants and

$$y = 10^{1.72858 - 0.0139928x + 0.00017646x^2}$$

for sleeping infants, where x is the age in months. ***Source: Pediatrics.***

a. What is the domain for each function?

b. For each respiratory rate, is the rate decreasing or increasing over the first 3 years of life? (*Hint:* Is the graph of the quadratic in the exponent opening upward or downward? Where is the vertex?)

c. Verify your answer to part b using a graphing calculator.

d. For a 1-year-old infant in the 95th percentile, how much higher is the waking respiratory rate than the sleeping respiratory rate?

113. Polar Bear Mass One formula for estimating the mass (in kg) of a polar bear is given by

$$m(g) = e^{0.02 + 0.062g - 0.000165g^2},$$

where g is the axillary girth in centimeters. It seems reasonable that as girth increases, so does the mass. What is the largest girth for which this formula gives a reasonable answer? What is the predicted mass of a polar bear with this girth? ***Source: Wildlife Management.***

*These data include only those infants born in the 25 states with confidential name-based HIV infection reporting.

114. Population Growth A population of 15,000 small deer in a specific region has grown exponentially to 17,000 in 4 years.

a. Write an exponential equation to express the population growth y in terms of time t in years.

b. At this rate, how long will it take for the population to reach 45,000?

115. Population Growth In 1960 in an article in *Science* magazine, H. Van Forester, P. M. Mora, and W. Amiot predicted that world population would be infinite in the year 2026. Their projection was based on the rational function defined by

$$p(t) = \frac{1.79 \times 10^{11}}{(2026.87 - t)^{0.99}},$$

where $p(t)$ gives population in year t. This function has provided a relatively good fit to the population until very recently. ***Source: Science.***

a. Estimate world population in 2010 using this function, and compare it with the estimate of 6.909 billion. ***Source: United Nations.***

b. What does the function predict for world population in 2020? 2025?

c. Discuss why this function is not realistic, despite its good fit to past data.

116. Intensity of Light The intensity of light (in appropriate units) passing through water decreases exponentially with the depth it penetrates beneath the surface according to the function

$$I(x) = 10e^{-0.3x},$$

where x is the depth in meters. A certain water plant requires light of an intensity of 1 unit. What is the greatest depth of water in which it will grow?

117. Drug Concentration The concentration of a certain drug in the bloodstream at time t (in minutes) is given by

$$c(t) = e^{-t} - e^{-2t}.$$

Use a graphing calculator to find the maximum concentration and the time when it occurs.

118. Glucose Concentration When glucose is infused into a person's bloodstream at a constant rate of c grams per minute, the glucose is converted and removed from the bloodstream at a rate proportional to the amount present. The amount of glucose in grams in the bloodstream at time t (in minutes) is given by

$$g(t) = \frac{c}{a} + \left(g_0 - \frac{c}{a}\right)e^{-at},$$

where a is a positive constant. Assume $g_0 = 0.08$, $c = 0.1$, and $a = 1.3$.

a. At what time is the amount of glucose a maximum? What is the maximum amount of glucose in the bloodstream?

b. When is the amount of glucose in the bloodstream 0.1 g?

c. What happens to the amount of glucose in the bloodstream after a very long time?

119. Species Biologists have long noticed a relationship between the area of a piece of land and the number of species found there. The following data shows a sample of the British Isles

and how many vascular plants are found on each. ***Source: Journal of Biogeography.***

Isle	Area (km^2)	Species
Ailsa	0.8	75
Fair	5.2	174
Iona	9.1	388
Man	571.6	765
N. Ronaldsay	7.3	131
Skye	1735.3	594
Stronsay	35.2	62
Wight	380.7	1008

a. One common model for this relationship is logarithmic. Using the logarithmic regression feature on a graphing calculator, find a logarithmic function that best fits the data.

b. An alternative to the logarithmic model is a power function of the form $S = b(A^c)$. Using the power regression feature on a graphing calculator, find a power function that best fits the data.

c. Graph both functions from parts a and b along with the data. Give advantages and drawbacks of both models.

d. Use both functions to predict the number of species found on the isle of Shetland, with an area of 984.2 km^2. Compare with the actual number of 421.

e. Describe one or more situations where being able to predict the number of species could be useful.

Physical Sciences

120. Oil Production The production of an oil well has decreased exponentially from 128,000 barrels per year 5 years ago to 100,000 barrels per year at present.

a. Letting $t = 0$ represent the present time, write an exponential equation for production y in terms of time t in years.

b. Find the time it will take for production to fall to 70,000 barrels per year.

121. Dating Rocks Geologists sometimes measure the age of rocks by using "atomic clocks." By measuring the amounts of potassium-40 and argon-40 in a rock, the age t of the specimen (in years) is found with the formula

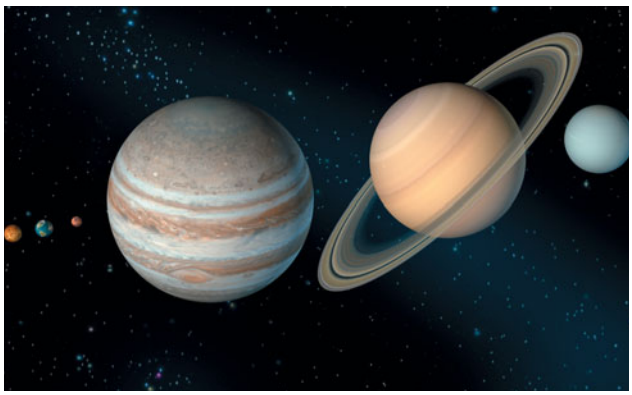
$$t = (1.26 \times 10^9)\frac{\ln[1 + 8.33(A/K)]}{\ln 2},$$

where A and K, respectively, are the numbers of atoms of argon-40 and potassium-40 in the specimen.

a. How old is a rock in which $A = 0$ and $K > 0$?

b. The ratio A/K for a sample of granite from New Hampshire is 0.212. How old is the sample?

c. Let $A/K = r$. What happens to t as r gets larger? Smaller?



122. Planets The following table contains the average distance D from the sun for the eight planets and their period P of revolution around the sun in years. ***Source: The Natural History of the Universe.***

Planet	Distance (D)	Period (P)
Mercury	0.39	0.24
Venus	0.72	0.62
Earth	1	1
Mars	1.52	1.89
Jupiter	5.20	11.9
Saturn	9.54	29.5
Uranus	19.2	84.0
Neptune	30.1	164.8

The distances are given in astronomical units (A.U.); 1 A.U. is the average distance from Earth to the sun. For example, since Jupiter's distance is 5.2 A.U., its distance from the sun is 5.2 times farther than Earth's.

a. Find functions of the form $P = kD^n$ for $n = 1$, 1.5, and 2 that fit the data at Neptune.

b. Use a graphing calculator to plot the data in the table and to graph the three functions found in part a. Which function best fits the data?

c. Use the best-fitting function from part b to predict the period of Pluto (which was removed from the list of planets in 2006), which has a distance from the sun of 39.5 A.U. Compare your answer to the true value of 248.5 years.

d. If you have a graphing calculator or computer program with a power regression feature, use it to find a power function (a function of the form $P = kD^n$) that approximately fits the data. How does this answer compare with the answer to part b?

EXTENDED APPLICATION

POWER FUNCTIONS

In this chapter we have seen several applications of power functions, which have the general form $y = ax^b$. Power functions are so named because the independent variable is raised to a power. (These should not be confused with exponential functions, in which the independent variable appears in the power.) We explored some special cases of power functions, such as $b = 2$ (a simple quadratic function) and $b = 1/2$ (a square root function). But applications of power functions vary greatly and are not limited to these special cases.

For example, in Exercise 86 in Section 10.5, we saw that the basal metabolism rate of marsupial carnivores is a power function of the body mass. In that exercise we also saw a way to verify that empirical data can be modeled with a power function. By taking the natural logarithm of both sides of the equation for a power function,

$$y = ax^b, \tag{1}$$

we obtain the equation

$$\ln y = \ln a + b \ln x. \tag{2}$$

Letting $Y = \ln y$, $X = \ln x$, and $A = \ln a$ results in the linear equation

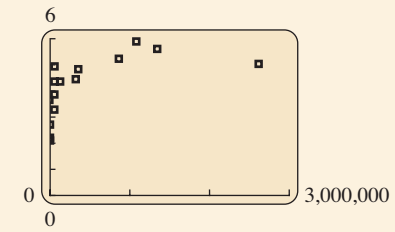
$$Y = A + bX. \tag{3}$$

Plotting the logarithm of the original data reveals whether a straight line approximates the data well. If it does, then a power function is a good fit to the original data.

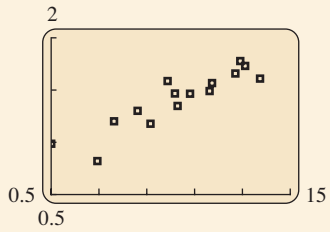
Here is another example. In an attempt to measure how the pace of city life is related to the population of the city, two researchers estimated the average speed of pedestrians in 15 cities by measuring the mean time it took them to walk 50 feet. Their results are shown in the table in the next column. ***Source: Nature.***

City	Population (x)	Speed (ft/sec) (y)
Brno, Czechoslovakia	341,948	4.81
Prague, Czechoslovakia	1,092,759	5.88
Corte, Corsica	5491	3.31
Bastia, France	49,375	4.90
Munich, Germany	1,340,000	5.62
Psychro, Crete	365	2.67
Itea, Greece	2500	2.27
Iráklion, Greece	78,200	3.85
Athens, Greece	867,023	5.21
Safed, Israel	14,000	3.70
Dimona, Israel	23,700	3.27
Netanya, Israel	70,700	4.31
Jerusalem, Israel	304,500	4.42
New Haven, U.S.A.	138,000	4.39
Brooklyn, U.S.A.	2,602,000	5.05

Figure 59(a) shows the speed (stored in the list L_2 on a TI-84 Plus) plotted against the population (stored in L_1). The natural logarithm of the data was then calculated and stored using the commands $\ln(L_1) \rightarrow L_3$ and $\ln(L_2) \rightarrow L_4$. A plot of the data in L_3 and L_4 is shown in Figure 59(b). Notice that the data lie fairly closely along a straight line, confirming that a power function is an appropriate model for the original data. (These calculations and plots could also be carried out on a spreadsheet.)



(a)



(b)

FIGURE 59

A power function that best fits the data according to the least squares principle of Section 1.3 is found with the TI-84 Plus command `PwrReg` L_1, L_2, Y_1. The result is

$$y = 1.363x^{0.09799}, \tag{4}$$

with a correlation coefficient of $r = 0.9081$. (For more on the correlation coefficient, see Section 1.3.) Because this value is close to 1, it indicates a good fit. Similarly, we can find a line that fits the data in Figure 59(b) with the command `LinReg(`$ax + b$`)` L_3, L_4, Y_2.

The result is

$$Y = 0.30985 + 0.09799X, \tag{5}$$

again with $r = 0.9081$. The identical correlation coefficient is not a surprise, since the two commands accomplish essentially the same thing. Comparing Equations (4) and (5) with Equations (1), (2), and (3), notice that $b = 0.09799$ in both Equations (4) and (5), and that $A = 0.30985 \approx \ln a = \ln 1.363$. (The slight difference is due to rounding.) Equations (4) and (5) are plotted on the same window as the data in Figure 60(a) and (b), respectively.

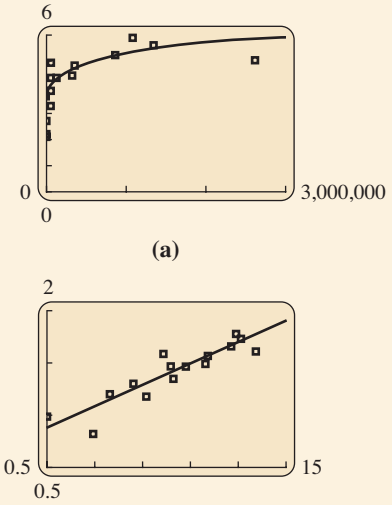


FIGURE 60

These results raise numerous questions worth exploring. What does this analysis tell you about the connection between the pace of city life and the population of a city? What might be some reasons for this connection?

A third example was considered in Review Exercise 119, where we explored the relationship between the area of each of the British isles and the number of species of vascular plants on the isles. In Figure 61(a) we have plotted the complete set of data from the original article (except for the Isle of Britain, whose area is so large that it doesn't fit on a graph that shows the other data in detail). ***Source: Journal of Biogeography.*** In Figure 61(b) we have plotted the natural logarithm of the data (again leaving out Britain). Notice that despite the large amount of scatter in the data, there is a linear trend. Figure 61(a) includes the best-fitting power function

$$y = 125.9x^{0.2088}, \tag{6}$$

while Figure 61 (b) includes the best-fitting line

$$Y = 4.836 + 0.2088X. \tag{7}$$

(We have included the data for Britain in both of these calculations.) Notice as before that the exponent of the power function equals the slope of the linear function, and that $A = 4.836 \approx \ln a = \ln 125.9$. The correlation is 0.6917 for both, indicating that there is a trend, but that the data is somewhat scattered about the best-fitting curve or line.

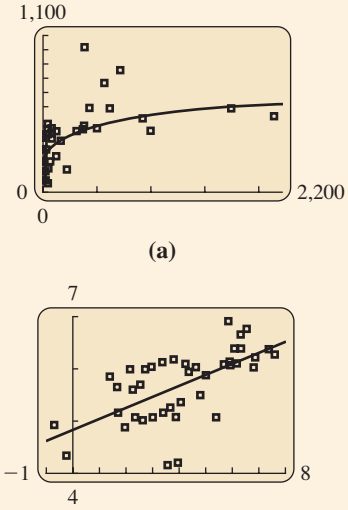


FIGURE 61

There are many more examples of data that fit a power function. Bohorquez et al. have found that the frequency of attacks in a war is a power function of the number of people killed in the attacks. ***Source: Nature.*** Amazingly, this holds true for a wide variety of different wars, with the average value of the exponent as $b \approx 2.5$. For a fragmented, fluid enemy with more groups, the value of b tends to be larger, and for a robust, stronger enemy with fewer groups, the value of b tends to be smaller.* You will explore other applications of power functions in the exercises.

EXERCISES

1. Gwartney et al. listed the following data relating the price of a monthly cellular bill (in dollars) and the demand (in millions of subscribers).† ***Source: Economics: Private and Public Choice.***

Quantity (in millions)	Price (in dollars)
2.1	123
3.5	107
5.3	92
7.6	79
11.0	73
16.0	63
24.1	56

**For a TED video on this phenomenon, see http://www.ted.com/talks/sean_gourley_on_the_mathematics_of_war.html.*

† The authors point out that these are actual prices and quantities annually for 1988 to 1994. If they could assume that other demand determinants, such as income, had remained constant during that period, this would give an accurate measurement of the demand function.

a. Using a graphing calculator, plot the natural logarithm of the price against the natural logarithm of the quantity. Does the relationship appear to be linear?

b. Find the best-fitting line to the natural logarithm of the data, as plotted in part a. Plot this line on the same axes as the data.

c. Plot the price against the quantity. What is different about the trend in these data from the trend in Figures 59(a) and 61(a)? What does this tell you about the exponent of the best-fitting power function for these data? What conclusions can you make about how demand varies with the price?

d. Find the best-fitting power function for the data plotted in part c. Verify that this function is equivalent to the least squares line through the logarithm of the data found in part b.

2. For many years researchers thought that the basal metabolic rate (BMR) in mammals was a power function of the mass, with disagreement on whether the power was 0.67 or 0.75. More recently, White et al. proposed that the power may vary for mammals with different evolutionary lineages. ***Source: Evolution.*** The following table shows a portion of their data containing the natural logarithm of the mass (in grams) and the natural logarithm of the BMR (in mL of oxygen per hour) for 12 species from the genus Peromyscus, or deer mouse.

Species	ln(mass)	ln(BMR)
Peromyscus boylii	3.1442	3.9943
Peromyscus californicus	3.8618	3.9506
Peromyscus crinitus	2.7663	3.2237
Peromyscus eremicus	3.0681	3.4998
Peromyscus gossypinus	3.0681	3.6104
Peromyscus leucopus	3.1355	3.8111
Peromyscus maniculatus	3.0282	3.6835
Peromyscus megalops	4.1927	4.5075
Peromyscus oreas	3.2019	3.7729
Peromyscus polionotus	2.4849	3.0671
Peromyscus sitkensis	3.3439	3.8447
Peromyscus truei	3.5041	4.0378

a. Plot ln(BMR) against ln(mass) using a graphing calculator. Does the relationship appear to be linear?

b. Find the least squares line for the data plotted in part a. Plot the line on the same axes as the data.

c. Calculate the mass and BMR for each species, and then find the best-fitting power function for these data. Plot this function on the same axes as the mass and BMR data.

d. What would you conclude about whether the deer mouse BMR can be modeled as a power function of the mass? What seems to be an approximate value of the power?

DIRECTIONS FOR GROUP PROJECT

Go to the section on "build your own tables" of the Human Development Reports website at http://hdrstats.undp.org/en/buildtables. *Select a group of countries, as well as two indicators that you think might be related by a power function. For example, you might choose "GDP per capita" and "Population not using an improved water source (%)" Click on "Display indicators in Row" and then "Show results." Then click on "Export to Excel." From the Excel spreadsheet, create a scatterplot of the original data, as well as a scatterplot of the natural logarithm of the data. Find data for which the natural logarithm is roughly a straight line, and find the least squares line. Then convert this to a power function modeling the original data. Present your results in a report, describing in detail what your analysis tells you about the countries under consideration and any other conclusions that you can make.*

The Derivative

11.1 Limits

11.2 Continuity

11.3 Rates of Change

11.4 Definition of the Derivative

11.5 Graphical Differentiation

Chapter 11 Review

Extended Application: A Model for Drugs Administered Intravenously

The population of the United States has been increasing since 1790, when the first census was taken. Over the past few decades, the population has not only been increasing, but the level of diversity has also been increasing. This fact is important to school districts, businesses, and government officials. Using examples in the third section of this chapter, we explore two rates of change related to the increase in minority population. In the first example, we calculate an average rate of change; in the second, we calculate the rate of change at a particular time. This latter rate is an example of a derivative, the subject of this chapter.

The algebraic problems considered in earlier chapters dealt with *static* situations:

- What is the revenue when 100 items are sold?
- How much interest is earned in three years?
- What is the equilibrium price?

Calculus, on the other hand, deals with *dynamic* situations:

- At what rate is the demand for a product changing?
- How fast is a car moving after 2 hours?
- When does the growth of a population begin to slow down?

The techniques of calculus allow us to answer these questions, which deal with rates of change.

The key idea underlying calculus is the concept of limit, so we will begin by studying limits.

11.1 Limits

APPLY IT **What happens to the demand of an essential commodity as its price continues to increase?**

We will find an answer to this question in Exercise 82 using the concept of limit.

The limit is one of the tools that we use to describe the behavior of a function as the values of x approach, or become closer and closer to, some particular number.

EXAMPLE 1 Finding a Limit

What happens to $f(x) = x^2$ when x is a number *very close* to (but not equal to) 2?

SOLUTION We can construct a table with x values getting closer and closer to 2 and find the corresponding values of $f(x)$.

x approaches 2 from left ↓ x approaches 2 from right

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	3.61	3.9601	3.996001	3.99960001	4	4.00040001	4.004001	4.0401	4.41

$f(x)$ approaches 4 ↑ $f(x)$ approaches 4

The table suggests that, as x gets closer and closer to 2 from either side, $f(x)$ gets closer and closer to 4. In fact, you can use a calculator to show the values of $f(x)$ can be made as close as you want to 4 by taking values of x close enough to 2. This is not surprising since the value of the function at $x = 2$ is $f(x) = 4$. We can observe this fact by looking at the graph $y = x^2$, as shown in Figure 1. In such a case, we say "the limit of $f(x)$ as x approaches 2 is 4," which is written as

$$\lim_{x \to 2} f(x) = 4.$$

YOUR TURN 1
Find $\lim_{x \to 1} (x^2 + 2)$.

TRY YOUR TURN 1

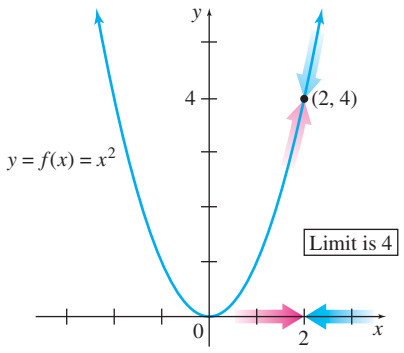


FIGURE 1

The phrase "x approaches 2 from the left" is written $x \to 2^-$. Similarly, "x approaches 2 from the right" is written $x \to 2^+$. These expressions are used to write **one-sided limits**. The **limit from the left** (as x approaches 2 from the negative direction) is written

$$\lim_{x \to 2^-} f(x) = 4,$$

and shown in red in Figure 1. The **limit from the right** (as x approaches 2 from the positive direction) is written

$$\lim_{x \to 2^+} f(x) = 4,$$

and shown in blue in Figure 1. A **two-sided limit**, such as

$$\lim_{x \to 2} f(x) = 4,$$

exists only if both one-sided limits exist and are the same; that is, if $f(x)$ approaches the same number as x approaches a given number from *either* side.

CAUTION Notice that $\lim_{x \to a^-} f(x)$ does not mean to take negative values of x, nor does it mean to choose values of x to the right of a and then move in the negative direction. It means to use values less than a ($x < a$) that get closer and closer to a.

The previous example suggests the following informal definition.

Limit of a Function

Let f be a function and let a and L be real numbers. If

1. as x takes values closer and closer (but not equal) to a on both sides of a, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L; and
2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a;

then L is the **limit** of $f(x)$ as x approaches a, written

$$\lim_{x \to a} f(x) = L.$$

This definition is informal because the expressions "closer and closer to" and "as close as desired" have not been defined. A more formal definition would be needed to prove the rules for limits given later in this section.*

*The limit is the key concept from which all the ideas of calculus flow. Calculus was independently discovered by the English mathematician Isaac Newton (1642–1727) and the German mathematician Gottfried Wilhelm Leibniz (1646–1716). For the next century, supporters of each accused the other of plagiarism, resulting in a lack of communication between mathematicians in England and on the European continent. Neither Newton nor Leibniz developed a mathematically rigorous definition of the limit (and we have no intention of doing so here). More than 100 years passed before the French mathematician Augustin-Louis Cauchy (1789–1857) accomplished this feat.

The definition of a limit describes what happens to $f(x)$ when x is near, but not at, the value a. It is not affected by how (or even whether) $f(a)$ is defined. Also the definition implies that the function values cannot approach two different numbers, so that if a limit exists, it is unique. These ideas are illustrated in the following examples.

EXAMPLE 2 Finding a Limit

Find $\lim_{x\to 2} g(x)$, where $g(x) = \dfrac{x^3 - 2x^2}{x - 2}$.

Method 1 Using a Table

SOLUTION

The function $g(x)$ is undefined when $x = 2$ since the value $x = 2$ makes the denominator 0. However, in determining the limit as x approaches 2 we are concerned only with the values of $g(x)$ when x is close to but *not equal to* 2. To determine if the limit exists, consider the value of g at some numbers close to but not equal to 2, as shown in the following table.

x approaches 2 from left | x approaches 2 from right

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$g(x)$	3.61	3.9601	3.996001	3.99960001		4.00040001	4.004001	4.0401	4.41

$f(x)$ approaches 4 | Undefined | $f(x)$ approaches 4

Notice that this table is almost identical to the previous table, except that g is undefined at $x = 2$. This suggests that $\lim_{x\to 2} g(x) = 4$, in spite of the fact that the function g does not exist at $x = 2$.

Method 2 Using Algebra

A second approach to this limit is to analyze the function. By factoring the numerator,

$$x^3 - 2x^2 = x^2(x - 2),$$

$g(x)$ simplifies to

$$g(x) = \frac{x^2(x-2)}{x-2} = x^2, \qquad \text{provided } x \neq 2.$$

The graph of $g(x)$, as shown in Figure 2, is almost the same as the graph of $y = x^2$, except that it is undefined at $x = 2$ (illustrated by the "hole" in the graph).

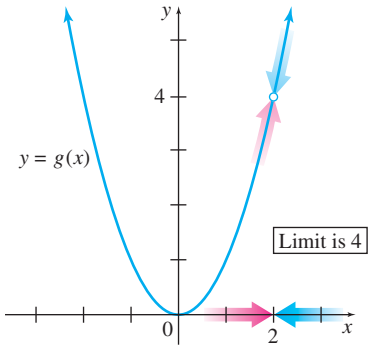


FIGURE 2

Since we are looking at the limit as x approaches 2, we look at values of the function for x close to but not equal to 2. Thus, the limit is

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} x^2 = 4.$$

YOUR TURN 2

Find $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2}$.

TRY YOUR TURN 2

TECHNOLOGY NOTE

We can use the TRACE feature on a graphing calculator to determine the limit. Figure 3 shows the graph of the function in Example 2 drawn with a graphing calculator. Notice that the function has a small gap at the point (2, 4), which agrees with our previous observation that the function is undefined at $x = 2$, where the limit is 4. (Due to the limitations of the graphing calculator, this gap may vanish when the viewing window is changed very slightly.)

The result after pressing the TRACE key is shown in Figure 4. The cursor is already located at $x = 2$; if it were not, we could use the right or left arrow key to move the cursor there. The calculator does not give a y-value because the function is undefined at $x = 2$. Moving the cursor back a step gives $x = 1.98$, $y = 3.92$. Moving the cursor forward two steps gives $x = 2.02$, $y = 4.09$. It seems that as x approaches 2, y approaches 4, or at least something close to 4. Zooming in on the point (2, 4) (such as using the window [1.9, 2.1] by [3.9, 4.1]) allows the limit to be estimated more accurately and helps ensure that the graph has no unexpected behavior very close to $x = 2$.

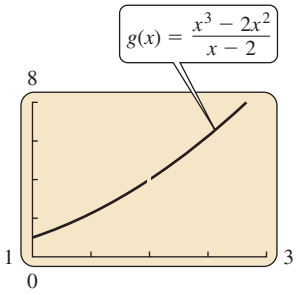


FIGURE 3

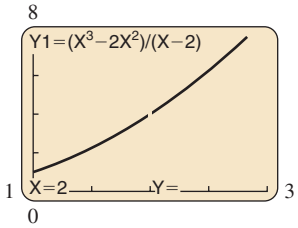


FIGURE 4

EXAMPLE 3 Finding a Limit

Determine $\lim_{x \to 2} h(x)$ for the function h defined by

$$h(x) = \begin{cases} x^2, & \text{if } x \neq 2, \\ 1, & \text{if } x = 2. \end{cases}$$

SOLUTION A function defined by two or more cases is called a **piecewise function**. The domain of h is all real numbers, and its graph is shown in Figure 5. Notice that $h(2) = 1$, but $h(x) = x^2$ when $x \neq 2$. To determine the limit as x approaches 2, we are concerned only with the values of $h(x)$ when x is close but not equal to 2. Once again,

$$\lim_{x \to 2} h(x) = \lim_{x \to 2} x^2 = 4.$$

TRY YOUR TURN 3

YOUR TURN 3
Find $\lim_{x \to 3} f(x)$ if

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 1 & \text{if } x = 3. \end{cases}$$

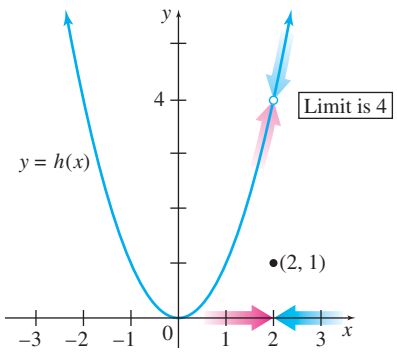


FIGURE 5

EXAMPLE 4 Finding a Limit

Find $\lim_{x \to -2} f(x)$, where

$$f(x) = \frac{3x + 2}{2x + 4}.$$

SOLUTION The graph of the function is shown in Figure 6. A table with the values of $f(x)$ as x gets closer and closer to -2 is given below.

x approaches -2 from left — x approaches -2 from right

x	-2.1	-2.01	-2.001	-2.0001	-1.9999	-1.999	-1.99	-1.9
$f(x)$	21.5	201.5	2001.5	20,001.5	$-19{,}998.5$	-1998.5	-198.5	-18.5

Both the graph and the table suggest that as x approaches -2 from the left, $f(x)$ becomes larger and larger without bound. This happens because as x approaches -2, the denominator approaches 0, while the numerator approaches -4, and -4 divided by a smaller and smaller number becomes larger and larger. When this occurs, we say that "the limit as x approaches -2 from the left is infinity," and we write

$$\lim_{x \to -2^-} f(x) = \infty.$$

Because ∞ is not a real number, the limit in this case does not exist.

In the same way, the behavior of the function as x approaches -2 from the right is indicated by writing

$$\lim_{x \to -2^+} f(x) = -\infty,$$

since $f(x)$ becomes more and more negative without bound. Since there is no real number that $f(x)$ approaches as x approaches -2 (from either side), nor does $f(x)$ approach either ∞ or $-\infty$, we simply say

$$\lim_{x \to -2} \frac{3x + 2}{2x + 4} \text{ does not exist.}$$

TRY YOUR TURN 4

YOUR TURN 4
Find $\lim_{x \to 0} \frac{2x - 1}{x}$.

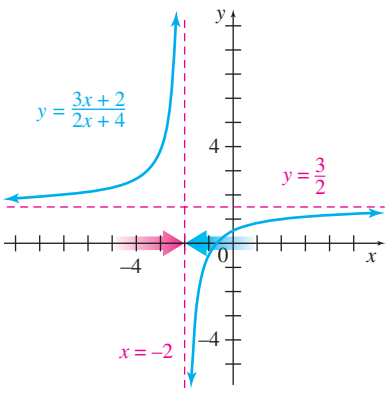


FIGURE 6

NOTE In general, if both the limit from the left and from the right approach ∞, so that $\lim_{x \to a} f(x) = \infty$, the limit would not exist because ∞ is not a real number. It is customary, however, to give ∞ as the answer since it describes how the function is behaving near $x = a$. Likewise, if $\lim_{x \to a} f(x) = -\infty$, we give $-\infty$ as the answer.

We have shown three methods for determining limits: (1) using a table of numbers, (2) using algebraic simplification, and (3) tracing the graph on a graphing calculator. Which method you choose depends on the complexity of the function and the accuracy required by the application. Algebraic simplification gives the exact answer, but it can be difficult or even impossible to use in some situations. Calculating a table of numbers or tracing the graph may be easier when the function is complicated, but be careful, because the results could be inaccurate, inconclusive, or misleading. A graphing calculator does not tell us what happens between or beyond the points that are plotted.

EXAMPLE 5 Finding a Limit

Find $\lim_{x \to 0} \dfrac{|x|}{x}$.

SOLUTION

Method 1
Algebraic Approach

The function $f(x) = |x|/x$ is not defined when $x = 0$. When $x > 0$, the definition of absolute value says that $|x| = x$, so $f(x) = |x|/x = x/x = 1$. When $x < 0$, then $|x| = -x$ and $f(x) = |x|/x = -x/x = -1$. Therefore,

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$

Since the limits from the left and from the right are different, the limit does not exist.

Method 2
Graphing Calculator Approach

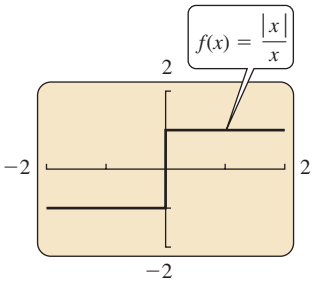


FIGURE 7

A calculator graph of f is shown in Figure 7.

As x approaches 0 from the right, x is always positive and the corresponding value of $f(x)$ is 1, so

$$\lim_{x \to 0^+} f(x) = 1.$$

But as x approaches 0 from the left, x is always negative and the corresponding value of $f(x)$ is -1, so

$$\lim_{x \to 0^-} f(x) = -1.$$

As in the algebraic approach, the limits from the left and from the right are different, so the limit does not exist.

The discussion up to this point can be summarized as follows.

Existence of Limits

The limit of f as x approaches a may not exist.

1. If $f(x)$ becomes infinitely large in magnitude (positive or negative) as x approaches the number a from either side, we write $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$. In either case, the limit does not exist.
2. If $f(x)$ becomes infinitely large in magnitude (positive) as x approaches a from one side and infinitely large in magnitude (negative) as x approaches a from the other side, then $\lim_{x \to a} f(x)$ does not exist.
3. If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = M$, and $L \neq M$, then $\lim_{x \to a} f(x)$ does not exist.

Figure 8 illustrates these three facts.

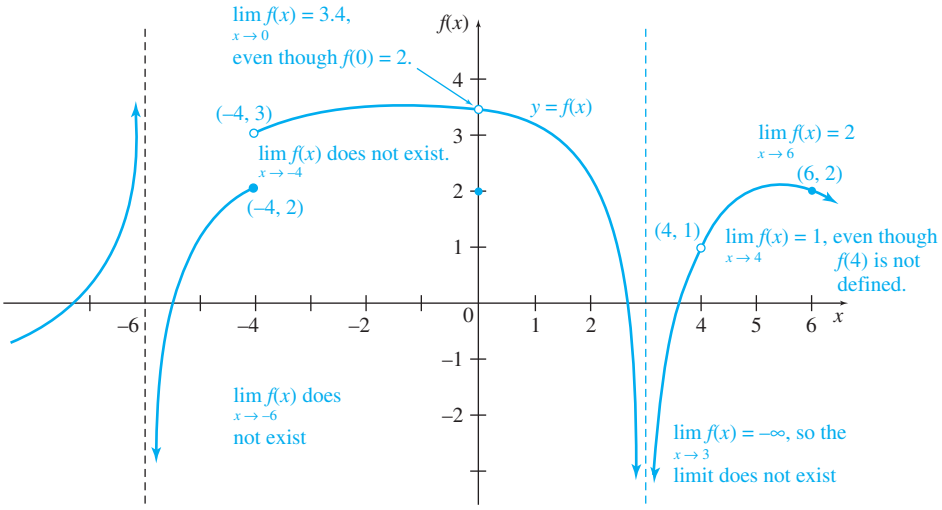


FIGURE 8

Rules for Limits As shown by the preceding examples, tables and graphs can be used to find limits. However, it is usually more efficient to use the rules for limits given below. (Proofs of these rules require a formal definition of limit, which we have not given.)

Rules for Limits

Let a, A, and B be real numbers, and let f and g be functions such that

$$\lim_{x\to a} f(x) = A \qquad \text{and} \qquad \lim_{x\to a} g(x) = B.$$

1. If k is a constant, then $\lim_{x\to a} k = k$ and $\lim_{x\to a} [k \cdot f(x)] = k \cdot \lim_{x\to a} f(x) = k \cdot A$.
2. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$
   (The limit of a sum or difference is the sum or difference of the limits.)
3. $\lim_{x\to a} [f(x) \cdot g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)] = A \cdot B$
   (The limit of a product is the product of the limits.)
4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{A}{B}$ if $B \neq 0$
   (The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.)
5. If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.
6. For any real number k, $\lim_{x\to a} [f(x)]^k = [\lim_{x\to a} f(x)]^k = A^k$, provided this limit exists.*
7. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$.
8. For any real number $b > 0$, $\lim_{x\to a} b^{f(x)} = b^{[\lim_{x\to a} f(x)]} = b^A$.
9. For any real number b such that $0 < b < 1$ or $1 < b$,
   $\lim_{x\to a} [\log_b f(x)] = \log_b [\lim_{x\to a} f(x)] = \log_b A$ if $A > 0$.

*This limit does not exist, for example, when $A < 0$ and $k = 1/2$, or when $A = 0$ and $k \leq 0$.

This list may seem imposing, but these limit rules, once understood, agree with common sense. For example, Rule 3 says that if $f(x)$ becomes close to A as x approaches a, and if $g(x)$ becomes close to B, then $f(x) \cdot g(x)$ should become close to $A \cdot B$, which seems plausible.

EXAMPLE 6 Rules for Limits

Suppose $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = 4$. Use the limit rules to find the following limits.

(a) $\lim_{x\to 2} [f(x) + 5g(x)]$

SOLUTION

$$\begin{aligned}
\lim_{x\to 2} [f(x) + 5g(x)] &= \lim_{x\to 2} f(x) + \lim_{x\to 2} 5g(x) && \text{Rule 2}\\
&= \lim_{x\to 2} f(x) + 5 \lim_{x\to 2} g(x) && \text{Rule 1}\\
&= 3 + 5(4)\\
&= 23
\end{aligned}$$

(b) $\lim_{x\to 2} \dfrac{[f(x)]^2}{\ln g(x)}$

SOLUTION

$$\begin{aligned}
\lim_{x\to 2} \frac{[f(x)]^2}{\ln g(x)} &= \frac{\lim_{x\to 2} [f(x)]^2}{\lim_{x\to 2} \ln g(x)} && \text{Rule 4}\\
&= \frac{[\lim_{x\to 2} f(x)]^2}{\ln[\lim_{x\to 2} g(x)]} && \text{Rule 6 and Rule 9}\\
&= \frac{3^2}{\ln 4}\\
&\approx \frac{9}{1.38629} \approx 6.492
\end{aligned}$$

YOUR TURN 5 Find $\lim_{x\to 2} [f(x) + g(x)]^2$.

TRY YOUR TURN 5

EXAMPLE 7 Finding a Limit

Find $\lim_{x\to 3} \dfrac{x^2 - x - 1}{\sqrt{x+1}}$.

SOLUTION

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - x - 1}{\sqrt{x+1}} &= \frac{\lim_{x\to 3} (x^2 - x - 1)}{\lim_{x\to 3} \sqrt{x+1}} && \text{Rule 4}\\
&= \frac{\lim_{x\to 3} (x^2 - x - 1)}{\sqrt{\lim_{x\to 3} (x+1)}} && \text{Rule 6 } (\sqrt{a} = a^{1/2})\\
&= \frac{3^2 - 3 - 1}{\sqrt{3+1}} && \text{Rule 5}\\
&= \frac{5}{\sqrt{4}}\\
&= \frac{5}{2}
\end{aligned}$$

As Examples 6 and 7 suggest, the rules for limits actually mean that many limits can be found simply by evaluation. This process is valid for polynomials, rational functions, exponential functions, logarithmic functions, and roots and powers, as long as this does not involve an illegal operation, such as division by 0 or taking the logarithm of a negative number. Division by 0 presents particular problems that can often be solved by algebraic simplification, as the following example shows.

EXAMPLE 8 Finding a Limit

Find $\lim_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$.

SOLUTION Rule 4 cannot be used here, since

$$\lim_{x\to 2}(x - 2) = 0.$$

The numerator also approaches 0 as x approaches 2, and $0/0$ is meaningless. For $x \neq 2$, we can, however, simplify the function by rewriting the fraction as

$$\frac{x^2 + x - 6}{x - 2} = \frac{(x + 3)(x - 2)}{x - 2} = x + 3.$$

Now Rule 7 can be used.

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2}(x + 3) = 2 + 3 = 5$$

YOUR TURN 6 Find $\lim_{x\to -3} \dfrac{x^2 - x - 12}{x + 3}$.

TRY YOUR TURN 6

NOTE Mathematicians often refer to a limit that gives $0/0$, as in Example 8, as an *indeterminate form*. This means that when the numerator and denominator are polynomials, they must have a common factor, which is why we factored the numerator in Example 8.

EXAMPLE 9 Finding a Limit

Find $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4}$.

SOLUTION As $x \to 4$, the numerator approaches 0 and the denominator also approaches 0, giving the meaningless expression $0/0$. In an expression such as this involving square roots, rather than trying to factor, you may find it simpler to use algebra to rationalize the numerator by multiplying both the numerator and the denominator by $\sqrt{x} + 2$. This gives

$$\frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{(\sqrt{x})^2 - 2^2}{(x - 4)(\sqrt{x} + 2)} \qquad (a - b)(a + b) = a^2 - b^2$$

$$= \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \frac{1}{\sqrt{x} + 2}$$

if $x \neq 4$. Now use the rules for limits.

$$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

YOUR TURN 7 Find $\lim_{x\to 1} \dfrac{\sqrt{x} - 1}{x - 1}$.

TRY YOUR TURN 7

CAUTION Simply because the expression in a limit is approaching $0/0$, as in Examples 8 and 9, does *not* mean that the limit is 0 or that the limit does not exist. For such a limit, try to simplify the expression using the following principle: **To calculate the limit of $f(x)/g(x)$ as x approaches a, where $f(a) = g(a) = 0$, you should attempt to factor $x - a$ from both the numerator and the denominator.**

EXAMPLE 10 Finding a Limit

Find $\lim_{x \to 1} \dfrac{x^2 - 2x + 1}{(x - 1)^3}$.

Method 1 Algebraic Approach

SOLUTION

Again, Rule 4 cannot be used, since $\lim_{x \to 1} (x - 1)^3 = 0$. If $x \neq 1$, the function can be rewritten as

$$\frac{x^2 - 2x + 1}{(x - 1)^3} = \frac{(x - 1)^2}{(x - 1)^3} = \frac{1}{x - 1}.$$

Then

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{(x - 1)^3} = \lim_{x \to 1} \frac{1}{x - 1}$$

by Rule 7. None of the rules can be used to find

$$\lim_{x \to 1} \frac{1}{x - 1},$$

but as x approaches 1, the denominator approaches 0 while the numerator stays at 1, making the result larger and larger in magnitude. If $x > 1$, both the numerator and denominator are positive, so $\lim_{x \to 1^+} 1/(x - 1) = \infty$. If $x < 1$, the denominator is negative, so $\lim_{x \to 1^-} 1/(x - 1) = -\infty$. Therefore,

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{(x - 1)^3} = \lim_{x \to 1} \frac{1}{x - 1} \text{ does not exist.}$$

Method 2 Graphing Calculator Approach

Using the TABLE feature on a TI-84 Plus, we can produce the table of numbers shown in Figure 9, where Y_1 represents the function $y = 1/(x - 1)$. Figure 10 shows a graphing calculator view of the function on $[0, 2]$ by $[-10, 10]$. The behavior of the function indicates a vertical asymptote at $x = 1$, with the limit approaching $-\infty$ from the left and ∞ from the right, so

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{(x - 1)^3} = \lim_{x \to 1} \frac{1}{x - 1} \text{ does not exist.}$$

Both the table and the graph can be easily generated using a spreadsheet. Consult the *Graphing Calculator and Excel Spreadsheet Manual*, available with this text, for details.

X	Y1
.9	-10
.99	-100
.999	-1000
.9999	-10000
1.0001	10000
1.001	1000
1.1	10

X=1.1

FIGURE 9

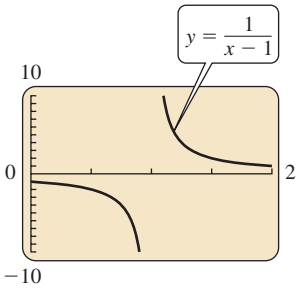


FIGURE 10

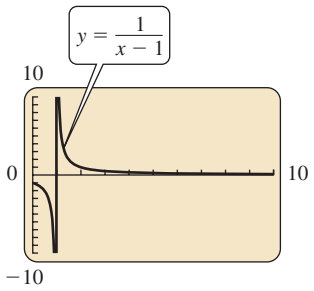


FIGURE 11

CAUTION A graphing calculator can give a deceptive view of a function. Figure 11 shows the result if we graph the previous function on $[0, 10]$ by $[-10, 10]$. Near $x = 1$, the graph appears to be a steep line connecting the two pieces. The graph in Figure 10 is more representative of the function near $x = 1$. When using a graphing calculator, you may need to experiment with the viewing window, guided by what you have learned about functions and limits, to get a good picture of a function. On many calculators, extraneous lines connecting parts of the graph can be avoided by using DOT mode rather than CONNECTED mode.

NOTE Another way to understand the behavior of the function in the previous example near $x = 1$ is to recall from the section on Polynomial and Rational Functions that a rational function often has a vertical asymptote at a value of x where the denominator is 0, although it may not if the numerator there is also 0. In this example, we see after simplifying that the function has a vertical asymptote at $x = 1$ because that would make the denominator of $1/(x - 1)$ equal to 0, while the numerator is 1.

Limits at Infinity

Sometimes it is useful to examine the behavior of the values of $f(x)$ as x gets larger and larger (or more and more negative). The phrase "x approaches infinity," written $x \to \infty$, expresses the fact that x becomes larger without bound. Similarly, the phrase "x approaches negative infinity" (symbolically, $x \to -\infty$) means that x becomes more and more negative without bound (such as -10, -1000, $-10{,}000$, etc.). The next example illustrates a **limit at infinity**.

EXAMPLE 11 Oxygen Concentration

Suppose a small pond normally contains 12 units of dissolved oxygen in a fixed volume of water. Suppose also that at time $t = 0$ a quantity of organic waste is introduced into the pond, with the oxygen concentration t weeks later given by

$$f(t) = \frac{12t^2 - 15t + 12}{t^2 + 1}.$$

As time goes on, what will be the ultimate concentration of oxygen? Will it return to 12 units?

SOLUTION After 2 weeks, the pond contains

$$f(2) = \frac{12 \cdot 2^2 - 15 \cdot 2 + 12}{2^2 + 1} = \frac{30}{5} = 6$$

units of oxygen, and after 4 weeks, it contains

$$f(4) = \frac{12 \cdot 4^2 - 15 \cdot 4 + 12}{4^2 + 1} \approx 8.5$$

units. Choosing several values of t and finding the corresponding values of $f(t)$, or using a graphing calculator or computer, leads to the table and graph in Figure 12.

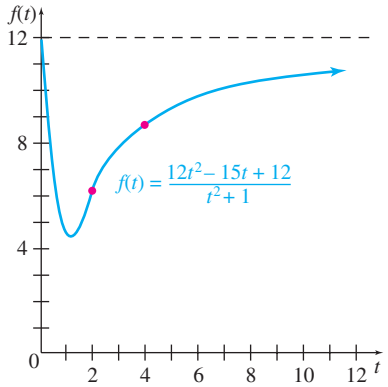
The graph suggests that, as time goes on, the oxygen level gets closer and closer to the original 12 units. If so, the line $y = 12$ is a horizontal asymptote. The table suggests that

$$\lim_{t \to \infty} f(t) = 12.$$

Thus, the oxygen concentration will approach 12, but it will never be *exactly* 12.

x	$f(x)$
10	10.515
100	11.85
1000	11.985
10,000	11.9985
100,000	11.99985

(a)



(b)

FIGURE 12

As we saw in the previous example, *limits at infinity or negative infinity, if they exist, correspond to horizontal asymptotes of the graph of the function*. In the previous chapter, we saw one way to find horizontal asymptotes. We will now show a more precise way, based upon some simple limits at infinity. The graphs of $f(x) = 1/x$ (in red) and $g(x) = 1/x^2$ (in blue) shown in Figure 13, as well as the table there, indicate that $\lim_{x \to \infty} 1/x = 0$, $\lim_{x \to -\infty} 1/x = 0$, $\lim_{x \to \infty} 1/x^2 = 0$, and $\lim_{x \to -\infty} 1/x^2 = 0$, suggesting the following rule.

x	$\frac{1}{x}$	$\frac{1}{x^2}$
−100	−0.01	0.0001
−10	−0.1	0.01
−1	−1	1
1	1	1
10	0.1	0.01
100	0.01	0.0001

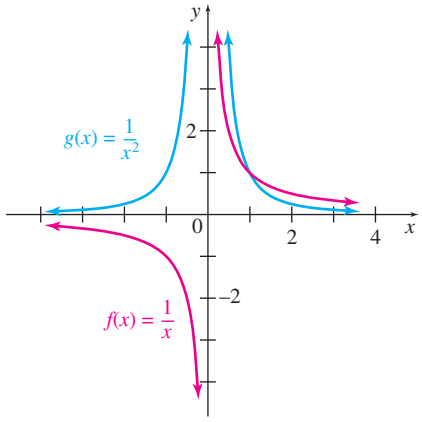


FIGURE 13

Limits at Infinity

For any positive real number n,

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0.^*$$

*If x is negative, x^n does not exist for certain values of n, so the second limit is undefined for those values of n.

The rules for limits given earlier remain unchanged when a is replaced with ∞ or $-\infty$.

To evaluate the limit at infinity of a rational function, divide the numerator and denominator by the largest power of the variable that appears in the denominator, t^2 here, and then use these results. In the previous example, we find that

$$\lim_{t\to\infty}\frac{12t^2-15t+12}{t^2+1}=\lim_{t\to\infty}\frac{\frac{12t^2}{t^2}-\frac{15t}{t^2}+\frac{12}{t^2}}{\frac{t^2}{t^2}+\frac{1}{t^2}}$$

$$=\lim_{t\to\infty}\frac{12-15\cdot\frac{1}{t}+12\cdot\frac{1}{t^2}}{1+\frac{1}{t^2}}.$$

FOR REVIEW

In Section 10.3, we saw a way to find horizontal asymptotes by considering the behavior of the function as x (or t) gets large. For large t, $12t^2-15t+12\approx 12t^2$, because the t-term and the constant term are small compared with the t^2-term when t is large. Similarly, $t^2+1\approx t^2$. Thus, for large t, $f(t)=\dfrac{12t^2-15t+12}{t^2+1}\approx\dfrac{12t^2}{t^2}=12$. Thus the function f has a horizontal asymptote at $y=12$.

Now apply the limit rules and the fact that $\lim_{t\to\infty}1/t^n=0$.

$$\frac{\lim_{t\to\infty}\left(12-15\cdot\frac{1}{t}+12\cdot\frac{1}{t^2}\right)}{\lim_{t\to\infty}\left(1+\frac{1}{t^2}\right)}$$

$$=\frac{\lim_{t\to\infty}12-\lim_{t\to\infty}15\cdot\frac{1}{t}+\lim_{t\to\infty}12\cdot\frac{1}{t^2}}{\lim_{t\to\infty}1+\lim_{t\to\infty}\frac{1}{t^2}}$$ Rules 4 and 2

$$=\frac{12-15\left(\lim_{t\to\infty}\frac{1}{t}\right)+12\left(\lim_{t\to\infty}\frac{1}{t^2}\right)}{1+\lim_{t\to\infty}\frac{1}{t^2}}$$ Rule 1

$$=\frac{12-15\cdot 0+12\cdot 0}{1+0}=12.$$ Limits at infinity

EXAMPLE 12 Limits at Infinity

Find each limit.

(a) $\lim_{x\to\infty}\dfrac{8x+6}{3x-1}$

SOLUTION We can use the rule $\lim_{x\to\infty}1/x^n=0$ to find this limit by first dividing the numerator and denominator by x, as follows.

$$\lim_{x\to\infty}\frac{8x+6}{3x-1}=\lim_{x\to\infty}\frac{\frac{8x}{x}+\frac{6}{x}}{\frac{3x}{x}-\frac{1}{x}}=\lim_{x\to\infty}\frac{8+6\cdot\frac{1}{x}}{3-\frac{1}{x}}=\frac{8+0}{3-0}=\frac{8}{3}$$

(b) $\lim_{x\to\infty}\dfrac{3x+2}{4x^3-1}=\lim_{x\to\infty}\dfrac{3\cdot\frac{1}{x^2}+2\cdot\frac{1}{x^3}}{4-\frac{1}{x^3}}=\dfrac{0+0}{4-0}=\dfrac{0}{4}=0$

Here, the highest power of x in the denominator is x^3, which is used to divide each term in the numerator and denominator.

(c) $\lim\limits_{x\to\infty}\dfrac{3x^2+2}{4x-3}=\lim\limits_{x\to\infty}\dfrac{3x+\dfrac{2}{x}}{4-\dfrac{3}{x}}$

The highest power of x in the denominator is x (to the first power). There is a higher power of x in the numerator, but we don't divide by this. Notice that the denominator approaches 4, while the numerator becomes infinitely large, so

$$\lim_{x\to\infty}\frac{3x^2+2}{4x-3}=\infty.$$

(d) $\lim\limits_{x\to\infty}\dfrac{5x^2-4x^3}{3x^2+2x-1}=\lim\limits_{x\to\infty}\dfrac{5-4x}{3+\dfrac{2}{x}-\dfrac{1}{x^2}}$

The highest power of x in the denominator is x^2. The denominator approaches 3, while the numerator becomes a negative number that is larger and larger in magnitude, so

$$\lim_{x\to\infty}\frac{5x^2-4x^3}{3x^2+2x-1}=-\infty.$$

YOUR TURN 8 Find $\lim\limits_{x\to\infty}\dfrac{2x^2+3x-4}{6x^2-5x+7}$.

TRY YOUR TURN 8

The method used in Example 12 is a useful way to rewrite expressions with fractions so that the rules for limits at infinity can be used.

Finding Limits at Infinity

If $f(x)=p(x)/q(x)$, for polynomials $p(x)$ and $q(x)$, $q(x)\neq 0$, $\lim\limits_{x\to-\infty}f(x)$ and $\lim\limits_{x\to\infty}f(x)$ can be found as follows.

1. Divide $p(x)$ and $q(x)$ by the highest power of x in $q(x)$.
2. Use the rules for limits, including the rules for limits at infinity,

$$\lim_{x\to\infty}\frac{1}{x^n}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x^n}=0,$$

to find the limit of the result from step 1.

For an alternate approach to finding limits at infinity, see Exercise 81.

11.1 EXERCISES

In Exercises 1–4, choose the best answer for each limit.

1. If $\lim\limits_{x\to2^-}f(x)=5$ and $\lim\limits_{x\to2^+}f(x)=6$, then $\lim\limits_{x\to2}f(x)$

a. is 5. **b.** is 6.

c. does not exist. **d.** is infinite.

2. If $\lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^+}f(x)=-1$, but $f(2)=1$, then $\lim\limits_{x\to2}f(x)$

a. is -1. **b.** does not exist.

c. is infinite. **d.** is 1.

3. If $\lim\limits_{x\to4^-}f(x)=\lim\limits_{x\to4^+}f(x)=6$, but $f(4)$ does not exist, then $\lim\limits_{x\to4}f(x)$

a. does not exist. **b.** is 6.

c. is $-\infty$. **d.** is ∞.

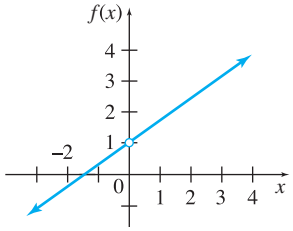
4. If $\lim\limits_{x\to1^-}f(x)=-\infty$ and $\lim\limits_{x\to1^+}f(x)=-\infty$, then $\lim\limits_{x\to1}f(x)$

a. is ∞. **b.** is $-\infty$.

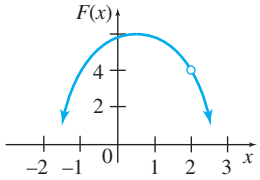
c. does not exist. **d.** is 1.

Decide whether each limit exists. If a limit exists, estimate its value.

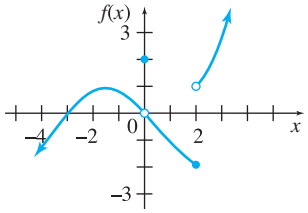
5. a. $\lim_{x\to 3} f(x)$ **b.** $\lim_{x\to 0} f(x)$



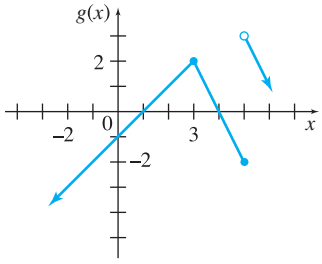
6. a. $\lim_{x\to 2} F(x)$ **b.** $\lim_{x\to -1} F(x)$



7. a. $\lim_{x\to 0} f(x)$ **b.** $\lim_{x\to 2} f(x)$

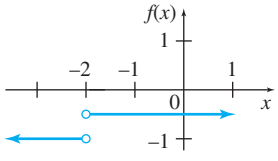


8. a. $\lim_{x\to 3} g(x)$ **b.** $\lim_{x\to 5} g(x)$

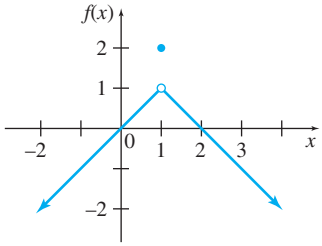


In Exercises 9 and 10, use the graph to find (i) $\lim_{x\to a} f(x)$, (ii) $\lim_{x\to a^-} f(x)$, (iii) $\lim_{x\to a^+} f(x)$, and (iv) $f(a)$ if it exists.

9. a. $a = -2$ **b.** $a = -1$

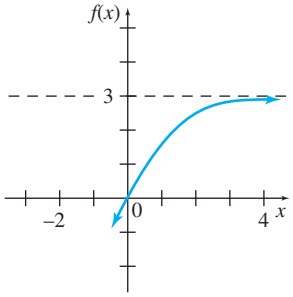


10. a. $a = 1$ **b.** $a = 2$

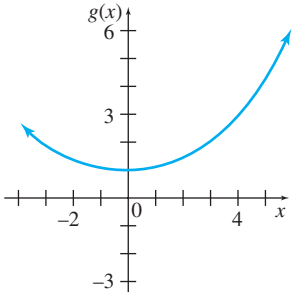


Decide whether each limit exists. If a limit exists, find its value.

11. $\lim_{x\to\infty} f(x)$



12. $\lim_{x\to -\infty} g(x)$



13. Explain why $\lim_{x\to 2} F(x)$ in Exercise 6 exists, but $\lim_{x\to -2} f(x)$ in Exercise 9 does not.

14. In Exercise 10, why does $\lim_{x\to 1} f(x) = 1$, even though $f(1) = 2$?

15. Use the table of values to estimate $\lim_{x\to 1} f(x)$.

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x)$	3.9	3.99	3.999	3.9999	4.0001	4.001	4.01	4.1

Complete the tables and use the results to find the indicated limits.

16. If $f(x) = 2x^2 - 4x + 7$, find $\lim_{x\to 1} f(x)$.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$			5.000002	5.000002		

17. If $k(x) = \dfrac{x^3 - 2x - 4}{x - 2}$, find $\lim_{x \to 2} k(x)$.

x	1.9	1.99	1.999	2.001	2.01	2.1
$k(x)$						

18. If $f(x) = \dfrac{2x^3 + 3x^2 - 4x - 5}{x + 1}$, find $\lim_{x \to -1} f(x)$.

x	−1.1	−1.01	−1.001	−0.999	−0.99	−0.9
$f(x)$						

19. If $h(x) = \dfrac{\sqrt{x} - 2}{x - 1}$, find $\lim_{x \to 1} h(x)$.

x	0.9	0.99	0.999	1.001	1.01	1.1
$h(x)$						

20. If $f(x) = \dfrac{\sqrt{x} - 3}{x - 3}$, find $\lim_{x \to 3} f(x)$.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

Let $\lim_{x \to 4} f(x) = 9$ and $\lim_{x \to 4} g(x) = 27$. Use the limit rules to find each limit.

21. $\lim_{x \to 4}[f(x) - g(x)]$

22. $\lim_{x \to 4}[g(x) \cdot f(x)]$

23. $\lim_{x \to 4} \dfrac{f(x)}{g(x)}$

24. $\lim_{x \to 4} \log_3 f(x)$

25. $\lim_{x \to 4} \sqrt{f(x)}$

26. $\lim_{x \to 4} \sqrt[3]{g(x)}$

27. $\lim_{x \to 4} 2^{f(x)}$

28. $\lim_{x \to 4}[1 + f(x)]^2$

29. $\lim_{x \to 4} \dfrac{f(x) + g(x)}{2g(x)}$

30. $\lim_{x \to 4} \dfrac{5g(x) + 2}{1 - f(x)}$

Use the properties of limits to help decide whether each limit exists. If a limit exists, find its value.

31. $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

32. $\lim_{x \to -2} \dfrac{x^2 - 4}{x + 2}$

33. $\lim_{x \to 1} \dfrac{5x^2 - 7x + 2}{x^2 - 1}$

34. $\lim_{x \to -3} \dfrac{x^2 - 9}{x^2 + x - 6}$

35. $\lim_{x \to -2} \dfrac{x^2 - x - 6}{x + 2}$

36. $\lim_{x \to 5} \dfrac{x^2 - 3x - 10}{x - 5}$

37. $\lim_{x \to 0} \dfrac{1/(x + 3) - 1/3}{x}$

38. $\lim_{x \to 0} \dfrac{-1/(x + 2) + 1/2}{x}$

39. $\lim_{x \to 25} \dfrac{\sqrt{x} - 5}{x - 25}$

40. $\lim_{x \to 36} \dfrac{\sqrt{x} - 6}{x - 36}$

41. $\lim_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

42. $\lim_{h \to 0} \dfrac{(x + h)^3 - x^3}{h}$

43. $\lim_{x \to \infty} \dfrac{3x}{7x - 1}$

44. $\lim_{x \to -\infty} \dfrac{8x + 2}{4x - 5}$

45. $\lim_{x \to -\infty} \dfrac{3x^2 + 2x}{2x^2 - 2x + 1}$

46. $\lim_{x \to \infty} \dfrac{x^2 + 2x - 5}{3x^2 + 2}$

47. $\lim_{x \to \infty} \dfrac{3x^3 + 2x - 1}{2x^4 - 3x^3 - 2}$

48. $\lim_{x \to \infty} \dfrac{2x^2 - 1}{3x^4 + 2}$

49. $\lim_{x \to \infty} \dfrac{2x^3 - x - 3}{6x^2 - x - 1}$

50. $\lim_{x \to \infty} \dfrac{x^4 - x^3 - 3x}{7x^2 + 9}$

51. $\lim_{x \to \infty} \dfrac{2x^2 - 7x^4}{9x^2 + 5x - 6}$

52. $\lim_{x \to \infty} \dfrac{-5x^3 - 4x^2 + 8}{6x^2 + 3x + 2}$

53. Let $f(x) = \begin{cases} x^3 + 2 & \text{if } x \neq -1 \\ 5 & \text{if } x = -1 \end{cases}$. Find $\lim_{x \to -1} f(x)$.

54. Let $g(x) = \begin{cases} 0 & \text{if } x = -2 \\ \frac{1}{2}x^2 - 3 & \text{if } x \neq -2 \end{cases}$. Find $\lim_{x \to -2} g(x)$.

55. Let $f(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 2 & \text{if } 3 \leq x \leq 5 \\ x + 3 & \text{if } x > 5 \end{cases}$.

a. Find $\lim_{x \to 3} f(x)$. **b.** Find $\lim_{x \to 5} f(x)$.

56. Let $g(x) = \begin{cases} 5 & \text{if } x < 0 \\ x^2 - 2 & \text{if } 0 \leq x \leq 3 \\ 7 & \text{if } x > 3 \end{cases}$.

a. Find $\lim_{x \to 0} g(x)$. **b.** Find $\lim_{x \to 3} g(x)$.

In Exercises 57–60, calculate the limit in the specified exercise, using a table such as in Exercises 15–20. Verify your answer by using a graphing calculator to zoom in on the point on the graph.

57. Exercise 31

58. Exercise 32

59. Exercise 33

60. Exercise 34

61. Let $F(x) = \dfrac{3x}{(x + 2)^3}$.

a. Find $\lim_{x \to -2} F(x)$.

b. Find the vertical asymptote of the graph of $F(x)$.

c. Compare your answers for parts a and b. What can you conclude?

62. Let $G(x) = \dfrac{-6}{(x - 4)^2}$.

a. Find $\lim_{x \to 4} G(x)$.

b. Find the vertical asymptote of the graph of $G(x)$.

c. Compare your answers for parts a and b. Are they related? How?

63. How can you tell that the graph in Figure 10 is more representative of the function $f(x) = 1/(x - 1)$ than the graph in Figure 11?

64. A friend who is confused about limits wonders why you investigate the value of a function closer and closer to a point, instead of just finding the value of a function at the point. How would you respond?

65. Use a graph of $f(x) = e^x$ to answer the following questions.

a. Find $\lim_{x \to -\infty} e^x$.

b. Where does the function e^x have a horizontal asymptote?

66. Use a graphing calculator to answer the following questions.

a. From a graph of $y = xe^{-x}$, what do you think is the value of $\lim_{x \to \infty} xe^{-x}$? Support this by evaluating the function for several large values of x.

b. Repeat part a, this time using the graph of $y = x^2e^{-x}$.

c. Based on your results from parts a and b, what do you think is the value of $\lim_{x \to \infty} x^ne^{-x}$, where n is a positive integer? Support this by experimenting with other positive integers n.

67. Use a graph of $f(x) = \ln x$ to answer the following questions.

a. Find $\lim_{x \to 0^+} \ln x$.

b. Where does the function $\ln x$ have a vertical asymptote?

68. Use a graphing calculator to answer the following questions.

a. From a graph of $y = x \ln x$, what do you think is the value of $\lim_{x \to 0^+} x \ln x$? Support this by evaluating the function for several small values of x.

b. Repeat part a, this time using the graph of $y = x(\ln x)^2$.

c. Based on your results from parts a and b, what do you think is the value of $\lim_{x \to 0^+} x(\ln x)^n$, where n is a positive integer? Support this by experimenting with other positive integers n.

69. Explain in your own words why the rules for limits at infinity should be true.

70. Explain in your own words what Rule 4 for limits means.

Find each of the following limits (a) by investigating values of the function near the x-value where the limit is taken, and (b) using a graphing calculator to view the function near that value of x.

71. $\lim_{x \to 1} \dfrac{x^4 + 4x^3 - 9x^2 + 7x - 3}{x - 1}$

72. $\lim_{x \to 2} \dfrac{x^4 + x - 18}{x^2 - 4}$

73. $\lim_{x \to -1} \dfrac{x^{1/3} + 1}{x + 1}$

74. $\lim_{x \to 4} \dfrac{x^{3/2} - 8}{x + x^{1/2} - 6}$

Use a graphing calculator to graph the function. (a) Determine the limit from the graph. (b) Explain how your answer could be determined from the expression for $f(x)$.

75. $\lim_{x \to \infty} \dfrac{\sqrt{9x^2 + 5}}{2x}$

76. $\lim_{x \to -\infty} \dfrac{\sqrt{9x^2 + 5}}{2x}$

77. $\lim_{x \to -\infty} \dfrac{\sqrt{36x^2 + 2x + 7}}{3x}$

78. $\lim_{x \to \infty} \dfrac{\sqrt{36x^2 + 2x + 7}}{3x}$

79. $\lim_{x \to \infty} \dfrac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5}$

80. $\lim_{x \to -\infty} \dfrac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5}$

81. Explain why the following rules can be used to find $\lim_{x \to \infty} [p(x)/q(x)]$:

a. If the degree of $p(x)$ is less than the degree of $q(x)$, the limit is 0.

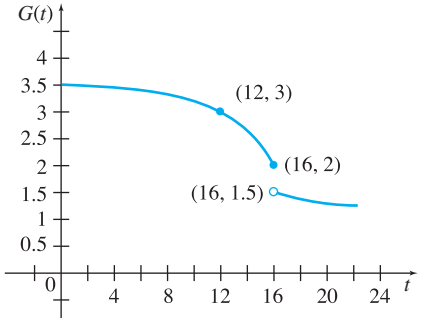
b. If the degree of $p(x)$ is equal to the degree of $q(x)$, the limit is A/B, where A and B are the leading coefficients of $p(x)$ and $q(x)$, respectively.

c. If the degree of $p(x)$ is greater than the degree of $q(x)$, the limit is ∞ or $-\infty$.

APPLICATIONS

Business and Economics

82. APPLY IT Consumer Demand When the price of an essential commodity (such as gasoline) rises rapidly, consumption drops slowly at first. If the price continues to rise, however, a "tipping" point may be reached, at which consumption takes a sudden substantial drop. Suppose the accompanying graph shows the consumption of gasoline, $G(t)$, in millions of gallons, in a certain area. We assume that the price is rising rapidly. Here t is time in months after the price began rising. Use the graph to find the following.



a. $\lim_{t \to 12} G(t)$ **b.** $\lim_{t \to 16} G(t)$

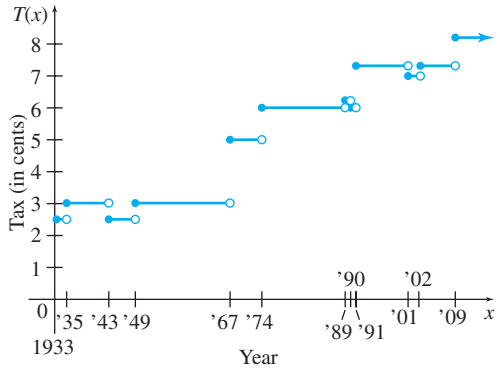
c. $G(16)$ **d.** The tipping point (in months)

83. Sales Tax Officials in California tend to raise the sales tax in years in which the state faces a budget deficit and then cut the tax when the state has a surplus. The graph below shows the California state sales tax since it was first established in 1933. Let $T(x)$ represent the sales tax per dollar spent in year x. Find the following. ***Source: California State.***

a. $\lim_{x \to 53} T(x)$ **b.** $\lim_{x \to 09^-} T(x)$

c. $\lim_{x \to 09^+} T(x)$ **d.** $\lim_{x \to 09} T(x)$

e. $T(09)$



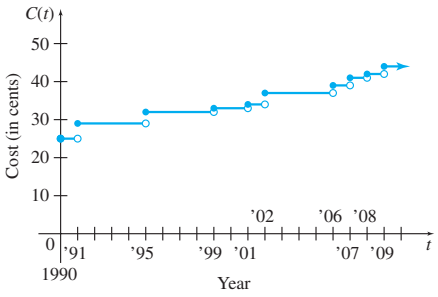
84. Postage The graph below shows how the postage required to mail a letter in the United States has changed in recent years. Let $C(t)$ be the cost to mail a letter in the year t. Find the following. ***Source: United States Postal Service***.

a. $\lim_{t \to 2009^-} C(t)$

b. $\lim_{t \to 2009^+} C(t)$

c. $\lim_{t \to 2009} C(t)$

d. $C(2009)$



85. Average Cost The cost (in dollars) for manufacturing a particular DVD is

$$C(x) = 15{,}000 + 6x,$$

where x is the number of DVDs produced. Recall from the previous chapter that the average cost per DVD, denoted by $\overline{C}(x)$, is found by dividing $C(x)$ by x. Find and interpret $\lim_{x \to \infty} \overline{C}(x)$.

86. Average Cost In Chapter 1, we saw that the cost to fly x miles on American Airlines could be approximated by the equation

$$C(x) = 0.0738x + 111.83.$$

Recall from the previous chapter that the average cost per mile, denoted by $\overline{C}(x)$, is found by dividing $C(x)$ by x. Find and interpret $\lim_{x \to \infty} \overline{C}(x)$. ***Source: American Airlines***.

87. Employee Productivity A company training program has determined that, on the average, a new employee produces $P(s)$ items per day after s days of on-the-job training, where

$$P(s) = \frac{63s}{s + 8}.$$

Find and interpret $\lim_{s \to \infty} P(s)$.

88. Preferred Stock In business finance, an annuity is a series of equal payments received at equal intervals for a finite period of time. The *present value* of an n-period annuity takes the form

$$P = R\left[\frac{1 - (1 + i)^{-n}}{i}\right],$$

where R is the amount of the periodic payment and i is the fixed interest rate per period. Many corporations raise money by issuing preferred stock. Holders of the preferred stock, called a *perpetuity*, receive payments that take the form of an annuity in that the amount of the payment never changes. However, normally the payments for preferred stock do not end but theoretically continue forever. Find the limit of this present value equation as n approaches infinity to derive a formula for the present value of a share of preferred stock paying a periodic dividend R. ***Source: Robert D. Campbell***.

89. Growing Annuities For some annuities encountered in business finance, called *growing annuities*, the amount of the periodic payment is not constant but grows at a constant periodic rate. Leases with escalation clauses can be examples of growing annuities. The present value of a growing annuity takes the form

$$P = \frac{R}{i - g}\left[1 - \left(\frac{1 + g}{1 + i}\right)^n\right],$$

where

R = amount of the next annuity payment,
g = expected constant annuity growth rate,
i = required periodic return at the time the annuity is evaluated,
n = number of periodic payments.

A corporation's common stock may be thought of as a claim on a growing annuity where the annuity is the company's annual dividend. However, in the case of common stock, these payments have no contractual end but theoretically continue forever. Compute the limit of the expression above as n approaches infinity to derive the Gordon–Shapiro Dividend Model popularly used to estimate the value of common stock. Make the reasonable assumption that $i > g$. (*Hint:* What happens to a^n as $n \to \infty$ if $0 < a < 1$?) ***Source: Robert D. Campbell***.

Life Sciences

90. Alligator Teeth Researchers have developed a mathematical model that can be used to estimate the number of teeth $N(t)$ at time t (days of incubation) for *Alligator mississippiensis*, where

$$N(t) = 71.8e^{-8.96e^{-0.0685t}}.$$

Source: Journal of Theoretical Biology.

a. Find $N(65)$, the number of teeth of an alligator that hatched after 65 days.

b. Find $\lim_{t \to \infty} N(t)$ and use this value as an estimate of the number of teeth of a newborn alligator. (*Hint:* See Exercise 65.) Does this estimate differ significantly from the estimate of part a?

91. Sediment To develop strategies to manage water quality in polluted lakes, biologists must determine the depths of sediments and the rate of sedimentation. It has been determined that the depth of sediment $D(t)$ (in centimeters) with respect to time (in years before 1990) for Lake Coeur d'Alene, Idaho, can be estimated by the equation

$$D(t) = 155(1 - e^{-0.0133t}).$$

Source: Mathematics Teacher.

a. Find $D(20)$ and interpret.

b. Find $\lim_{t \to \infty} D(t)$ and interpret.

92. Drug Concentration The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{0.17h}{h^2 + 2}.$$

Find and interpret $\lim_{h\to\infty} A(h)$.

Social Sciences

93. Legislative Voting Members of a legislature often must vote repeatedly on the same bill. As time goes on, members may change their votes. Suppose that p_0 is the probability that an individual legislator favors an issue before the first roll call vote, and suppose that p is the probability of a change in position from one vote to the next. Then the probability that the legislator will vote "yes" on the nth roll call is given by

$$p_n = \frac{1}{2} + \left(p_0 - \frac{1}{2}\right)(1 - 2p)^n.$$

For example, the chance of a "yes" on the third roll call vote is

$$p_3 = \frac{1}{2} + \left(p_0 - \frac{1}{2}\right)(1 - 2p)^3.$$

Source: Mathematics in the Behavioral and Social Sciences.

Suppose that there is a chance of $p_0 = 0.7$ that Congressman Stephens will favor the budget appropriation bill before the first roll call, but only a probability of $p = 0.2$ that he will change his mind on the subsequent vote. Find and interpret the following.

a. p_2 **b.** p_4

c. p_8 **d.** $\lim_{n\to\infty} p_n$

YOUR TURN ANSWERS

1. 3 **2.** 4 **3.** 5 **4.** Does not exist. **5.** 49
6. -7 **7.** 1/2 **8.** 1/3

11.2 Continuity

APPLY IT **How does the average cost per day of a rental car change with the number of days the car is rented?**

We will answer this question in Exercise 38.

In 2009, Congress passed legislation raising the federal minimum wage for the third time in three years. Figure 14 below shows how that wage has varied since it was instituted in 1938. We will denote this function by $f(t)$, where t is the year. ***Source: U.S. Department of Labor.***

Notice from the graph that $\lim_{t\to 1997^-} f(t) = 4.75$ and that $\lim_{t\to 1997^+} f(t) = 5.15$, so that $\lim_{t\to 1997} f(t)$ does not exist. Notice also that $f(1997) = 5.15$. A point such as this, where a function has a sudden sharp break, is a point where the function is *discontinuous*. In this case, the discontinuity is caused by the jump in the minimum wage from \$4.75 per hour to \$5.15 per hour in 1997.

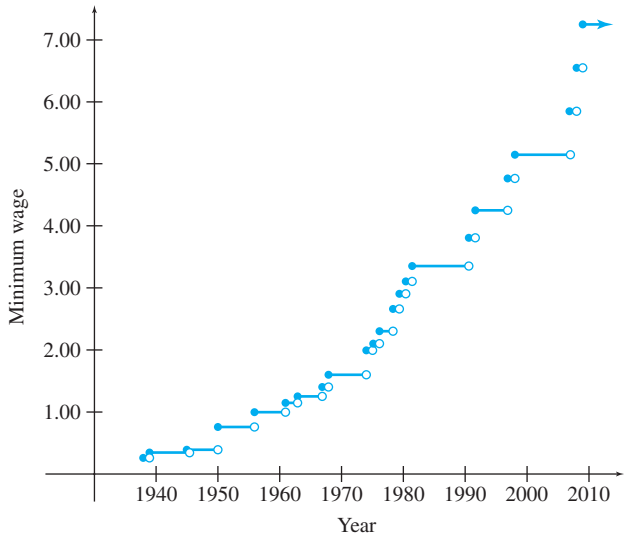


FIGURE 14

Intuitively speaking, a function is *continuous* at a point if you can draw the graph of the function in the vicinity of that point without lifting your pencil from the paper. As we already mentioned, this would not be possible in Figure 14 if it were drawn correctly; there would be a break in the graph at $t = 1997$, for example. Conversely, a function is discontinuous at any x-value where the pencil *must* be lifted from the paper in order to draw the graph on both sides of the point. A more precise definition is as follows.

Continuity at $x = c$

A function f is **continuous** at $x = c$ if the following three conditions are satisfied:

1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

If f is not continuous at c, it is **discontinuous** there.

The following example shows how to check a function for continuity at a specific point. We use a three-step test, and if any step of the test fails, the function is not continuous at that point.

EXAMPLE 1 Continuity

Determine if each function is continuous at the indicated x-value.

(a) $f(x)$ in Figure 15 at $x = 3$

SOLUTION

Step 1 Does the function exist at $x = 3$?

The open circle on the graph of Figure 15 at the point where $x = 3$ means that $f(x)$ does not exist at $x = 3$. Since the function does not pass the first test, it is discontinuous at $x = 3$, and there is no need to proceed to Step 2.

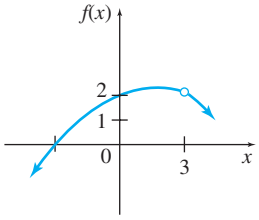


FIGURE 15

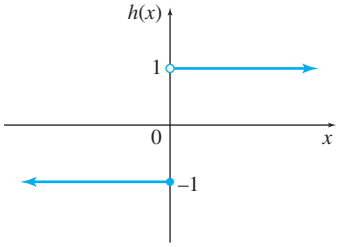


FIGURE 16

(b) $h(x)$ in Figure 16 at $x = 0$

SOLUTION

Step 1 Does the function exist at $x = 0$?

According to the graph in Figure 16, $h(0)$ exists and is equal to -1.

Step 2 Does the limit exist at $x = 0$?

As x approaches 0 from the left, $h(x)$ is -1. As x approaches 0 from the right, however, $h(x)$ is 1. In other words,

$$\lim_{x \to 0^-} h(x) = -1,$$

while

$$\lim_{x \to 0^+} h(x) = 1.$$

Since no single number is approached by the values of $h(x)$ as x approaches 0, the limit $\lim_{x \to 0} h(x)$ does not exist. Since the function does not pass the second test, it is discontinuous at $x = 0$, and there is no need to proceed to Step 3.

(c) $g(x)$ in Figure 17 at $x = 4$

SOLUTION

Step 1 Is the function defined at $x = 4$?

In Figure 17, the heavy dot above 4 shows that $g(4)$ is defined. In fact, $g(4) = 1$.

Step 2 Does the limit exist at $x = 4$?

The graph shows that

$$\lim_{x \to 4^-} g(x) = -2, \text{ and } \lim_{x \to 4^+} g(x) = -2.$$

Therefore, the limit exists at $x = 4$ and

$$\lim_{x \to 4} g(x) = -2.$$

Step 3 Does $g(4) = \lim_{x \to 4} g(x)$?

Using the results of Step 1 and Step 2, we see that $g(4) \neq \lim_{x \to 4} g(x)$. Since the function does not pass the third test, it is discontinuous at $x = 4$.

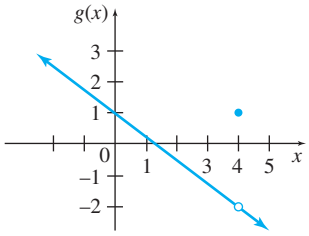


FIGURE 17

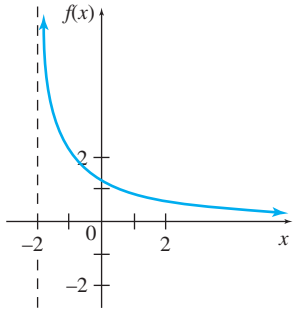


FIGURE 18

(d) $f(x)$ in Figure 18 at $x = -2$.

SOLUTION

Step 1 Does the function exist at $x = -2$?

The function f graphed in Figure 18 is not defined at $x = -2$. Since the function does not pass the first test, it is discontinuous at $x = -2$. (Function f is continuous at any value of x greater than –2, however.)

Notice that the function in part (a) of Example 1 could be made continuous simply by defining $f(3) = 2$. Similarly, the function in part (c) could be made continuous by redefining $g(4) = -2$. In such cases, when the function can be made continuous at a specific point simply by defining or redefining it at that point, the function is said to have a **removable discontinuity**.

A function is said to be **continuous on an open interval** if it is continuous at every x-value in the interval. Continuity on a closed interval is slightly more complicated because

we must decide what to do with the endpoints. We will say that a function f is **continuous from the right** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$. A function f is **continuous from the left** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$. With these ideas, we can now define continuity on a closed interval.

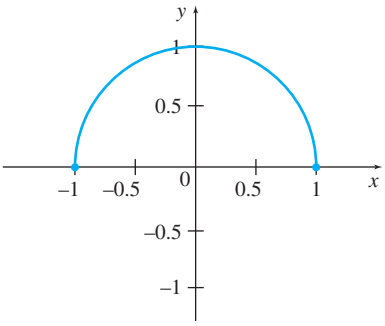


FIGURE 19

Continuity on a Closed Interval

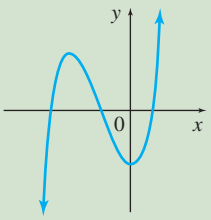
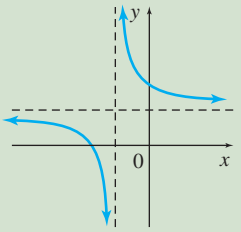
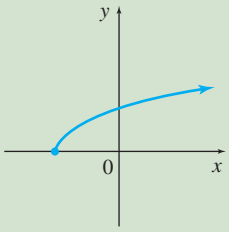
A function is **continuous on a closed interval** $[a, b]$ if

1. it is continuous on the open interval (a, b),
2. it is continuous from the right at $x = a$, and
3. it is continuous from the left at $x = b$.

For example, the function $f(x) = \sqrt{1 - x^2}$, shown in Figure 19, is continuous on the closed interval $[-1, 1]$. By defining continuity on a closed interval in this way, we need not worry about the fact that $\sqrt{1 - x^2}$ does not exist to the left of $x = -1$ or to the right of $x = 1$.

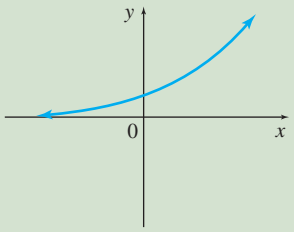
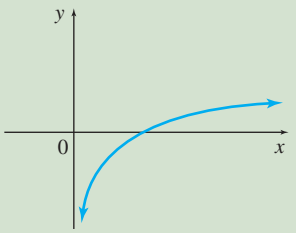
The table below lists some key functions and tells where each is continuous.

Continuous Functions

Type of Function	Where It Is Continuous	Graphic Example
Polynomial Function $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where a_n $a_{n-1}, \ldots, a_1, a_0$ are real numbers, not all 0	For all x	
Rational Function $y = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$	For all x where $q(x) \neq 0$	
Root Function $y = \sqrt{ax + b}$, where a and b are real numbers, with $a \neq 0$ and $ax + b \geq 0$	For all x where $ax + b \geq 0$	

(*continued*)

Continuous Functions (cont.)

Type of Function	Where It Is Continuous	Graphic Example
Exponential Function $y = a^x$ where $a > 0$	For all x	
Logarithmic Function $y = \log_a x$ where $a > 0$, $a \neq 1$	For all $x > 0$	

Continuous functions are nice to work with because finding $\lim_{x \to c} f(x)$ is simple if f is continuous: just evaluate $f(c)$.

When a function is given by a graph, any discontinuities are clearly visible. When a function is given by a formula, it is usually continuous at all x-values except those where the function is undefined or possibly where there is a change in the defining formula for the function, as shown in the following examples.

EXAMPLE 2 Continuity

Find all values $x = a$ where the function is discontinuous.

(a) $f(x) = \dfrac{4x - 3}{2x - 7}$

SOLUTION This rational function is discontinuous wherever the denominator is zero. There is a discontinuity when $a = 7/2$.

(b) $g(x) = e^{2x-3}$

SOLUTION This exponential function is continuous for all x. TRY YOUR TURN 1

YOUR TURN 1 Find all values $x = a$ where the function is discontinuous.

$f(x) = \sqrt{5x + 3}$

EXAMPLE 3 Continuity

Find all values of x where the following piecewise function is discontinuous.

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \le x \le 3. \\ 5 - x & \text{if } x > 3 \end{cases}$$

SOLUTION Since each piece of this function is a polynomial, the only x-values where f might be discontinuous here are 1 and 3. We investigate at $x = 1$ first. From the left, where x-values are less than 1,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 1) = 1 + 1 = 2.$$

From the right, where x-values are greater than 1,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 3x + 4) = 1^2 - 3 + 4 = 2.$$

Furthermore, $f(1) = 1^2 - 3 + 4 = 2$, so $\lim_{x \to 1} f(x) = f(1) = 2$. Thus f is continuous at $x = 1$ since $f(1) = \lim_{x \to 1} f(x)$.

Now let us investigate $x = 3$. From the left,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 3x + 4) = 3^2 - 3(3) + 4 = 4.$$

From the right,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5 - x) = 5 - 3 = 2.$$

Because $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$, the limit $\lim_{x \to 3} f(x)$ does not exist, so f is discontinuous at $x = 3$, regardless of the value of $f(3)$.

YOUR TURN 2 Find all values of x where the piecewise function is discontinuous.

$$f(x) = \begin{cases} 5x - 4 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 3 \\ x + 6 & \text{if } x > 3 \end{cases}$$

The graph of $f(x)$ can be drawn by considering each of the three parts separately. In the first part, the line $y = x + 1$ is drawn including only the section of the line to the left of $x = 1$. The other two parts are drawn similarly, as illustrated in Figure 20. We can see by the graph that the function is continuous at $x = 1$ and discontinuous at $x = 3$, which confirms our solution above. TRY YOUR TURN 2

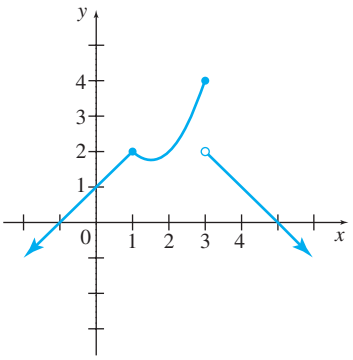


FIGURE 20

TECHNOLOGY NOTE

Some graphing calculators have the ability to draw piecewise functions. On the TI-84 Plus, letting

$$Y_1 = (X + 1)(X < 1) + (X^2 - 3X + 4)(1 \le X)(X \le 3) + (5 - X)(X > 3)$$

produces the graph shown in Figure 21(a).

CAUTION It is important here that the graphing mode be set on DOT rather than CONNECTED. Otherwise, the calculator will show a line segment at $x = 3$ connecting the parabola to the line, as in Figure 21(b), although such a segment does not really exist.

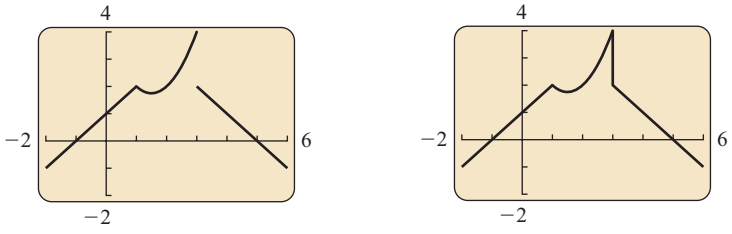


FIGURE 21

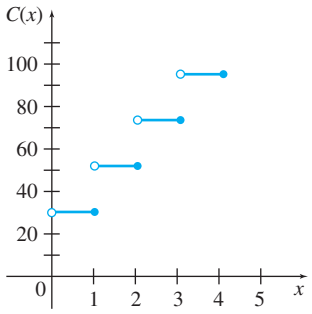


FIGURE 22

EXAMPLE 4 Cost Analysis

A trailer rental firm charges a flat \$8 to rent a hitch. The trailer itself is rented for \$22 per day or fraction of a day. Let $C(x)$ represent the cost of renting a hitch and trailer for x days.

(a) Graph C.

SOLUTION The charge for one day is \$8 for the hitch and \$22 for the trailer, or \$30. In fact, if $0 < x \le 1$, then $C(x) = 30$. To rent the trailer for more than one day, but not more than two days, the charge is $8 + 2 \cdot 22 = 52$ dollars. For any value of x satisfying $1 < x \le 2$, the cost is $C(x) = 52$. Also, if $2 < x \le 3$, then $C(x) = 74$. These results lead to the graph in Figure 22.

(b) Find any values of x where C is discontinuous.

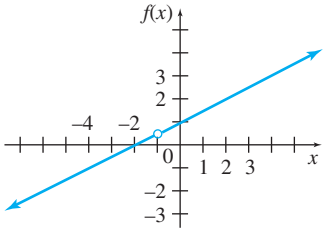
SOLUTION As the graph suggests, C is discontinuous at $x = 1, 2, 3, 4$, and all other positive integers.

One application of continuity is the **Intermediate Value Theorem**, which says that if a function is continuous on a closed interval $[a, b]$, the function takes on every value between $f(a)$ and $f(b)$. For example, if $f(1) = -3$ and $f(2) = 5$, then f must take on every value between -3 and 5 as x varies over the interval $[1, 2]$. In particular (in this case), there must be a value of x in the interval $(1, 2)$ such that $f(x) = 0$. If f were discontinuous, however, this conclusion would not necessarily be true. This is important because, if we are searching for a solution to $f(x) = 0$ in $[1, 2]$, we would like to know that a solution exists.

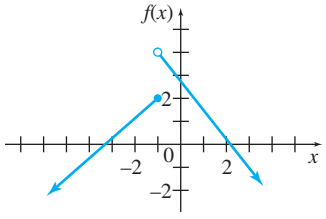
11.2 EXERCISES

In Exercises 1–6, find all values $x = a$ where the function is discontinuous. For each point of discontinuity, give (a) $f(a)$ if it exists, (b) $\lim_{x \to a} f(x)$, (c) $\lim_{x \to a^+} f(x)$, (d) $\lim_{x \to a} f(x)$, and (e) identify which conditions for continuity are not met. Be sure to note when the limit doesn't exist.

1.



2.



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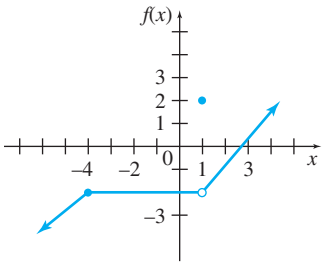

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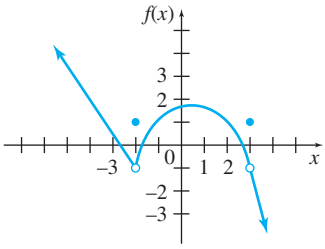

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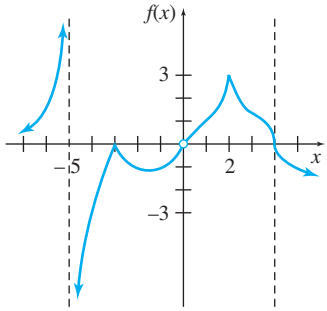

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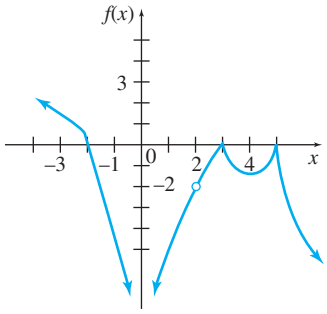


Find all values $x = a$ where the function is discontinuous. For each value of x, give the limit of the function as x approaches a. Be sure to note when the limit doesn't exist.

7. $f(x) = \dfrac{5 + x}{x(x - 2)}$

8. $f(x) = \dfrac{-2x}{(2x + 1)(3x + 6)}$

9. $f(x) = \dfrac{x^2 - 4}{x - 2}$

10. $f(x) = \dfrac{x^2 - 25}{x + 5}$

11. $p(x) = x^2 - 4x + 11$

12. $q(x) = -3x^3 + 2x^2 - 4x + 1$

13. $p(x) = \dfrac{|x + 2|}{x + 2}$

14. $r(x) = \dfrac{|5 - x|}{x - 5}$

15. $k(x) = e^{\sqrt{x-1}}$

16. $j(x) = e^{1/x}$

17. $r(x) = \ln\left|\dfrac{x}{x - 1}\right|$

18. $j(x) = \ln\left|\dfrac{x + 2}{x - 3}\right|$

In Exercises 19–24, (a) graph the given function, (b) find all values of x where the function is discontinuous, and (c) find the limit from the left and from the right at any values of x found in part b.

19. $f(x) = \begin{cases} 1 & \text{if } x < 2 \\ x + 3 & \text{if } 2 \le x \le 4 \\ 7 & \text{if } x > 4 \end{cases}$

20. $f(x) = \begin{cases} x - 1 & \text{if } x < 1 \\ 0 & \text{if } 1 \le x \le 4 \\ x - 2 & \text{if } x > 4 \end{cases}$

21. $g(x) = \begin{cases} 11 & \text{if } x < -1 \\ x^2 + 2 & \text{if } -1 \le x \le 3 \\ 11 & \text{if } x > 3 \end{cases}$

22. $g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 - 5x & \text{if } 0 \le x \le 5 \\ 5 & \text{if } x > 5 \end{cases}$

23. $h(x) = \begin{cases} 4x + 4 & \text{if } x \le 0 \\ x^2 - 4x + 4 & \text{if } x > 0 \end{cases}$

24. $h(x) = \begin{cases} x^2 + x - 12 & \text{if } x \le 1 \\ 3 - x & \text{if } x > 1 \end{cases}$

In Exercises 25–28, find the value of the constant k that makes the function continuous.

25. $f(x) = \begin{cases} kx^2 & \text{if } x \le 2 \\ x + k & \text{if } x > 2 \end{cases}$

26. $g(x) = \begin{cases} x^3 + k & \text{if } x \le 3 \\ kx - 5 & \text{if } x > 3 \end{cases}$

27. $g(x) = \begin{cases} \dfrac{2x^2 - x - 15}{x - 3} & \text{if } x \ne 3 \\ kx - 1 & \text{if } x = 3 \end{cases}$

28. $h(x) = \begin{cases} \dfrac{3x^2 + 2x - 8}{x + 2} & \text{if } x \ne -2 \\ 3x + k & \text{if } x = -2 \end{cases}$

29. Explain in your own words what the Intermediate Value Theorem says and why it seems plausible.

30. Explain why $\lim\limits_{x \to 2}(3x^2 + 8x)$ can be evaluated by substituting $x = 2$.

In Exercises 31–32, (a) use a graphing calculator to tell where the rational function $P(x)/Q(x)$ is discontinuous, and (b) verify your answer from part (a) by using the graphing calculator to plot $Q(x)$ and determine where $Q(x) = 0$. You will need to choose the viewing window carefully.

31. $f(x) = \dfrac{x^2 + x + 2}{x^3 - 0.9x^2 + 4.14x - 5.4}$

32. $f(x) = \dfrac{x^2 + 3x - 2}{x^3 - 0.9x^2 + 4.14x + 5.4}$

33. Let $g(x) = \dfrac{x + 4}{x^2 + 2x - 8}$. Determine all values of x at which g is discontinuous, and for each of these values of x, define g in such a manner so as to remove the discontinuity, if possible. Choose one of the following. ***Source: Society of Actuaries.***

a. g is discontinuous only at -4 and 2.
Define $g(-4) = -\frac{1}{6}$ to make g continuous at -4.
$g(2)$ cannot be defined to make g continuous at 2.

b. g is discontinuous only at -4 and 2.
Define $g(-4) = -\frac{1}{6}$ to make g continuous at -4.
Define $g(2) = 6$ to make g continuous at 2.

c. g is discontinuous only at -4 and 2.
$g(-4)$ cannot be defined to make g continuous at -4.
$g(2)$ cannot be defined to make g continuous at 2.

d. g is discontinuous only at 2.
Define $g(2) = 6$ to make g continuous at 2.

e. g is discontinuous only at 2.
$g(2)$ cannot be defined to make g continuous at 2.

34. Tell at what values of x the function $f(x)$ in Figure 8 from the previous section is discontinuous. Explain why it is discontinuous at each of these values.

APPLICATIONS

Business and Economics

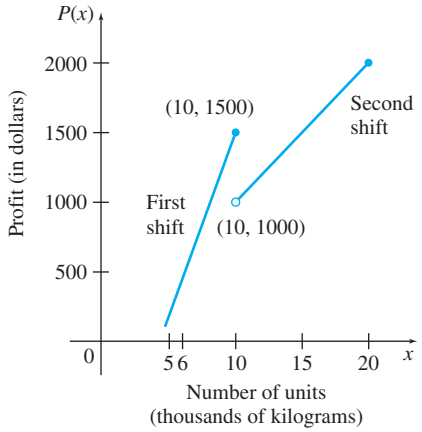
35. Production The graph shows the profit from the daily production of x thousand kilograms of an industrial chemical. Use the graph to find the following limits.

a. $\lim_{x\to 6} P(x)$ **b.** $\lim_{x\to 10^-} P(x)$ **c.** $\lim_{x\to 10^+} P(x)$

d. $\lim_{x\to 10} P(x)$

e. Where is the function discontinuous? What might account for such a discontinuity?

f. Use the graph to estimate the number of units of the chemical that must be produced before the second shift is as profitable as the first.



36. Cost Analysis The cost to transport a mobile home depends on the distance, x, in miles that the home is moved. Let $C(x)$ represent the cost to move a mobile home x miles. One firm charges as follows.

Cost per Mile	Distance in Miles
\$4.00	$0 < x \le 150$
\$3.00	$150 < x \le 400$
\$2.50	$400 < x$

Find the cost to move a mobile home the following distances.

a. 130 miles **b.** 150 miles **c.** 210 miles

d. 400 miles **e.** 500 miles

f. Where is C discontinuous?

37. Cost Analysis A company charges \$1.25 per lb for a certain fertilizer on all orders 100 lb. or less, and \$1 per lb for orders over 100 lb. Let $F(x)$ represent the cost for buying x lb of the fertilizer. Find the cost of buying the following.

a. 80 lb **b.** 150 lb **c.** 100 lb

d. Where is F discontinuous?

38. APPLY IT Car Rental Recently, a car rental firm charged \$36 per day or portion of a day to rent a car for a period of 1 to 5 days. Days 6 and 7 were then free, while the charge for days 8 through 12 was again \$36 per day. Let $A(t)$ represent the average cost to rent the car for t days, where $0 < t \le 12$. Find the average cost of a rental for the following number of days.

a. 4 **b.** 5 **c.** 6 **d.** 7 **e.** 8

f. Find $\lim_{x\to 5^-} A(t)$. **g.** Find $\lim_{x\to 5^+} A(t)$.

h. Where is A discontinuous on the given interval?

39. Postage To send international first class mail (large envelopes) from the United States to Australia in 2010, it cost \$1.24 for the first ounce, \$0.84 for each additional ounce up to a total of 8 oz, and \$1.72 for each additional four ounces after that up to a total of 64 oz. Let $C(x)$ be the cost to mail x ounces. Find the following. ***Source: U.S. Postal Service.***

a. $\lim_{x\to 3^-} C(x)$ **b.** $\lim_{x\to 3^+} C(x)$ **c.** $\lim_{x\to 3} C(x)$

d. $C(3)$ **e.** $\lim_{x\to 14^+} C(x)$ **f.** $\lim_{x\to 14^-} C(x)$

g. $\lim_{x\to 14} C(x)$ **h.** $C(14)$

i. Find all values on the interval $(0, 64)$ where the function C is discontinuous.

Life Sciences

40. Pregnancy A woman's weight naturally increases during the course of a pregnancy. When she delivers, her weight immediately decreases by the approximate weight of the child. Suppose that a 120-lb woman gains 27 lb during pregnancy, delivers a 7-lb baby, and then, through diet and exercise, loses the remaining weight during the next 20 weeks.

a. Graph the weight gain and loss during the pregnancy and the 20 weeks following the birth of the baby. Assume that the pregnancy lasts 40 weeks, that delivery occurs immediately after this time interval, and that the weight gain/loss before and after birth is linear.

b. Is this a continuous function? If not, then find the value(s) of t where the function is discontinuous.

41. Poultry Farming Researchers at Iowa State University and the University of Arkansas have developed a piecewise function that can be used to estimate the body weight (in grams) of a male broiler during the first 56 days of life according to

$$W(t) = \begin{cases} 48 + 3.64t + 0.6363t^2 + 0.00963t^3 & \text{if } 1 \le t \le 28, \\ -1004 + 65.8t & \text{if } 28 < t \le 56, \end{cases}$$

where t is the age of the chicken (in days). ***Source: Poultry Science.***

a. Determine the weight of a male broiler that is 25 days old.

b. Is $W(t)$ a continuous function?

c. Use a graphing calculator to graph $W(t)$ on $[1, 56]$ by $[0, 3000]$. Comment on the accuracy of the graph.

d. Comment on why researchers would use two different types of functions to estimate the weight of a chicken at various ages.

YOUR TURN ANSWERS

1. Discontinuous when $a < -3/5$.

2. Discontinuous at $x = 0$.

11.3 Rates of Change

APPLY IT **How does the manufacturing cost of a DVD change as the number of DVDs manufactured changes?**

This question will be answered in Example 4 of this section as we develop a method for finding the rate of change of one variable with respect to a unit change in another variable.

Average Rate of Change

One of the main applications of calculus is determining how one variable changes in relation to another. A marketing manager wants to know how profit changes with respect to the amount spent on advertising, while a physician wants to know how a patient's reaction to a drug changes with respect to the dose.

For example, suppose we take a trip from San Francisco driving south. Every half-hour we note how far we have traveled, with the following results for the first three hours.

	Distance Traveled						
Time in Hours	0	0.5	1	1.5	2	2.5	3
Distance in Miles	0	30	55	80	104	124	138

If s is the function whose rule is

$$s(t) = \text{Distance from San Francisco at time } t,$$

then the table shows, for example, that $s(0) = 0$, $s(1) = 55$, $s(2.5) = 124$, and so on. The distance traveled during, say, the second hour can be calculated by $s(2) - s(1) = 104 - 55 = 49$ miles.

Distance equals time multiplied by rate (or speed); so the distance formula is $d = rt$. Solving for rate gives $r = d/t$, or

$$\text{Average speed} = \frac{\text{Distance}}{\text{Time}}.$$

For example, the average speed over the time interval from $t = 0$ to $t = 3$ is

$$\text{Average speed} = \frac{s(3) - s(0)}{3 - 0} = \frac{138 - 0}{3} = 46,$$

or 46 mph. We can use this formula to find the average speed for any interval of time during the trip, as shown below.

Time Interval	Average Speed $= \frac{\text{Distance}}{\text{Time}}$
	Average Speed
$t = 0.5$ to $t = 1$	$\frac{s(1) - s(0.5)}{1 - 0.5} = \frac{25}{0.5} = 50$
$t = 0.5$ to $t = 1.5$	$\frac{s(1.5) - s(0.5)}{1.5 - 0.5} = \frac{50}{1} = 50$
$t = 1$ to $t = 2$	$\frac{s(2) - s(1)}{2 - 1} = \frac{49}{1} = 49$
$t = 1$ to $t = 3$	$\frac{s(3) - s(1)}{3 - 1} = \frac{83}{2} = 41.5$
$t = a$ to $t = b$	$\frac{s(b) - s(a)}{b - a}$

FOR REVIEW

Recall from Section 1.1 the formula for the slope of a line through two points (x_1, y_1) and (x_2, y_2):

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Find the slopes of the lines through the following points.

$(0.5, 30)$ and $(1, 55)$
$(0.5, 30)$ and $(1.5, 80)$
$(1, 55)$ and $(2, 104)$

Compare your answers to the average speeds shown in the table.

The analysis of the average speed or *average rate of change* of distance s with respect to t can be extended to include any function defined by $f(x)$ to get a formula for the average rate of change of f with respect to x.

Average Rate of Change

The **average rate of change** of $f(x)$ with respect to x for a function f as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}.$$

NOTE
The formula for the average rate of change is the same as the formula for the slope of the line through $(a, f(a))$ and $(b, f(b))$. This connection between slope and rate of change will be examined more closely in the next section.

In Figure 23 we have plotted the distance vs. time for our trip from San Francisco, connecting the points with straight line segments. Because the change in y gives the change in distance, and the change in x gives the change in time, the slope of each line segment gives the average speed over that time interval:

$$\text{Slope} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\text{Change in distance}}{\text{Change in time}} = \text{Average speed.}$$

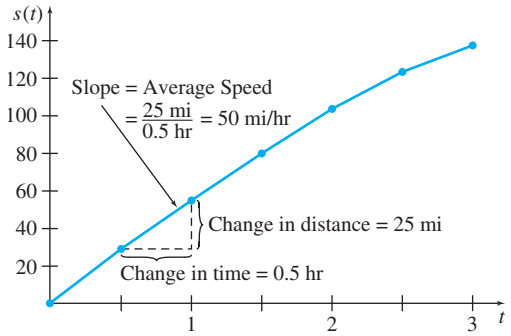


FIGURE 23

EXAMPLE 1 Minority Population

The United States population is becoming more diverse. Based on the U.S. Census population projections for 2000 to 2050, the projected Hispanic population (in millions) can be modeled by the exponential function

$$H(t) = 37.791(1.021)^t,$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. Use H to estimate the average rate of change in the Hispanic population from 2000 to 2010. ***Source: U.S. Census Bureau.***

SOLUTION On the interval from $t = 0$ (2000) to $t = 10$ (2010), the average rate of change is

$$\frac{H(10) - H(0)}{10 - 0} = \frac{37.791(1.021)^{10} - 37.791(1.021)^0}{10}$$

$$\approx \frac{46.521 - 37.791}{10} = \frac{8.73}{10}$$

$$= 0.873,$$

or 0.873 million. Based on this model, it is estimated that the Hispanic population in the United States increased, on average, at a rate of about 873,000 people per year between 2000 and 2010.

YOUR TURN 1 The projected U.S. Asian population (in millions) for this same time period is $A(t) = 11.14(1.023)^t$. Use A to estimate the average rate of change from 2000 to 2010.

TRY YOUR TURN 1

EXAMPLE 2 Household Telephones

Some U.S. households are substituting wireless telephones for traditional landline telephones. The graph in Figure 24 shows the percent of households in the United States with landline telephones for the years 2005 to 2009. Find the average rate of change in the percent of households with a landline between 2005 and 2009. ***Source: Centers for Disease Control and Prevention.***

SOLUTION Let $L(t)$ be the percent of U.S. households with landlines in the year t. Then the average rate of change between 2005 and 2009 was

$$\frac{L(2009) - L(2005)}{2009 - 2005} = \frac{73.5 - 89.7}{4} = \frac{-16.2}{4} = -4.05,$$

or -4.05%. On average, the percent of U.S. households with landline telephones decreased by about 4.05% per year during this time period. TRY YOUR TURN 2

YOUR TURN 2 In Example 2, find the average rate of change in percent of households with a landline between 2007 and 2009.

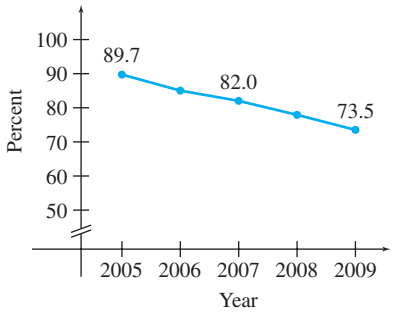


FIGURE 24

Instantaneous Rate of Change

Suppose a car is stopped at a traffic light. When the light turns green, the car begins to move along a straight road. Assume that the distance traveled by the car is given by the function

$$s(t) = 3t^2, \quad \text{for} \quad 0 \le t \le 15,$$

where t is the time in seconds and $s(t)$ is the distance in feet. We have already seen how to find the *average* speed of the car over any time interval. We now turn to a different problem, that of determining the exact speed of the car at a particular instant, say $t = 10$.

The intuitive idea is that the exact speed at $t = 10$ is very close to the average speed over a very short time interval near $t = 10$. If we take shorter and shorter time intervals near $t = 10$, the average speeds over these intervals should get closer and closer to the exact speed at $t = 10$. In other words, the exact speed at $t = 10$ is the limit of the average speeds over shorter and shorter time intervals near $t = 10$. The following chart illustrates this idea. The values in the chart are found using $s(t) = 3t^2$, so that, for example, $s(10) = 3(10)^2 = 300$ and $s(10.1) = 3(10.1)^2 = 306.03$.

Approximation of Speed at 10 Seconds	
Interval	**Average Speed**
$t = 10$ to $t = 10.1$	$\frac{s(10.1) - s(10)}{10.1 - 10} = \frac{306.03 - 300}{0.1} = 60.3$
$t = 10$ to $t = 10.01$	$\frac{s(10.01) - s(10)}{10.01 - 10} = \frac{300.6003 - 300}{0.01} = 60.03$
$t = 10$ to $t = 10.001$	$\frac{s(10.001) - s(10)}{10.001 - 10} = \frac{300.060003 - 300}{0.001} = 60.003$

The results in the chart suggest that the exact speed at $t = 10$ is 60 ft/sec. We can confirm this by computing the average speed from $t = 10$ to $t = 10 + h$, where h is a small, but nonzero, number that represents a small change in time. (The chart does this for $h = 0.1$, $h = 0.01$, and $h = 0.001$.) The average speed from $t = 10$ to $t = 10 + h$ is then

$$\begin{aligned}\frac{s(10+h)-s(10)}{(10+h)-10} &= \frac{3(10+h)^2 - 3\cdot 10^2}{h}\\ &= \frac{3(100+20h+h^2)-300}{h}\\ &= \frac{300+60h+3h^2-300}{h}\\ &= \frac{60h+3h^2}{h}\\ &= \frac{h(60+3h)}{h}\\ &= 60+3h,\end{aligned}$$

where h is not equal to 0. Saying that the time interval from 10 to $10 + h$ gets shorter and shorter is equivalent to saying that h gets closer and closer to 0. Therefore, the exact speed at $t = 10$ is the limit, as h approaches 0, of the average speed over the interval from $t = 10$ to $t = 10 + h$; that is,

$$\begin{aligned}\lim_{h\to 0}\frac{s(10+h)-s(10)}{h} &= \lim_{h\to 0}(60+3h)\\ &= 60 \text{ ft/sec}.\end{aligned}$$

This example can be easily generalized to any function f. Let a be a specific x-value, such as 10 in the example. Let h be a (small) number, which represents the distance between the two values of x, namely, a and $a + h$. The average rate of change of f as x changes from a to $a + h$ is

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h},$$

which is often called the **difference quotient**. Observe that the difference quotient is equivalent to the average rate of change formula, which can be verified by letting $b = a + h$ in the average rate of change formula. Furthermore, the exact rate of change of f at $x = a$, called the *instantaneous rate of change of f at* $x = a$, is the limit of this difference quotient.

Instantaneous Rate of Change

The **instantaneous rate of change** for a function f when $x = a$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

provided this limit exists.

CAUTION Remember that $f(x + h) \neq f(x) + f(h)$. To find $f(x + h)$, replace x with $(x + h)$ in the expression for $f(x)$. For example, if $f(x) = x^2$,

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2,$$

but

$$f(x) + f(h) = x^2 + h^2.$$

In the example just discussed, with the car starting from the traffic light, we saw that the instantaneous rate of change gave the speed of the car. But speed is always positive, while

instantaneous rate of change can be positive or negative. Therefore, we will refer to **velocity** when we want to consider not only how fast something is moving but also in what direction it is moving. In any motion along a straight line, one direction is arbitrarily labeled as positive, so when an object moves in the opposite direction, its velocity is negative. In general, velocity is the same as the instantaneous rate of change of a function that gives position in terms of time.

In Figure 25, we have plotted the function $s(t) = 3t^2$, giving distance as a function of time. We have also plotted in green a line through the points (10, s(10)) and (15, s(15)). As we observed earlier, the slope of this line is the same as the average speed between $t = 10$ and $t = 15$. Finally, in red, we have plotted the line that results when the second point, (15, s(15)), moves closer and closer to the first point until the two coincide. The slope of this line corresponds to the instantaneous velocity at $t = 10$. We will explore these ideas further in the next section. Meanwhile, you might think about how to calculate the equations of these lines.

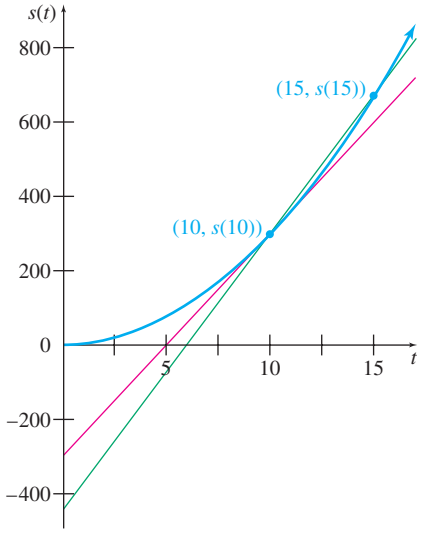


FIGURE 25

An alternate, but equivalent, approach is to let $a + h = b$ in the definition for instantaneous rate of change, so that $h = b - a$. This makes the instantaneous rate of change formula look more like the average rate of change formula.

Instantaneous Rate of Change (Alternate Form)

The **instantaneous rate of change** for a function f when $x = a$ can be written as

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a},$$

provided this limit exists.

EXAMPLE 3 Velocity

The distance in feet of an object from a starting point is given by $s(t) = 2t^2 - 5t + 40$, where t is time in seconds.

(a) Find the average velocity of the object from 2 seconds to 4 seconds.

SOLUTION The average velocity is

$$\frac{s(4) - s(2)}{4 - 2} = \frac{52 - 38}{2} = \frac{14}{2} = 7$$

ft per second.

(b) Find the instantaneous velocity at 4 seconds.

Method 1 Standard Form

SOLUTION

For $t = 4$, the instantaneous velocity is

$$\lim_{h\to 0}\frac{s(4+h)-s(4)}{h}$$

ft per second. We first calculate $s(4+h)$ and $s(4)$, that is,

$$\begin{aligned} s(\mathbf{4+h}) &= 2(\mathbf{4+h})^2 - 5(\mathbf{4+h}) + 40 \\ &= 2(16 + 8h + h^2) - 20 - 5h + 40 \\ &= 32 + 16h + 2h^2 - 20 - 5h + 40 \\ &= 2h^2 + 11h + 52, \end{aligned}$$

and

$$s(4) = 2(4)^2 - 5(4) + 40 = 52.$$

Therefore, the instantaneous velocity at $t = 4$ is

$$\begin{aligned} \lim_{h\to 0}\frac{(2h^2+11h+52)-52}{h} &= \lim_{h\to 0}\frac{2h^2+11h}{h} = \lim_{h\to 0}\frac{h(2h+11)}{h} \\ &= \lim_{h\to 0}(2h+11) = 11, \end{aligned}$$

or 11 ft per second.

Method 2 Alternate Form

SOLUTION

For $t = 4$, the instantaneous velocity is

$$\lim_{b\to 4}\frac{s(b)-s(4)}{b-4}$$

ft per second. We first calculate $s(b)$ and $s(4)$, that is,

$$s(b) = 2b^2 - 5b + 40$$

and

$$s(4) = 2(4)^2 - 5(4) + 40 = 52.$$

The instantaneous rate of change is then

$$\begin{aligned} \lim_{b\to 4}\frac{2b^2-5b+40-52}{b-4} &= \lim_{b\to 4}\frac{2b^2-5b-12}{b-4} && \text{Simplify the numerator.} \\ &= \lim_{b\to 4}\frac{(2b+3)(b-4)}{b-4} && \text{Factor the numerator.} \\ &= \lim_{b\to 4}(2b+3) && \text{Cancel the } b-4. \\ &= 11, && \text{Calculate the limit.} \end{aligned}$$

or 11 ft per second.

YOUR TURN 3 For the function in Example 3, find the instantaneous velocity at 2 seconds.

TRY YOUR TURN 3

EXAMPLE 4 Manufacturing

APPLY IT

A company determines that the cost in dollars to manufacture x cases of the DVD "Mathematicians Caught in Embarrassing Moments" is given by

$$C(x) = 100 + 15x - x^2 \quad (0 \le x \le 7).$$

(a) Find the average rate of change of cost per case for manufacturing between 1 and 5 cases.

SOLUTION Use the formula for average rate of change.

The average rate of change of cost is

$$\frac{C(5) - C(1)}{5 - 1} = \frac{150 - 114}{4} = 9.$$

Thus, on average, the cost increases at the rate of \$9 per case when production increases from 1 to 5 cases.

(b) Find the additional cost when production is increased from 1 to 2 cases.

SOLUTION The additional cost can be found by calculating the cost to produce 2 cases, and subtracting the cost to produce 1 case; that is,

$$C(2) - C(1) = 126 - 114 = 12.$$

The additional cost to produce the second case is \$12.

(c) Find the instantaneous rate of change of cost with respect to the number of cases produced when just one case is produced.

SOLUTION The instantaneous rate of change for $x = 1$ is given by

$$\begin{aligned}
&\lim_{h\to 0} \frac{C(1+h) - C(1)}{h} \\
&= \lim_{h\to 0} \frac{[100 + 15(1+h) - (1+h)^2] - [100 + 15(1) - 1^2]}{h} \\
&= \lim_{h\to 0} \frac{100 + 15 + 15h - 1 - 2h - h^2 - 114}{h} \\
&= \lim_{h\to 0} \frac{13h - h^2}{h} && \text{Combine terms.} \\
&= \lim_{h\to 0} \frac{h(13 - h)}{h} && \text{Factor.} \\
&= \lim_{h\to 0} (13 - h) && \text{Divide by } h. \\
&= 13. && \text{Calculate the limit.}
\end{aligned}$$

When 1 case is manufactured, the cost is increasing at the rate of \$13 per case. Notice that this is close to the value calculated in part (b).

As we mentioned in Chapter 1, economists sometimes define the marginal cost as the cost of producing one additional item and sometimes as the instantaneous rate of change of the cost function. These definitions are considered to be essentially equivalent. If a company (or an economy) produces millions of items, it makes little difference whether we let $h = 1$ or take the limit as h goes to 0, because 1 is very close to 0 when production is in the millions. The advantage of taking the instantaneous rate of change point of view is that it allows all the power of calculus to be used, including the Fundamental Theorem of Calculus, which is discussed later in this book.

Throughout this textbook, *we define the marginal cost to be the instantaneous rate of change of the cost function*. It can then be interpreted as the approximate cost of producing one additional item. For simplicity, we will make this interpretation even when production numbers are fairly small.

EXAMPLE 5 Manufacturing

For the cost function in the previous example, find the instantaneous rate of change of cost when 5 cases are made.

SOLUTION The instantaneous rate of change for $x = 5$ is given by

$$\lim_{h\to 0}\frac{C(5+h)-C(5)}{h}$$

$$=\lim_{h\to 0}\frac{[100+15(5+h)-(5+h)^2]-[100+15(5)-5^2]}{h}$$

$$=\lim_{h\to 0}\frac{100+75+15h-25-10h-h^2-150}{h}$$

$$=\lim_{h\to 0}\frac{5h-h^2}{h} \qquad \text{Combine terms.}$$

$$=\lim_{h\to 0}\frac{h(5-h)}{h} \qquad \text{Factor.}$$

$$=\lim_{h\to 0}(5-h) \qquad \text{Divide by } h.$$

$$=5. \qquad \text{Calculate the limit.}$$

When 5 cases are manufactured, the cost is increasing at the rate of \$5 per case; that is, the marginal cost when $x = 5$ is \$5. Notice that as the number of items produced goes up, the marginal cost goes down, as might be expected. TRY YOUR TURN 4

YOUR TURN 4 If the cost function is given by $C(x) = x^2 - 2x + 12$, find the instantaneous rate of change of cost when $x = 4$.

EXAMPLE 6 Minority Population

Estimate the instantaneous rate of change in 2010 in the Hispanic population of the United States.

SOLUTION We saw in Example 1 that the U.S. Hispanic population is approximately given by $H(t) = 37.791(1.021)^t$, where $t = 0$ corresponds to 2000. Unlike the previous example, in which the function was a polynomial, the function in this example is an exponential, making it harder to compute the limit directly using the formula for instantaneous rate of change at $t = 10$ (the year 2010):

$$\lim_{h\to 0}\frac{37.791(1.021)^{10+h}-37.791(1.021)^{10}}{h}.$$

Instead, we will approximate the instantaneous rate of change by using smaller and smaller values of h. See the following table. The limit seems to be approaching 0.96682 (million). Thus, the instantaneous rate of change in the U.S. Hispanic population is about 966,820 people per year in 2010. TRY YOUR TURN 5

YOUR TURN 5 Estimate the instantaneous rate of change in 2010 in the Asian population of the United States. An estimate of the U.S. Asian population is given by $A(t) = 11.14(1.023)^t$, where $t = 0$ corresponds to 2000.

Limit Calculations

h	$\dfrac{37.791(1.021)^{10+h}-37.791(1.021)^{10}}{h}$
1	0.97693
0.1	0.96782
0.01	0.96692
0.001	0.96683
0.0001	0.96682
0.00001	0.96682

TECHNOLOGY NOTE

The table could be created using the TABLE feature on a TI-84 Plus calculator by entering Y_1 as the function from Example 6, and $Y_2 = (Y_1(10 + X) - Y_1(10))/X$. (The calculator requires us to use X in place of h in the formula for instantaneous rate of change.) The result is shown in Figure 26. This table can also be generated using a spreadsheet.

X	Y2
1	.97693
.1	.96782
.01	.96692
.001	.96683
1E-4	.96682
1E-5	.96682

X=

FIGURE 26

EXAMPLE 7 Velocity

One day Musk, the friendly pit bull, escaped from the yard and ran across the street to see a neighbor, who was 50 ft away. An estimate of the distance Musk ran as a function of time is given by the following table.

Distance Traveled					
t (sec)	0	1	2	3	4
s (ft)	0	10	25	42	50

(a) Find Musk's average velocity during her 4-second trip.

SOLUTION The total distance she traveled is 50 ft, and the total time is 4 seconds, so her average velocity is $50/4 = 12.5$ ft per second.

(b) Estimate Musk's velocity at 2 seconds.

SOLUTION We could estimate her velocity by taking the short time interval from 2 to 3 seconds, for which the velocity is

$$\frac{42 - 25}{1} = 17 \text{ ft per second.}$$

Alternatively, we could estimate her velocity by taking the short time interval from 1 to 2 seconds, for which the velocity is

$$\frac{25 - 10}{1} = 15 \text{ ft per second.}$$

A better estimate is found by averaging these two values to get

$$\frac{17 + 15}{2} = 16 \text{ ft per second.}$$

Another way to get this same answer is to take the time interval from 1 to 3 seconds, for which the velocity is

$$\frac{42 - 10}{2} = 16 \text{ ft per second.}$$

This answer is reasonable if we assume Musk's velocity changes at a fairly steady rate and does not increase or decrease drastically from one second to the next. It is impossible to calculate Musk's exact velocity without knowing her position at times arbitrarily close to 2 seconds, or without a formula for her position as a function of time, or without a radar gun or speedometer on her. (In any case, she was very happy when she reached the neighbor.)

11.3 EXERCISES

Find the average rate of change for each function over the given interval.

1. $y = x^2 + 2x$ between $x = 1$ and $x = 3$

2. $y = -4x^2 - 6$ between $x = 2$ and $x = 6$

3. $y = -3x^3 + 2x^2 - 4x + 1$ between $x = -2$ and $x = 1$

4. $y = 2x^3 - 4x^2 + 6x$ between $x = -1$ and $x = 4$

5. $y = \sqrt{x}$ between $x = 1$ and $x = 4$

6. $y = \sqrt{3x - 2}$ between $x = 1$ and $x = 2$

7. $y = e^x$ between $x = -2$ and $x = 0$

8. $y = \ln x$ between $x = 2$ and $x = 4$

Suppose the position of an object moving in a straight line is given by $s(t) = t^2 + 5t + 2$. Find the instantaneous velocity at each time.

9. $t = 6$ **10.** $t = 1$

Suppose the position of an object moving in a straight line is given by $s(t) = 5t^2 - 2t - 7$. Find the instantaneous velocity at each time.

11. $t = 2$ **12.** $t = 3$

Suppose the position of an object moving in a straight line is given by $s(t) = t^3 + 2t + 9$. Find the instantaneous velocity at each time.

13. $t = 1$ **14.** $t = 4$

Find the instantaneous rate of change for each function at the given value.

15. $f(x) = x^2 + 2x$ at $x = 0$

16. $s(t) = -4t^2 - 6$ at $t = 2$

17. $g(t) = 1 - t^2$ at $t = -1$

18. $F(x) = x^2 + 2$ at $x = 0$

Use the formula for instantaneous rate of change, approximating the limit by using smaller and smaller values of h, to find the instantaneous rate of change for each function at the given value.

19. $f(x) = x^x$ at $x = 2$ **20.** $f(x) = x^x$ at $x = 3$

21. $f(x) = x^{\ln x}$ at $x = 2$ **22.** $f(x) = x^{\ln x}$ at $x = 3$

23. Explain the difference between the average rate of change of y as x changes from a to b, and the instantaneous rate of change of y at $x = a$.

24. If the instantaneous rate of change of $f(x)$ with respect to x is positive when $x = 1$, is f increasing or decreasing there?

APPLICATIONS

Business and Economics

25. Profit Suppose that the total profit in hundreds of dollars from selling x items is given by

$$P(x) = 2x^2 - 5x + 6.$$

Find the average rate of change of profit for the following changes in x.

a. 2 to 4 **b.** 2 to 3

c. Find and interpret the instantaneous rate of change of profit with respect to the number of items produced when $x = 2$. (This number is called the *marginal profit* at $x = 2$.)

d. Find the marginal profit at $x = 4$.

26. Revenue The revenue (in thousands of dollars) from producing x units of an item is

$$R(x) = 10x - 0.002x^2.$$

a. Find the average rate of change of revenue when production is increased from 1000 to 1001 units.

b. Find and interpret the instantaneous rate of change of revenue with respect to the number of items produced when 1000 units are produced. (This number is called the *marginal revenue* at $x = 1000$.)

c. Find the additional revenue if production is increased from 1000 to 1001 units.

d. Compare your answers for parts a and c. What do you find? How do these answers compare with your answer to part b?

27. Demand Suppose customers in a hardware store are willing to buy $N(p)$ boxes of nails at p dollars per box, as given by

$$N(p) = 80 - 5p^2, \quad 1 \le p \le 4.$$

a. Find the average rate of change of demand for a change in price from \$2 to \$3.

b. Find and interpret the instantaneous rate of change of demand when the price is \$2.

c. Find the instantaneous rate of change of demand when the price is \$3.

d. As the price is increased from \$2 to \$3, how is demand changing? Is the change to be expected? Explain.

28. Interest If \$1000 is invested in an account that pays 5% compounded annually, the total amount, $A(t)$, in the account after t years is

$$A(t) = 1000(1.05)^t.$$

a. Find the average rate of change per year of the total amount in the account for the first five years of the investment (from $t = 0$ to $t = 5$).

b. Find the average rate of change per year of the total amount in the account for the second five years of the investment (from $t = 5$ to $t = 10$).

c. Estimate the instantaneous rate of change for $t = 5$.

29. Interest If \$1000 is invested in an account that pays 5% compounded continuously, the total amount, $A(t)$, in the account after t years is

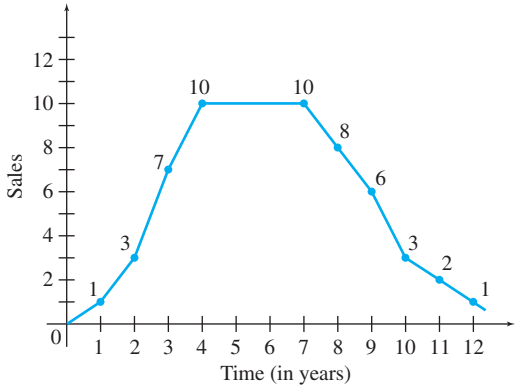
$$A(t) = 1000e^{0.05t}.$$

a. Find the average rate of change per year of the total amount in the account for the first five years of the investment (from $t = 0$ to $t = 5$).

b. Find the average rate of change per year of the total amount in the account for the second five years of the investment (from $t = 5$ to $t = 10$).

c. Estimate the instantaneous rate of change for $t = 5$.

30. Sales The graph shows annual sales (in thousands of dollars) of a Nintendo game at a particular store. Find the average annual rate of change in sales for the following changes in years.



a. 1 to 4 **b.** 4 to 7 **c.** 7 to 12

d. What do your answers for parts a–c tell you about the sales of this product?

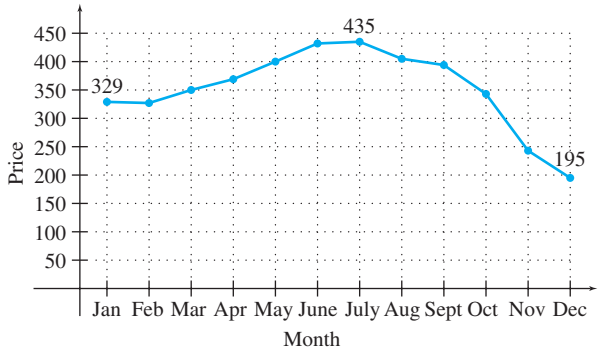
e. Give an example of another product that might have such a sales curve.

31. Gasoline Prices In 2008, the price of gasoline in the United States inexplicably spiked and then dropped. The average monthly price (in cents) per gallon of unleaded regular gasoline for 2008 is shown in the following chart. Find the average rate of change per month in the average price per gallon for each time period. ***Source: U.S. Energy Information Administration.***

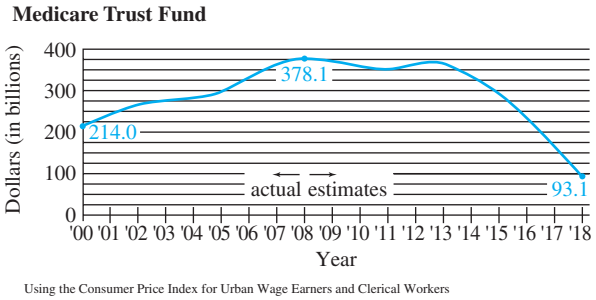
a. From January to July (the peak)

b. From July to December

c. From January to December



32. Medicare Trust Fund The graph shows the money remaining in the Medicare Trust Fund at the end of the fiscal year. ***Source: Social Security Administration.***



Find the approximate average rate of change in the trust fund for each time period.

a. From 2000 to 2008 (the peak) **b.** From 2008 to 2018

Life Sciences

33. Flu Epidemic Epidemiologists in College Station, Texas, estimate that t days after the flu begins to spread in town, the percent of the population infected by the flu is approximated by

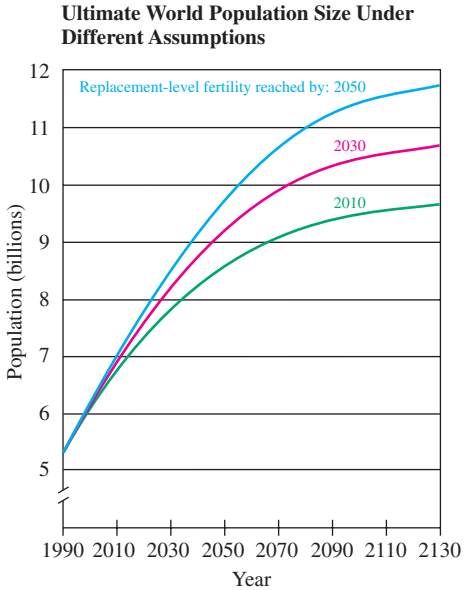
$$p(t) = t^2 + t$$

for $0 \le t \le 5$.

a. Find the average rate of change of p with respect to t over the interval from 1 to 4 days.

b. Find and interpret the instantaneous rate of change of p with respect to t at $t = 3$.

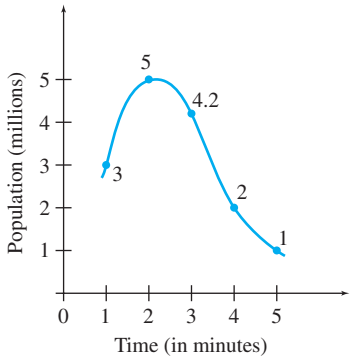
34. World Population Growth The future size of the world population depends on how soon it reaches replacement-level fertility, the point at which each woman bears on average about 2.1 children. The graph shows projections for reaching that point in different years. ***Source: Population Reference Bureau.***



a. Estimate the average rate of change in population for each projection from 1990 to 2050. Which projection shows the smallest rate of change in world population?

b. Estimate the average rate of change in population from 2090 to 2130 for each projection. Interpret your answer.

35. Bacteria Population The graph shows the population in millions of bacteria t minutes after an antibiotic is introduced into a culture. Find and interpret the average rate of change of population with respect to time for the following time intervals.



a. 1 to 2 **b.** 2 to 3 **c.** 3 to 4 **d.** 4 to 5

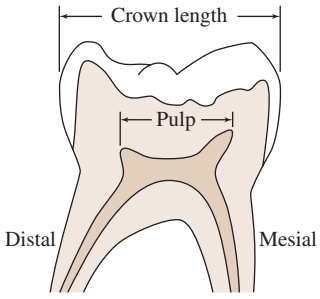
e. How long after the antibiotic was introduced did the population begin to decrease?

f. At what time did the rate of decrease of the population slow down?

36. Molars The crown length (as shown below) of first molars in fetuses is related to the postconception age of the tooth as

$$L(t) = -0.01t^2 + 0.788t - 7.048,$$

where $L(t)$ is the crown length, in millimeters, of the molar t weeks after conception. ***Source: American Journal of Physical Anthropology***.



a. Find the average rate of growth in crown length during weeks 22 through 28.

b. Find the instantaneous rate of growth in crown length when the tooth is exactly 22 weeks of age.

c. Graph the function on $[0, 50]$ by $[0, 9]$. Does a function that increases and then begins to decrease make sense for this particular application? What do you suppose is happening during the first 11 weeks? Does this function accurately model crown length during those weeks?

37. Thermic Effect of Food The metabolic rate of a person who has just eaten a meal tends to go up and then, after some time has passed, returns to a resting metabolic rate. This phenomenon is known as the thermic effect of food. Researchers have indicated that the thermic effect of food (in kJ/hr) for a particular person is

$$F(t) = -10.28 + 175.9te^{-t/1.3},$$

where t is the number of hours that have elapsed since eating a meal. ***Source: American Journal of Clinical Nutrition***.

a. Graph the function on $[0, 6]$ by $[-20, 100]$.

b. Find the average rate of change of the thermic effect of food during the first hour after eating.

c. Use a graphing calculator to find the instantaneous rate of change of the thermic effect of food exactly 1 hour after eating.

d. Use a graphing calculator to estimate when the function stops increasing and begins to decrease.

38. Mass of Bighorn Yearlings The body mass of yearling bighorn sheep on Ram Mountain in Alberta, Canada, can be estimated by

$$M(t) = 27.5 + 0.3t - 0.001t^2$$

where $M(t)$ is measured in kilograms and t is days since May 25. ***Source: Canadian Journal of Zoology***.

a. Find the average rate of change of the weight of a bighorn yearling between 105 and 115 days past May 25.

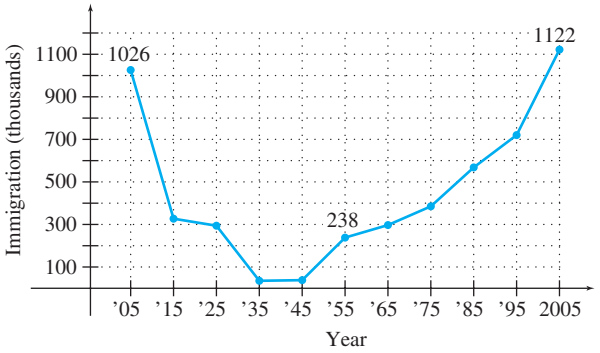
b. Find the instantaneous rate of change of weight for a bighorn yearling sheep whose age is 105 days past May 25.

c. Graph the function $M(t)$ on $[5, 125]$ by $[25, 65]$.

d. Does the behavior of the function past 125 days accurately model the mass of the sheep? Why or why not?

Social Sciences

39. Immigration The following graph shows how immigration (in thousands) to the United States has varied over the past century. ***Source: Homeland Security***.



a. Find the average annual rate of change in immigration for the first half of the century (from 1905 to 1955).

b. Find the average annual rate of change in immigration for the second half of the century (from 1955 to 2005).

c. Find the average annual rate of change in immigration for the entire century (from 1905 to 2005).

d. Average your answers to parts a and b, and compare the result with your answer from part c. Will these always be equal for any two time periods?

e. If the annual average rate of change for the entire century continues, predict the number of immigrants in 2009. Compare your answer to the actual number of 1,130,818 immigrants.

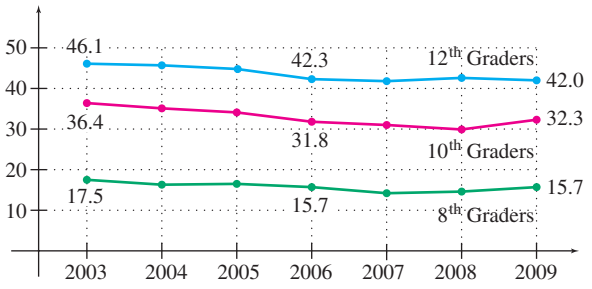
40. Drug Use The following chart shows how the percentage of eighth graders, tenth graders, and twelfth graders who have used marijuana in their lifetime has varied in recent years. *Source: National Institute of Health*.

a. Find the average annual rate of change in the percent of eighth graders who have used marijuana in their lifetime over the three-year period 2003–2006 and the three-year period 2006–2009. Then calculate the annual rate of change for 2003–2009.

b. Repeat part a for tenth graders.

c. Repeat part a for twelfth graders.

d. Discuss any similarities and differences between your answers to parts a through c, as well as possible reasons for these differences and similarities.



Physical Sciences

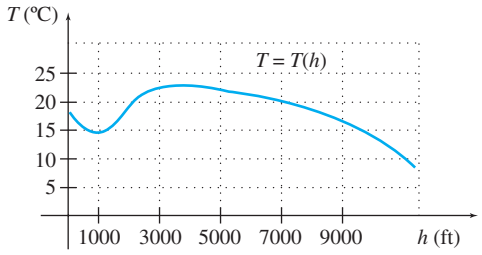
41. Temperature The graph shows the temperature T in degrees Celsius as a function of the altitude h in feet when an inversion layer is over Southern California. (An inversion layer is formed when air at a higher altitude, say 3000 ft, is warmer than air at sea level, even though air normally is cooler with increasing altitude.) Estimate and interpret the average rate of change in temperature for the following changes in altitude.

a. 1000 to 3000 ft **b.** 1000 to 5000 ft

c. 3000 to 9000 ft **d.** 1000 to 9000 ft

e. At what altitude at or below 7000 ft is the temperature highest? Lowest? How would your answer change if 7000 ft is changed to 10,000 ft?

f. At what altitude is the temperature the same as it is at 1000 ft?



42. Velocity A car is moving along a straight test track. The position in feet of the car, $s(t)$, at various times t is measured, with the following results.

t (sec)	0	2	4	6	8	10
$s(t)$ (ft)	0	10	14	20	30	36

Find and interpret the average velocities for the following changes in t.

a. 0 to 2 seconds **b.** 2 to 4 seconds

c. 4 to 6 seconds **d.** 6 to 8 seconds

e. Estimate the instantaneous velocity at 4 seconds

i. by finding the average velocity between 2 and 6 seconds, and

ii. by averaging the answers for the average velocity in the two seconds before and the two seconds after (that is, the answers to parts b and c).

f. Estimate the instantaneous velocity at 6 seconds using the two methods in part e.

g. Notice in parts e and f that your two answers are the same. Discuss whether this will always be the case, and why or why not.

43. Velocity Consider the example at the beginning of this section regarding the car traveling from San Francisco.

a. Estimate the instantaneous velocity at 1 hour. Assume that the velocity changes at a steady rate from one half-hour to the next.

b. Estimate the instantaneous velocity at 2 hours.

44. Velocity The distance of a particle from some fixed point is given by

$$s(t) = t^2 + 5t + 2,$$

where t is time measured in seconds. Find the average velocity of the particle over the following intervals.

a. 4 to 6 seconds

b. 4 to 5 seconds

c. Find the instantaneous velocity of the particle when $t = 4$.

YOUR TURN ANSWERS

1. Increase, on average, by 284,000 people per year
2. Decrease, on average, of 4.25% per year
3. 3 ft per second
4. $6 per unit
5. About 0.318 million, or 318,000 people per year

11.4 Definition of the Derivative

APPLY IT **How does the risk of chromosomal abnormality in a child change with the mother's age?**

We will answer this question in Example 3, using the concept of the derivative.

In the previous section, the formula

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

was used to calculate the instantaneous rate of change of a function f at the point where $x = a$. Now we will give a geometric interpretation of this limit.

The Tangent Line

In geometry, a *tangent line* to a circle is defined as a line that touches the circle at only one point, as at the point P in Figure 27 (which shows the top half of a circle). If you think of this half-circle as part of a curving road on which you are driving at night, then the tangent line indicates the direction of the light beam from your headlights as you pass through the point P. (We are not considering the new type of headlights on some cars that follow the direction of the curve.) Intuitively, the tangent line to an arbitrary curve at a point P on the curve should touch the curve at P, but not at any points nearby, and should indicate the direction of the curve. In Figure 28, for example, the lines through P_1 and P_3 are tangent lines, while the lines through P_2 and P_5 are not. The tangent lines just touch the curve and indicate the direction of the curve, while the other lines pass through the curve heading in some other direction. To decide about the line at P_4, we need to define the idea of a tangent line to the graph of a function more carefully.

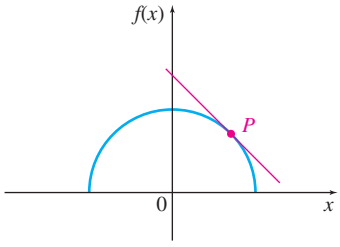


FIGURE 27

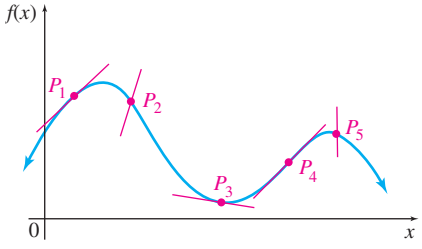


FIGURE 28

To see how we might define the slope of a line tangent to the graph of a function f at a given point, let R be a fixed point with coordinates $(a, f(a))$ on the graph of a function $y = f(x)$, as in Figure 29 on the next page. Choose a different point S on the graph and draw the line through R and S; this line is called a **secant line.** If S has coordinates $(a + h, f(a + h))$, then by the definition of slope, the slope of the secant line RS is given by

$$\text{Slope of secant} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}.$$

This slope corresponds to the average rate of change of y with respect to x over the interval from a to $a + h$. As h approaches 0, point S will slide along the curve, getting closer and closer to the fixed point R. See Figure 30, which shows successive positions S_1, S_2, S_3, and

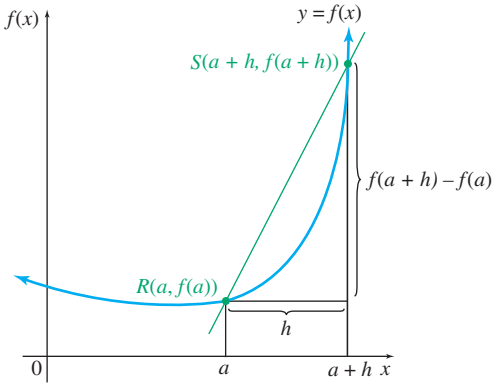


FIGURE 29

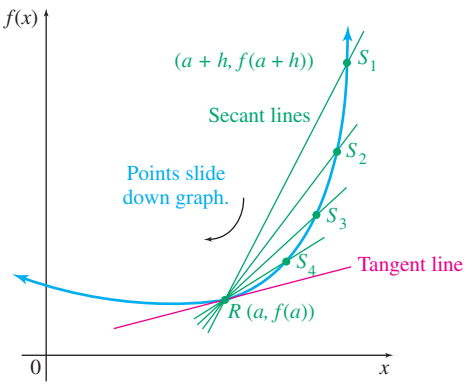


FIGURE 30

S_4 of the point S. If the slopes of the corresponding secant lines approach a limit as h approaches 0, then this limit is defined to be the slope of the tangent line at point R.

Slope of the Tangent Line

The **tangent line** of the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point having slope

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

Notice that the definition of the slope of the tangent line is identical to that of the instantaneous rate of change discussed in the previous section and is calculated by the same procedure.

The slope of the tangent line at a point is also called the **slope of the curve** at the point and corresponds to the instantaneous rate of change of y with respect to x at the point. It indicates the direction of the curve at that point.

FOR REVIEW

In Section 1.1, we saw that the equation of a line can be found with the point-slope form $y - y_1 = m(x - x_1)$, if the slope m and the coordinates (x_1, y_1) of a point on the line are known. Use the point-slope form to find the equation of the line with slope 3 that goes through the point $(-1, 4)$.

Let $m = 3$, $x_1 = -1$, $y_1 = 4$. Then

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= 3(x - (-1)) \\ y - 4 &= 3x + 3 \\ y &= 3x + 7. \end{aligned}$$

EXAMPLE 1 Tangent Line

Consider the graph of $f(x) = x^2 + 2$.

(a) Find the slope and equation of the secant line through the points where $x = -1$ and $x = 2$.

SOLUTION Use the formula for slope as the change in y over the change in x, where y is given by $f(x)$. Since $f(-1) = (-1)^2 + 2 = 3$ and $f(2) = 2^2 + 2 = 6$, we have

$$\text{Slope of secant line} = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{6 - 3}{3} = 1.$$

The slope of the secant line through $(-1, f(-1)) = (-1, 3)$ and $(2, f(2)) = (2, 6)$ is 1.

The equation of the secant line can be found with the point-slope form of the equation of a line from Chapter 1. We'll use the point $(-1, 3)$, although we could have just as well used the point $(2, 6)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= 1[x - (-1)] \\ y - 3 &= x + 1 \\ y &= x + 4 \end{aligned}$$

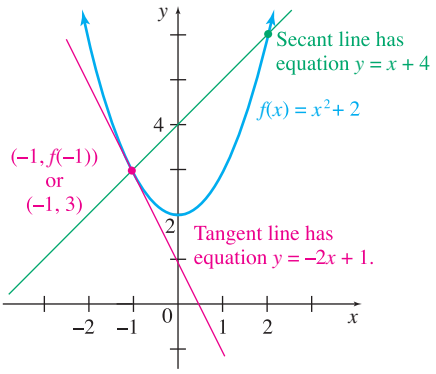


FIGURE 31

Figure 31 shows a graph of $f(x) = x^2 + 2$, along with a graph of the secant line (in green) through the points where $x = -1$ and $x = 2$.

(b) Find the slope and equation of the tangent line at $x = -1$.

SOLUTION Use the definition given previously, with $f(x) = x^2 + 2$ and $a = -1$. The slope of the tangent line is given by

$$\begin{aligned}
\text{Slope of tangent} &= \lim_{h\to 0} \frac{f(\boldsymbol{a} + h) - f(\boldsymbol{a})}{h} \\
&= \lim_{h\to 0} \frac{[(\boldsymbol{-1} + h)^2 + 2] - [(\boldsymbol{-1})^2 + 2]}{h} \\
&= \lim_{h\to 0} \frac{[1 - 2h + h^2 + 2] - [1 + 2]}{h} \\
&= \lim_{h\to 0} \frac{-2h + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(-2 + h)}{h} \\
&= \lim_{h\to 0} (-2 + h) = \boldsymbol{-2}.
\end{aligned}$$

The slope of the tangent line at $(-1, f(-1)) = (-1, 3)$ is -2.

The equation of the tangent line can be found with the point-slope form of the equation of a line from Chapter 1.

$$\begin{aligned}
y - y_1 &= \boldsymbol{m}(x - x_1) \\
y - \boldsymbol{3} &= \boldsymbol{-2}[x - (\boldsymbol{-1})] \\
y - 3 &= -2(x + 1) \\
y - 3 &= -2x - 2 \\
y &= -2x + 1
\end{aligned}$$

The tangent line at $x = -1$ is shown in red in Figure 31.

YOUR TURN 1 For the graph of $f(x) = x^2 - x$, (a) find the equation of the secant line through the points where $x = -2$ and $x = 1$, and (b) find the equation of the tangent line at $x = -2$.

TRY YOUR TURN 1

Figure 32 shows the result of zooming in on the point $(-1, 3)$ in Figure 31. Notice that in this closeup view, the graph and its tangent line appear virtually identical. This gives us another interpretation of the tangent line. Suppose, as we zoom in on a function, the graph appears to become a straight line. Then this line is the tangent line to the graph at that point. In other words, the tangent line captures the behavior of the function very close to the point under consideration. (This assumes, of course, that the function when viewed close up is approximately a straight line. As we will see later in this section, this may not occur.)

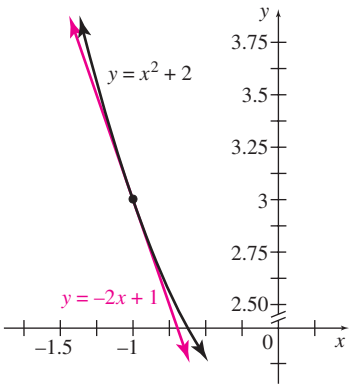


FIGURE 32

If it exists, the tangent line at $x = a$ is a good approximation of the graph of a function near $x = a$.

Consequently, another way to approximate the slope of the curve is to zoom in on the function using a graphing calculator until it appears to be a straight line (the tangent line). Then find the slope using any two points on that line.

TECHNOLOGY

EXAMPLE 2 Slope (Using a Graphing Calculator)

Use a graphing calculator to find the slope of the graph of $f(x) = x^x$ at $x = 1$.

SOLUTION The slope would be challenging to evaluate algebraically using the limit definition. Instead, using a graphing calculator on the window $[0, 2]$ by $[0, 2]$, we see the graph in Figure 33.

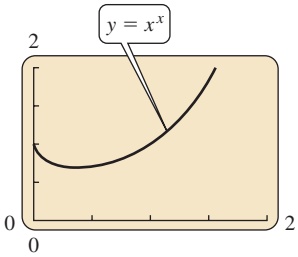


FIGURE 33

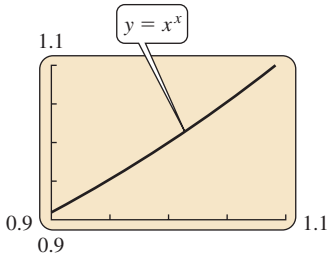


FIGURE 34

Zooming in gives the view in Figure 34. Using the TRACE key, we find two points on the line to be $(1, 1)$ and $(1.0021277, 1.0021322)$. Therefore, the slope is approximately

$$\frac{1.0021322 - 1}{1.0021277 - 1} \approx 1.$$

TECHNOLOGY NOTE

In addition to the method used in Example 2, there are other ways to use a graphing calculator to determine slopes of tangent lines and estimate instantaneous rates. Some alternate methods are listed below.

1. The Tangent command (under the DRAW menu) on a TI-84 Plus allows the tangent line to be drawn on a curve, giving an easy way to generate a graph with its tangent line similar to Figure 31.
2. Rather than using a graph, we could use a TI-84 Plus to create a table, as we did in the previous section, to estimate the instantaneous rate of change. Letting $Y_1 = X\text{^}X$ and

$Y_2 = (Y_1(1 + X) - Y_1(1))/X$, along with specific table settings, results in the table shown in Figure 35. Based on this table, we estimate that the slope of the graph of $f(x) = x^x$ at $x = 1$ is 1.

X	Y2
1	1.1053
.01	1.0101
.001	1.001
1E-4	1.0001
1E-5	1
1E-6	1

X=.1

FIGURE 35

3. An even simpler method on a TI-84 Plus is to use the `dy/dx` command (under the `CALC` menu) or the `nDeriv` command (under the `MATH` menu). We will use this method in Example 4(b). But be careful, because sometimes these commands give erroneous results. For an example, see the Caution at the end of this section. For more details on the `dy/dx` command or the `nDeriv` command, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

EXAMPLE 3 Genetics

Figure 36 shows how the risk of chromosomal abnormality in a child increases with the age of the mother. Find the rate that the risk is rising when the mother is 40 years old. ***Source: downsyndrome.about.com***.

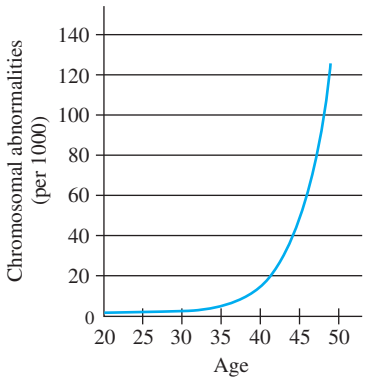


FIGURE 36

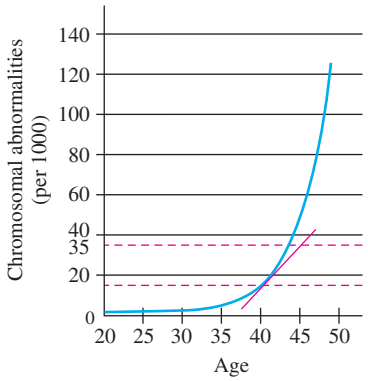


FIGURE 37

APPLY IT

SOLUTION In Figure 37, we have added the tangent line to the graph at the point where the age of the mother is 40. At that point, the risk is approximately 15 per 1000. Extending the line, we estimate that when the age is 45, the y-coordinate of the line is roughly 35. Thus, the slope of the line is

$$\frac{35 - 15}{45 - 40} = \frac{20}{5} = 4.$$

Therefore, at the age of 40, the risk of chromosomal abnormality in the child is increasing at the rate of about 4 per 1000 for each additional year of the mother's age.

The Derivative

If $y = f(x)$ is a function and a is a number in its domain, then we shall use the symbol $f'(a)$ to denote the special limit

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided that it exists. This means that for each number a we can assign the number $f'(a)$ found by calculating this limit. This assignment defines an important new function.

Derivative

The **derivative** of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided this limit exists.

NOTE

The derivative is a *function of* x, since $f'(x)$ varies as x varies. This differs from both the slope of the tangent line and the instantaneous rate of change, either of which is represented by the number $f'(a)$ that corresponds to a number a. Otherwise, the formula for the derivative is identical to the formula for the slope of the tangent line given earlier in this section and to the formula for instantaneous rate of change given in the previous section.

The notation $f'(x)$ is read "f-prime of x." The function $f'(x)$ is called the derivative of f with respect to x. If x is a value in the domain of f and if $f'(x)$ exists, then f is **differentiable** at x. The process that produces f' is called **differentiation**.

The derivative function has several interpretations, two of which we have discussed.

1. The function $f'(x)$ represents the *instantaneous rate of change* of $y = f(x)$ with respect to x. This instantaneous rate of change could be interpreted as marginal cost, revenue, or profit (if the original function represented cost, revenue, or profit) or velocity (if the original function described displacement along a line). From now on we will use *rate of change* to mean *instantaneous* rate of change.
2. The function $f'(x)$ represents the *slope* of the graph of $f(x)$ at any point x. If the derivative is evaluated at the point $x = a$, then it represents the slope of the curve, or the slope of the tangent line, at that point.

The following table compares the different interpretations of the difference quotient and the derivative.

The Difference Quotient and the Derivative

Difference Quotient	Derivative
$\dfrac{f(x + h) - f(x)}{h}$	$\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$
■ Slope of the secant line	■ Slope of the tangent line
■ Average rate of change	■ Instantaneous rate of change
■ Average velocity	■ Instantaneous velocity
■ Average rate of change in cost, revenue, or profit	■ Marginal cost, revenue, or profit

Just as we had an alternate definition in the previous section by using b instead of $a + h$, we now have an alternate definition by using b in place of $x + h$.

Derivative (Alternate Form)

The **derivative** of function f at x can be written as

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x},$$

provided this limit exists.

The next few examples show how to use the definition to find the derivative of a function by means of a four-step procedure.

EXAMPLE 4 Derivative

Let $f(x) = x^2$.

(a) Find the derivative.

Method 1 Original Definition

SOLUTION By definition, for all values of x where the following limit exists, the derivative is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Use the following sequence of steps to evaluate this limit.

Step 1 Find $f(x+h)$.

Replace x with $x + h$ in the equation for $f(x)$. Simplify the result.

$$f(x) = x^2$$
$$f(x+h) = (x+h)^2$$
$$= x^2 + 2xh + h^2$$

(Note that $f(x+h) \neq f(x) + h$, since $f(x) + h = x^2 + h$.)

Step 2 Find $f(x+h) - f(x)$.

Since $f(x) = x^2$,

$$f(x+h) - f(x) = (x^2 + 2xh + h^2) - x^2 = 2xh + h^2.$$

Step 3 Find and simplify the quotient $\dfrac{f(x+h) - f(x)}{h}$. We find that

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h} = 2x + h,$$

except that $2x + h$ is defined for all real numbers h, while $[f(x+h) - f(x)]/h$ is not defined at $h = 0$. But this makes no difference in the limit, which ignores the value of the expression at $h = 0$.

Step 4 Finally, find the limit as h approaches 0. In this step, h is the variable and x is fixed.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x + 0 = 2x$$

Method 2 Alternate Form

SOLUTION Use

$$f(b) = b^2$$

and

$$f(x) = x^2.$$

We apply the alternate definition of the derivative as follows.

$$\lim_{b \to x} \frac{f(b) - f(x)}{b - x} = \lim_{b \to x} \frac{b^2 - x^2}{b - x}$$
$$= \lim_{b \to x} \frac{(b+x)(b-x)}{b - x} \qquad \text{Factor the numerator.}$$
$$= \lim_{b \to x} (b + x) \qquad \text{Divide by } b - x.$$
$$= x + x \qquad \text{Calculate the limit.}$$
$$= 2x$$

The alternate method appears shorter here because factoring $b^2 - x^2$ may seem simpler than calculating $f(x + h) - f(x)$. In other problems, however, factoring may be harder, in which case the first method may be preferable. Thus, from now on, we will use only the first method.

(b) Calculate and interpret $f'(3)$.

Method 1
Algebraic Method

SOLUTION Since $f'(x) = 2x$, we have

$$f'(3) = 2 \cdot 3 = 6.$$

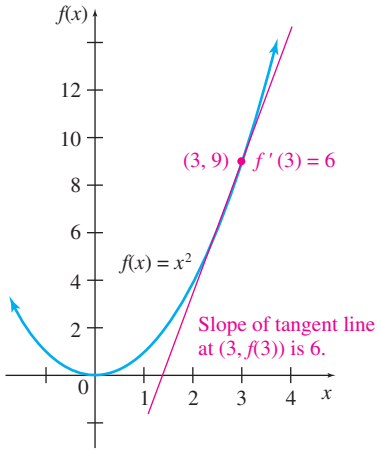
The number 6 is the slope of the tangent line to the graph of $f(x) = x^2$ at the point where $x = 3$, that is, at $(3, f(3)) = (3, 9)$. See Figure 38(a).

Method 2
Graphing Calculator

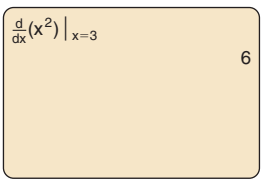
As we mentioned earlier, some graphing calculators can calculate the value of the derivative at a given x-value. For example, the TI-84 Plus uses the `nDeriv` command as shown in Figure 38(b), with the expression for $f(x)$, the variable, and the value of a entered to find $f'(3)$ for $f(x) = x^2$.

YOUR TURN 2 Let $f(x) = x^2 - x$. Find the derivative, and then find $f'(-2)$.

TRY YOUR TURN 2



(a)



(b)

FIGURE 38

CAUTION

1. In Example 4(a) notice that $f(x + h)$ is *not* equal to $f(x) + h$. In fact,

$$f(x + h) = (x + h)^2 = x^2 + 2xh + h^2,$$

but

$$f(x) + h = x^2 + h.$$

2. In Example 4(b), do not confuse $f(3)$ and $f'(3)$. The value $f(3)$ is the y-value that corresponds to $x = 3$. It is found by substituting 3 for x in $f(x)$; $f(3) = 3^2 = 9$. On the other hand, $f'(3)$ is the slope of the tangent line to the curve at $x = 3$; as Example 4(b) shows, $f'(3) = 2 \cdot 3 = 6$.

Finding $f'(x)$ from the Definition of Derivative

The four steps used to find the derivative $f'(x)$ for a function $y = f(x)$ are summarized here.

1. Find $f(x + h)$.
2. Find and simplify $f(x + h) - f(x)$.
3. Divide by h to get $\dfrac{f(x + h) - f(x)}{h}$.
4. Let $h \to 0$; $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$, if this limit exists.

We now have four equivalent expressions for the change in x, but each has its uses, as the following box shows. We emphasize that these expressions all represent the same concept.

Equivalent Expressions for the Change in x

$x_2 - x_1$	Useful for describing the equation of a line through two points
$b - a$	A way to write $x_2 - x_1$ without the subscripts
Δx	Useful for describing slope without referring to the individual points
h	A way to write Δx with just one symbol

EXAMPLE 5 Derivative

Let $f(x) = 2x^3 + 4x$. Find $f'(x)$, $f'(2)$, and $f'(-3)$.

SOLUTION Go through the four steps to find $f'(x)$.

Step 1 Find $f(x + h)$ by replacing x with $x + h$.

$$\begin{aligned} f(x + h) &= 2(x + h)^3 + 4(x + h) \\ &= 2(x^3 + 3x^2h + 3xh^2 + h^3) + 4(x + h) \\ &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 4x + 4h \end{aligned}$$

Step 2 $$\begin{aligned} f(x + h) - f(x) &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 4x + 4h \\ &\quad - 2x^3 - 4x \\ &= 6x^2h + 6xh^2 + 2h^3 + 4h \end{aligned}$$

Step 3 $$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{6x^2h + 6xh^2 + 2h^3 + 4h}{h} \\ &= \frac{h(6x^2 + 6xh + 2h^2 + 4)}{h} \\ &= 6x^2 + 6xh + 2h^2 + 4 \end{aligned}$$

Step 4 Now use the rules for limits to get

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} (6x^2 + 6xh + 2h^2 + 4) \\ &= 6x^2 + 6x(0) + 2(0)^2 + 4 \\ f'(x) &= 6x^2 + 4. \end{aligned}$$

YOUR TURN 3 Let $f(x) = x^3 - 1$. Find $f'(x)$ and $f'(-1)$.

Use this result to find $f'(2)$ and $f'(-3)$.

$$f'(2) = 6 \cdot 2^2 + 4 = 28$$
$$f'(-3) = 6 \cdot (-3)^2 + 4 = 58$$

TRY YOUR TURN 3

TECHNOLOGY NOTE

One way to support the result in Example 5 is to plot $[f(x + h) - f(x)]/h$ on a graphing calculator with a small value of h. Figure 39 shows a graphing calculator screen of $y = [f(x + 0.1) - f(x)]/0.1$, where f is the function $f(x) = 2x^3 + 4x$, and $y = 6x^2 + 4$, which was just found to be the derivative of f. The two functions, plotted on the window $[-2, 2]$ by $[0, 30]$, appear virtually identical. If $h = 0.01$ had been used, the two functions would be indistinguishable.

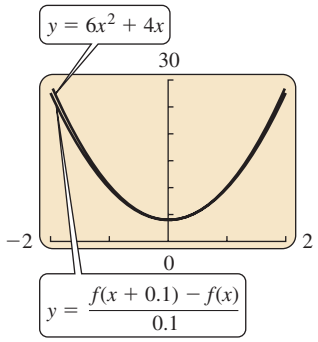


FIGURE 39

EXAMPLE 6 Derivative

Let $f(x) = \dfrac{4}{x}$. Find $f'(x)$.

SOLUTION

Step 1 $f(x + h) = \dfrac{4}{x + h}$

Step 2 $f(x + h) - f(x) = \dfrac{4}{x + h} - \dfrac{4}{x}$

$$= \frac{4x - 4(x + h)}{x(x + h)}$$ Find a common denominator.

$$= \frac{4x - 4x - 4h}{x(x + h)}$$ Simplify the numerator.

$$= \frac{-4h}{x(x + h)}$$

Step 3 $\dfrac{f(x + h) - f(x)}{h} = \dfrac{\dfrac{-4h}{x(x + h)}}{h}$

$$= \frac{-4h}{x(x + h)} \cdot \frac{1}{h}$$ Invert and multiply.

$$= \frac{-4}{x(x + h)}$$

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{-4}{x(x + h)}$$

$$= \frac{-4}{x(x + 0)}$$

$$f'(x) = \frac{-4}{x(x)} = \frac{-4}{x^2}$$

TRY YOUR TURN 4

YOUR TURN 4 Let $f(x) = -\dfrac{2}{x}$. Find $f'(x)$.

Notice that in Example 6 neither $f(x)$ nor $f'(x)$ is defined when $x = 0$. Look at a graph of $f(x) = 4/x$ to see why this is true.

EXAMPLE 7 Weight Gain

A mathematics professor found that, after introducing his dog Django to a new brand of food, Django's weight began to increase. After x weeks on the new food, Django's weight (in pounds) was approximately given by $w(x) = \sqrt{x} + 40$ for $0 \le x \le 6$. Find the rate of change of Django's weight after x weeks.

SOLUTION

Step 1 $w(x + h) = \sqrt{x + h} + 40$

Step 2 $w(x + h) - w(x) = \sqrt{x + h} + 40 - (\sqrt{x} + 40)$

$$= \sqrt{x + h} - \sqrt{x}$$

Step 3 $$\frac{w(x + h) - w(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h}$$

In order to be able to divide by h, multiply both numerator and denominator by $\sqrt{x + h} + \sqrt{x}$; that is, rationalize the *numerator*.

$$\frac{w(x + h) - w(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}$$ Rationalize the numerator

$$= \frac{(\sqrt{x + h})^2 - (\sqrt{x})^2}{h(\sqrt{x + h} + \sqrt{x})}$$ $(a - b)(a + b) = a^2 - b^2$.

$$= \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x + h} + \sqrt{x})}$$ Simplify.

$$= \frac{1}{\sqrt{x + h} + \sqrt{x}}$$ Divide by h.

Step 4 $$w'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

This tells us, for example, that after 4 weeks, when Django's weight is $w(4) = \sqrt{4} + 40 = 42$ lb, her weight is increasing at a rate of $w'(4) = 1/(2\sqrt{4}) = 1/4$ lb per week.

TRY YOUR TURN 5

YOUR TURN 5 Let $f(x) = 2\sqrt{x}$. Find $f'(x)$.

EXAMPLE 8 Cost Analysis

The cost in dollars to manufacture x graphing calculators is given by $C(x) = -0.005x^2 + 20x + 150$ when $0 \le x \le 2000$. Find the rate of change of cost with respect to the number manufactured when 100 calculators are made and when 1000 calculators are made.

SOLUTION The rate of change of cost is given by the derivative of the cost function,

$$C'(x) = \lim_{h \to 0} \frac{C(x + h) - C(x)}{h}.$$

Going through the steps for finding $C'(x)$ gives

$$C'(x) = -0.01x + 20.$$

When $x = 100$,

$$C'(100) = -0.01(100) + 20 = 19.$$

This rate of change of cost per calculator gives the marginal cost at $x = 100$, which means the approximate cost of producing the 101st calculator is \$19.

When 1000 calculators are made, the marginal cost is

$$C'(1000) = -0.01(1000) + 20 = 10,$$

or \$10.

YOUR TURN 6 If cost is given by $C(x) = 10x - 0.002x^2$, find the rate of change when $x = 100$.

TRY YOUR TURN 6

We can use the notation for the derivative to write the equation of the tangent line. Using the point-slope form, $y - y_1 = m(x - x_1)$, and letting $y_1 = f(x_1)$ and $m = f'(x_1)$, we have the following formula.

Equation of the Tangent Line

The tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is given by the equation

$$y - f(x_1) = f'(x_1)(x - x_1),$$

provided $f'(x)$ exists.

EXAMPLE 9 Tangent Line

Find the equation of the tangent line to the graph of $f(x) = 4/x$ at $x = 2$.

SOLUTION From the answer to Example 6, we have $f'(x) = -4/x^2$, so $f'(x_1) = f'(2) = -4/2^2 = -1$. Also $f(x_1) = f(2) = 4/2 = 2$. Then the equation of the tangent line is

$$y - 2 = (-1)(x - 2),$$

or

$$y = -x + 4$$

after simplifying.

YOUR TURN 7 Find the equation of the tangent line to the graph $f(x) = 2\sqrt{x}$ at $x = 4$.

TRY YOUR TURN 7

Existence of the Derivative

The definition of the derivative included the phrase "provided this limit exists." If the limit used to define the derivative does not exist, then of course the derivative does not exist. For example, a derivative cannot exist at a point where the function itself is not defined. If there is no function value for a particular value of x, there can be no tangent line for that value. This was the case in Example 6—there was no tangent line (and no derivative) when $x = 0$.

Derivatives also do not exist at "corners" or "sharp points" on a graph. For example, the function graphed in Figure 40 is the *absolute value function*, defined previously as

$$f(x) = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0, \end{cases}$$

and written $f(x) = |x|$. By the definition of derivative, the derivative at any value of x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided this limit exists. To find the derivative at 0 for $f(x) = |x|$, replace x with 0 and $f(x)$ with $|0|$ to get

$$f'(0) = \lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

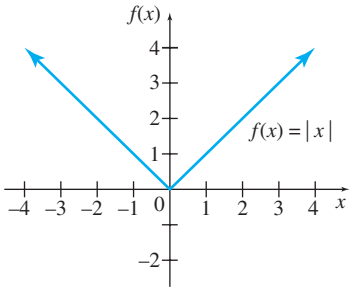


FIGURE 40

In Example 5 in the first section of this chapter, we showed that

$$\lim_{h \to 0} \frac{|h|}{h} \text{ does not exist;}$$

therefore, the derivative does not exist at 0. However, the derivative does exist for all values of x other than 0.

CAUTION The command `nDeriv(abs(X),X,0)` on a TI-84 Plus calculator gives the answer 0, which is wrong. It does this by investigating a point slightly to the left of 0 and slightly to the right of 0. Since the function has the same value at these two points, it assumes that the function must be flat around 0, which is false in this case because of the sharp corner at 0. Be careful about naively trusting your calculator; think about whether the answer is reasonable.

In Figure 41, we have zoomed in on the origin in Figure 40. Notice that the graph looks essentially the same. The corner is still sharp, and the graph does not resemble a straight line any more than it originally did. As we observed earlier, the derivative only exists at a point when the function more and more resembles a straight line as we zoom in on the point.

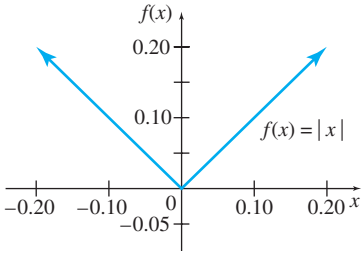


FIGURE 41

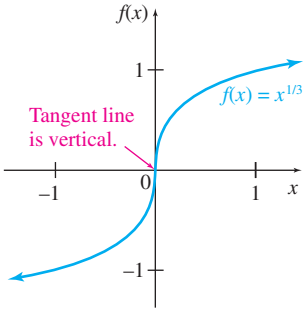


FIGURE 42

A graph of the function $f(x) = x^{1/3}$ is shown in Figure 42. As the graph suggests, the tangent line is vertical when $x = 0$. Since a vertical line has an undefined slope, the derivative of $f(x) = x^{1/3}$ cannot exist when $x = 0$. Use the fact that $\lim_{h \to 0} h^{1/3}/h = \lim_{h \to 0} 1/h^{2/3}$ does not exist and the definition of the derivative to verify that $f'(0)$ does not exist for $f(x) = x^{1/3}$.

Figure 43 summarizes the various ways that a derivative can fail to exist. Notice in Figure 43 that at a point where the function is discontinuous, such as x_3, x_4, and x_6, the

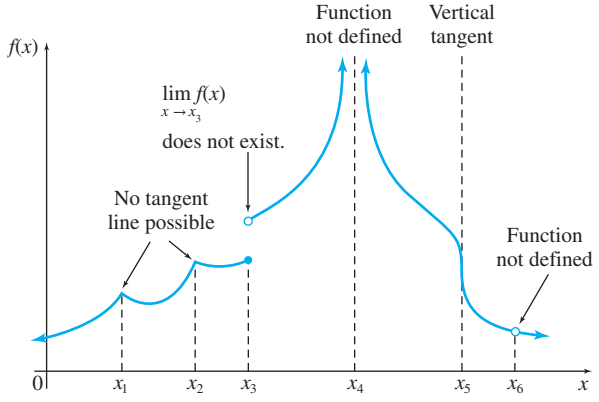


FIGURE 43

derivative does not exist. A function must be continuous at a point for the derivative to exist there. But just because a function is continuous at a point does not mean the derivative necessarily exists. For example, observe that the function in Figure 43 is continuous at x_1 and x_2, but the derivative does not exist at those values of x because of the sharp corners, making a tangent line impossible. This is exactly what happens with the function $f(x) = |x|$ at $x = 0$, as we saw in Figure 40. Also, the function is continuous at x_5, but the derivative doesn't exist there because the tangent line is vertical, and the slope of a vertical line is undefined.

We summarize conditions for the derivative to exist or not below.

Existence of the Derivative

The derivative exists when a function f satisfies *all* of the following conditions at a point.

1. f is continuous,
2. f is smooth, and
3. f does not have a vertical tangent line.

The derivative does *not* exist when *any* of the following conditions are true for a function at a point.

1. f is discontinuous,
2. f has a sharp corner, or
3. f has a vertical tangent line.

EXAMPLE 10 Astronomy

A nova is a star whose brightness suddenly increases and then gradually fades. The cause of the sudden increase in brightness is thought to be an explosion of some kind. The intensity of light emitted by a nova as a function of time is shown in Figure 44. Find where the function is not differentiable. ***Source: Astronomy: The Structure of the Universe.***

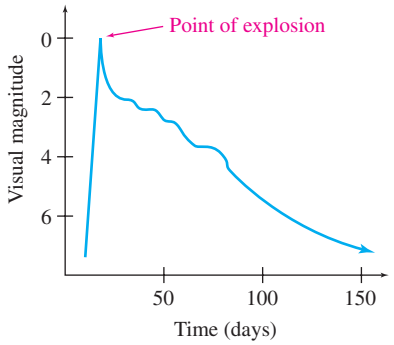


FIGURE 44

SOLUTION Notice that although the graph is a continuous curve, it is not differentiable at the point of explosion.

11.4 EXERCISES

1. By considering, but not calculating, the slope of the tangent line, give the derivative of the following.

a. $f(x) = 5$ **b.** $f(x) = x$

c. $f(x) = -x$ **d.** The line $x = 3$

e. The line $y = mx + b$

2. a. Suppose $g(x) = \sqrt[3]{x}$. Use the graph of $g(x)$ to find $g'(0)$.

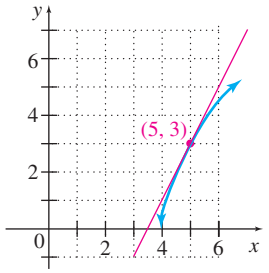
b. Explain why the derivative of a function does not exist at a point where the tangent line is vertical.

3. If $f(x) = \dfrac{x^2 - 1}{x + 2}$, where is f not differentiable?

4. If the rate of change of $f(x)$ is zero when $x = a$, what can be said about the tangent line to the graph of $f(x)$ at $x = a$?

Estimate the slope of the tangent line to each curve at the given point (x, y).

5.



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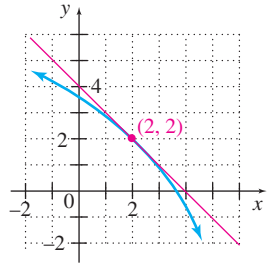

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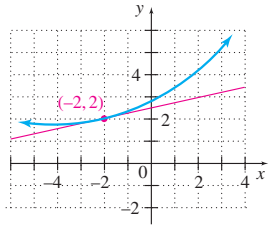

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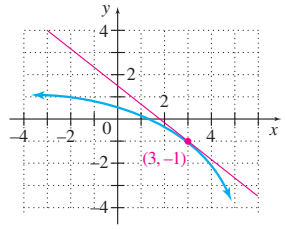

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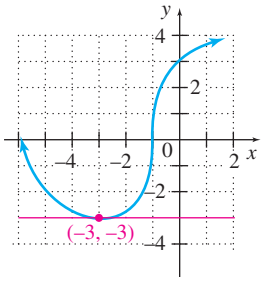

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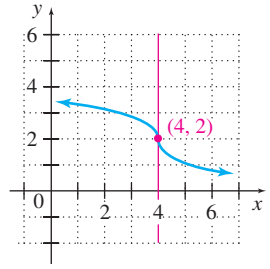


Using the definition of the derivative, find $f'(x)$. Then find $f'(-2)$, $f'(0)$, and $f'(3)$ when the derivative exists. (Hint for Exercises 17 and 18: In Step 3, multiply numerator and denominator by $\sqrt{x + h} + \sqrt{x}$.)

11. $f(x) = 3x - 7$ **12.** $f(x) = -2x + 5$

13. $f(x) = -4x^2 + 9x + 2$ **14.** $f(x) = 6x^2 - 5x - 1$

15. $f(x) = 12/x$ **16.** $f(x) = 3/x$

17. $f(x) = \sqrt{x}$ **18.** $f(x) = -3\sqrt{x}$

19. $f(x) = 2x^3 + 5$ **20.** $f(x) = 4x^3 - 3$

For each function, find (a) the equation of the secant line through the points where x has the given values, and (b) the equation of the tangent line when x has the first value.

21. $f(x) = x^2 + 2x$; $x = 3$, $x = 5$

22. $f(x) = 6 - x^2$; $x = -1$, $x = 3$

23. $f(x) = 5/x$; $x = 2$, $x = 5$

24. $f(x) = -3/(x + 1)$; $x = 1$, $x = 5$

25. $f(x) = 4\sqrt{x}$; $x = 9$, $x = 16$

26. $f(x) = \sqrt{x}$; $x = 25$, $x = 36$

Use a graphing calculator to find $f'(2)$, $f'(16)$, and $f'(-3)$ for the following when the derivative exists.

27. $f(x) = -4x^2 + 11x$ **28.** $f(x) = 6x^2 - 4x$

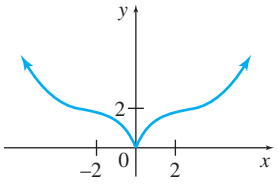
29. $f(x) = e^x$ **30.** $f(x) = \ln|x|$

31. $f(x) = -\dfrac{2}{x}$ **32.** $f(x) = \dfrac{6}{x}$

33. $f(x) = \sqrt{x}$ **34.** $f(x) = -3\sqrt{x}$

Find the x-values where the following do not have derivatives.

35.



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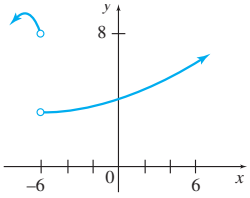

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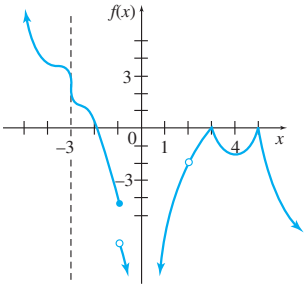
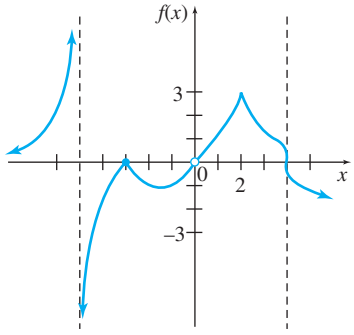
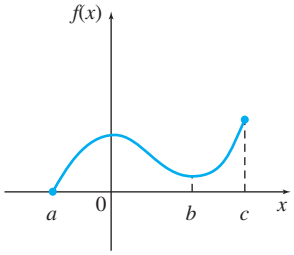


38.



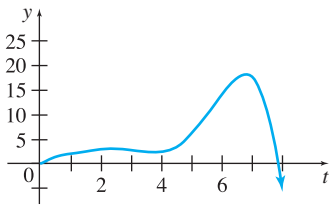
39. For the function shown in the sketch, give the intervals or points on the x-axis where the rate of change of $f(x)$ with respect to x is

a. positive; **b.** negative; **c.** zero.

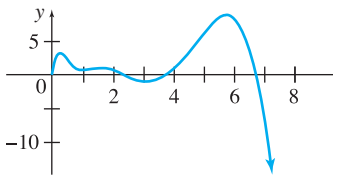


In Exercises 40 and 41, tell which graph, a or b, represents velocity and which represents distance from a starting point. (Hint: Consider where the derivative is zero, positive, or negative.)

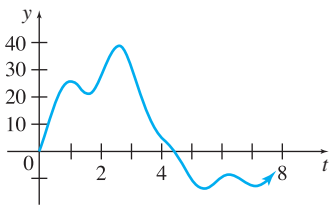
40. a.



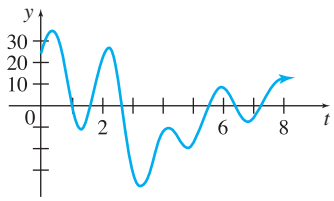
b.



41. a.



b.



In Exercises 42–45, find the derivative of the function at the given point.

a. Approximate the definition of the derivative with small values of h.

b. Use a graphing calculator to zoom in on the function until it appears to be a straight line, and then find the slope of that line.

42. $f(x) = x^x$; $a = 2$ **43.** $f(x) = x^x$; $a = 3$

44. $f(x) = x^{1/x}$; $a = 2$ **45.** $f(x) = x^{1/x}$; $a = 3$

46. For each function in Column A, graph $[f(x + h) - f(x)]/h$ for a small value of h on the window $[-2, 2]$ by $[-2, 8]$. Then graph each function in Column B on the same window. Compare the first set of graphs with the second set to choose from Column B the derivative of each of the functions in Column A.

Column A	Column B
$\ln \lvert x \rvert$	e^x
e^x	$3x^2$
x^3	$\dfrac{1}{x}$

47. Explain why

$$\frac{f(x + h) - f(x - h)}{2h}$$

should give a reasonable approximation of $f'(x)$ when $f'(x)$ exists and h is small.

48. a. For the function $f(x) = -4x^2 + 11x$, find the value of $f'(3)$, as well as the approximation using

$$\frac{f(x + h) - f(x)}{h}$$

and using the formula in Exercise 47 with $h = 0.1$.

b. Repeat part a using $h = 0.01$.

c. Repeat part a using the function $f(x) = -2/x$ and $h = 0.1$.

d. Repeat part c using $h = 0.01$.

e. Repeat part a using the function $f(x) = \sqrt{x}$ and $h = 0.1$.

f. Repeat part e using $h = 0.01$.

g. Using the results of parts a through f, discuss which approximation formula seems to give better accuracy.

APPLICATIONS

Business and Economics

49. Demand Suppose the demand for a certain item is given by $D(p) = -2p^2 - 4p + 300$, where p represents the price of the item in dollars.

a. Find the rate of change of demand with respect to price.

b. Find and interpret the rate of change of demand when the price is \$10.

50. Profit The profit (in thousands of dollars) from the expenditure of x thousand dollars on advertising is given by $P(x) = 1000 + 32x - 2x^2$. Find the marginal profit at the following expenditures. In each case, decide whether the firm should increase the expenditure.

a. \$8000 **b.** \$6000 **c.** \$12,000 **d.** \$20,000

51. Revenue The revenue in dollars generated from the sale of x picnic tables is given by $R(x) = 20x - \dfrac{x^2}{500}$.

a. Find the marginal revenue when 1000 tables are sold.

b. Estimate the revenue from the sale of the 1001st table by finding $R'(1000)$.

c. Determine the actual revenue from the sale of the 1001st table.

d. Compare your answers for parts b and c. What do you find?

52. Cost The cost in dollars of producing x tacos is $C(x) = -0.00375x^2 + 1.5x + 1000$, for $0 \le x \le 180$.

a. Find the marginal cost.

b. Find and interpret the marginal cost at a production level of 100 tacos.

c. Find the exact cost to produce the 101st taco.

d. Compare the answers to parts b and c. How are they related?

e. Show that whenever $C(x) = ax^2 + bx + c$, $[C(x+1) - C(x)] - C'(x) = a$. ***Source: The College Mathematics Journal.***

f. Show that whenever $C(x) = ax^2 + bx + c$,

$$C(x+1) - C(x) = C'\left(x + \frac{1}{2}\right).$$

53. Social Security Assets The table gives actual and projected year-end assets in Social Security trust funds, in trillions of current dollars, where Year represents the number of years since 2000. ***Source: Social Security Administration.***

The polynomial function defined by

$$f(x) = 0.0000329x^3 - 0.00450x^2 + 0.0613x + 2.34$$

models the data quite well.

a. To verify the fit of the model, find $f(10)$, $f(20)$, and $f(30)$.

b. Use a graphing calculator with a command such as `nDeriv` to find the slope of the tangent line to the graph of f at the following x-values: 0, 10, 20, 30, and 35.

c. Use your results in part b to describe the graph of f and interpret the corresponding changes in Social Security assets.

Year	Trillions of Dollars
10	2.45
20	2.34
30	1.03
40	−0.57
50	−1.86

Life Sciences

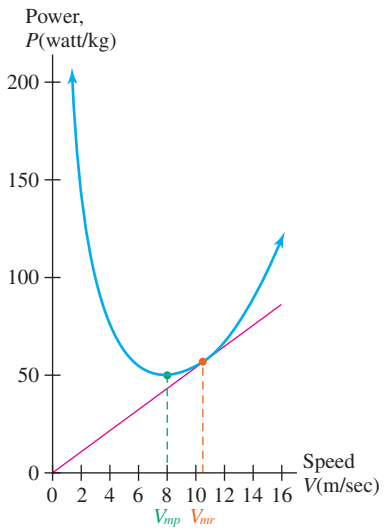
54. Flight Speed Many birds, such as cockatiels or the Arctic terns shown below, have flight muscles whose expenditure of power varies with the flight speed in a manner similar to the graph shown in the next column. The horizontal axis of the graph shows flight speed in meters per second, and the vertical axis shows power in watts per kilogram. ***Source: Biolog-e: The Undergraduate Bioscience Research Journal.***

a. The speed V_{mp} minimizes energy costs per unit of time. What is the slope of the line tangent to the curve at the point corresponding to V_{mp}? What is the physical significance of the slope at that point?

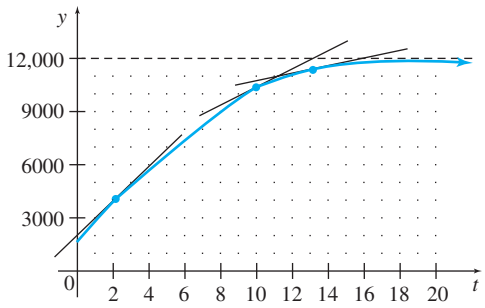
b. The speed V_{mr} minimizes the energy costs per unit of distance covered. Estimate the slope of the curve at the point corresponding to V_{mr}. Give the significance of the slope at that point.

c. By looking at the shape of the curve, describe how the power level decreases and increases for various speeds.

d. Notice that the slope of the lines found in parts a and b represents the power divided by speed. Power is measured in energy per unit time per unit weight of the bird, and speed is distance per unit time, so the slope represents energy per unit distance per unit weight of the bird. If a line is drawn from the origin to a point on the graph, at which point is the slope of the line (representing energy per unit distance per unit weight of the bird) smallest? How does this compare with your answers to parts a and b?



55. Shellfish Population In one research study, the population of a certain shellfish in an area at time t was closely approximated by the following graph. Estimate and interpret the derivative at each of the marked points.



56. Eating Behavior The eating behavior of a typical human during a meal can be described by

$$I(t) = 27 + 72t - 1.5t^2,$$

where t is the number of minutes since the meal began, and $I(t)$ represents the amount (in grams) that the person has eaten at time t. ***Source: Appetite.***

a. Find the rate of change of the intake of food for this particular person 5 minutes into a meal and interpret.

b. Verify that the rate in which food is consumed is zero 24 minutes after the meal starts.

c. Comment on the assumptions and usefulness of this function after 24 minutes. Given this fact, determine a logical domain for this function.

57. Quality Control of Cheese It is often difficult to evaluate the quality of products that undergo a ripening or maturation process. Researchers have successfully used ultrasonic velocity to determine the maturation time of Mahon cheese. The age can be determined by

$$M(v) = 0.0312443v^2 - 101.39v + 82{,}264, \quad v \geq 1620,$$

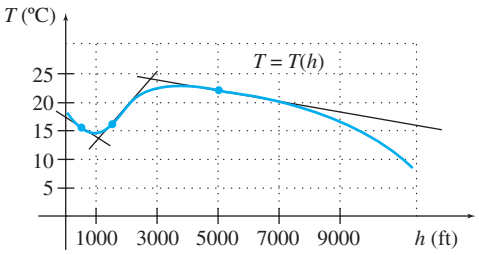
where $M(v)$ is the estimated age of the cheese (in days) for a velocity v (m per second). ***Source: Journal of Food Science.***

a. If Mahon cheese ripens in 150 days, determine the velocity of the ultrasound that one would expect to measure. (*Hint:* Set $M(v) = 150$ and solve for v.)

b. Determine the derivative of this function when $v = 1700$ m per second and interpret.

Physical Sciences

58. Temperature The graph shows the temperature in degrees Celsius as a function of the altitude h in feet when an inversion layer is over Southern California. (See Exercise 41 in the previous section.) Estimate and interpret the derivatives of $T(h)$ at the marked points.



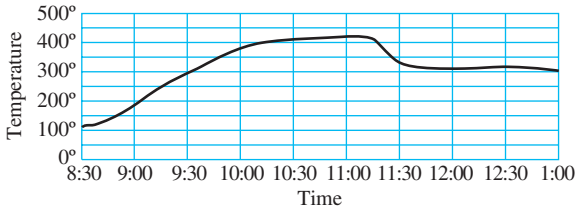
59. Oven Temperature The graph in the next column shows the temperature one Christmas day of a Heat-Kit Bakeoven, a wood-burning oven for baking. ***Source: Heatkit.com.*** The oven was lit at 8:30 a.m. Let $T(x)$ be the temperature x hours after 8:30 a.m.

a. Find and interpret $T'(0.5)$.

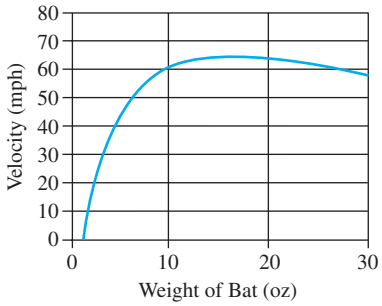
b. Find and interpret $T'(3)$.

c. Find and interpret $T'(4)$.

d. At a certain time a Christmas turkey was put into the oven, causing the oven temperature to drop. Estimate when the turkey was put into the oven.



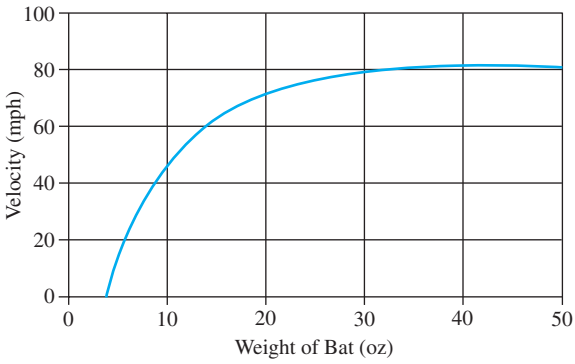
60. Baseball The graph shows how the velocity of a baseball that was traveling at 40 miles per hour when it was hit by a Little League baseball player varies with respect to the weight of the bat. ***Source: Biological Cybernetics.***



a. Estimate and interpret the derivative for a 16-oz and 25-oz bat.

b. What is the optimal bat weight for this player?

61. Baseball The graph shows how the velocity of a baseball that was traveling at 90 miles per hour when it was hit by a Major League baseball player varies with respect to the weight of the bat. ***Source: Biological Cybernetics.***



a. Estimate and interpret the derivative for a 40-oz and 30-oz bat.

b. What is the optimal bat weight for this player?

YOUR TURN ANSWERS

1. a. $y = -2x + 2$ **b.** $y = -5x - 4$

2. $f'(x) = 2x - 1$ and $f'(-2) = -5$.

3. $f'(x) = 3x^2$ and $f'(-1) = 3$.

4. $f'(x) = \dfrac{2}{x^2}$.

5. $f'(x) = \dfrac{1}{\sqrt{x}}$

6. \$9.60

7. $y = \frac{1}{2}x + 2$

11.5 Graphical Differentiation

APPLY IT **Given a graph of the production function, how can we find the graph of the marginal production function?**

We will answer this question in Example 1 using graphical differentiation.

In the previous section, we estimated the derivative at various points of a graph by estimating the slope of the tangent line at those points. We will now extend this process to show how to sketch the graph of the derivative given the graph of the original function. This is important because, in many applications, a graph is all we have, and it is easier to find the derivative graphically than to find a formula that fits the graph and take the derivative of that formula.

EXAMPLE 1 Production of Landscape Mulch

In Figure 45(a), the graph shows total production (TP), measured in cubic yards of landscape mulch per week, as a function of labor used, measured in workers hired by a small business. The graph in Figure 45(b) shows the marginal production curve (MP_L), which is the derivative of the total production function. Verify that the graph of the marginal production curve (MP_L) is the graph of the derivative of the total production curve (TP).

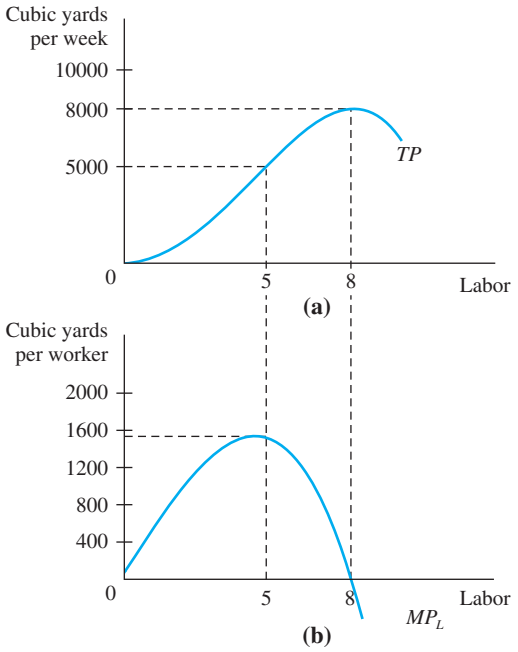


FIGURE 45

APPLY IT **SOLUTION** Let q refer to the quantity of labor. We begin by choosing a point where estimating the derivative of TP is simple. Observe that when $q = 8$, TP has a horizontal tangent line, so its derivative is 0. This explains why the graph of MP_L equals 0 when $q = 8$.

Now, observe in Figure 45(a) that when $q < 8$, the tangent lines of TP have positive slope and the slope is steepest when $q = 5$. This means that the derivative should be positive for $q < 8$ and largest when $q = 5$. Verify that the graph of MP_L has this property.

Finally, as Figure 45(a) shows, the tangent lines of TP have negative slope when $q > 8$, so its derivative, represented by the graph of MP_L, should also be negative there. Verify that the graph of MP_L, in Figure 45(b), has this property as well.

In Example 1, we saw how the general shape of the graph of the derivative could be found from the graph of the original function. To get a more accurate graph of the derivative, we need to estimate the slope of the tangent line at various points, as we did in the previous section.

EXAMPLE 2 Temperature

Figure 46 gives the temperature in degrees Celsius as a function, $T(h)$, of the altitude h in feet when an inversion layer is over Southern California. Sketch the graph of the derivative of the function. (This graph appeared in Exercise 58 in the previous section.)

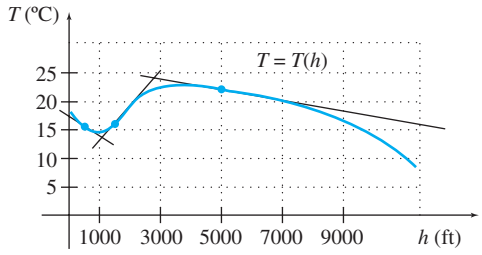


FIGURE 46

SOLUTION First, observe that when $h = 1000$ and $h = 3500$, $T(h)$ has horizontal tangent lines, so $T'(1000) = 0$ and $T'(3500) = 0$.

Notice that the tangent lines would have a negative slope for $0 < h < 1000$. Thus, the graph of the derivative should be negative (below the x-axis) there. Then, for $1000 < h < 3500$, the tangent lines have positive slope, so the graph of the derivative should be positive (above the x-axis) there. Notice from the graph of $T(h)$ that the slope is largest when $h = 1500$. Finally, for $h > 3500$, the tangent lines would be negative again, forcing the graph of the derivative back down below the x-axis to take on negative values.

Now that we have a general shape of the graph, we can estimate the value of the derivative at several points to improve the accuracy of the graph. To estimate the derivative, find two points on each tangent line and compute the slope between the two points. The estimates at selected points are given in the table to the left. (Your answers may be slightly different, since estimation from a picture can be inexact.) Figure 47 shows a graph of these values of $T'(h)$.

Estimates of $T'(h)$	
h	$T'(h)$
500	−0.005
1000	0
1500	0.008
3500	0
5000	−0.00125

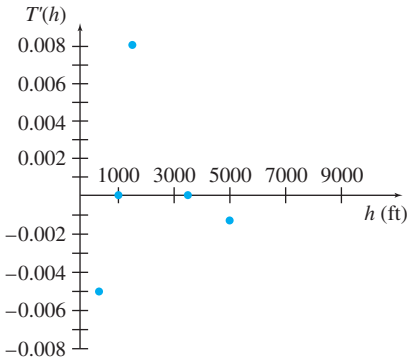


FIGURE 47

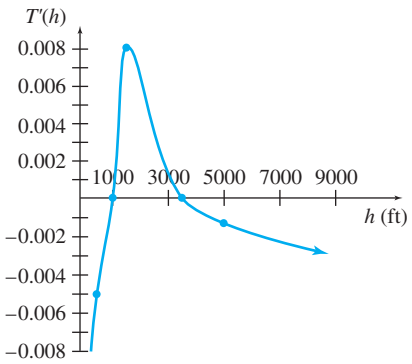
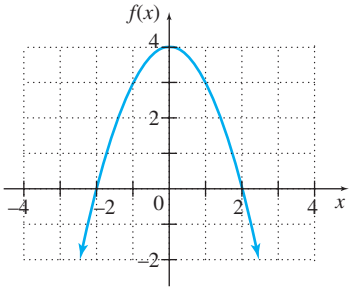


FIGURE 48

Using all of these facts, we connect the points in the graph $T'(h)$ smoothly, with the result shown in Figure 48.

TRY YOUR TURN 1

YOUR TURN 1 Sketch the graph of the derivative of the function $f(x)$.



CAUTION Remember that when you graph the derivative, you are graphing the *slope* of the original function. Do not confuse the slope of the original function with the y-value of the original function. In fact, the slope of the original function is equal to the y-value of its derivative.

Sometimes the original function is not smooth or even continuous, so the graph of the derivative may also be discontinuous.

EXAMPLE 3 Graphing a Derivative

Sketch the graph of the derivative of the function shown in Figure 49.

SOLUTION Notice that when $x < -2$, the slope is 1, and when $-2 < x < 0$, the slope is -1. At $x = -2$, the derivative does not exist due to the sharp corner in the graph. The derivative also does not exist at $x = 0$ because the function is discontinuous there. Using this information, the graph of $f'(x)$ on $x < 0$ is shown in Figure 50.

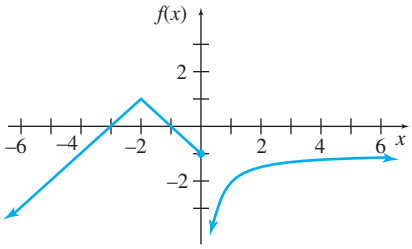


FIGURE 49

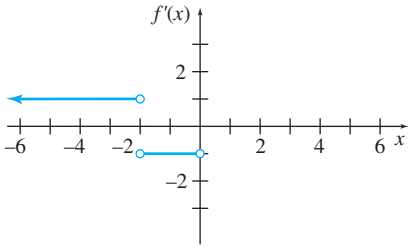


FIGURE 50

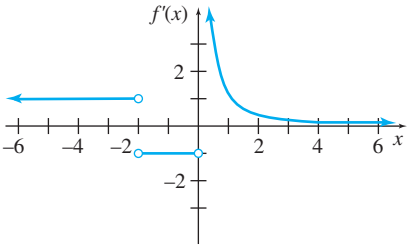
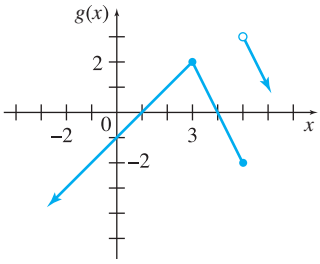


FIGURE 51

YOUR TURN 2 Sketch the graph of the derivative of the function $g(x)$.



For $x > 0$, the derivative is positive. If you draw a tangent line at $x = 1$, you should find that the slope of this line is roughly 1. As x approaches 0 from the right, the derivative becomes larger and larger. As x approaches infinity, the tangent lines become more and more horizontal, so the derivative approaches 0. The resulting sketch of the graph of $y = f'(x)$ is shown in Figure 51. TRY YOUR TURN 2

Finding the derivative graphically may seem difficult at first, but with practice you should be able to quickly sketch the derivative of any function graphed. Your answers to the exercises may not look exactly like those in the back of the book, because estimating the slope accurately can be difficult, but your answers should have the same general shape.

Figures 52(a), (b), and (c) show the graphs of $y = x^2$, $y = x^4$, and $y = x^{4/3}$ on a graphing calculator. When finding the derivative graphically, all three seem to have the same behavior: negative derivative for $x < 0$, 0 derivative at $x = 0$, and positive derivative for $x > 0$. Beyond these general features, however, the derivatives look quite different, as you

can see from Figures 53(a), (b), and (c), which show the graphs of the derivatives. When finding derivatives graphically, detailed information can only be found by very carefully measuring the slope of the tangent line at a large number of points.

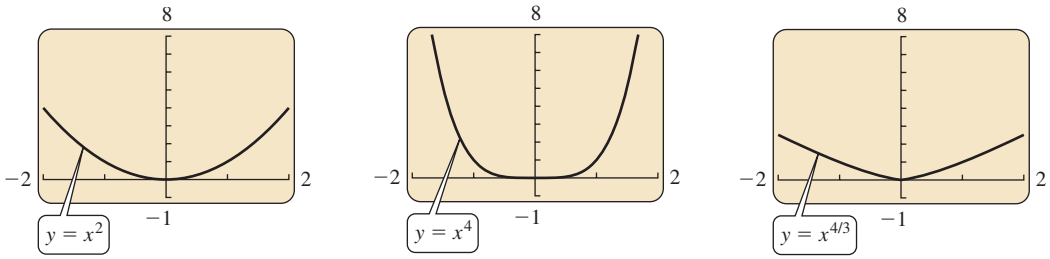


FIGURE 52

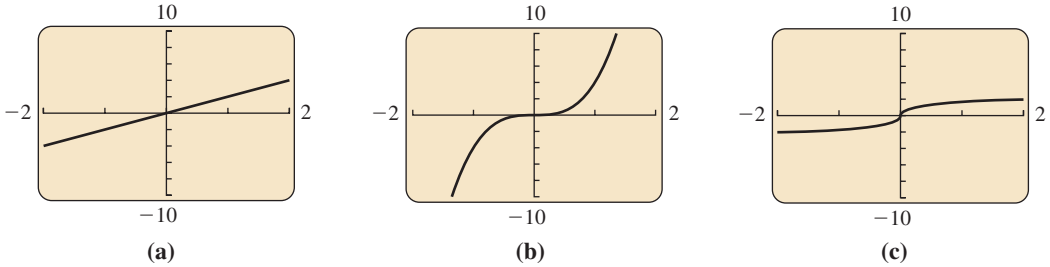


FIGURE 53

TECHNOLOGY NOTE

On many calculators, the graph of the derivative can be plotted if a formula for the original function is known. For example, the graphs in Figure 53 were drawn on a TI-84 Plus by using the `nDeriv` command. Define $Y_2 = \frac{d}{dx}(Y_1)\big|_{x=x}$ after entering the original function into Y_1. You can use this feature to practice finding the derivative graphically. Enter a function into Y_1, sketch the graph on the graphing calculator, and use it to draw by hand the graph of the derivative. Then use `nDeriv` to draw the graph of the derivative, and compare it with your sketch.

EXAMPLE 4 Graphical Differentiation

Figure 54 shows the graph of a function f and its derivative function f'. Use slopes to decide which graph is that of f and which is the graph of f'.

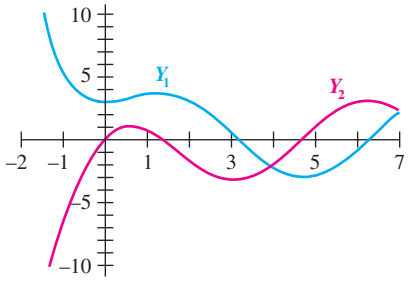


FIGURE 54

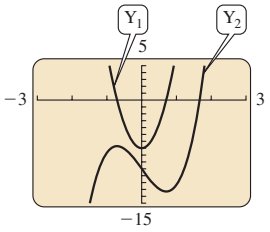
SOLUTION Look at the places where each graph crosses the x-axis; that is, the x-intercepts, since x-intercepts occur on the graph of f' whenever the graph of f has a horizontal tangent line or slope of zero. Also, a decreasing graph corresponds to negative slope or a negative derivative, while an increasing graph corresponds to positive slope or a positive derivative. Y_1 has zero slope near $x = 0$, $x = 1$, and $x = 5$; Y_2 has x-intercepts near these values of x. Y_1 decreases on $(-2, 0)$ and $(1, 5)$; Y_2 is negative on those intervals. Y_1 increases on $(0, 1)$ and $(5, 7)$; Y_2 is positive there. Thus, Y_1 is the graph of f and Y_2 is the graph of f'.

11.5 EXERCISES

1. Explain how to graph the derivative of a function given the graph of the function.
2. Explain how to graph a function given the graph of the derivative function. Note that the answer is not unique.

Each graphing calculator window shows the graph of a function $f(x)$ and its derivative function $f'(x)$. Decide which is the graph of the function and which is the graph of the derivative.

3.



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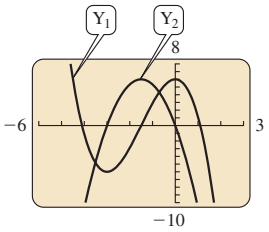

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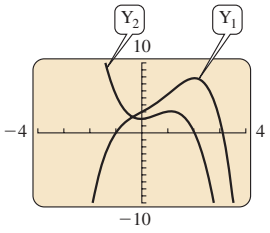

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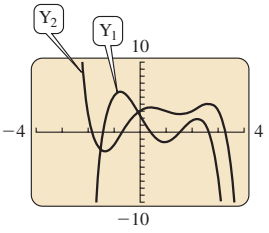
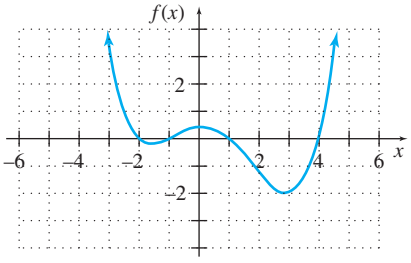


Sketch the graph of the derivative for each function shown.

7.



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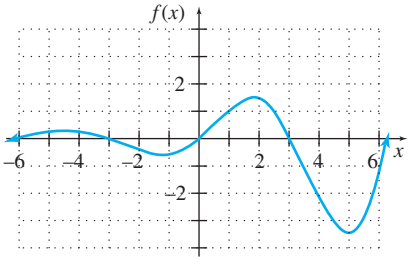

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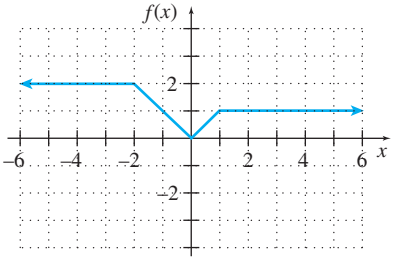

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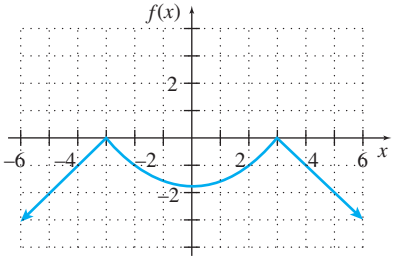

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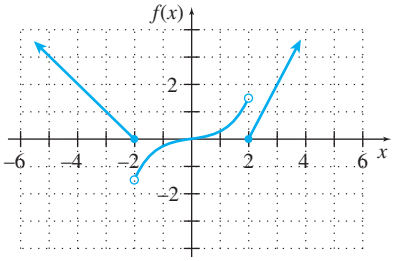

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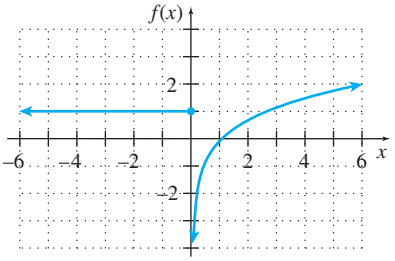

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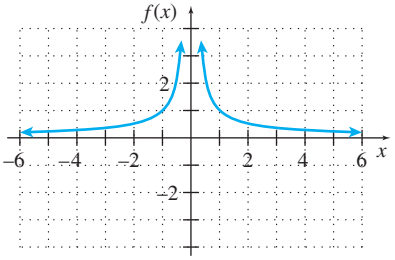
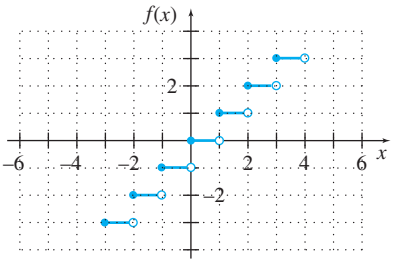


14.



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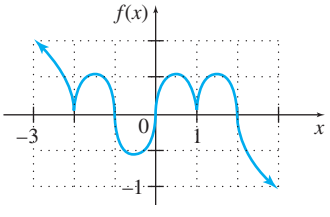

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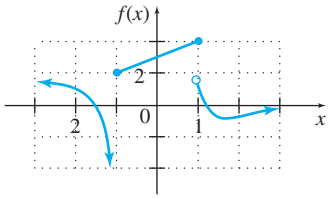
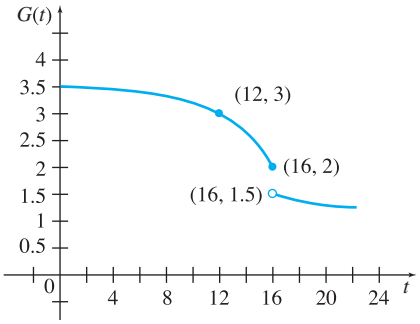


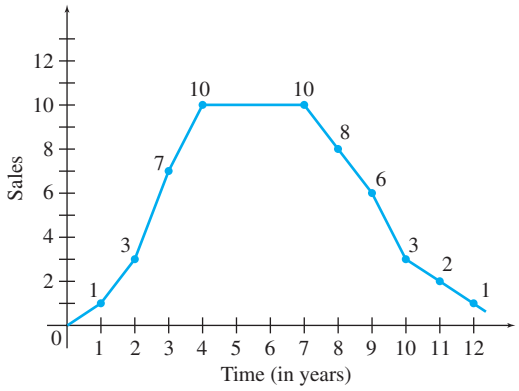
APPLICATIONS

Business and Economics

17. Consumer Demand When the price of an essential commodity rises rapidly, consumption drops slowly at first. If the price continues to rise, however, a "tipping" point may be reached, at which consumption takes a sudden substantial drop. Suppose the accompanying graph shows the consumption of gasoline, $G(t)$, in millions of gallons, in a certain area. We assume that the price is rising rapidly. Here t is the time in months after the price began rising. Sketch a graph of the rate of change in consumption as a function of time.

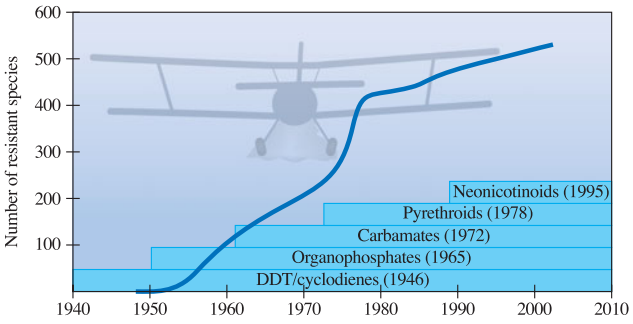


18. Sales The graph shows annual sales (in thousands of dollars) of a Nintendo game at a particular store. Sketch a graph of the rate of change of sales as a function of time.



Life Sciences

19. Insecticide The graph in the next column shows how the number of arthropod species resistant to insecticides has varied with time. Sketch a graph of the rate of change of the insecticide–resistant species as a function of time. ***Source: Science***.

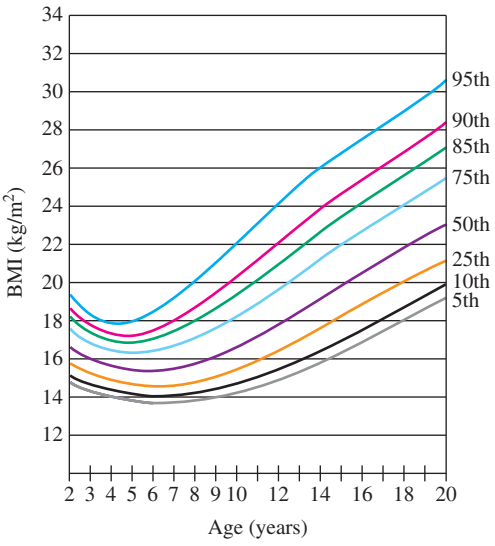


20. Body Mass Index The following graph shows how the body mass index-for-age percentile for boys varies from the age of 2 to 20 years. ***Source: Centers for Disease Control***.

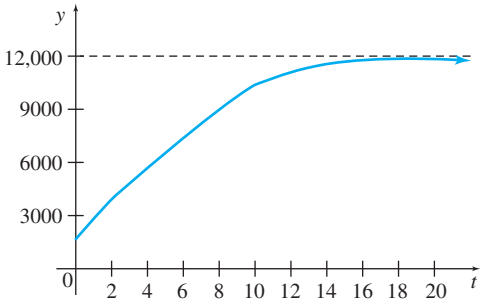
a. Sketch a graph of the rate of change of the 95th percentile as a function of age.

b. Sketch a graph of the rate of change of the 50th percentile as a function of age.

Body Mass Index-for-Age Percentiles: Boys, 2 to 20 years

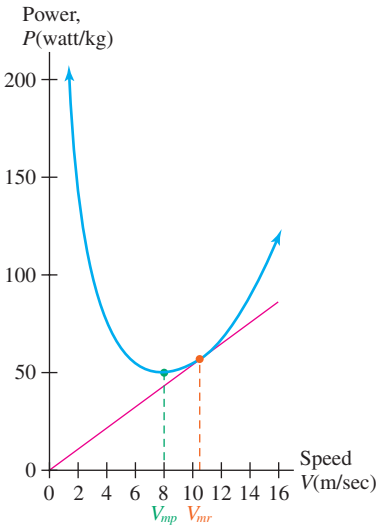


21. Shellfish Population In one research study, the population of a certain shellfish in an area at time t was closely approximated by the following graph. Sketch a graph of the growth rate of the population.

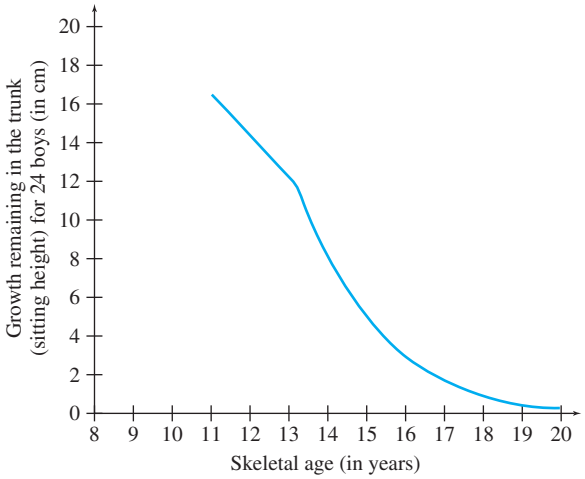


22. Flight Speed The graph on the next page shows the relationship between the speed of a bird in flight and the required

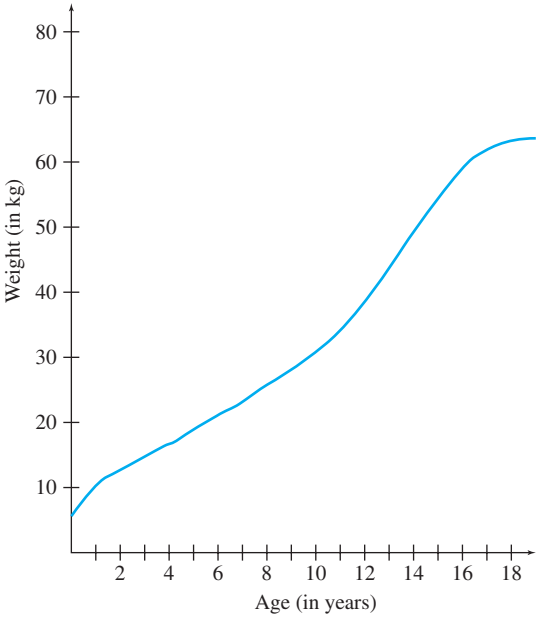
power expended by flight muscles. Sketch the graph of the rate of change of the power as a function of the speed. ***Source: Biolog-e: The Undergraduate Bioscience Research Journal***.



23. Human Growth The growth remaining in sitting height at consecutive skeletal age levels is indicated below for boys. ***Source: Standards in Pediatric Orthopedics: Tables, Charts, and Graphs Illustrating Growth***. Sketch a graph showing the rate of change of growth remaining for the indicated years. Use the graph and your sketch to estimate the remaining growth and the rate of change of remaining growth for a 14-year-old boy.

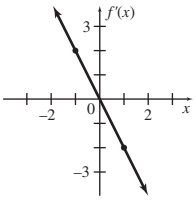


24. Weight Gain The graph below shows the typical weight (in kilograms) of an English boy for his first 18 years of life. Sketch the graph of the rate of change of weight with respect to time. ***Source: Human Growth After Birth***.



YOUR TURN ANSWERS

1.



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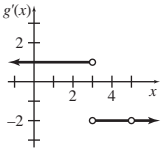


CHAPTER REVIEW

SUMMARY

In this chapter we introduced the ideas of limit and continuity of functions and then used these ideas to explore calculus. We saw that the difference quotient can represent

- the average rate of change,
- the slope of the secant line, and
- the average velocity.

We saw that the derivative can represent

- the instantaneous rate of change,
- the slope of the tangent line, and
- the instantaneous velocity.

We also learned how to estimate the value of the derivative using graphical differentiation. In the next chapter, we will take a closer look at the definition of the derivative to develop a set of rules to quickly and easily calculate the derivative of a wide range of functions without the need to directly apply the definition of the derivative each time.

Limit of a Function

Let f be a function and let a and L be real numbers. If

1. as x takes values closer and closer (but not equal) to a on both sides of a, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L; and
2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a;

then L is the limit of $f(x)$ as x approaches a, written

$$\lim_{x \to a} f(x) = L.$$

Existence of Limits

The limit of f as x approaches a may not exist.

1. If $f(x)$ becomes infinitely large in magnitude (positive or negative) as x approaches the number a from either side, we write $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$. In either case, the limit does not exist.
2. If $f(x)$ becomes infinitely large in magnitude (positive) as x approaches a from one side and infinitely large in magnitude (negative) as x approaches a from the other side, then $\lim_{x \to a} f(x)$ does not exist.
3. If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = M$, and $L \neq M$, then $\lim_{x \to a} f(x)$ does not exist.

Limits at Infinity

For any positive real number n,

$$\lim_{x \to \infty} \frac{1}{x^n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0.$$

Finding Limits at Infinity

If $f(x) = p(x)/q(x)$ for polynomials $p(x)$ and $q(x)$, $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ can be found by dividing $p(x)$ and $q(x)$ by the highest power of x in $q(x)$.

Continuity

A function f is continuous at c if

1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

Average Rate of Change

The average rate of change of $f(x)$ with respect to x as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}.$$

Difference Quotient

The average rate of change can also be written as

$$\frac{f(x + h) - f(x)}{h}.$$

Derivative

The derivative of $f(x)$ with respect to x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

KEY TERMS

To understand the concepts presented in this chapter, you should know the meaning and use of the following terms. For easy reference, the section in the chapter where a word (or expression) was first used is provided.

11.1
limit
limit from the left/right
one-/two-sided limit
piecewise function
limit at infinity

11.2
continuous
discontinuous
removable discontinuity
continuous on an open/closed interval
continuous from the right/left
Intermediate Value Theorem

11.3
average rate of change
difference quotient
instantaneous rate of change
velocity

11.4
secant line
tangent line
slope of the curve
derivative
differentiable
differentiation

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The limit of a product is the product of the limits when each of the limits exists.

2. The limit of a function may not exist at a point even though the function is defined there.

3. If a rational function has a polynomial in the denominator of higher degree than the polynomial in the numerator, then the limit at infinity must equal zero.

4. If the limit of a function exists at a point, then the function is continuous there.

5. A polynomial function is continuous everywhere.

6. A rational function is continuous everywhere.

7. The derivative gives the average rate of change of a function.

8. The derivative gives the instantaneous rate of change of a function.

9. The instantaneous rate of change is a limit.

10. The derivative is a function.

11. The slope of the tangent line gives the average rate of change.

12. The derivative of a function exists wherever the function is continuous.

PRACTICE AND EXPLORATIONS

13. Is a derivative always a limit? Is a limit always a derivative? Explain.

14. Is every continuous function differentiable? Is every differentiable function continuous? Explain.

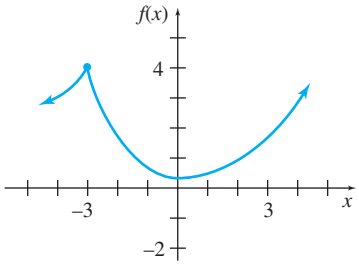
15. Describe how to tell when a function is discontinuous at the real number $x = a$.

16. Give two applications of the derivative

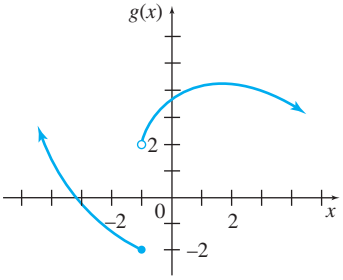
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Decide whether the limits in Exercises 17–34 exist. If a limit exists, find its value.

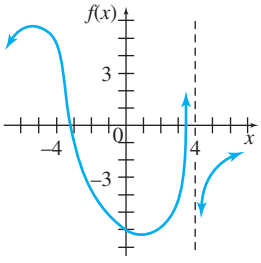
17. a. $\lim_{x \to -3^-} f(x)$ **b.** $\lim_{x \to -3^+} f(x)$ **c.** $\lim_{x \to -3} f(x)$ **d.** $f(-3)$



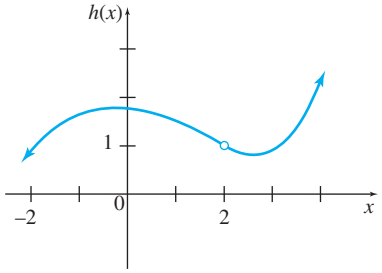
18. a. $\lim_{x \to -1^-} g(x)$ **b.** $\lim_{x \to -1^+} g(x)$ **c.** $\lim_{x \to -1} g(x)$ **d.** $g(-1)$



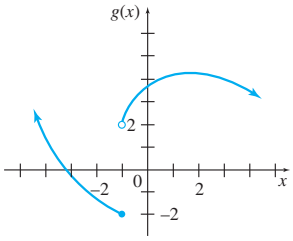
19. a. $\lim_{x \to 4^-} f(x)$ **b.** $\lim_{x \to 4^+} f(x)$ **c.** $\lim_{x \to 4} f(x)$ **d.** $f(4)$



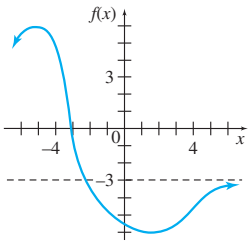
20. **a.** $\lim_{x\to 2^-} h(x)$ **b.** $\lim_{x\to 2^+} h(x)$ **c.** $\lim_{x\to 2} h(x)$ **d.** $h(2)$



21. $\lim_{x\to -\infty} g(x)$



22. $\lim_{x\to \infty} f(x)$



23. $\lim_{x\to 6} \dfrac{2x+7}{x+3}$

24. $\lim_{x\to -3} \dfrac{2x+5}{x+3}$

25. $\lim_{x\to 4} \dfrac{x^2-16}{x-4}$

26. $\lim_{x\to 2} \dfrac{x^2+3x-10}{x-2}$

27. $\lim_{x\to -4} \dfrac{2x^2+3x-20}{x+4}$

28. $\lim_{x\to 3} \dfrac{3x^2-2x-21}{x-3}$

29. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

30. $\lim_{x\to 16} \dfrac{\sqrt{x}-4}{x-16}$

31. $\lim_{x\to \infty} \dfrac{2x^2+5}{5x^2-1}$

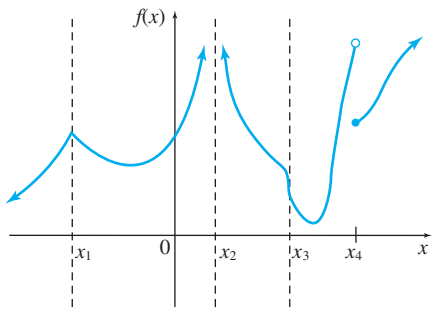
32. $\lim_{x\to \infty} \dfrac{x^2+6x+8}{x^3+2x+1}$

33. $\lim_{x\to -\infty} \left(\dfrac{3}{8}+\dfrac{3}{x}-\dfrac{6}{x^2}\right)$

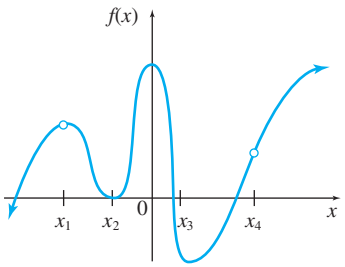
34. $\lim_{x\to -\infty} \left(\dfrac{9}{x^4}+\dfrac{10}{x^2}-6\right)$

Identify the *x*-values where *f* is discontinuous.

35.



36.



Find all *x*-values where the function is discontinuous. For each such value, give $f(a)$ and $\lim_{x\to a} f(x)$ or state that it does not exist.

37. $f(x) = \dfrac{-5+x}{3x(3x+1)}$

38. $f(x) = \dfrac{7-3x}{(1-x)(3+x)}$

39. $f(x) = \dfrac{x-6}{x+5}$

40. $f(x) = \dfrac{x^2-9}{x+3}$

41. $f(x) = x^2+3x-4$

42. $f(x) = 2x^2-5x-3$

In Exercises 43 and 44, (a) graph the given function, (b) find all values of *x* where the function is discontinuous, and (c) find the limit from the left and from the right at any values of *x* found in part *b*.

43. $f(x) = \begin{cases} 1-x & \text{if } x<1 \\ 2 & \text{if } 1\le x\le 2 \\ 4-x & \text{if } x>2 \end{cases}$

44. $f(x) = \begin{cases} 2 & \text{if } x<0 \\ -x^2+x+2 & \text{if } 0\le x\le 2 \\ 1 & \text{if } x>2 \end{cases}$

Find each limit (a) by investigating values of the function near the point where the limit is taken and (b) by using a graphing calculator to view the function near the point.

45. $\lim_{x\to 1} \dfrac{x^4+2x^3+2x^2-10x+5}{x^2-1}$

46. $\lim_{x\to -2} \dfrac{x^4+3x^3+7x^2+11x+2}{x^3+2x^2-3x-6}$

Find the average rate of change for the following on the given interval. Then find the instantaneous rate of change at the first *x*-value.

47. $y = 6x^3+2$ from $x=1$ to $x=4$

48. $y = -2x^3-3x^2+8$ from $x=-2$ to $x=6$

49. $y = \dfrac{-6}{3x-5}$ from $x=4$ to $x=9$

50. $y = \dfrac{x+4}{x-1}$ from $x=2$ to $x=5$

For each function, find (a) the equation of the secant line through the points where *x* has the given values, and (b) the equation of the tangent line when *x* has the first value.

51. $f(x) = 3x^2-5x+7;\ x=2, x=4$

52. $f(x) = \dfrac{1}{x};\ x=1/2, x=3$

53. $f(x) = \dfrac{12}{x-1};\ x=3, x=7$

54. $f(x) = 2\sqrt{x-1};\ x=5, x=10$

Use the definition of the derivative to find the derivative of the following.

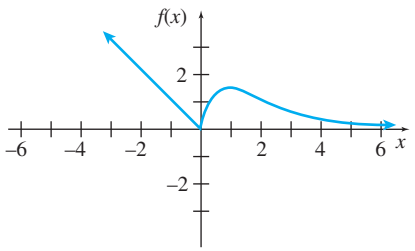
55. $y = 4x^2 + 3x - 2$ **56.** $y = 5x^2 - 6x + 7$

In Exercises 57 and 58, find the derivative of the function at the given point (a) by approximating the definition of the derivative with small values of h and (b) by using a graphing calculator to zoom in on the function until it appears to be a straight line, and then finding the slope of that line.

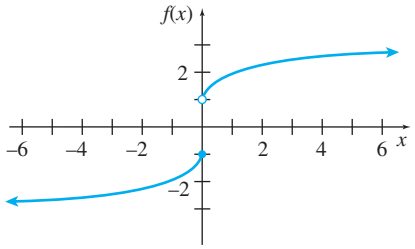
57. $f(x) = (\ln x)^x$; $x_0 = 3$ **58.** $f(x) = x^{\ln x}$; $x_0 = 2$

Sketch the graph of the derivative for each function shown.

59.



60.



61. Let f and g be differentiable functions such that

$$\lim_{x\to\infty} f(x) = c$$
$$\lim_{x\to\infty} g(x) = d$$

where $c \neq d$. Determine

$$\lim_{x\to\infty} \frac{cf(x) - dg(x)}{f(x) - g(x)}.$$

(Choose one of the following.) ***Source: Society of Actuaries***.

a. 0 **b.** $\dfrac{cf'(0) - dg'(0)}{f'(0) - g'(0)}$

c. $f'(0) - g'(0)$ **d.** $c - d$ **e.** $c + d$

APPLICATIONS

Business and Economics

62. Revenue Waverly Products has found that its revenue is related to advertising expenditures by the function

$$R(x) = 5000 + 16x - 3x^2,$$

where $R(x)$ is the revenue in dollars when x hundred dollars are spent on advertising.

a. Find the marginal revenue function.

b. Find and interpret the marginal revenue when \$1000 is spent on advertising.

63. Cost Analysis A company charges \$1.50 per lb when a certain chemical is bought in lots of 125 lb or less, with a price per pound of \$1.35 if more than 125 lb are purchased. Let $C(x)$ represent the cost of x lb. Find the cost for the following numbers of pounds.

a. 100 **b.** 125 **c.** 140 **d.** Graph $y = C(x)$.

e. Where is C discontinuous?

Find the average cost per pound if the following number of pounds are bought.

f. 100 **g.** 125 **h.** 140

Find and interpret the marginal cost (that is, the instantaneous rate of change of the cost) for the following numbers of pounds.

i. 100 **j.** 140

64. Marginal Analysis Suppose the profit (in cents) from selling x lb of potatoes is given by

$$P(x) = 15x + 25x^2.$$

Find the average rate of change in profit from selling each of the following amounts.

a. 6 lb to 7 lb **b.** 6 lb to 6.5 lb **c.** 6 lb to 6.1 lb

Find the marginal profit (that is, the instantaneous rate of change of the profit) from selling the following amounts.

d. 6 lb **e.** 20 lb **f.** 30 lb

g. What is the domain of x?

h. Is it possible for the marginal profit to be negative here? What does this mean?

i. Find the average profit function. (Recall that average profit is given by total profit divided by the number produced, or $\overline{P}(x) = P(x)/x$.)

j. Find the marginal average profit function (that is, the function giving the instantaneous rate of change of the average profit function).

k. Is it possible for the marginal average profit to vary here? What does this mean?

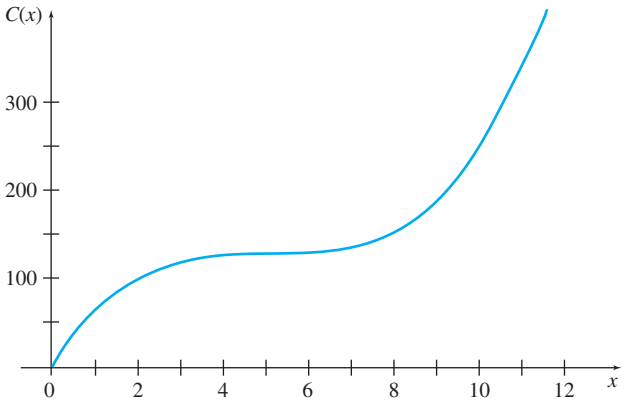
l. Discuss whether this function describes a realistic situation.

65. Average Cost The graph on the next page shows the total cost $C(x)$ to produce x tons of cement. (Recall that average cost is given by total cost divided by the number produced, or $\overline{C}(x) = C(x)/x$.)

a. Draw a line through $(0, 0)$ and $(5, C(5))$. Explain why the slope of this line represents the average cost per ton when 5 tons of cement are produced.

b. Find the value of x for which the average cost is smallest.

c. What can you say about the marginal cost at the point where the average cost is smallest?



66. Tax Rates A simplified income tax considered in the U.S. Senate in 1986 had two tax brackets. Married couples earning $29,300 or less would pay 15% of their income in taxes. Those earning more than $29,300 would pay $4350 plus 27% of the income over $29,300 in taxes. Let $T(x)$ be the amount of taxes paid by someone earning x dollars in a year. ***Source: Wall Street Journal***.

a. Find $\lim_{x \to 29{,}300^-} T(x)$. **b.** Find $\lim_{x \to 29{,}300^+} T(x)$.

c. Find $\lim_{x \to 29{,}300} T(x)$. **d.** Sketch a graph of $T(x)$.

e. Identify any x-values where T is discontinuous.

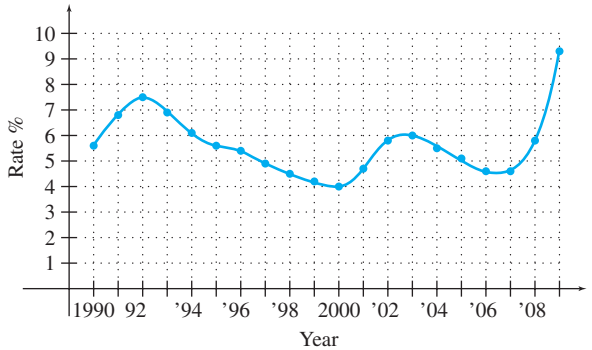
f. Let $A(x) = T(x)/x$ be the average tax rate, that is, the amount paid in taxes divided by the income. Find a formula for $A(x)$. (*Note:* The formula will have two parts: one for $x \leq 29{,}300$ and one for $x > 29{,}300$.)

g. Find $\lim_{x \to 29{,}300^-} A(x)$. **h.** Find $\lim_{x \to 29{,}300^+} A(x)$.

i. Find $\lim_{x \to 29{,}300} A(x)$. **j.** Find $\lim_{x \to \infty} A(x)$.

k. Sketch the graph of $A(x)$.

67. Unemployment The annual unemployment rates of the U.S. civilian noninstitutional population for 1990–2009 are shown in the graph. Sketch a graph showing the rate of change of the annual unemployment rates for this period. Use the given graph and your sketch to estimate the annual unemployment rate and rate of change of the unemployment rate in 2008. ***Source: Bureau of Labor Statistics***.

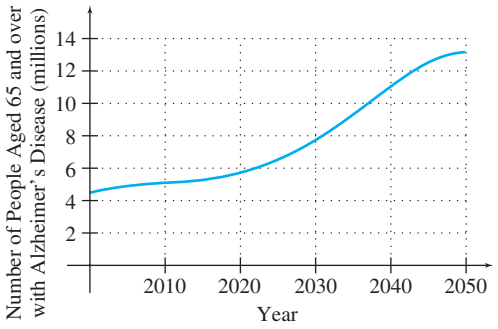


Life Sciences

68. Alzheimer's Disease The graph below shows the projected number of people aged 65 and over in the United States with Alzheimer's Disease. ***Source: Alzheimer's Disease Facts and Figures***. Estimate and interpret the derivative in each of the following years

a. 2000 **b.** 2040

c. Find the average rate of change between 2000 and 2040 in the number of people aged 65 and over in the United States with Alzheimer's disease.



69. Spread of a Virus The spread of a virus is modeled by

$$V(t) = -t^2 + 6t - 4,$$

where $V(t)$ is the number of people (in hundreds) with the virus and t is the number of weeks since the first case was observed.

a. Graph $V(t)$.

b. What is a reasonable domain of t for this problem?

c. When does the number of cases reach a maximum? What is the maximum number of cases?

d. Find the rate of change function.

e. What is the rate of change in the number of cases at the maximum?

f. Give the sign (+ or −) of the rate of change up to the maximum and after the maximum.

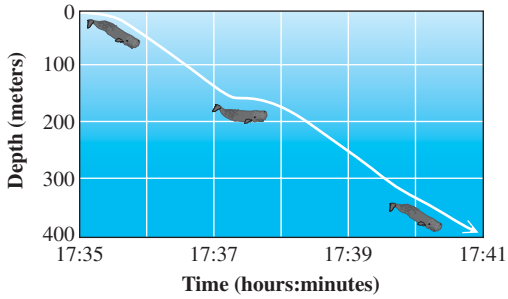
70. Whales Diving The following figure, already shown in the section on Properties of Functions, shows the depth of a sperm whale as a function of time, recorded by researchers at the Woods Hole Oceanographic Institution in Massachusetts. ***Source: Peter Tyack, Woods Hole Oceanographic Institution***.

a. Find the rate that the whale was descending at the following times.

i. 17 hours and 37 minutes

ii. 17 hours and 39 minutes

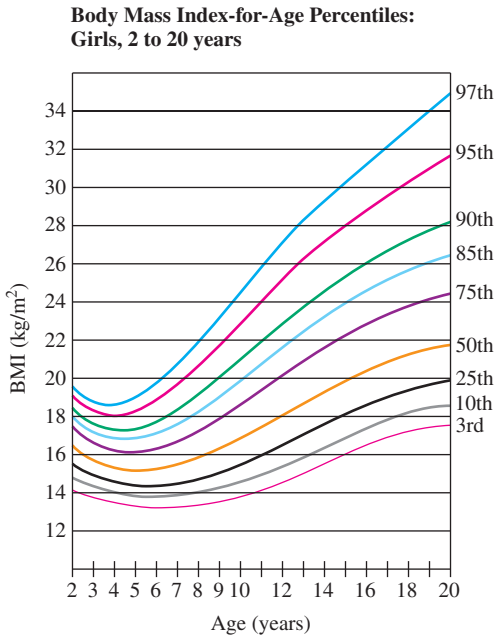
b. Sketch a graph of the rate the whale was descending as a function of time.



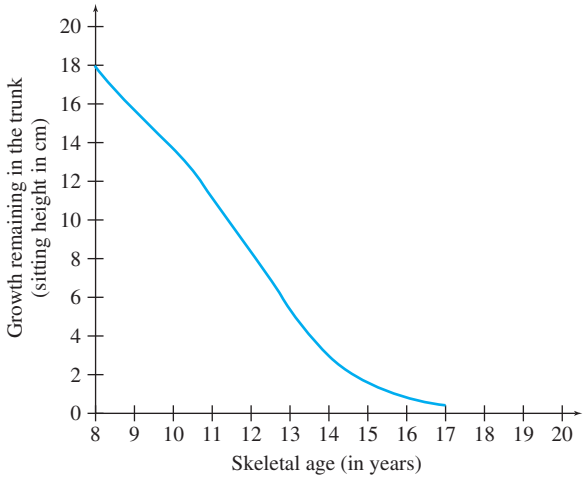
71. Body Mass Index The following graph shows how the body mass index-for-age percentile for girls varies from the age of 2 to 20 years. ***Source: Centers for Disease Control***.

a. Sketch a graph of the rate of change of the 95th percentile as a function of age.

b. Sketch a graph of the rate of change of the 50th percentile as a function of age.



72. Human Growth The growth remaining in sitting height at consecutive skeletal age levels is indicated below for girls. Sketch a graph showing the rate of change of growth remaining for the indicated years. Use the graph and your sketch to estimate the remaining growth and the rate of change of remaining growth for a 10-year-old girl. ***Source: Standards in Pediatric Orthopedics: Tables, Charts, and Graphs Illustrating Growth***.



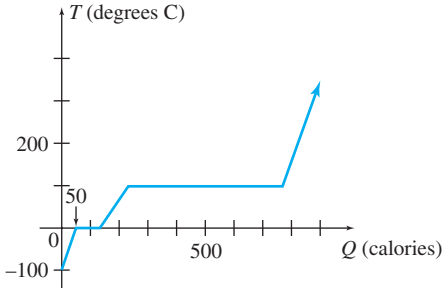
Physical Sciences

73. Temperature Suppose a gram of ice is at a temperature of −100°C. The graph shows the temperature of the ice as increasing numbers of calories of heat are applied. It takes 80 calories to melt one gram of ice at 0°C into water, and 540 calories to boil one gram of water at 100°C into steam.

a. Where is this graph discontinuous?

b. Where is this graph not differentiable?

c. Sketch the graph of the derivative.



EXTENDED APPLICATION

A MODEL FOR DRUGS ADMINISTERED INTRAVENOUSLY

When a drug is administered intravenously it enters the bloodstream immediately, producing an immediate effect for the patient. The drug can be either given as a single rapid injection or given at a constant drip rate. The latter is commonly referred to as an intravenous (IV) infusion. Common drugs administered intravenously include morphine for pain, diazepam (or Valium) to control a seizure, and digoxin for heart failure.

SINGLE RAPID INJECTION

With a single rapid injection, the amount of drug in the bloodstream reaches its peak immediately and then the body eliminates the drug exponentially. The larger the amount of drug there is in the body, the faster the body eliminates it. If a lesser amount of drug is in the body, it is eliminated more slowly.

The amount of drug in the bloodstream t hours after a single rapid injection can be modeled using an exponential decay function, like those found in the chapter on Nonlinear Functions, as follows:

$$A(t) = De^{kt},$$

where D is the size of the dose administered and k is the exponential decay constant for the drug.

EXAMPLE 1 Rapid Injection

The drug labetalol is used for the control of blood pressure in patients with severe hypertension. The half-life of labetalol is 4 hours. Suppose a 35-mg dose of the drug is administered to a patient by rapid injection.

(a) Find a model for the amount of drug in the bloodstream t hours after the drug is administered.

SOLUTION Since $D = 35$ mg, the function has the form

$$A(t) = 35e^{kt}.$$

Recall from the chapter on Nonlinear Functions that the general equation giving the half-life T in terms of the decay constant k was

$$T = -\frac{\ln 2}{k}.$$

Solving this equation for k, we get

$$k = -\frac{\ln 2}{T}.$$

Since the half-life of this drug is 4 hours,

$$k = -\frac{\ln 2}{4} \approx -0.17.$$

Therefore, the model is

$$A(t) = 35e^{-0.17t}.$$

The graph of $A(t)$ is given in Figure 55.

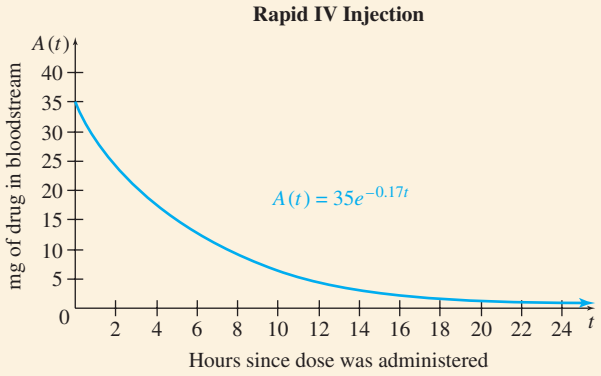


FIGURE 55

(b) Find the average rate of change of drug in the bloodstream between $t = 0$ and $t = 2$. Repeat for $t = 4$ and $t = 6$.

SOLUTION The average rate of change from $t = 0$ to $t = 2$ is

$$\frac{A(2) - A(0)}{2 - 0} \approx \frac{25 - 35}{2} = -5 \text{ mg/hr}.$$

The average rate of change from $t = 4$ to $t = 6$ is

$$\frac{A(6) - A(4)}{6 - 4} \approx \frac{13 - 18}{2} = -2.5 \text{ mg/hr}.$$

Notice that since the half-life of the drug is 4 hours, the average rate of change from $t = 4$ to $t = 6$ is half of the average rate of change from $t = 0$ to $t = 2$. What would the average rate of change be from $t = 8$ to $t = 10$?

(c) What happens to the amount of drug in the bloodstream as t increases? (i.e., What is the limit of the function as t approaches ∞?)

SOLUTION Looking at the graph of $A(t)$, we can see that

$$\lim_{t \to \infty} A(t) = 0.$$

An advantage of an intravenous rapid injection is that the amount of drug in the body reaches a high level immediately. Suppose,

however, that the effective level of this drug is between 30 mg and 40 mg. From the graph, we can see that it only takes an hour after the dose is given for the amount of drug in the body to fall below the effective level.

INTRAVENOUS INFUSION

With an IV infusion, the amount of drug in the bloodstream starts at zero and increases until the rate the drug is entering the body equals the rate the drug is being eliminated from the body. At this point, the amount of drug in the bloodstream levels off. This model is a limited growth function, like those from Chapter 10.

The amount of drug in the bloodstream t hours after an IV infusion begins can be modeled using a limited growth function, as follows.

$$A(t) = \frac{r}{-k}(1 - e^{kt}),$$

where r is the rate of infusion per hour and k is the exponential decay constant for the drug.

EXAMPLE 2 IV Infusion

The same drug used in Example 1 is given to a patient by IV infusion at a drip rate of 6 mg/hr. Recall that the half-life of this drug is 4 hours.

(a) Find a model for the amount of drug in the bloodstream t hours after the IV infusion begins.

SOLUTION Since $r = 6$ and $k = -0.17$, the function has the form

$$A(t) = 35(1 - e^{-0.17t}).$$

The graph of $A(t)$ is given in Figure 56.

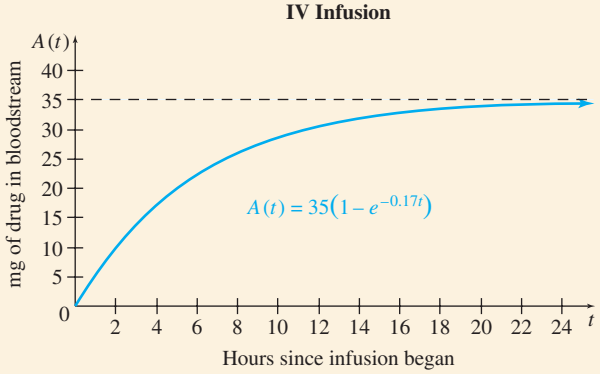


FIGURE 56

(b) Find the average rate of change of drug in the bloodstream between $t = 0$ and $t = 2$. Repeat for $t = 4$ and $t = 6$.

SOLUTION The average rate of change from $t = 0$ to $t = 2$ is

$$\frac{A(2) - A(0)}{2 - 0} \approx \frac{10 - 0}{2} = 5 \text{ mg/hr}.$$

The average rate of change from $t = 4$ to $t = 6$ is

$$\frac{A(6) - A(4)}{6 - 4} \approx \frac{22 - 17}{2} = 2.5 \text{ mg/hr}.$$

Recall that the average rate of change from $t = 0$ to $t = 2$ for the rapid injection of this drug was -5 mg/hr and the average rate of change from $t = 4$ to $t = 6$ was -2.5 mg/hr. In fact, at any given time, the rapid injection function is decreasing at the same rate the IV infusion function is increasing.

(c) What happens to the amount of drug in the bloodstream as t increases? (i.e., What is the limit of the function as t approaches ∞?)

SOLUTION Looking at the graph of $A(t)$ in Figure 56 and the formula for $A(t)$ in part (a), we can see that

$$\lim_{t \to \infty} A(t) = 35.$$

An advantage of an IV infusion is that a dose can be given such that the limit of $A(t)$ as t approaches ∞ is an effective level. Once the amount of drug has reached this effective level, it will remain there as long as the infusion continues. However, using this method of administration, it may take a while for the amount of drug in the body to reach an effective level. For our example, the effective level is between 30 mg and 40 mg. Looking at the graph, you can see that it takes about 11 hours to reach an effective level. If this patient were experiencing dangerously high blood pressure, you wouldn't want to wait 11 hours for the drug to reach an effective level.

SINGLE RAPID INJECTION FOLLOWED BY AN INTRAVENOUS INFUSION

Giving a patient a single rapid injection immediately followed by an intravenous infusion allows a patient to experience the advantages of both methods. The single rapid injection immediately produces an effective drug level in the patient's bloodstream. While the amount of drug in the bloodstream from the rapid infusion is decreasing, the amount of drug in the system from the IV infusion is increasing.

The amount of drug in the bloodstream t hours after the injection is given and infusion has started can be calculated by finding the sum of the two models.

$$A(t) = De^{kt} + \frac{r}{-k}(1 - e^{kt}).$$

EXAMPLE 3 Combination Model

A 35-mg dose of labetalol is administered to a patient by rapid injection. Immediately thereafter, the patient is given an IV infusion at a drip rate of 6 mg/hr. Find a model for the amount of drug in the bloodstream t hours after the drug is administered.

SOLUTION Recall from Example 1, the amount of drug in the bloodstream t hours after the rapid injection was found to be

$$A(t) = 35e^{-0.17t}.$$

From Example 2, the amount of drug in the bloodstream t hours after the IV infusion began was found to be

$$A(t) = 35(1 - e^{-0.17t}).$$

Therefore, t hours after administering both the rapid injection and the IV infusion, the amount of drug in the bloodstream is

$$\begin{aligned} A(t) &= 35e^{-0.17t} + 35(1-e^{-0.17t}) \\ &= 35 \text{ mg}. \end{aligned}$$

The graph of $A(t)$ is given in Figure 57.

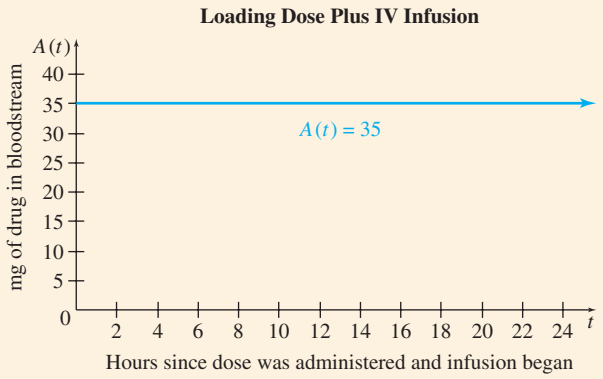


FIGURE 57

Notice that the constant multiple of the rapid injection function, 35, is equal to the constant multiple of the IV infusion function. When this is the case, the sum of the two functions will be that constant.

EXAMPLE 4 Combination Model

A drug with a half-life of 3 hours is found to be effective when the amount of drug in the bloodstream is 58 mg. A 58-mg loading dose is given by rapid injection followed by an IV infusion. What should the rate of infusion be to maintain this level of drug in the bloodstream?

SOLUTION Recall that the amount of drug in the bloodstream t hours after both a rapid injection and IV infusion are administered is given by

$$A(t) = De^{kt} + \frac{r}{-k}(1 - e^{kt}).$$

The rapid injection dose, D, is 58 mg. The half-life of the drug is three hours; therefore,

$$k = -\frac{\ln 2}{3} \approx -0.23.$$

It follows that

$$A(t) = 58e^{-0.23t} + \frac{r}{0.23}(1 - e^{-0.23t}).$$

Since we want the sum of the rapid injection function and the IV infusion function to be 58 mg, it follows that

$$\frac{r}{0.23} = 58.$$

Solving for r, we get

$$r = 13.34 \text{ mg/hr}.$$

EXERCISES

1. A 500-mg dose of a drug is administered by rapid injection to a patient. The half-life of the drug is 9 hours.

a. Find a model for the amount of drug in the bloodstream t hours after the drug is administered.

b. Find the average rate of change of drug in the bloodstream between $t = 0$ and $t = 2$. Repeat for $t = 9$ and $t = 11$.

2. A drug is given to a patient by IV infusion at a drip rate of 350 mg/hr. The half-life of this drug is 3 hours.

a. Find a model of the amount of drug in the bloodstream t hours after the IV infusion begins.

b. Find the average rate of change of drug in the bloodstream between $t = 0$ and $t = 3$. Repeat for $t = 3$ and $t = 6$.

3. A drug with a half-life of 9 hours is found to be effective when the amount of drug in the bloodstream is 250 mg. A 250-mg loading dose is given by rapid injection followed by an IV infusion. What should the rate of infusion be to maintain this level of drug in the bloodstream?

4. Use the table feature on a graphing calculator or a spreadsheet to develop a table that shows how much of the drug is present in a patient's system at the end of each 1/2 hour time interval for 24 hours for the model found in Exercise 1a. A chart such as this provides the health care worker with immediate information about patient drug levels.

5. Use the table feature on a graphing calculator or a spreadsheet to develop a table that shows how much of the drug is present in a patient's system at the end of each 1/2 hour time interval for 10 hours for the model found in Exercise 2a. A chart such as this provides the health care worker with immediate information about patient drug levels.

6. Use the table feature on a graphing calculator or a spreadsheet to develop a table that shows how much of the drug is present in a patient's system at the end of each 1/2 hour time interval for 10 hours for the model found in Exercise 3. Are your results surprising?

DIRECTIONS FOR GROUP PROJECT

Choose a drug that is commonly prescribed by physicians for a common ailment. Develop an analysis for this drug that is similar to the analysis for labetalol in Examples 1 through 3. You can obtain information on the drug from the Internet or from advertisements found in various media. Once you complete the analysis, prepare a professional presentation that can be delivered at a public forum. The presentation should summarize the facts presented in this extended application but at a level that is understandable to a typical layperson.

12 Calculating the Derivative

12.1 Techniques for Finding Derivatives

12.2 Derivatives of Products and Quotients

12.3 The Chain Rule

12.4 Derivatives of Exponential Functions

12.5 Derivatives of Logarithmic Functions

Chapter 12 Review

Extended Application: Electric Potential and Electric Field

By differentiating the function defining a mathematical model we can see how the model's output changes with the input. In an exercise in Section 2 we explore a rational-function model for the length of the rest period needed to recover from vigorous exercise such as riding a bike. The derivative indicates how the rest required changes with the work expended in kilocalories per minute.

In the previous chapter, we found the derivative to be a useful tool for describing the rate of change, velocity, and the slope of a curve. Taking the derivative by using the definition, however, can be difficult. To take full advantage of the power of the derivative, we need faster ways of calculating the derivative. That is the goal of this chapter.

12.1 Techniques for Finding Derivatives

APPLY IT

How can a manager determine the best production level if the relationship between profit and production is known? How fast is the number of Americans who are expected to be over 100 years old growing?

These questions can be answered by finding the derivative of an appropriate function. We shall return to them at the end of this section in Examples 8 and 9.

Using the definition to calculate the derivative of a function is a very involved process even for simple functions. In this section we develop rules that make the calculation of derivatives much easier. Keep in mind that even though the process of finding a derivative will be greatly simplified with these rules, *the interpretation of the derivative will not change*. But first, a few words about notation are in order.

In addition to $f'(x)$, there are several other commonly used notations for the derivative.

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)], \quad \text{or} \quad D_x[f(x)].$$

The dy/dx notation for the derivative (read "the derivative of y with respect to x") is sometimes referred to as *Leibniz notation,* named after one of the co-inventors of calculus, Gottfried Wilhelm von Leibniz (1646–1716). (The other was Sir Isaac Newton, 1642–1727.)

With the above notation, the derivative of $y = f(x) = 2x^3 + 4x$, for example, which was found in Example 5 of Section 11.4 to be $f'(x) = 6x^2 + 4$, would be written

$$\frac{dy}{dx} = 6x^2 + 4$$

$$\frac{d}{dx}(2x^3 + 4x) = 6x^2 + 4$$

$$D_x(2x^3 + 4x) = 6x^2 + 4.$$

A variable other than x is often used as the independent variable. For example, if $y = f(t)$ gives population growth as a function of time, then the derivative of y with respect to t could be written

$$f'(t), \quad \frac{dy}{dt}, \quad \frac{d}{dt}[f(t)], \quad \text{or} \quad D_t[f(t)].$$

Other variables also may be used to name the function, as in $g(x)$ or $h(t)$.

Now we will use the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

to develop some rules for finding derivatives more easily than by the four-step process given in the previous chapter.

The first rule tells how to find the derivative of a constant function defined by $f(x) = k$, where k is a constant real number. Since $f(x + h)$ is also k, by definition $f'(x)$ is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0, \end{aligned}$$

establishing the following rule.

Constant Rule

If $f(x) = k$, where k is any real number, then

$$\boldsymbol{f'(x) = 0}.$$

(The derivative of a constant is 0.)

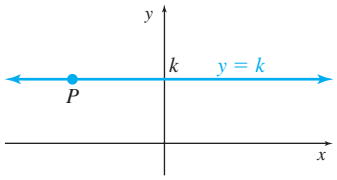


FIGURE 1

This rule is logical because the derivative represents rate of change, and a constant function, by definition, does not change. Figure 1 illustrates this constant rule geometrically; it shows a graph of the horizontal line $y = k$. At any point P on this line, the tangent line at P is the line itself. Since a horizontal line has a slope of 0, the slope of the tangent line is 0. This agrees with the result above: The derivative of a constant is 0.

EXAMPLE 1 Derivative of a Constant

(a) If $f(x) = 9$, then $f'(x) = 0$.

(b) If $h(t) = \pi$, then $D_t[h(t)] = 0$.

(c) If $y = 2^3$, then $dy/dx = 0$.

Functions of the form $y = x^n$, where n is a fixed real number, are very common in applications. To obtain a rule for finding the derivative of such a function, we can use the definition to work out the derivatives for various special values of n. This was done in Section 11.4 in Example 4 to show that for $f(x) = x^2$, $f'(x) = 2x$.

For $f(x) = x^3$, the derivative is found as follows.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}. \end{aligned}$$

The binomial theorem (discussed in most intermediate and college algebra texts) was used to expand $(x + h)^3$ in the last step. Now, the limit can be determined.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \to 0} (3x^2 + 3xh + h^2) \\ &= 3x^2. \end{aligned}$$

The results in the following table were found in a similar way, using the definition of the derivative. (These results are modifications of some of the examples and exercises from the previous chapter.)

Derivative of $f(x) = x^n$		
Function	n	**Derivative**
$f(x) = x$	1	$f'(x) = 1 = 1x^0$
$f(x) = x^2$	2	$f'(x) = 2x = 2x^1$
$f(x) = x^3$	3	$f'(x) = 3x^2$
$f(x) = x^4$	4	$f'(x) = 4x^3$
$f(x) = x^{-1}$	-1	$f'(x) = -1 \cdot x^{-2} = \dfrac{-1}{x^2}$
$f(x) = x^{1/2}$	$1/2$	$f'(x) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2x^{1/2}}$

These results suggest the following rule.

Power Rule

If $f(x) = x^n$ for any real number n, then

$$\boldsymbol{f'(x) = nx^{n-1}}.$$

(The derivative of $f(x) = x^n$ is found by multiplying by the exponent n and decreasing the exponent on x by 1.)

While the power rule is true for every real-number value of n, a proof is given here only for positive integer values of n. This proof follows the steps used above in finding the derivative of $f(x) = x^3$.

For any real numbers p and q, by the binomial theorem,

$$\boldsymbol{(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{2}p^{n-2}q^2 + \cdots + npq^{n-1} + q^n.}$$

Replacing p with x and q with h gives

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n,$$

from which

$$(x+h)^n - x^n = nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n.$$

Dividing each term by h yields

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}.$$

Use the definition of derivative, and the fact that each term except the first contains h as a factor and thus approaches 0 as h approaches 0, to get

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\ &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}0 + \cdots + nx0^{n-2} + 0^{n-1} \\ &= nx^{n-1}. \end{aligned}$$

This shows that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$, proving the power rule for positive integer values of n.

EXAMPLE 2 Power Rule

(a) If $f(x) = x^6$, find $f'(x)$.

SOLUTION $f'(x) = 6x^{6-1} = 6x^5$

(b) If $y = t = t^1$, find $\dfrac{dy}{dt}$.

SOLUTION $\dfrac{dy}{dt} = 1t^{1-1} = t^0 = 1$

(c) If $y = 1/x^3$, find dy/dx.

SOLUTION Use a negative exponent to rewrite this equation as $y = x^{-3}$; then

$$\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} \quad \text{or} \quad \frac{-3}{x^4}.$$

(d) Find $D_x(x^{4/3})$.

SOLUTION $D_x(x^{4/3}) = \dfrac{4}{3}x^{4/3-1} = \dfrac{4}{3}x^{1/3}$

(e) If $y = \sqrt{z}$, find dy/dz.

SOLUTION Rewrite this as $y = z^{1/2}$; then

$$\frac{dy}{dz} = \frac{1}{2}z^{1/2-1} = \frac{1}{2}z^{-1/2} \quad \text{or} \quad \frac{1}{2z^{1/2}} \quad \text{or} \quad \frac{1}{2\sqrt{z}}.$$

TRY YOUR TURN 1

FOR REVIEW

At this point you may wish to turn back to Sections R.6 and R.7 for a review of negative exponents and rational exponents. The relationship between powers, roots, and rational exponents is explained there.

YOUR TURN 1 If $f(t) = \dfrac{1}{\sqrt{t}}$, find $f'(t)$.

The next rule shows how to find the derivative of the product of a constant and a function.

Constant Times a Function

Let k be a real number. If $g'(x)$ exists, then the derivative of $f(x) = k \cdot g(x)$ is

$$f'(x) = k \cdot g'(x).$$

(The derivative of a constant times a function is the constant times the derivative of the function.)

This rule is proved with the definition of the derivative and rules for limits.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{kg(x+h) - kg(x)}{h} \\ &= \lim_{h \to 0} k \frac{[g(x+h) - g(x)]}{h} && \text{Factor out } k. \\ &= k \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} && \text{Limit rule 1} \\ &= k \cdot g'(x) && \text{Definition of derivative} \end{aligned}$$

EXAMPLE 3 Derivative of a Constant Times a Function

(a) If $y = 8x^4$, find $\dfrac{dy}{dx}$.

SOLUTION $\dfrac{dy}{dx} = 8(4x^3) = 32x^3$

(b) If $y = -\dfrac{3}{4}x^{12}$, find dy/dx.

SOLUTION $\dfrac{dy}{dx} = -\dfrac{3}{4}(12x^{11}) = -9x^{11}$

(c) Find $D_t(-8t)$.

SOLUTION $D_t(-8t) = -8(1) = -8$

(d) Find $D_p(10p^{3/2})$.

SOLUTION $D_p(10p^{3/2}) = 10\left(\frac{3}{2}p^{1/2}\right) = 15p^{1/2}$

(e) If $y = \frac{6}{x}$, find $\frac{dy}{dx}$.

SOLUTION Rewrite this as $y = 6x^{-1}$; then

$$\frac{dy}{dx} = 6(-1x^{-2}) = -6x^{-2} \quad \text{or} \quad \frac{-6}{x^2}.$$

YOUR TURN 2 If $y = 3\sqrt{x}$, find dy/dx.

TRY YOUR TURN 2

EXAMPLE 4 Beagles

Researchers have determined that the daily energy requirements of female beagles who are at least 1 year old change with respect to age according to the function

$$E(t) = 753t^{-0.1321},$$

where $E(t)$ is the daily energy requirements $(\text{in kJ/W}^{0.67})$ for a dog that is t years old. ***Source: Journal of Nutrition.***

(a) Find $E'(t)$.

SOLUTION Using the rules of differentiation we find that

$$E'(t) = 753(-0.1321)t^{-0.1321-1} = -99.4713t^{-1.1321}.$$

(b) Determine the rate of change of the daily energy requirements of a 2-year-old female beagle.

SOLUTION $E'(2) = -99.4713(2)^{-1.1321} \approx -45.4$

Thus, the daily energy requirements of a 2-year-old female beagle are decreasing at the rate of $45.4\ \text{kJ/W}^{0.67}$ per year.

The final rule in this section is for the derivative of a function that is a sum or difference of terms.

Sum or Difference Rule

If $f(x) = u(x) \pm v(x)$, and if $u'(x)$ and $v'(x)$ exist, then

$$\boldsymbol{f'(x) = u'(x) \pm v'(x).}$$

(The derivative of a sum or difference of functions is the sum or difference of the derivatives.)

The proof of the sum part of this rule is as follows: If $f(x) = u(x) + v(x)$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h\to 0} \frac{[u(x+h) - u(x)] + [v(x+h) - v(x)]}{h} \\ &= \lim_{h\to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) + v'(x). \end{aligned}$$

A similar proof can be given for the difference of two functions.

EXAMPLE 5 Derivative of a Sum

Find the derivative of each function.

(a) $y = 6x^3 + 15x^2$

SOLUTION Let $u(x) = 6x^3$ and $v(x) = 15x^2$; then $y = u(x) + v(x)$. Since $u'(x) = 18x^2$ and $v'(x) = 30x$,

$$\frac{dy}{dx} = 18x^2 + 30x.$$

(b) $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

SOLUTION Rewrite $p(t)$ as $p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$; then

$$p'(t) = 48t^3 - 3t^{-1/2} - 5t^{-2}.$$

Also, $p'(t)$ may be written as $p'(t) = 48t^3 - \frac{3}{\sqrt{t}} - \frac{5}{t^2}$.

(c) $f(x) = \frac{x^3 + 3\sqrt{x}}{x}$

SOLUTION Rewrite $f(x)$ as $f(x) = \frac{x^3}{x} + \frac{3x^{1/2}}{x} = x^2 + 3x^{-1/2}$. Then

$$D_x[f(x)] = 2x - \frac{3}{2}x^{-3/2},$$

or

$$D_x[f(x)] = 2x - \frac{3}{2\sqrt{x^3}}.$$

(d) $f(x) = (4x^2 - 3x)^2$

SOLUTION Rewrite $f(x)$ as $f(x) = 16x^4 - 24x^3 + 9x^2$ using the fact that $(a - b)^2 = a^2 - 2ab + b^2$; then

$$f'(x) = 64x^3 - 72x^2 + 18x.$$ TRY YOUR TURN 3

YOUR TURN 3 If $h(t) = -3t^2 + 2\sqrt{t} + \frac{5}{t^4} - 7$, find $h'(t)$.

TECHNOLOGY NOTE

Some computer programs and calculators have built-in methods for taking derivatives symbolically, which is what we have been doing in this section, as opposed to approximating the derivative numerically by using a small number for h in the definition of the derivative. In the computer program Maple, we would do part (a) of Example 5 by entering

```
> diff(6*x^3+15*x^2,x);
```

where the x after the comma tells what variable the derivative is with respect to. Maple would respond with

```
18*x^2+30*x.
```

Similarly, on the TI-89, we would enter `d(6x^3+15x^2,x)` and the calculator would give "`18·x^2+30·x.`"

Other graphing calculators, such as the TI-84 Plus, do not have built-in methods for taking derivatives symbolically. As we saw in the last chapter, however, they do have the ability to calculate the derivative of a function at a particular point and to simultaneously graph a function and its derivative.

Recall that, on the TI-84 Plus, we could use the `nDeriv` Command, as shown in Figure 2, to approximate the value of the derivative when $x = 1$. Figure 3(a) and Figure 3(b) indicate how to

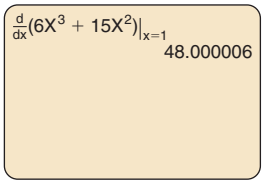


FIGURE 2

input the functions into the calculator and the corresponding graphs of both the function and its derivative. Consult the *Graphing Calculator and Excel Spreadsheet Manual*, available with this book, for assistance.

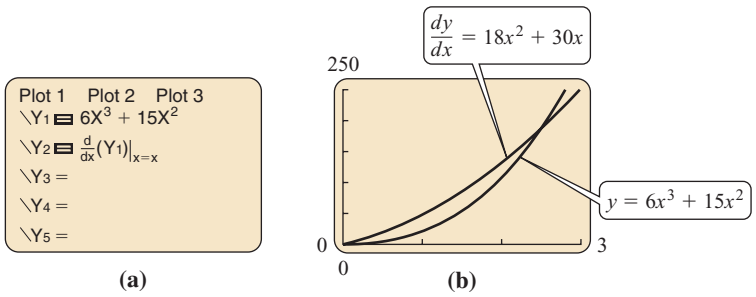


FIGURE 3

The rules developed in this section make it possible to find the derivative of a function more directly, so that applications of the derivative can be dealt with more effectively. The following examples illustrate some business applications.

Marginal Analysis

In previous sections we discussed the concepts of marginal cost, marginal revenue, and marginal profit. These concepts of **marginal analysis** are summarized here.

In business and economics the rates of change of such variables as cost, revenue, and profit are important considerations. Economists use the word *marginal* to refer to rates of change. For example, *marginal cost* refers to the rate of change of cost. Since the derivative of a function gives the rate of change of the function, a marginal cost (or revenue, or profit) function is found by taking the derivative of the cost (or revenue, or profit) function. Roughly speaking, the marginal cost at some level of production x is the cost to produce the $(x + 1)$st item. (Similar statements could be made for revenue or profit.)

To see why it is reasonable to say that the marginal cost function is approximately the cost of producing one more unit, look at Figure 4, where $C(x)$ represents the cost of producing x units of some item. Then the cost of producing $x + 1$ units is $C(x + 1)$. The cost of the $(x + 1)$st unit is, therefore, $C(x + 1) - C(x)$. This quantity is shown in the graph in Figure 4.

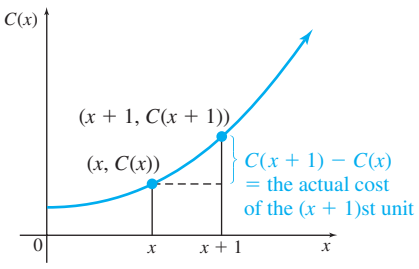


FIGURE 4

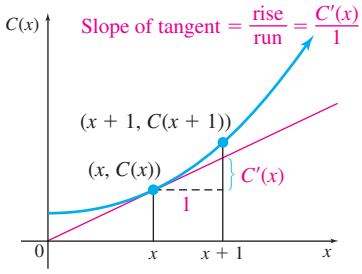


FIGURE 5

Now if $C(x)$ is the cost function, then the marginal cost $C'(x)$ represents the slope of the tangent line at any point $(x, C(x))$. The graph in Figure 5 shows the cost function $C(x)$ and the tangent line at a point $(x, C(x))$. Remember what it means for a line to have a given slope. If the slope of the line is $C'(x)$, then

$$\frac{\Delta y}{\Delta x} = C'(x) = \frac{C'(x)}{1},$$

and beginning at any point on the line and moving 1 unit to the right requires moving $C'(x)$ units up to get back to the line again. The vertical distance from the horizontal line to the tangent line shown in Figure 5 is therefore $C'(x)$.

Superimposing the graphs from Figures 4 and 5 as in Figure 6 shows that $C'(x)$ is indeed very close to $C(x + 1) - C(x)$. The two values are closest when x is very large, so that 1 unit is relatively small.

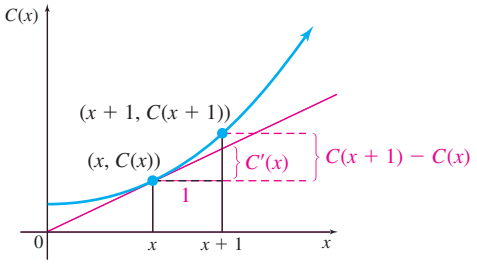


FIGURE 6

EXAMPLE 6 Marginal Cost

Suppose that the total cost in hundreds of dollars to produce x thousand barrels of a beverage is given by

$$C(x) = 4x^2 + 100x + 500.$$

Find the marginal cost for the following values of x.

(a) $x = 5$

SOLUTION To find the marginal cost, first find $C'(x)$, the derivative of the total cost function.

$$C'(x) = 8x + 100$$

When $x = 5$,

$$C'(5) = 8(5) + 100 = 140.$$

After 5 thousand barrels of the beverage have been produced, the cost to produce one thousand more barrels will be *approximately* 140 hundred dollars, or \$14,000.

The *actual* cost to produce one thousand more barrels is $C(6) - C(5)$:

$$\begin{aligned} C(6) - C(5) &= (4 \cdot 6^2 + 100 \cdot 6 + 500) - (4 \cdot 5^2 + 100 \cdot 5 + 500) \\ &= 1244 - 1100 = 144, \end{aligned}$$

144 hundred dollars, or \$14,400.

(b) $x = 30$

SOLUTION After 30 thousand barrels have been produced, the cost to produce one thousand more barrels will be approximately

$$C'(30) = 8(30) + 100 = 340,$$

or \$34,000. Notice that the cost to produce an additional thousand barrels of beverage has increased by approximately \$20,000 at a production level of 30,000 barrels compared to a production level of 5000 barrels.

YOUR TURN 4 If the cost function is given by $C(x) = 5x^3 - 10x^2 + 75$, find the marginal cost when $x = 100$.

TRY YOUR TURN 4

Demand Functions

The demand function, defined by $p = D(q)$, relates the number of units q of an item that consumers are willing to purchase to the price p. (Demand functions were also discussed in Chapter 1.) The total revenue $R(q)$ is related to price per unit and the amount demanded (or sold) by the equation

$$R(q) = qp = q \cdot D(q).$$

EXAMPLE 7 Marginal Revenue

The demand function for a certain product is given by

$$p = \frac{50{,}000 - q}{25{,}000}.$$

Find the marginal revenue when $q = 10{,}000$ units and p is in dollars.

SOLUTION From the given function for p, the revenue function is given by

$$\begin{aligned} R(q) &= qp \\ &= q\left(\frac{50{,}000 - q}{25{,}000}\right) \\ &= \frac{50{,}000q - q^2}{25{,}000} \\ &= 2q - \frac{1}{25{,}000}q^2. \end{aligned}$$

The marginal revenue is

$$R'(q) = 2 - \frac{2}{25{,}000}q.$$

When $q = 10{,}000$, the marginal revenue is

$$R'(10{,}000) = 2 - \frac{2}{25{,}000}(10{,}000) = 1.2,$$

or \$1.20 per unit. Thus, the next item sold (at sales of 10,000) will produce additional revenue of about \$1.20.

YOUR TURN 5 If the demand function is given by $p = 16 - 1.25q$, find the marginal revenue when $q = 5$.

TRY YOUR TURN 5

Management must be careful to keep track of marginal costs and revenue. If the marginal cost of producing an extra unit exceeds the marginal revenue received from selling it, then the company will lose money on that unit.

EXAMPLE 8 Marginal Profit

Suppose that the cost function for the product in Example 7 is given by

$$C(q) = 2100 + 0.25q, \qquad \text{where } 0 \le q \le 30{,}000.$$

Find the marginal profit from the production of the following numbers of units.

(a) 15,000

APPLY IT

SOLUTION From Example 7, the revenue from the sale of x units is

$$R(q) = 2q - \frac{1}{25{,}000}q^2.$$

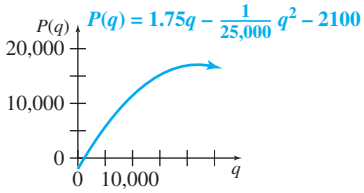


FIGURE 7

Since profit, P, is given by $P = R - C$,

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= \left(2q - \frac{1}{25{,}000}q^2\right) - (2100 + 0.25q) \\ &= 2q - \frac{1}{25{,}000}q^2 - 2100 - 0.25q \\ &= 1.75q - \frac{1}{25{,}000}q^2 - 2100. \end{aligned}$$

See Figure 7.

The marginal profit from the sale of q units is

$$P'(q) = 1.75 - \frac{2}{25{,}000}q = 1.75 - \frac{1}{12{,}500}q.$$

At $q = 15{,}000$ the marginal profit is

$$P'(15{,}000) = 1.75 - \frac{1}{12{,}500}(15{,}000) = 0.55,$$

or \$0.55 per unit.

(b) 21,875

SOLUTION When $q = 21{,}875$, the marginal profit is

$$P'(21{,}875) = 1.75 - \frac{1}{12{,}500}(21{,}875) = 0.$$

(c) 25,000

SOLUTION When $q = 25{,}000$, the marginal profit is

$$P'(25{,}000) = 1.75 - \frac{1}{12{,}500}(25{,}000) = -0.25,$$

or $-\$0.25$ per unit.

As shown by parts (b) and (c), if more than 21,875 units are sold, the marginal profit is negative. This indicates that increasing production beyond that level will *reduce* profit.

The final example shows an application of the derivative to a problem of demography.

EXAMPLE 9 Centenarians

The number of Americans (in thousands) who are expected to be over 100 years old can be approximated by the function

$$f(t) = 0.00943t^3 - 0.470t^2 + 11.085t + 23.441,$$

where t is the year, with $t = 0$ corresponding to 2000, and $0 \le t \le 50$. ***Source: U.S. Census Bureau.***

(a) Find a formula giving the rate of change of the number of Americans over 100 years old.

APPLY IT

SOLUTION Using the techniques for finding the derivative, we have

$$f'(t) = 0.02829t^2 - 0.940t + 11.085.$$

This tells us the rate of change in the number of Americans over 100 years old.

(b) Find the rate of change in the number of Americans who are expected to be over 100 years old in the year 2015.

SOLUTION The year 2015 corresponds to $t = 15$.

$$f'(15) = 0.02829(15)^2 - 0.940(15) + 11.085 = 3.35025$$

The number of Americans over 100 years old is expected to grow at a rate of about 3.35 thousand, or about 3350, per year in the year 2015.

12.1 EXERCISES

Find the derivative of each function defined as follows.

1. $y = 12x^3 - 8x^2 + 7x + 5$
2. $y = 8x^3 - 5x^2 - \dfrac{x}{12}$
3. $y = 3x^4 - 6x^3 + \dfrac{x^2}{8} + 5$
4. $y = 5x^4 + 9x^3 + 12x^2 - 7x$
5. $f(x) = 6x^{3.5} - 10x^{0.5}$
6. $f(x) = -2x^{1.5} + 12x^{0.5}$
7. $y = 8\sqrt{x} + 6x^{3/4}$
8. $y = -100\sqrt{x} - 11x^{2/3}$
9. $y = 10x^{-3} + 5x^{-4} - 8x$
10. $y = 5x^{-5} - 6x^{-2} + 13x^{-1}$
11. $f(t) = \dfrac{7}{t} - \dfrac{5}{t^3}$
12. $f(t) = \dfrac{14}{t} + \dfrac{12}{t^4} + \sqrt{2}$
13. $y = \dfrac{6}{x^4} - \dfrac{7}{x^3} + \dfrac{3}{x} + \sqrt{5}$
14. $y = \dfrac{3}{x^6} + \dfrac{1}{x^5} - \dfrac{7}{x^2}$
15. $p(x) = -10x^{-1/2} + 8x^{-3/2}$
16. $h(x) = x^{-1/2} - 14x^{-3/2}$
17. $y = \dfrac{6}{\sqrt[4]{x}}$
18. $y = \dfrac{-2}{\sqrt[3]{x}}$
19. $f(x) = \dfrac{x^3 + 5}{x}$
20. $g(x) = \dfrac{x^3 - 4x}{\sqrt{x}}$
21. $g(x) = (8x^2 - 4x)^2$
22. $h(x) = (x^2 - 1)^3$
23. Which of the following describes the derivative function $f'(x)$ of a quadratic function $f(x)$?

    a. Quadratic **b.** Linear **c.** Constant **d.** Cubic (third degree)
24. Which of the following describes the derivative function $f'(x)$ of a cubic (third degree) function $f(x)$?

    a. Quadratic **b.** Linear **c.** Constant **d.** Cubic
25. Explain the relationship between the slope and the derivative of $f(x)$ at $x = a$.
26. Which of the following do *not* equal $\dfrac{d}{dx}(4x^3 - 6x^{-2})$?

    a. $\dfrac{12x^2 + 12}{x^3}$ **b.** $\dfrac{12x^5 + 12}{x^3}$ **c.** $12x^2 + \dfrac{12}{x^3}$

    d. $12x^3 + 12x^{-3}$

Find each derivative.

27. $D_x\left[9x^{-1/2} + \dfrac{2}{x^{3/2}}\right]$
28. $D_x\left[\dfrac{8}{\sqrt[4]{x}} - \dfrac{3}{\sqrt{x^3}}\right]$
29. $f'(-2)$ if $f(x) = \dfrac{x^4}{6} - 3x$
30. $f'(3)$ if $f(x) = \dfrac{x^3}{9} - 7x^2$

In Exercises 31–34, find the slope of the tangent line to the graph of the given function at the given value of x. Find the equation of the tangent line in Exercises 31 and 32.

31. $y = x^4 - 5x^3 + 2; \quad x = 2$
32. $y = -3x^5 - 8x^3 + 4x^2; \quad x = 1$
33. $y = -2x^{1/2} + x^{3/2}; \quad x = 9$
34. $y = -x^{-3} + x^{-2}; \quad x = 2$
35. Find all points on the graph of $f(x) = 9x^2 - 8x + 4$ where the slope of the tangent line is 0.
36. Find all points on the graph of $f(x) = x^3 + 9x^2 + 19x - 10$ where the slope of the tangent line is -5.

In Exercises 37–40, for each function find all values of x where the tangent line is horizontal.

37. $f(x) = 2x^3 + 9x^2 - 60x + 4$

38. $f(x) = x^3 + 15x^2 + 63x - 10$

39. $f(x) = x^3 - 4x^2 - 7x + 8$

40. $f(x) = x^3 - 5x^2 + 6x + 3$

41. At what points on the graph of $f(x) = 6x^2 + 4x - 9$ is the slope of the tangent line -2?

42. At what points on the graph of $f(x) = 2x^3 - 9x^2 - 12x + 5$ is the slope of the tangent line 12?

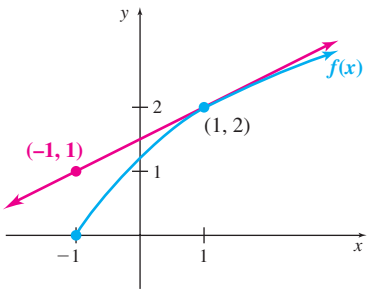
43. At what points on the graph of $f(x) = x^3 + 6x^2 + 21x + 2$ is the slope of the tangent line 9?

44. If $g'(5) = 12$ and $h'(5) = -3$, find $f'(5)$ for $f(x) = 3g(x) - 2h(x) + 3$.

45. If $g'(2) = 7$ and $h'(2) = 14$, find $f'(2)$ for $f(x) = \frac{1}{2}g(x) + \frac{1}{4}h(x)$.

46. Use the information given in the figure to find the following values.

a. $f(1)$ **b.** $f'(1)$ **c.** The domain of f **d.** The range of f



47. Explain the concept of marginal cost. How does it relate to cost? How is it found?

48. In Exercises 43–46 of Section 10.2, the effect of a when graphing $y = af(x)$ was discussed. Now describe how this relates to the fact that $D_x[af(x)] = af'(x)$.

49. Show that, for any constant k,

$$\frac{d}{dx}\left[\frac{f(x)}{k}\right] = \frac{f'(x)}{k}.$$

50. Use the differentiation feature on your graphing calculator to solve the problems (to 2 decimal places) below, where $f(x)$ is defined as follows:

$$f(x) = 1.25x^3 + 0.01x^2 - 2.9x + 1.$$

a. Find $f'(4)$.

b. Find all values of x where $f'(x) = 0$.

APPLICATIONS

Business and Economics

51. Revenue Assume that a demand equation is given by $q = 5000 - 100p$. Find the marginal revenue for the following production levels (values of q). (*Hint:* Solve the demand equation for p and use $R(q) = qp$.)

a. 1000 units **b.** 2500 units **c.** 3000 units

52. Profit Suppose that for the situation in Exercise 51 the cost of producing q units is given by $C(q) = 3000 - 20q + 0.03q^2$. Find the marginal profit for the following production levels.

a. 500 units **b.** 815 units **c.** 1000 units

53. Revenue If the price in dollars of a stereo system is given by

$$p(q) = \frac{1000}{q^2} + 1000,$$

where q represents the demand for the product, find the marginal revenue when the demand is 10.

54. Profit Suppose that for the situation in Exercise 53 the cost in dollars of producing q stereo systems is given by $C(q) = 0.2q^2 + 6q + 50$. Find the marginal profit when the demand is 10.

55. Sales Often sales of a new product grow rapidly at first and then level off with time. This is the case with the sales represented by the function

$$S(t) = 100 - 100t^{-1},$$

where t represents time in years. Find the rate of change of sales for the following numbers of years.

a. 1 **b.** 10

56. Profit An analyst has found that a company's costs and revenues in dollars for one product are given by

$$C(x) = 2x \quad \text{and} \quad R(x) = 6x - \frac{x^2}{1000},$$

respectively, where x is the number of items produced.

a. Find the marginal cost function.

b. Find the marginal revenue function.

c. Using the fact that profit is the difference between revenue and costs, find the marginal profit function.

d. What value of x makes marginal profit equal 0?

e. Find the profit when the marginal profit is 0.

(As we shall see in the next chapter, this process is used to find *maximum* profit.)

57. Postal Rates U.S. postal rates have steadily increased since 1932. Using data depicted in the table for the years 1932–2009, the cost in cents to mail a single letter can be modeled using a quadratic formula as follows:

$$C(t) = 0.008446t^2 - 0.08924t + 1.254$$

where t is the number of years since 1932. ***Source: U.S. Postal Service.***

Year	Cost	Year	Cost
1932	3	1988	25
1958	4	1991	29
1963	5	1995	32
1968	6	1999	33
1971	8	2001	34
1974	10	2002	37
1975	13	2006	39
1978	15	2007	41
1981	18	2008	42
1981	20	2009	44
1985	22		

a. Find the predicted cost of mailing a letter in 1982 and 2002 and compare these estimates with the actual rates.

b. Find the rate of change of the postage cost for the years 1982 and 2002 and interpret your results.

c. Using the regression feature on a graphing calculator, find a cubic function that models this data, letting $t = 0$ correspond to the year 1932. Then use your answer to find the rate of change of the postage cost for the years 1982 and 2002.

d. Discuss whether the quadratic or cubic function best describes the data. Do the answers from part b or from part c best describe the rate that postage was going up in the years 1982 and 2002?

e. Explore other functions that could be used to model the data, using the various regression features on a graphing calculator, and discuss to what extent any of them are useful descriptions of the data.

58. Money The total amount of money in circulation for the years 1950–2009 can be closely approximated by

$$M(t) = 0.005209t^3 - 0.04159t^2 - 0.3664t + 34.49$$

where t represents the number of years since 1950 and $M(t)$ is in billions of dollars. Find the derivative of $M(t)$ and use it to find the rate of change of money in circulation in the following years. ***Source: U.S. Treasury.***

a. 1960 **b.** 1980 **c.** 1990 **d.** 2000

e. What do your answers to parts a–d tell you about the amount of money in circulation in those years?

Life Sciences

59. Cancer Insulation workers who were exposed to asbestos and employed before 1960 experienced an increased likelihood of lung cancer. If a group of insulation workers has a cumulative total of 100,000 years of work experience with their first date of employment t years ago, then the number of lung cancer cases occurring within the group can be modeled using the function

$$N(t) = 0.00437t^{3.2}.$$

Find the rate of growth of the number of workers with lung cancer in a group as described by the following first dates of employment. ***Source: Observation and Inference: An Introduction to the Methods of Epidemiology.***

a. 5 years ago **b.** 10 years ago

60. Blood Sugar Level Insulin affects the glucose, or blood sugar, level of some diabetics according to the function

$$G(x) = -0.2x^2 + 450,$$

where $G(x)$ is the blood sugar level 1 hour after x units of insulin are injected. (This mathematical model is only approximate, and it is valid only for values of x less than about 40.) Find the blood sugar level after the following numbers of units of insulin are injected.

a. 0 **b.** 25

Find the rate of change of blood sugar level after injection of the following numbers of units of insulin.

c. 10 **d.** 25

61. Bighorn Sheep The cumulative horn volume for certain types of bighorn rams, found in the Rocky Mountains, can be described by the quadratic function

$$V(t) = -2159 + 1313t - 60.82t^2,$$

where $V(t)$ is the horn volume (in cm^3) and t is the year of growth, $2 \le t \le 9$. ***Source: Conservation Biology.***

a. Find the horn volume for a 3-year-old ram.

b. Find the rate at which the horn volume of a 3-year-old ram is changing.

62. Brain Mass The brain mass of a human fetus during the last trimester can be accurately estimated from the circumference of the head by

$$m(c) = \frac{c^3}{100} - \frac{1500}{c},$$

where $m(c)$ is the mass of the brain (in grams) and c is the circumference (in centimeters) of the head. ***Source: Early Human Development.***

a. Estimate the brain mass of a fetus that has a head circumference of 30 cm.

b. Find the rate of change of the brain mass for a fetus that has a head circumference of 30 cm and interpret your results.

63. Velocity of Marine Organism The typical velocity (in centimeters per second) of a marine organism of length l (in centimeters) is given by $v = 2.69l^{1.86}$. Find the rate of change of the velocity with respect to the length of the organism. ***Source: Mathematical Topics in Population Biology Morphogenesis and Neurosciences.***

64. Heart The left ventricular length (viewed from the front of the heart) of a fetus that is at least 18 weeks old can be estimated by

$$l(x) = -2.318 + 0.2356x - 0.002674x^2,$$

where $l(x)$ is the ventricular length (in centimeters) and x is the age (in weeks) of the fetus. ***Source: American Journal of Cardiology.***

a. Determine a meaningful domain for this function.

b. Find $l'(x)$.

c. Find $l'(25)$.

65. Track and Field In 1906 Kennelly developed a simple formula for predicting an upper limit on the fastest time that humans could ever run distances from 100 yards to 10 miles. His formula is given by

$$t = 0.0588s^{1.125},$$

where s is the distance in meters and t is the time to run that distance in seconds. ***Source: Proceedings of the American Academy of Arts and Sciences.***

a. Find Kennelly's estimate for the fastest mile. (*Hint:* 1 mile $\approx$ 1609 meters.)

b. Find dt/ds when $s = 100$ and interpret your answer.

c. Compare this and other estimates to the current world records. Have these estimates been surpassed?

66. Human Cough To increase the velocity of the air flowing through the trachea when a human coughs, the body contracts the windpipe, producing a more effective cough. Tuchinsky formulated that the velocity of air that is flowing through the trachea during a cough is

$$V = C(R_0 - R)R^2,$$

where C is a constant based on individual body characteristics, R_0 is the radius of the windpipe before the cough, and R is the radius of the windpipe during the cough. It can be shown that the maximum velocity of the cough occurs when $dV/dR = 0$. Find the value of R that maximizes the velocity.* ***Source: COMAP, Inc.***

67. Body Mass Index The body mass index (BMI) is a number that can be calculated for any individual as follows: Multiply weight by 703 and divide by the person's height squared. That is,

$$BMI = \frac{703w}{h^2},$$

where w is in pounds and h is in inches. The National Heart, Lung, and Blood Institute uses the BMI to determine whether a person is "overweight" $(25 \le BMI < 30)$ or "obese" $(BMI \ge 30)$. ***Source: The National Institutes of Health.***

a. Calculate the BMI for Lebron James, basketball player for the Miami Heat, who is 250 lb. and 6'8" tall.

b. How much weight would Lebron James have to lose until he reaches a BMI of 24.9 and is no longer "overweight"? Comment on whether BMI cutoffs are appropriate for athletes with considerable muscle mass.

c. For a 125-lb female, what is the rate of change of BMI with respect to height? (*Hint:* Take the derivative of the function: $f(h) = 703(125)/h^2$.)

d. Calculate and interpret the meaning of $f'(65)$.

e. Use the TABLE feature on your graphing calculator to construct a table for BMI for various weights and heights.

Physical Sciences

Velocity We saw in the previous chapter that if a function $s(t)$ gives the position of an object at time t, the derivative gives the velocity, that is, $v(t) = s'(t)$. For each position function in Exercises 68–71, find (a) $v(t)$ and (b) the velocity when $t = 0$, $t = 5$, and $t = 10$.

68. $s(t) = 11t^2 + 4t + 2$

69. $s(t) = 18t^2 - 13t + 8$

70. $s(t) = 4t^3 + 8t^2 + t$

71. $s(t) = -3t^3 + 4t^2 - 10t + 5$

72. Velocity If a rock is dropped from a 144-ft building, its position (in feet above the ground) is given by $s(t) = -16t^2 + 144$, where t is the time in seconds since it was dropped.

a. What is its velocity 1 second after being dropped? 2 seconds after being dropped?

b. When will it hit the ground?

c. What is its velocity upon impact?

73. Velocity A ball is thrown vertically upward from the ground at a velocity of 64 ft per second. Its distance from the ground at t seconds is given by $s(t) = -16t^2 + 64t$.

a. How fast is the ball moving 2 seconds after being thrown? 3 seconds after being thrown?

b. How long after the ball is thrown does it reach its maximum height?

c. How high will it go?

74. Dead Sea Researchers who have been studying the alarming rate in which the level of the Dead Sea has been dropping have shown that the density $d(x)$ (in g per cm^3) of the Dead Sea brine during evaporation can be estimated by the function

$$d(x) = 1.66 - 0.90x + 0.47x^2,$$

where x is the fraction of the remaining brine, $0 \le x \le 1$. ***Source: Geology.***

a. Estimate the density of the brine when 50% of the brine remains.

b. Find and interpret the instantaneous rate of change of the density when 50% of the brine remains.

75. Dog's Human Age From the data printed in the following table from the *Minneapolis Star Tribune* on September 20, 1998, a dog's age when compared to a human's age can be modeled using either a linear formula or a quadratic formula as follows:

$$y_1 = 4.13x + 14.63$$
$$y_2 = -0.033x^2 + 4.647x + 13.347,$$

where y_1 and y_2 represent a dog's human age for each formula and x represents a dog's actual age. ***Source: Mathematics Teacher.***

Dog Age	Human Age
1	16
2	24
3	28
5	36
7	44
9	52
11	60
13	68
15	76

*Interestingly, Tuchinsky also states that X-rays indicate that the body naturally contracts the windpipe to this radius during a cough.

a. Find y_1 and y_2 when $x = 5$.

b. Find dy_1/dx and dy_2/dx when $x = 5$ and interpret your answers.

c. If the first two points are eliminated from the table, find the equation of a line that perfectly fits the reduced set of data. Interpret your findings.

d. Of the three formulas, which do you prefer?

YOUR TURN ANSWERS

1. $f'(t) = -\frac{1}{2}t^{-\frac{3}{2}}$ or $f'(t) = -\frac{1}{2t^{\frac{3}{2}}}$

2. $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}$ or $\frac{dy}{dx} = \frac{3}{2\sqrt{x}}$

3. $h'(t) = -6t + t^{-\frac{1}{2}} - 20t^{-5}$ or $h'(t) = -6t + \frac{1}{\sqrt{t}} - \frac{20}{t^5}$

4. \$148,000

5. \$3.50

12.2 Derivatives of Products and Quotients

APPLY IT

A manufacturer of small motors wants to make the average cost per motor as small as possible. How can this be done?

We show how the derivative is used to solve a problem like this in Example 5, later in this section.

In the previous section we saw that the derivative of a sum of two functions is found from the sum of the derivatives. What about products? Is the derivative of a product equal to the product of the derivatives? For example, if

$$u(x) = 2x + 3 \quad \text{and} \quad v(x) = 3x^2,$$

then

$$u'(x) = 2 \quad \text{and} \quad v'(x) = 6x.$$

Let $f(x)$ be the product of u and v; that is, $f(x) = (2x + 3)(3x^2) = 6x^3 + 9x^2$. By the rules of the preceding section, $f'(x) = 18x^2 + 18x = 18x(x + 1)$. On the other hand, $u'(x) \cdot v'(x) = 2(6x) = 12x \neq f'(x)$. In this example, the derivative of a product is *not* equal to the product of the derivatives, nor is this usually the case.

The rule for finding derivatives of products is as follows.

Product Rule

If $f(x) = u(x) \cdot v(x)$, and if $u'(x)$ and $v'(x)$ both exist, then

$$\boldsymbol{f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).}$$

(The derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.)

FOR REVIEW

This proof uses several of the rules for limits given in the first section of the previous chapter. You may want to review them at this time.

To sketch the method used to prove the product rule, let

$$f(x) = u(x) \cdot v(x).$$

Then $f(x + h) = u(x + h) \cdot v(x + h)$, and, by definition, $f'(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x + h) \cdot v(x + h) - u(x) \cdot v(x)}{h}.$$

Now subtract and add $u(x+h)\cdot v(x)$ in the numerator, giving

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{u(x+h)\cdot v(x+h) - \boldsymbol{u(x+h)\cdot v(x) + u(x+h)\cdot v(x)} - u(x)\cdot v(x)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h)[v(x+h)-v(x)] + v(x)[u(x+h)-u(x)]}{h} \\ &= \lim_{h\to 0} u(x+h)\left[\frac{v(x+h)-v(x)}{h}\right] + \lim_{h\to 0} v(x)\left[\frac{u(x+h)-u(x)}{h}\right] \\ &= \lim_{h\to 0} u(x+h)\cdot \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} + \lim_{h\to 0} v(x)\cdot \lim_{h\to 0} \frac{u(x+h)-u(x)}{h}. \end{aligned} \tag{1}$$

If $u'(x)$ and $v'(x)$ both exist, then

$$\lim_{h\to 0} \frac{u(x+h)-u(x)}{h} = u'(x) \qquad \text{and} \qquad \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} = v'(x).$$

The fact that $u'(x)$ exists can be used to prove

$$\lim_{h\to 0} u(x+h) = u(x),$$

and since no h is involved in $v(x)$,

$$\lim_{h\to 0} v(x) = v(x).$$

Substituting these results into Equation (1) gives

$$f'(x) = u(x)\cdot v'(x) + v(x)\cdot u'(x),$$

the desired result.

To help see why the product rule is true, consider the special case in which u and v are positive functions. Then $u(x)\cdot v(x)$ represents the area of a rectangle, as shown in Figure 8. If we assume that u and v are increasing, then $u(x+h)\cdot v(x+h)$ represents the area of a slightly larger rectangle when h is a small positive number, as shown in the figure. The change in the area of the rectangle is given by the pink rectangle, with an area of $u(x)$ times the amount v has changed, plus the blue rectangle, with an area of $v(x)$ times the amount u has changed, plus the small green rectangle. As h becomes smaller and smaller, the green rectangle becomes negligibly small, and the change in the area is essentially $u(x)$ times the change in v plus $v(x)$ times the change in u.

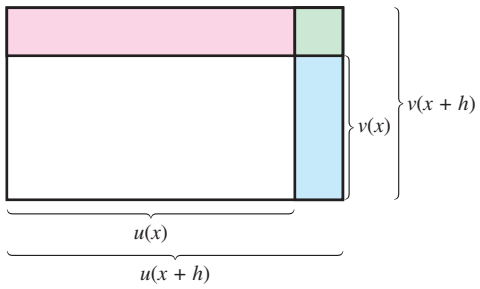


FIGURE 8

EXAMPLE 1 Product Rule

Let $f(x) = (2x + 3)(3x^2)$. Use the product rule to find $f'(x)$.

SOLUTION Here f is given as the product of $u(x) = 2x + 3$ and $v(x) = 3x^2$. By the product rule and the fact that $u'(x) = 2$ and $v'(x) = 6x$,

$$\begin{aligned} f'(x) &= u(x) \cdot v'(x) + v(x) \cdot u'(x) \\ &= (2x + 3)(6x) + (3x^2)(2) \\ &= 12x^2 + 18x + 6x^2 = 18x^2 + 18x = 18x(x + 1). \end{aligned}$$

This result is the same as that found at the beginning of the section.

EXAMPLE 2 Product Rule

Find the derivative of $y = (\sqrt{x} + 3)(x^2 - 5x)$.

SOLUTION Let $u(x) = \sqrt{x} + 3 = x^{1/2} + 3$, and $v(x) = x^2 - 5x$. Then

$$\begin{aligned} \frac{dy}{dx} &= u(x) \cdot v'(x) + v(x) \cdot u'(x) \\ &= (x^{1/2} + 3)(2x - 5) + (x^2 - 5x)\left(\frac{1}{2}x^{-1/2}\right). \end{aligned}$$

Simplify by multiplying and combining terms.

$$\begin{aligned} \frac{dy}{dx} &= (2x)(x^{1/2}) + 6x - 5x^{1/2} - 15 + (x^2)\left(\frac{1}{2}x^{-1/2}\right) - (5x)\left(\frac{1}{2}x^{-1/2}\right) \\ &= 2x^{3/2} + 6x - 5x^{1/2} - 15 + \frac{1}{2}x^{3/2} - \frac{5}{2}x^{1/2} \\ &= \frac{5}{2}x^{3/2} + 6x - \frac{15}{2}x^{1/2} - 15 \end{aligned}$$

YOUR TURN 1 Find the derivative of $y = (x^3 + 7)(4 - x^2)$.

TRY YOUR TURN 1

We could have found the derivatives above by multiplying out the original functions. The product rule then would not have been needed. In the next section, however, we shall see products of functions where the product rule is essential.

What about *quotients* of functions? To find the derivative of the quotient of two functions, use the next rule.

Quotient Rule

If $f(x) = u(x)/v(x)$, if all indicated derivatives exist, and if $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

(The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.)

The proof of the quotient rule is similar to that of the product rule and is left for the exercises. (See Exercises 37 and 38.)

FOR REVIEW

You may want to consult the Rational Expressions section of the Algebra Reference chapter (Section 3) to help you work with the fractions in the section.

CAUTION Just as the derivative of a product is *not* the product of the derivatives, the derivative of a quotient is *not* the quotient of the derivatives. If you are asked to take the derivative of a product or a quotient, it is essential that you recognize that the function contains a product or quotient and then use the appropriate rule.

EXAMPLE 3 Quotient Rule

Find $f'(x)$ if $f(x) = \dfrac{2x - 1}{4x + 3}$.

SOLUTION Let $u(x) = 2x - 1$, with $u'(x) = 2$. Also, let $v(x) = 4x + 3$, with $v'(x) = 4$. Then, by the quotient rule,

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

$$= \frac{(4x + 3)(2) - (2x - 1)(4)}{(4x + 3)^2}$$

$$= \frac{8x + 6 - 8x + 4}{(4x + 3)^2}$$

$$= \frac{10}{(4x + 3)^2}.$$

YOUR TURN 2 Find $f'(x)$ if $f(x) = \dfrac{3x + 2}{5 - 2x}$.

TRY YOUR TURN 2

CAUTION In the second step of Example 3, we had the expression

$$\frac{(4x + 3)(2) - (2x - 1)(4)}{(4x + 3)^2}.$$

Students often incorrectly "cancel" the $4x + 3$ in the numerator with one factor of the denominator. Because the numerator is a *difference* of two products, however, you must multiply and combine terms *before* looking for common factors in the numerator and denominator.

EXAMPLE 4 Product and Quotient Rules

Find $D_x\left[\dfrac{(3 - 4x)(5x + 1)}{7x - 9}\right]$.

SOLUTION This function has a product within a quotient. Instead of multiplying the factors in the numerator first (which is an option), we can use the quotient rule together with the product rule, as follows. Use the quotient rule first to get

$$D_x\left[\frac{(3 - 4x)(5x + 1)}{7x - 9}\right] = \frac{(7x - 9)D_x[(3 - 4x)(5x + 1)] - [(3 - 4x)(5x + 1)D_x(7x - 9)]}{(7x - 9)^2}.$$

Now use the product rule to find $D_x[(3 - 4x)(5x + 1)]$ in the numerator.

$$= \frac{(7x - 9)[(3 - 4x)5 + (5x + 1)(-4)] - (3 + 11x - 20x^2)(7)}{(7x - 9)^2}$$

$$= \frac{(7x - 9)(15 - 20x - 20x - 4) - (21 + 77x - 140x^2)}{(7x - 9)^2}$$

$$= \frac{(7x - 9)(11 - 40x) - 21 - 77x + 140x^2}{(7x - 9)^2}$$

$$= \frac{-280x^2 + 437x - 99 - 21 - 77x + 140x^2}{(7x - 9)^2}$$

$$= \frac{-140x^2 + 360x - 120}{(7x - 9)^2}$$

YOUR TURN 3 Find $D_x\left[\dfrac{(5x - 3)(2x + 7)}{3x + 7}\right]$.

TRY YOUR TURN 3

Average Cost

Suppose $y = C(x)$ gives the total cost to manufacture x items. As mentioned earlier, the average cost per item is found by dividing the total cost by the number of items. The rate of change of average cost, called the *marginal average cost*, is the derivative of the average cost.

Marginal Average Cost

If the total cost to manufacture x items is given by $C(x)$, then the average cost per item is $\overline{C}(x) = C(x)/x$. The **marginal average cost** is the derivative of the average cost function, $\overline{C}'(x)$.

Similarly, the marginal average revenue function, $\overline{R}'(x)$, is defined as the derivative of the average revenue function, $\overline{R}(x) = R(x)/x$, and the marginal average profit function, $\overline{P}'(x)$, is defined as the derivative of the average profit function, $\overline{P}(x) = P(x)/x$.

A company naturally would be interested in making the average cost as small as possible. The next chapter will show that this can be done by using the derivative of $C(x)/x$. This derivative often can be found by means of the quotient rule, as in the next example.

EXAMPLE 5 Minimum Average Cost

Suppose the cost in dollars of manufacturing x hundred small motors is given by

$$C(x) = \frac{3x^2 + 120}{2x + 1}, \quad 10 \le x \le 200.$$

(a) Find the average cost per hundred motors.

SOLUTION The average cost is defined by

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{3x^2 + 120}{2x + 1} \cdot \frac{1}{x} = \frac{3x^2 + 120}{2x^2 + x}.$$

(b) Find the marginal average cost.

SOLUTION The marginal average cost is given by

$$\begin{aligned}\frac{d}{dx}[\overline{C}(x)] &= \frac{(2x^2 + x)(6x) - (3x^2 + 120)(4x + 1)}{(2x^2 + x)^2} \\ &= \frac{12x^3 + 6x^2 - 12x^3 - 480x - 3x^2 - 120}{(2x^2 + x)^2} \\ &= \frac{3x^2 - 480x - 120}{(2x^2 + x)^2}.\end{aligned}$$

(c) As we shall see in the next chapter, average cost is generally minimized when the marginal average cost is zero. Find the level of production that minimizes average cost.

APPLY IT

SOLUTION Set the derivative $\overline{C}'(x) = 0$ and solve for x.

$$\begin{aligned}\frac{3x^2 - 480x - 120}{(2x^2 + x)^2} &= 0 \\ 3x^2 - 480x - 120 &= 0 \\ 3(x^2 - 160x - 40) &= 0\end{aligned}$$

Use the quadratic formula to solve this quadratic equation. Discarding the negative solution leaves $x = (160 + \sqrt{(160)^2 + 160})/2 \approx 160$ as the solution. Since x is in hundreds, production of 160 hundred or 16,000 motors will minimize average cost.

YOUR TURN 4 Suppose cost is given by $C(x) = \dfrac{4x + 50}{x + 2}$. Find the marginal average cost.

TRY YOUR TURN 4

12.2 EXERCISES

Use the product rule to find the derivative of the following. (Hint for Exercises 3–6: Write the quantity as a product.)

1. $y = (3x^2 + 2)(2x - 1)$

2. $y = (5x^2 - 1)(4x + 3)$

3. $y = (2x - 5)^2$

4. $y = (7x - 6)^2$

5. $k(t) = (t^2 - 1)^2$

6. $g(t) = (3t^2 + 2)^2$

7. $y = (x + 1)(\sqrt{x} + 2)$

8. $y = (2x - 3)(\sqrt{x} - 1)$

9. $p(y) = (y^{-1} + y^{-2})(2y^{-3} - 5y^{-4})$

10. $q(x) = (x^{-2} - x^{-3})(3x^{-1} + 4x^{-4})$

Use the quotient rule to find the derivative of the following.

11. $f(x) = \dfrac{6x + 1}{3x + 10}$

12. $f(x) = \dfrac{8x - 11}{7x + 3}$

13. $y = \dfrac{5 - 3t}{4 + t}$

14. $y = \dfrac{9 - 7t}{1 - t}$

15. $y = \dfrac{x^2 + x}{x - 1}$

16. $y = \dfrac{x^2 - 4x}{x + 3}$

17. $f(t) = \dfrac{4t^2 + 11}{t^2 + 3}$

18. $y = \dfrac{-x^2 + 8x}{4x^2 - 5}$

19. $g(x) = \dfrac{x^2 - 4x + 2}{x^2 + 3}$

20. $k(x) = \dfrac{x^2 + 7x - 2}{x^2 - 2}$

21. $p(t) = \dfrac{\sqrt{t}}{t - 1}$

22. $r(t) = \dfrac{\sqrt{t}}{2t + 3}$

23. $y = \dfrac{5x + 6}{\sqrt{x}}$

24. $y = \dfrac{4x - 3}{\sqrt{x}}$

25. $h(z) = \dfrac{z^{2.2}}{z^{3.2} + 5}$

26. $g(y) = \dfrac{y^{1.4} + 1}{y^{2.5} + 2}$

27. $f(x) = \dfrac{(3x^2 + 1)(2x - 1)}{5x + 4}$

28. $g(x) = \dfrac{(2x^2 + 3)(5x + 2)}{6x - 7}$

29. If $g(3) = 4$, $g'(3) = 5$, $f(3) = 9$, and $f'(3) = 8$, find $h'(3)$ when $h(x) = f(x)g(x)$.

30. If $g(3) = 4$, $g'(3) = 5$, $f(3) = 9$, and $f'(3) = 8$, find $h'(3)$ when $h(x) = f(x)/g(x)$.

31. Find the error in the following work.

$$D_x\left(\frac{2x + 5}{x^2 - 1}\right) = \frac{(2x + 5)(2x) - (x^2 - 1)2}{(x^2 - 1)^2} = \frac{4x^2 + 10x - 2x^2 + 2}{(x^2 - 1)^2} = \frac{2x^2 + 10x + 2}{(x^2 - 1)^2}$$

32. Find the error in the following work.

$$D_x\left(\frac{x^2 - 4}{x^3}\right) = x^3(2x) - (x^2 - 4)(3x^2) = 2x^4 - 3x^4 + 12x^2 = -x^4 + 12x^2$$

33. Find an equation of the line tangent to the graph of $f(x) = x/(x - 2)$ at $(3, 3)$.

34. Find an equation of the line tangent to the graph of $f(x) = (2x - 1)(x + 4)$ at $(1, 5)$.

35. Consider the function

$$f(x) = \frac{3x^3 + 6}{x^{2/3}}.$$

a. Find the derivative using the quotient rule.

b. Find the derivative by first simplifying the function to

$$f(x) = \frac{3x^3}{x^{2/3}} + \frac{6}{x^{2/3}} = 3x^{7/3} + 6x^{-2/3}$$

and using the rules from the previous section.

c. Compare your answers from parts a and b and explain any discrepancies.

36. What is the result of applying the product rule to the function

$$f(x) = kg(x),$$

where k is a constant? Compare with the rule for differentiating a constant times a function from the previous section.

37. Following the steps used to prove the product rule for derivatives, prove the quotient rule for derivatives.

38. Use the fact that $f(x) = u(x)/v(x)$ can be rewritten as $f(x)v(x) = u(x)$ and the product rule for derivatives to verify the quotient rule for derivatives. (*Hint:* After applying the product rule, substitute $u(x)/v(x)$ for $f(x)$ and simplify.)

For each function, find the value(s) of x in which $f'(x) = 0$, to 3 decimal places.

39. $f(x) = (x^2 - 2)(x^2 - \sqrt{2})$

40. $f(x) = \dfrac{x - 2}{x^2 + 4}$

APPLICATIONS

Business and Economics

41. Average Cost The total cost (in hundreds of dollars) to produce x units of perfume is

$$C(x) = \frac{3x + 2}{x + 4}.$$

Find the average cost for each production level.

a. 10 units **b.** 20 units **c.** x units

d. Find the marginal average cost function.

42. Average Profit The total profit (in tens of dollars) from selling x self-help books is

$$P(x) = \frac{5x - 6}{2x + 3}.$$

Find the average profit from each sales level.

a. 8 books **b.** 15 books **c.** x books

d. Find the marginal average profit function.

e. Is this a reasonable function for profit? Why or why not?

43. Employee Training A company that manufactures bicycles has determined that a new employee can assemble $M(d)$ bicycles per day after d days of on-the-job training, where

$$M(d) = \frac{100d^2}{3d^2 + 10}.$$

a. Find the rate of change function for the number of bicycles assembled with respect to time.

b. Find and interpret $M'(2)$ and $M'(5)$.

44. Marginal Revenue Suppose that the demand function is given by $p = D(q)$, where q is the quantity that consumers demand when the price is p. Show that the marginal revenue is given by

$$R'(q) = D(q) + qD'(q).$$

45. Marginal Average Cost Suppose that the average cost function is given by $\overline{C}(x) = C(x)/x$, where x is the number of items produced. Show that the marginal average cost function is given by

$$\overline{C}'(x) = \frac{xC'(x) - C(x)}{x^2}.$$

46. Revenue Suppose that at the beginning of the year, a Vermont maple syrup distributor found that the demand for maple syrup, sold at \$15 a quart, was 500 quarts each month. At that time, the price was going up at a rate of \$0.50 a month, but despite this, the demand was going up at a rate of 30 quarts a month due to increased advertising. How fast was the revenue increasing?

47. Average Cost A gasoline refinery found that the cost to produce 12,500 gallons of gasoline last month was \$27,000. At that time, the cost was going up at a rate of \$1200 per month, while the number of gallons of gasoline the refinery produced was going up at a rate of 350 gallons per month. At what rate was the average cost to produce a gallon of gasoline increasing or decreasing last month?

Life Sciences

48. Muscle Reaction When a certain drug is injected into a muscle, the muscle responds by contracting. The amount of contraction, s (in millimeters) is related to the concentration of the drug, x (in milliliters) by

$$s(x) = \frac{x}{m + nx},$$

where m and n are constants.

a. Find $s'(x)$.

b. Find the rate of contraction when the concentration of the drug is 50 ml, $m = 10$, and $n = 3$.

49. Growth Models In Exercise 58 of Section 10.3, the formula for the growth rate of a population in the presence of a quantity x of food was given as

$$f(x) = \frac{Kx}{A + x}.$$

This was referred to as Michaelis-Menten kinetics.

a. Find the rate of change of the growth rate with respect to the amount of food.

b. The quantity A in the formula for $f(x)$ represents the quantity of food for which the growth rate is half of its maximum. Using your answer from part a, find the rate of change of the growth rate when $x = A$.

50. Bacteria Population Assume that the total number (in millions) of bacteria present in a culture at a certain time t (in hours) is given by

$$N(t) = 3t(t - 10)^2 + 40.$$

a. Find $N'(t)$.

Find the rate at which the population of bacteria is changing at the following times.

b. 8 hours **c.** 11 hours

d. The answer in part b is negative, and the answer in part c is positive. What does this mean in terms of the population of bacteria?

51. Work/Rest Cycles Murrell's formula for calculating the total amount of rest, in minutes, required after performing a particular type of work activity for 30 minutes is given by the formula

$$R(w) = \frac{30(w - 4)}{w - 1.5},$$

where w is the work expended in kilocalories per minute, kcal/min. ***Source: Human Factors in Engineering and Design.***

a. A value of 5 for w indicates light work, such as riding a bicycle on a flat surface at 10 mph. Find $R(5)$.

b. A value of 7 for w indicates moderate work, such as mowing grass with a pushmower on level ground. Find $R(7)$.

c. Find $R'(5)$ and $R'(7)$ and compare your answers. Explain whether these answers make sense.

52. Optimal Foraging Using data collected by zoologist Reto Zach, the work done by a crow to break open a whelk (large marine snail) can be estimated by the function

$$W = \left(1 + \frac{20}{H - 0.93}\right)H,$$

where H is the height (in meters) of the whelk when it is dropped. ***Source: Mathematics Teacher***.

a. Find dW/dH.

b. One can show that the amount of work is minimized when $dW/dH = 0$. Find the value of H that minimizes W.

c. Interestingly, Zach observed the crows dropping the whelks from an average height of 5.23 m. What does this imply?

Social Sciences

53. Memory Retention Some psychologists contend that the number of facts of a certain type that are remembered after t hours is given by

$$f(t) = \frac{90t}{99t - 90}.$$

Find the rate at which the number of facts remembered is changing after the following numbers of hours.

a. 1 **b.** 10

General Interest

54. Vehicle Waiting Time The average number of vehicles waiting in a line to enter a parking ramp can be modeled by the function

$$f(x) = \frac{x^2}{2(1 - x)},$$

where x is a quantity between 0 and 1 known as the traffic intensity. Find the rate of change of the number of vehicles in line with respect to the traffic intensity for the following values of the intensity. ***Source: Principles of Highway Engineering and Traffic Control***.

a. $x = 0.1$ **b.** $x = 0.6$

YOUR TURN ANSWERS

1. $\dfrac{dy}{dx} = -5x^4 + 12x^2 - 14x$

2. $f'(x) = \dfrac{19}{(5 - 2x)^2}$

3. $\dfrac{30x^2 + 140x + 266}{(3x + 7)^2}$

4. $\dfrac{-4x^2 - 100x - 100}{(x^2 + 2x)^2}$

12.3 The Chain Rule

APPLY IT **Suppose we know how fast the radius of a circular oil slick is growing, and we know how much the area of the oil slick is growing per unit of change in the radius. How fast is the area growing?**
We will answer this question in Example 4 using the chain rule for derivatives.

Before discussing the chain rule, we consider the composition of functions. Many of the most useful functions for modeling are created by combining simpler functions. Viewing complex functions as combinations of simpler functions often makes them easier to understand and use.

Composition of Functions

Suppose a function f assigns to each element x in set X some element $y = f(x)$ in set Y. Suppose also that a function g takes each element in set Y and assigns to it a value $z = g[f(x)]$ in set Z. By using both f and g, an element x in X is assigned to an element z in Z, as illustrated in Figure 9. The result of this process is a new function called the *composition* of functions g and f and defined as follows.

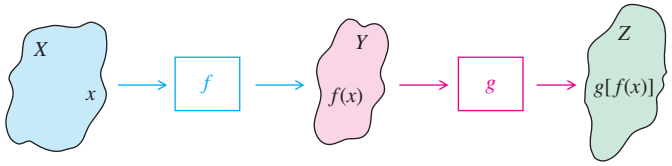


FIGURE 9

FOR REVIEW

You may want to review how to find the domain of a function. Domain was discussed in Section 10.1 on Properties of Functions.

Composite Function

Let f and g be functions. The **composite function**, or **composition**, of g and f is the function whose values are given by $g[f(x)]$ for all x in the domain of f such that $f(x)$ is in the domain of g. (Read $g[f(x)]$ as "g of f of x".)

EXAMPLE 1 Composite Functions

Let $f(x) = 2x - 1$ and $g(x) = \sqrt{3x + 5}$. Find the following.

(a) $g[f(4)]$

SOLUTION Find $f(4)$ first.

$$f(4) = 2 \cdot 4 - 1 = 8 - 1 = 7$$

Then

$$g[f(4)] = g[7] = \sqrt{3 \cdot 7 + 5} = \sqrt{26}.$$

(b) $f[g(4)]$

SOLUTION Since $g(4) = \sqrt{3 \cdot 4 + 5} = \sqrt{17}$,

$$f[g(4)] = 2 \cdot \sqrt{17} - 1 = 2\sqrt{17} - 1.$$

(c) $f[g(-2)]$

SOLUTION $f[g(-2)]$ does not exist since -2 is not in the domain of g.

YOUR TURN 1 For the functions in Example 1, find $f[g(0)]$ and $g[f(0)]$.

TRY YOUR TURN 1

EXAMPLE 2 Composition of Functions

Let $f(x) = 2x^2 + 5x$ and $g(x) = 4x + 1$. Find the following.

(a) $f[g(x)]$

SOLUTION Using the given functions, we have

$$\begin{aligned} f[g(x)] &= f[4x + 1] \\ &= 2(4x + 1)^2 + 5(4x + 1) \\ &= 2(16x^2 + 8x + 1) + 20x + 5 \\ &= 32x^2 + 16x + 2 + 20x + 5 \\ &= 32x^2 + 36x + 7. \end{aligned}$$

(b) $g[f(x)]$

SOLUTION By the definition above, with f and g interchanged,

$$\begin{aligned} g[f(x)] &= g[2x^2 + 5x] \\ &= 4(2x^2 + 5x) + 1 \\ &= 8x^2 + 20x + 1. \end{aligned}$$

YOUR TURN 2 Let $f(x) = 2x - 3$ and $g(x) = x^2 + 1$. Find $g[f(x)]$.

TRY YOUR TURN 2

As Example 2 shows, it is not always true that $f[g(x)] = g[f(x)]$. In fact, it is rare to find two functions f and g such that $f[g(x)] = g[f(x)]$. The domain of both composite functions given in Example 2 is the set of all real numbers.

EXAMPLE 3 Composition of Functions

Write each function as the composition of two functions f and g so that $h(x) = f[g(x)]$.

(a) $h(x) = 2(4x + 1)^2 + 5(4x + 1)$

SOLUTION Let $f(x) = 2x^2 + 5x$ and $g(x) = 4x + 1$. Then $f[g(x)] = f(4x + 1) = 2(4x + 1)^2 + 5(4x + 1)$. Notice that $h(x)$ here is the same as $f[g(x)]$ in Example 2(a).

(b) $h(x) = \sqrt{1 - x^2}$

SOLUTION One way to do this is to let $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$. Another choice is to let $f(x) = \sqrt{1 - x}$ and $g(x) = x^2$. Verify that with either choice, $f[g(x)] = \sqrt{1 - x^2}$. For the purposes of this section, the first choice is better; it is useful to think of f as being the function on the outer layer and g as the function on the inner layer. With this function h, we see a square root on the outer layer, and when we peel that away we see $1 - x^2$ on the inside. TRY YOUR TURN 3

YOUR TURN 3 Write $h(x) = (2x - 3)^3$ as a composition of two functions f and g so that $h(x) = f[g(x)]$.

The Chain Rule

Suppose $f(x) = x^2$ and $g(x) = 5x^3 + 2$. What is the derivative of $h(x) = f[g(x)] = (5x^3 + 2)^2$? At first you might think the answer is just $h'(x) = 2(5x^3 + 2) = 10x^3 + 4$ by using the power rule. You can check this answer by multiplying out $h(x) = (5x^3 + 2)^2 = 25x^6 + 20x^3 + 4$. Now calculate $h'(x) = 150x^5 + 60x^2$. The guess using the power rule was clearly wrong! The error is that the power rule applies to x raised to a power, not to some other function of x raised to a power.

How, then, could we take the derivative of $p(x) = (5x^3 + 2)^{20}$? This seems far too difficult to multiply out. Fortunately, there is a way. Notice from the previous paragraph that $h'(x) = 150x^5 + 60x^2 = 2(5x^3 + 2)15x^2$. So the original guess was almost correct, except it was missing the factor of $15x^2$, which just happens to be $g'(x)$. This is not a coincidence. To see why the derivative of $f[g(x)]$ involves taking the derivative of f and then multiplying by the derivative of g, let us consider a realistic example, the question from the beginning of this section.

EXAMPLE 4 Area of an Oil Slick

A leaking oil well off the Gulf Coast is spreading a circular film of oil over the water surface. At any time t (in minutes) after the beginning of the leak, the radius of the circular oil slick (in feet) is given by

$$r(t) = 4t.$$

Find the rate of change of the area of the oil slick with respect to time.

APPLY IT

SOLUTION We first find the rate of change in the radius over time by finding dr/dt:

$$\frac{dr}{dt} = 4.$$

This value indicates that the radius is increasing by 4 ft each minute.

The area of the oil slick is given by

$$A(r) = \pi r^2, \quad \text{with} \quad \frac{dA}{dr} = 2\pi r.$$

The derivative, dA/dr, gives the rate of change in area per unit increase in the radius.

As these derivatives show, the radius is increasing at a rate of 4 ft/min, and for each foot that the radius increases, the area increases by $2\pi r$ ft^2. It seems reasonable, then, that the area is increasing at a rate of

$$2\pi r \text{ ft}^2/\text{ft} \times 4 \text{ ft/min} = 8\pi r \text{ ft}^2/\text{min}.$$

That is,

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot 4 = 8\pi r.$$

Notice that because area (A) is a function of radius (r), which is a function of time (t), area as a function of time is a composition of two functions, written $A(r(t))$. The last step, then, can also be written as

$$\frac{dA}{dt} = \frac{d}{dt}A[r(t)] = A'[r(t)] \cdot r'(t) = 2\pi r \cdot 4 = 8\pi r.$$

Finally, we can substitute $r(t) = 4t$ to get the derivative in terms of t:

$$\frac{dA}{dt} = 8\pi r = 8\pi(4t) = 32\pi t.$$

The rate of change of the area of the oil slick with respect to time is $32\pi t$ ft^2/min.

To check the result of Example 4, use the fact that $r = 4t$ and $A = \pi r^2$ to get the same result:

$$A = \pi(4t)^2 = 16\pi t^2, \qquad \text{with} \qquad \frac{dA}{dt} = 32\pi t.$$

The product used in Example 4,

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt},$$

is an example of the **chain rule**, which is used to find the derivative of a composite function.

Chain Rule

If y is a function of u, say $y = f(u)$, and if u is a function of x, say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}.$$

One way to remember the chain rule is to pretend that dy/du and du/dx are fractions, with du "canceling out." The proof of the chain rule requires advanced concepts and, therefore, is not given here.

EXAMPLE 5 Chain Rule

Find dy/dx if $y = (3x^2 - 5x)^{1/2}$.

SOLUTION Let $y = u^{1/2}$, and $u = 3x^2 - 5x$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2}u^{-1/2} \cdot (6x - 5).$$

YOUR TURN 4 Find dy/dx if $y = (5x^2 - 6x)^{-2}$.

Replacing u with $3x^2 - 5x$ gives

$$\frac{dy}{dx} = \frac{1}{2}(3x^2 - 5x)^{-1/2}(6x - 5) = \frac{6x - 5}{2(3x^2 - 5x)^{1/2}}.$$

TRY YOUR TURN 4

The following alternative version of the chain rule is stated in terms of composite functions.

Chain Rule (Alternate Form)
If $y = f[g(x)]$, then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

(To find the derivative of $f[g(x)]$, find the derivative of $f(x)$, replace each x with $g(x)$, and then multiply the result by the derivative of $g(x)$.)

In words, the chain rule tells us to first take the derivative of the outer function, then multiply it by the derivative of the inner function.

EXAMPLE 6 Chain Rule

Use the chain rule to find $D_x(x^2 + 5x)^8$.

SOLUTION As in Example 3(b), think of this as a function with layers. The outer layer is something being raised to the 8th power, so let $f(x) = x^8$. Once this layer is peeled away, we see that the inner layer is $x^2 + 5x$, so $g(x) = x^2 + 5x$. Then $(x^2 + 5x)^8 = f[g(x)]$ and

$$D_x(x^2 + 5x)^8 = f'[g(x)]g'(x).$$

Here $f'(x) = 8x^7$, with $f'[g(x)] = 8[g(x)]^7 = 8(x^2 + 5x)^7$ and $g'(x) = 2x + 5$.

$$D_x(x^2 + 5x)^8 = f'[g(x)]g'(x)$$
$$= 8[g(x)]^7 g'(x)$$
$$= 8(x^2 + 5x)^7(2x + 5)$$

YOUR TURN 5 Find $D_x(x^2 - 7)^{10}$.

TRY YOUR TURN 5

CAUTION

(a) A common error is to forget to multiply by $g'(x)$ when using the chain rule. Remember, the derivative must involve a "chain," or product, of derivatives.

(b) Another common mistake is to write the derivative as $f'[g'(x)]$. Remember to leave $g(x)$ unchanged in $f'[g(x)]$ and then to multiply by $g'(x)$.

One way to avoid both of the errors described above is to remember that the chain rule is a two-step process. In Example 6, the first step was taking the derivative of the power, and the second step was multiplying by $g'(x)$. Forgetting to multiply by $g'(x)$ would be an erroneous one-step process. The other erroneous one-step process is to take the derivative inside the power, getting $f'[g'(x)]$, or $8(2x + 5)^7$ in Example 6.

Sometimes both the chain rule and either the product or quotient rule are needed to find a derivative, as the next examples show.

EXAMPLE 7 Derivative Rules

Find the derivative of $y = 4x(3x + 5)^5$.

SOLUTION Write $4x(3x + 5)^5$ as the product

$$(4x) \cdot (3x + 5)^5.$$

To find the derivative of $(3x + 5)^5$, let $g(x) = 3x + 5$, with $g'(x) = 3$. Now use the product rule and the chain rule.

Derivative of $(3x + 5)^5$ Derivative of $4x$

$$\frac{dy}{dx} = 4x[5(3x + 5)^4 \cdot 3] + (3x + 5)^5(4)$$

$$= 60x(3x + 5)^4 + 4(3x + 5)^5 \qquad \text{Factor out the greatest common factor, } 4(3x + 5)^4.$$

$$= 4(3x + 5)^4[15x + (3x + 5)^1] \qquad \text{Simplify inside brackets.}$$

$$= 4(3x + 5)^4(18x + 5)$$

TRY YOUR TURN 6

YOUR TURN 6 Find the derivative of $y = x^2(5x - 1)^3$.

EXAMPLE 8 Derivative Rules

Find $D_x\left[\dfrac{(3x + 2)^7}{x - 1}\right]$.

SOLUTION Use the quotient rule and the chain rule.

$$D_x\left[\frac{(3x + 2)^7}{x - 1}\right] = \frac{(x - 1)[7(3x + 2)^6 \cdot 3] - (3x + 2)^7(1)}{(x - 1)^2}$$

$$= \frac{21(x - 1)(3x + 2)^6 - (3x + 2)^7}{(x - 1)^2}$$

$$= \frac{(3x + 2)^6[21(x - 1) - (3x + 2)]}{(x - 1)^2} \qquad \text{Factor out the greatest common factor, } (3x + 2)^6.$$

$$= \frac{(3x + 2)^6[21x - 21 - 3x - 2]}{(x - 1)^2} \qquad \text{Simplify inside brackets.}$$

$$= \frac{(3x + 2)^6(18x - 23)}{(x - 1)^2}$$

TRY YOUR TURN 7

YOUR TURN 7 Find $D_x\left[\dfrac{(4x - 1)^3}{x + 3}\right]$.

Some applications requiring the use of the chain rule are illustrated in the next two examples.

EXAMPLE 9 City Revenue

The revenue realized by a small city from the collection of fines from parking tickets is given by

$$R(n) = \frac{8000n}{n + 2},$$

where n is the number of work-hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work-hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work-hours per day. How fast is revenue from parking fines decreasing at the outbreak of the epidemic?

SOLUTION We want to find dR/dt, the change in revenue with respect to time. By the chain rule,

$$\frac{dR}{dt} = \frac{dR}{dn} \cdot \frac{dn}{dt}.$$

First find dR/dn, using the quotient rule, as follows.

$$\frac{dR}{dn} = \frac{(n+2)(8000) - 8000n(1)}{(n+2)^2} = \frac{16{,}000}{(n+2)^2}$$

Since 30 work-hours were used at the outbreak of the epidemic, $n = 30$, so $dR/dn = 16{,}000/(30+2)^2 = 15.625$. Also, $dn/dt = -6$. Thus,

$$\frac{dR}{dt} = \frac{dR}{dn} \cdot \frac{dn}{dt} = (15.625)(-6) = -93.75.$$

Revenue is being lost at the rate of about \$94 per day at the outbreak of the epidemic.

EXAMPLE 10 Compound Interest

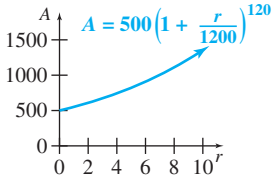


FIGURE 10

Suppose a sum of \$500 is deposited in an account with an interest rate of r percent per year compounded monthly. At the end of 10 years, the balance in the account (as illustrated in Figure 10) is given by

$$A = 500\left(1 + \frac{r}{1200}\right)^{120}.$$

Find the rate of change of A with respect to r if $r = 5$ or 7.*

SOLUTION First find dA/dr using the chain rule.

$$\frac{dA}{dr} = (120)(500)\left(1 + \frac{r}{1200}\right)^{119}\left(\frac{1}{1200}\right)$$

$$= 50\left(1 + \frac{r}{1200}\right)^{119}$$

If $r = 5$,

$$\frac{dA}{dr} = 50\left(1 + \frac{5}{1200}\right)^{119}$$

$$\approx 82.01,$$

or \$82.01 per percentage point. If $r = 7$,

$$\frac{dA}{dr} = 50\left(1 + \frac{7}{1200}\right)^{119}$$

$$\approx 99.90,$$

or \$99.90 per percentage point.

NOTE One lesson to learn from this section is that a derivative is always with respect to some variable. In the oil slick example, notice that the derivative of the area with respect to the radius is $2\pi r$, while the derivative of the area with respect to time is $8\pi r$. As another example, consider the velocity of a conductor walking at 2 mph on a train car. Her velocity with respect to the ground may be 50 mph, but the earth on which the train is running is moving about the sun at 1.6 million mph. The derivative of her position function might be 2, 50, or 1.6 million mph, depending on what variable it is with respect to.

*Notice that r is given here as an integer percent, rather than as a decimal, which is why the formula for compound interest has 1200 where you would expect to see 12. This leads to a simpler interpretation of the derivative.

12.3 EXERCISES

Let $f(x) = 5x^2 - 2x$ and $g(x) = 8x + 3$. Find the following.

1. $f[g(2)]$ **2.** $f[g(-5)]$ **3.** $g[f(2)]$

4. $g[f(-5)]$ **5.** $f[g(k)]$ **6.** $g[f(5z)]$

In Exercises 7–14, find $f[g(x)]$ and $g[f(x)]$.

7. $f(x) = \frac{x}{8} + 7; \quad g(x) = 6x - 1$

8. $f(x) = -8x + 9; \quad g(x) = \frac{x}{5} + 4$

9. $f(x) = \frac{1}{x}; \quad g(x) = x^2$

10. $f(x) = \frac{2}{x^4}; \quad g(x) = 2 - x$

11. $f(x) = \sqrt{x + 2}; \quad g(x) = 8x^2 - 6$

12. $f(x) = 9x^2 - 11x; \quad g(x) = 2\sqrt{x + 2}$

13. $f(x) = \sqrt{x + 1}; \quad g(x) = \frac{-1}{x}$

14. $f(x) = \frac{8}{x}; \quad g(x) = \sqrt{3 - x}$

Write each function as the composition of two functions. (There may be more than one way to do this.)

15. $y = (5 - x^2)^{3/5}$ **16.** $y = (3x^2 - 7)^{2/3}$

17. $y = -\sqrt{13 + 7x}$ **18.** $y = \sqrt{9 - 4x}$

19. $y = (x^2 + 5x)^{1/3} - 2(x^2 + 5x)^{2/3} + 7$

20. $y = (x^{1/2} - 3)^2 + (x^{1/2} - 3) + 5$

Find the derivative of each function defined as follows.

21. $y = (8x^4 - 5x^2 + 1)^4$ **22.** $y = (2x^3 + 9x)^5$

23. $k(x) = -2(12x^2 + 5)^{-6}$ **24.** $f(x) = -7(3x^4 + 2)^{-4}$

25. $s(t) = 45(3t^3 - 8)^{3/2}$ **26.** $s(t) = 12(2t^4 + 5)^{3/2}$

27. $g(t) = -3\sqrt{7t^3 - 1}$ **28.** $f(t) = 8\sqrt{4t^2 + 7}$

29. $m(t) = -6t(5t^4 - 1)^4$ **30.** $r(t) = 4t(2t^5 + 3)^4$

31. $y = (3x^4 + 1)^4(x^3 + 4)$ **32.** $y = (x^3 + 2)(x^2 - 1)^4$

33. $q(y) = 4y^2(y^2 + 1)^{5/4}$ **34.** $p(z) = z(6z + 1)^{4/3}$

35. $y = \frac{-5}{(2x^3 + 1)^2}$ **36.** $y = \frac{1}{(3x^2 - 4)^5}$

37. $r(t) = \frac{(5t - 6)^4}{3t^2 + 4}$ **38.** $p(t) = \frac{(2t + 3)^3}{4t^2 - 1}$

39. $y = \frac{3x^2 - x}{(2x - 1)^5}$ **40.** $y = \frac{x^2 + 4x}{(3x^3 + 2)^4}$

41. In your own words explain how to form the composition of two functions.

42. The generalized power rule says that if $g(x)$ is a function of x and $y = [g(x)]^n$ for any real number n, then

$$\frac{dy}{dx} = n \cdot [g(x)]^{n-1} \cdot g'(x).$$

Explain why the generalized power rule is a consequence of the chain rule and the power rule.

Consider the following table of values of the functions f and g and their derivatives at various points.

x	1	2	3	4
$f(x)$	2	4	1	3
$f'(x)$	-6	-7	-8	-9
$g(x)$	2	3	4	1
$g'(x)$	2/7	3/7	4/7	5/7

Find the following using the table above.

43. a. $D_x(f[g(x)])$ at $x = 1$ **b.** $D_x(f[g(x)])$ at $x = 2$

44. a. $D_x(g[f(x)])$ at $x = 1$ **b.** $D_x(g[f(x)])$ at $x = 2$

In Exercises 45–48, find the equation of the tangent line to the graph of the given function at the given value of x.

45. $f(x) = \sqrt{x^2 + 16}; \quad x = 3$

46. $f(x) = (x^3 + 7)^{2/3}; \quad x = 1$

47. $f(x) = x(x^2 - 4x + 5)^4; \quad x = 2$

48. $f(x) = x^2\sqrt{x^4 - 12}; \quad x = 2$

In Exercises 49 and 50, find all values of x for the given function where the tangent line is horizontal.

49. $f(x) = \sqrt{x^3 - 6x^2 + 9x + 1}$

50. $f(x) = \frac{x}{(x^2 + 4)^4}$

51. Katie and Sarah are working on taking the derivative of

$$f(x) = \frac{2x}{3x + 4}.$$

Katie uses the quotient rule to get

$$f'(x) = \frac{(3x + 4)2 - 2x(3)}{(3x + 4)^2} = \frac{8}{(3x + 4)^2}.$$

Sarah converts it into a product and uses the product rule and the chain rule:

$$f(x) = 2x(3x + 4)^{-1}$$
$$f'(x) = 2x(-1)(3x + 4)^{-2}(3) + 2(3x + 4)^{-1}$$
$$= 2(3x + 4)^{-1} - 6x(3x + 4)^{-2}.$$

Explain the discrepancies between the two answers. Which procedure do you think is preferable?

52. Margy and Nate are working on taking the derivative of

$$f(x) = \frac{2}{(3x + 1)^4}.$$

Margy uses the quotient rule and chain rule as follows:

$$f'(x) = \frac{(3x + 1)^4 \cdot 0 - 2 \cdot 4(3x + 1)^3 \cdot 3}{(3x + 1)^8}$$

$$= \frac{-24(3x + 1)^3}{(3x + 1)^8} = \frac{-24}{(3x + 1)^5}.$$

Nate rewrites the function and uses the power rule and chain rule as follows:

$$f(x) = 2(3x + 1)^{-4}$$

$$f'(x) = (-4)2(3x + 1)^{-5} \cdot 3 = \frac{-24}{(3x + 1)^5}.$$

Compare the two procedures. Which procedure do you think is preferable?

APPLICATIONS

Business and Economics

53. Demand Suppose the demand for a certain brand of vacuum cleaner is given by

$$D(p) = \frac{-p^2}{100} + 500,$$

where p is the price in dollars. If the price, in terms of the cost c, is expressed as

$$p(c) = 2c - 10,$$

find the demand in terms of the cost.

54. Revenue Assume that the total revenue (in dollars) from the sale of x television sets is given by

$$R(x) = 24(x^2 + x)^{2/3}.$$

Find the marginal revenue when the following numbers of sets are sold.

a. 100 **b.** 200 **c.** 300

d. Find the average revenue from the sale of x sets.

e. Find the marginal average revenue.

f. Write a paragraph covering the following questions. How does the revenue change over time? What does the marginal revenue function tell you about the revenue function? What does the average revenue function tell you about the revenue function?

55. Interest A sum of \$1500 is deposited in an account with an interest rate of r percent per year, compounded daily. At the end of 5 years, the balance in the account is given by

$$A = 1500\left(1 + \frac{r}{36{,}500}\right)^{1825}.$$

Find the rate of change of A with respect to r for the following interest rates.

a. 6% **b.** 8% **c.** 9%

56. Demand Suppose a demand function is given by

$$q = D(p) = 30\left(5 - \frac{p}{\sqrt{p^2 + 1}}\right),$$

where q is the demand for a product and p is the price per item in dollars. Find the rate of change in the demand for the product per unit change in price (i.e., find dq/dp).

57. Depreciation A certain truck depreciates according to the formula

$$V = \frac{60{,}000}{1 + 0.3t + 0.1t^2},$$

where V is the value of the truck (in dollars), t is time measured in years, and $t = 0$ represents the time of purchase (in years). Find the rate at which the value of the truck is changing at the following times.

a. 2 years **b.** 4 years

58. Cost Suppose the cost in dollars of manufacturing q items is given by

$$C = 2000q + 3500,$$

and the demand equation is given by

$$q = \sqrt{15{,}000 - 1.5p}.$$

In terms of the demand q,

a. find an expression for the revenue R;

b. find an expression for the profit P;

c. find an expression for the marginal profit.

d. Determine the value of the marginal profit when the price is \$5000.

Life Sciences

59. Fish Population Suppose the population P of a certain species of fish depends on the number x (in hundreds) of a smaller fish that serves as its food supply, so that

$$P(x) = 2x^2 + 1.$$

Suppose, also, that the number of the smaller species of fish depends on the amount a (in appropriate units) of its food supply, a kind of plankton. Specifically,

$$x = f(a) = 3a + 2.$$

A biologist wants to find the relationship between the population P of the large fish and the amount a of plankton available, that is, $P[f(a)]$. What is the relationship?

60. Oil Pollution An oil well off the Gulf Coast is leaking, with the leak spreading oil over the surface as a circle. At any time t (in minutes) after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = t^2$ feet. Let $A(r) = \pi r^2$ represent the area of a circle of radius r.

a. Find and interpret $A[r(t)]$.

b. Find and interpret $D_t A[r(t)]$ when $t = 100$.

61. Thermal Inversion When there is a thermal inversion layer over a city (as happens often in Los Angeles), pollutants cannot rise vertically but are trapped below the layer and must disperse horizontally. Assume that a factory smokestack begins emitting a pollutant at 8 A.M. Assume that the pollutant disperses horizontally, forming a circle. If t represents the time (in hours) since the factory began emitting pollutants ($t = 0$ represents 8 A.M.), assume that the radius of the circle of pollution is $r(t) = 2t$ miles. Let $A(r) = \pi r^2$ represent the area of a circle of radius r.

a. Find and interpret $A[r(t)]$.

b. Find and interpret $D_t A[r(t)]$ when $t = 4$.

62. Bacteria Population The total number of bacteria (in millions) present in a culture is given by

$$N(t) = 2t(5t + 9)^{1/2} + 12,$$

where t represents time (in hours) after the beginning of an experiment. Find the rate of change of the population of bacteria with respect to time for the following numbers of hours.

a. 0 **b.** 7/5 **c.** 8

63. Calcium Usage To test an individual's use of calcium, a researcher injects a small amount of radioactive calcium into the person's bloodstream. The calcium remaining in the bloodstream is measured each day for several days. Suppose the amount of the calcium remaining in the bloodstream (in milligrams per cubic centimeter) t days after the initial injection is approximated by

$$C(t) = \frac{1}{2}(2t + 1)^{-1/2}.$$

Find the rate of change of the calcium level with respect to time for the following numbers of days.

a. 0 **b.** 4 **c.** 7.5

d. Is C always increasing or always decreasing? How can you tell?

64. Drug Reaction The strength of a person's reaction to a certain drug is given by

$$R(Q) = Q\left(C - \frac{Q}{3}\right)^{1/2},$$

where Q represents the quantity of the drug given to the patient and C is a constant.

a. The derivative $R'(Q)$ is called the *sensitivity* to the drug. Find $R'(Q)$.

b. Find the sensitivity to the drug if $C = 59$ and a patient is given 87 units of the drug.

c. Is the patient's sensitivity to the drug increasing or decreasing when $Q = 87$?

General Interest

65. Candy The volume and surface area of a "jawbreaker" for any radius is given by the formulas

$$V(r) = \frac{4}{3}\pi r^3 \quad \text{and} \quad S(r) = 4\pi r^2,$$

respectively. Roger Guffey estimates the radius of a jawbreaker while in a person's mouth to be

$$r(t) = 6 - \frac{3}{17}t,$$

where $r(t)$ is in millimeters and t is in minutes. ***Source: Mathematics Teacher.***

a. What is the life expectancy of a jawbreaker?

b. Find dV/dt and dS/dt when $t = 17$ and interpret your answer.

c. Construct an analogous experiment using some other type of food or verify the results of this experiment.

66. Zenzizenzizenzic Zenzizenzizenzic is an obsolete word with the distinction of containing the most z's of any word found in the Oxford English Dictionary. It was used in mathematics, before powers were written as superscript numbers, to represent the square of the square of the square of a number. In symbols, zenzizenzizenzic is written as $((x^2)^2)^2$. ***Source: The Phrontistery.***

a. Use the chain rule twice to find the derivative.

b. Use the properties of exponents to first simplify the expression, and then find the derivative.

67. Zenzizenzicube Zenzizenzicube is another obsolete word (see Exercise 66) that represents the square of the square of a cube. In symbols, zenzizenzicube is written as $((x^3)^2)^2$. ***Source: The Phrontistery.***

a. Use the chain rule twice to find the derivative.

b. Use the properties of exponents to first simplify the expression, and then find the derivative.

YOUR TURN ANSWERS

1. $2\sqrt{5} - 1; \sqrt{2}$
2. $4x^2 - 12x + 10$
3. One possible answer is $g(x) = 2x - 3$ and $f(x) = x^3$
4. $\dfrac{dy}{dx} = \dfrac{-2(10x - 6)}{(5x^2 - 6x)^3}$
5. $20x(x^2 - 7)^9$
6. $\dfrac{dy}{dx} = x(5x - 1)^2(25x - 2)$
7. $\dfrac{(4x - 1)^2(8x + 37)}{(x + 3)^2}$

12.4 Derivatives of Exponential Functions

APPLY IT

Given a new product whose rate of growth is rapid at first and then slows, how can we find the rate of growth?

We will use a derivative to answer this question in Example 5 at the end of this section.

FOR REVIEW

Recall from Section 10.4 that e is a special irrational number whose value is approximately 2.718281828. It arises in many applications, such as continuously compounded interest, and it can be defined as

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.$$

We can find the derivative of the exponential function by using the definition of the derivative. Thus

$$\begin{aligned}\frac{d(e^x)}{dx} &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h - e^x}{h} && \text{Property 1 of exponents}\\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h}. && \text{Property 1 of limits}\end{aligned}$$

In the last step, since e^x does not involve h, we were able to bring e^x in front of the limit. The result says that the derivative of e^x is e^x times a constant, namely, $\lim_{h\to 0}(e^h-1)/h$. To investigate this limit, we evaluate the expression for smaller and smaller values of h, as shown in the table in the margin. Based on the table, it appears that $\lim_{h\to 0}(e^h-1)/h = 1$. This is proved in more advanced courses. We, therefore, have the following formula.

h	Approximation of $\lim_{h\to 0}\frac{e^h-1}{h}$: $\frac{e^h-1}{h}$
−0.1	0.9516
−0.01	0.9950
−0.001	0.9995
−0.0001	1.0000
0.00001	1.0000
0.0001	1.0001
0.001	1.0005
0.01	1.0050
0.1	1.0517

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

To find the derivative of the exponential function with a base other than e, use the change-of-base theorem for exponentials to rewrite a^x as $e^{(\ln a)x}$. Thus, for any positive constant $a \neq 1$,

$$\begin{aligned}\frac{d(a^x)}{dx} &= \frac{d[e^{(\ln a)x}]}{dx} && \text{Change-of-base theorem for exponentials}\\ &= e^{(\ln a)x}\ln a && \text{Chain rule}\\ &= (\ln a)a^x. && \text{Change-of-base theorem again}\end{aligned}$$

Derivative of a^x

For any positive constant $a \neq 1$,

$$\frac{d}{dx}(a^x) = (\ln a)a^x.$$

(The derivative of an exponential function is the original function times the natural logarithm of the base.)

We now see why e is the best base to work with: It has the simplest derivative of all the exponential functions. Even if we choose a different base, e appears in the derivative anyway through the $\ln a$ term. (Recall that $\ln a$ is the logarithm of a to the base e.) In fact,

of all the functions we have studied, e^x is the simplest to differentiate, because its derivative is just itself.*

The chain rule can be used to find the derivative of the more general exponential function $y = a^{g(x)}$. Let $y = f(u) = a^u$ and $u = g(x)$, so that $f[g(x)] = a^{g(x)}$. Then

$$f'[g(x)] = f'(u) = (\ln a)a^u = (\ln a)a^{g(x)},$$

and by the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= f'[g(x)] \cdot g'(x)\\ &= (\ln a)a^{g(x)} \cdot g'(x).\end{aligned}$$

As before, this formula becomes simpler when we use natural logarithms because $\ln e = 1$. We summarize these results next.

Derivative of $a^{g(x)}$ and $e^{g(x)}$

$$\frac{d}{dx}(a^{g(x)}) = (\ln a)a^{g(x)}g'(x)$$

and

$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)}g'(x)$$

You need not memorize the previous two formulas. They are simply the result of applying the chain rule to the formula for the derivative of a^x.

CAUTION Notice the difference between the derivative of a variable to a constant power, such as $D_x x^3 = 3x^2$, and a constant to a variable power, like $D_x 3^x = (\ln 3)3^x$. Remember, $D_x 3^x \neq x3^{x-1}$.

EXAMPLE 1 Derivatives of Exponential Functions

Find the derivative of each function.

(a) $y = e^{5x}$

SOLUTION Let $g(x) = 5x$, so $g'(x) = 5$. Then

$$\frac{dy}{dx} = 5e^{5x}.$$

(b) $s = 3^t$

SOLUTION

$$\frac{ds}{dt} = (\ln 3)3^t$$

(c) $y = 10e^{3x^2}$

SOLUTION

$$\frac{dy}{dx} = 10(e^{3x^2})(6x) = 60xe^{3x^2}$$

(d) $s = 8 \cdot 10^{1/t}$

SOLUTION

$$\begin{aligned}\frac{ds}{dt} &= 8(\ln 10)10^{1/t}\left(\frac{-1}{t^2}\right)\\ &= \frac{-8(\ln 10)10^{1/t}}{t^2}\end{aligned}$$

YOUR TURN 1 Find dy/dx for
(a) $y = 4^{3x}$,
(b) $y = e^{7x^3+5}$.

TRY YOUR TURN 1

*There is a joke about a deranged mathematician who frightened other inmates at an insane asylum by screaming at them, "I'm going to differentiate you!" But one inmate remained calm and simply responded, "I don't care; I'm e^x."

EXAMPLE 2 Derivative of an Exponential Function

Let $y = e^{x^2+1}\sqrt{5x+2}$. Find $\dfrac{dy}{dx}$.

SOLUTION Rewrite y as $y = e^{x^2+1}(5x+2)^{1/2}$, and then use the product rule and the chain rule.

$$\frac{dy}{dx} = e^{x^2+1}\cdot\frac{1}{2}(5x+2)^{-1/2}\cdot 5 + (5x+2)^{1/2}e^{x^2+1}\cdot 2x$$

$$= e^{x^2+1}(5x+2)^{-1/2}\left[\frac{5}{2} + (5x+2)\cdot 2x\right]$$

Factor out the greatest common factor, $e^{x^2+1}(5x+2)^{-1/2}$.

$$= e^{x^2+1}(5x+2)^{-1/2}\left[\frac{5+4x(5x+2)}{2}\right]$$

Least common denominator

$$= e^{x^2+1}(5x+2)^{-1/2}\left[\frac{5+20x^2+8x}{2}\right]$$

$$= \frac{e^{x^2+1}(20x^2+8x+5)}{2\sqrt{5x+2}}$$

Simplify.

YOUR TURN 2 Let $y = (x^2+1)^2e^{2x}$. Find dy/dx.

TRY YOUR TURN 2

EXAMPLE 3 Derivative of an Exponential Function

Let $f(x) = \dfrac{100{,}000}{1+100e^{-0.3x}}$. Find $f'(x)$.

SOLUTION Use the quotient rule.

$$f'(x) = \frac{(1+100e^{-0.3x})(0) - 100{,}000(-30e^{-0.3x})}{(1+100e^{-0.3x})^2}$$

$$= \frac{3{,}000{,}000e^{-0.3x}}{(1+100e^{-0.3x})^2}$$

YOUR TURN 3 Let $f(x) = \dfrac{100}{5+2e^{-0.01x}}$. Find $f'(x)$.

TRY YOUR TURN 3

In the previous example, we could also have taken the derivative by writing $f(x) = 100{,}000(1+100e^{-0.3x})^{-1}$, from which we have

$$f'(x) = -100{,}000(1+100e^{-0.3x})^{-2}100e^{-0.3x}(-0.3).$$

This simplifies to the same expression as in Example 3.

EXAMPLE 4 Radioactivity

The amount in grams in a sample of uranium 239 after t years is given by

$$A(t) = 100e^{-0.362t}.$$

Find the rate of change of the amount present after 3 years.

SOLUTION The rate of change is given by the derivative dA/dt.

$$\frac{dA}{dt} = 100(e^{-0.362t})(-0.362) = -36.2e^{-0.362t}$$

After 3 years ($t = 3$), the rate of change is

$$\frac{dA}{dt} = -36.2e^{-0.362(3)} = -36.2e^{-1.086} \approx -12.2$$

grams per year.

YOUR TURN 4 The quantity (in grams) of a radioactive substance present after t years is $Q(t) = 100e^{-0.421t}$. Find the rate of change of the quantity present after 2 years.

TRY YOUR TURN 4

Frequently a population, or the sales of a certain product, will start growing slowly, then grow more rapidly, and then gradually level off. Such growth can often be approximated by a mathematical model known as the **logistic function**:

$$G(t) = \frac{mG_0}{G_0 + (m - G_0)e^{-kmt}},$$

where t represents time in appropriate units, G_0 is the initial number present, m is the maximum possible size of the population, k is a positive constant, and $G(t)$ is the population at time t. It is sometimes simpler to divide the numerator and denominator of the logistic function by G_0, writing the result as

$$G(t) = \frac{m}{1 + \left(\frac{m}{G_0} - 1\right)e^{-kmt}}.$$

Notice that

$$\lim_{t\to\infty} G(t) = \frac{m}{1 + 0} = m$$

because $\lim_{t\to\infty} e^{-kmt} = 0$.

EXAMPLE 5 Product Sales

A company sells 990 units of a new product in the first year and 3213 units in the fourth year. They expect that sales can be approximated by a logistic function, leveling off at around 100,000 in the long run.

(a) Find a formula $S(t)$ for the sales as a function of time.

SOLUTION We already know that $S_0 = 990$ and $m = 100{,}000$, so

$$S(t) = \frac{100{,}000}{1 + \left(\frac{100{,}000}{990} - 1\right)e^{-k100{,}000t}}$$

$$= \frac{100{,}000}{1 + 100.01e^{-k100{,}000t}}.$$

To find k, use the fact that $S(4) = 3213$.

$$3213 = \frac{100{,}000}{1 + 100.01e^{-k100{,}000 \cdot 4}}$$

$$3213 = \frac{100{,}000}{1 + 100.01e^{-k400{,}000}}$$

$$3213(1 + 100.01e^{-k400{,}000}) = 100{,}000 \qquad \text{Cross multiply.}$$

$$3213 + 321{,}332e^{-k400{,}000} = 100{,}000$$

$$321{,}332e^{-k400{,}000} = 96{,}787 \qquad \text{Subtract 3213 from both sides.}$$

$$e^{-k400{,}000} = 0.3012 \qquad \text{Divide both sides by 321,332.}$$

$$-k400{,}000 = \ln 0.3012 \qquad \text{Take the natural logarithm of both sides.}$$

$$k = -\ln 0.3012/400{,}000$$

$$k \approx 3 \times 10^{-6}$$

Rounding 100.01 to 100 and simplifying $k100{,}000 = (3 \times 10^{-6})100{,}000 = 0.3$,

$$S(t) = \frac{100{,}000}{1 + 100e^{-k100{,}000t}}$$

$$= \frac{100{,}000}{1 + 100e^{-0.3t}}.$$

(b) Find the rate of change of sales after 4 years.

APPLY IT

SOLUTION The derivative of this sales function, which gives the rate of change of sales, was found in Example 3. Using that derivative,

$$S'(4) = \frac{3{,}000{,}000e^{-0.3(4)}}{[1 + 100e^{-0.3(4)}]^2} = \frac{3{,}000{,}000e^{-1.2}}{(1 + 100e^{-1.2})^2}.$$

Using a calculator, $e^{-1.2} \approx 0.3012$, and

$$\begin{aligned} S'(4) &\approx \frac{3{,}000{,}000(0.3012)}{[1 + 100(0.3012)]^2} \\ &\approx \frac{903{,}600}{(1 + 30.12)^2} \\ &\approx \frac{903{,}600}{968.5} \approx 933. \end{aligned}$$

The rate of change of sales after 4 years is about 933 units per year. The positive number indicates that sales are increasing at this time.

t	$S(t)$
0	990
5	4300
10	17,000
15	47,000
20	80,000
30	99,000

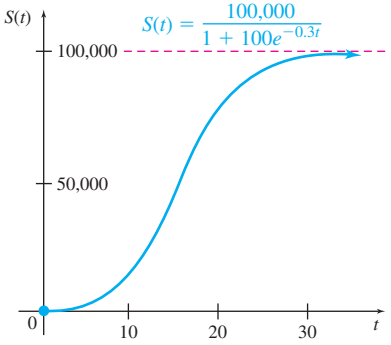


FIGURE 11

The graph of the function in Example 5 is shown in Figure 11.

12.4 EXERCISES

Find derivatives of the functions defined as follows.

1. $y = e^{4x}$

2. $y = e^{-2x}$

3. $y = -8e^{3x}$

4. $y = 1.2e^{5x}$

5. $y = -16e^{2x+1}$

6. $y = -4e^{-0.3x}$

7. $y = e^{x^2}$

8. $y = e^{-x^2}$

9. $y = 3e^{2x^2}$

10. $y = -5e^{4x^3}$

11. $y = 4e^{2x^2-4}$

12. $y = -3e^{3x^2+5}$

13. $y = xe^x$

14. $y = x^2e^{-2x}$

15. $y = (x + 3)^2e^{4x}$

16. $y = (3x^3 - 4x)e^{-5x}$

17. $y = \dfrac{x^2}{e^x}$

18. $y = \dfrac{e^x}{2x + 1}$

19. $y = \dfrac{e^x + e^{-x}}{x}$

20. $y = \dfrac{e^x - e^{-x}}{x}$

21. $p = \dfrac{10{,}000}{9 + 4e^{-0.2t}}$

22. $p = \dfrac{500}{12 + 5e^{-0.5t}}$

23. $f(z) = (2z + e^{-z^2})^2$

24. $f(t) = (e^{t^2} + 5t)^3$

25. $y = 7^{3x+1}$

26. $y = 4^{-5x+2}$

27. $y = 3 \cdot 4^{x^2+2}$

28. $y = -10^{3x^2-4}$

29. $s = 2 \cdot 3^{\sqrt{t}}$

30. $s = 5 \cdot 2^{\sqrt{t-2}}$

31. $y = \dfrac{te^t + 2}{e^{2t} + 1}$

32. $y = \dfrac{t^2e^{2t}}{t + e^{3t}}$

33. $f(x) = e^{x\sqrt{3x+2}}$

34. $f(x) = e^{x^2/(x^3+2)}$

35. Prove that if $y = y_0e^{kt}$, where y_0 and k are constants, then $dy/dt = ky$. (This says that for exponential growth and decay, the rate of change of the population is proportional to the size of the population, and the constant of proportionality is the growth or decay constant.)

36. Use a graphing calculator to sketch the graph of $y = [f(x + h) - f(x)]/h$ using $f(x) = e^x$ and $h = 0.0001$. Compare it with the graph of $y = e^x$ and discuss what you observe.

37. Use graphical differentiation to verify that $\dfrac{d}{dx}(e^x) = e^x$.

APPLICATIONS

Business and Economics

38. Sales The sales of a new personal computer (in thousands) are given by

$$S(t) = 100 - 90e^{-0.3t},$$

where t represents time in years. Find the rate of change of sales at each time.

a. After 1 year **b.** After 5 years

c. What is happening to the rate of change of sales as time goes on?

d. Does the rate of change of sales ever equal zero?

39. Cost The cost in dollars to produce x DVDs can be approximated by

$$C(x) = \sqrt{900 - 800 \cdot 1.1^{-x}}.$$

Find the marginal cost when the following quantities are made.

a. 0 **b.** 20

c. What happens to the marginal cost as the number produced becomes larger and larger?

40. Product Awareness After the introduction of a new product for tanning without sun exposure, the percent of the public that is aware of the product is approximated by

$$A(t) = 10t^2 2^{-t},$$

where t is the time in months. Find the rate of change of the percent of the public that is aware of the product after the following numbers of months.

a. 2 **b.** 4

c. Notice that the answer to part a is positive and the answer to part b is negative. What does this tell you about how public awareness of the product has changed?

41. Product Durability Using data in a car magazine, we constructed the mathematical model

$$y = 100e^{-0.03045t}$$

for the percent of cars of a certain type still on the road after t years. Find the percent of cars on the road after the following numbers of years.

a. 0 **b.** 2 **c.** 4 **d.** 6

Find the rate of change of the percent of cars still on the road after the following numbers of years.

e. 0 **f.** 2

g. Interpret your answers to parts e and f.

42. Investment The value of a particular investment changes over time according to the function

$$S(t) = 5000e^{0.1(e^{0.25t})},$$

where $S(t)$ is the value after t years. Calculate the rate at which the value of the investment is changing after 8 years. (Choose one of the following.) ***Source: Society of Actuaries***.

a. 618 **b.** 1934 **c.** 2011 **d.** 7735 **e.** 10,468

43. Internet Users The growth in the number (in millions) of Internet users in the United States between 1990 and 2015 can be approximated by a logistic function with $k = 0.0018$, where t is the number of years since 1990. In 1990 (when $t = 0$), the number of users was about 2 million, and the number is expected to level out around 250 million. ***Source: World Bank***.

a. Find the growth function $G(t)$ for the number of Internet users in the United States.

Estimate the number of Internet users in the United States and the rate of growth for the following years.

b. 1995

c. 2000

d. 2010

e. What happens to the rate of growth over time?

Life Sciences

44. Population Growth In Section 10.4, Exercise 47, the growth in world population (in millions) was approximated by the exponential function

$$A(t) = 3100e^{0.0166t},$$

where t is the number of years since 1960. Find the instantaneous rate of change in the world population at the following times. ***Source: United Nations***.

a. 2010 **b.** 2015

45. Minority Population In Section 10.4, Exercise 49, we saw that the projected Hispanic population in the United States (in millions) can be approximated by the function

$$h(t) = 37.79(1.021)^t$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. ***Source: U.S. Census Bureau***.

a. Estimate the Hispanic population in the United States for the year 2015.

b. What is the instantaneous rate of change of the Hispanic population in the United States when $t = 15$? Interpret your answer.

46. Insect Growth The growth of a population of rare South American beetles is given by the logistic function with $k = 0.00001$ and t in months. Assume that there are 200 beetles initially and that the maximum population size is 10,000.

a. Find the growth function $G(t)$ for these beetles.

Find the population and rate of growth of the population after the following times.

b. 6 months **c.** 3 years **d.** 7 years

e. What happens to the rate of growth over time?

47. Clam Population The population of a bed of clams in the Great South Bay off Long Island is described by the logistic function with $k = 0.0001$ and t in years. Assume that there are 400 clams initially and that the maximum population size is 5200.

a. Find the growth function $G(t)$ for the clams.

Find the population and rate of growth of the population after the following times.

b. 1 year **c.** 4 years **d.** 10 years

e. What happens to the rate of growth over time?

48. Pollution Concentration The concentration of pollutants (in grams per liter) in the east fork of the Big Weasel River is approximated by

$$P(x) = 0.04e^{-4x},$$

where x is the number of miles downstream from a paper mill that the measurement is taken. Find the following values.

a. The concentration of pollutants 0.5 mile downstream

b. The concentration of pollutants 1 mile downstream

c. The concentration of pollutants 2 miles downstream

Find the rate of change of concentration with respect to distance for the following distances.

d. 0.5 mile **e.** 1 mile **f.** 2 miles

49. Breast Cancer It has been observed that the following formula accurately models the relationship between the size of a breast tumor and the amount of time that it has been growing.

$$V(t) = 1100[1023e^{-0.02415t} + 1]^{-4},$$

where t is in months and $V(t)$ is measured in cubic centimeters. ***Source: Cancer.***

a. Find the tumor volume at 240 months.

b. Assuming that the shape of a tumor is spherical, find the radius of the tumor from part a. (*Hint:* The volume of a sphere is given by the formula $V = (4/3)\pi r^3$.)

c. If a tumor of size 0.5 cm^3 is detected, according to the formula, how long has it been growing? What does this imply?

d. Find $\lim_{t\to\infty} V(t)$ and interpret this value. Explain whether this makes sense.

e. Calculate the rate of change of tumor volume at 240 months and interpret.

50. Mortality The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t},$$

where t is the age of the person in years. ***Source: U.S. Vital Statistics.***

a. Find $P(25)$, $P(50)$, and $P(75)$.

b. Find $P'(25)$, $P'(50)$, and $P'(75)$.

c. Interpret your answers for parts a and b. Are there any limitations of this formula?

51. Medical Literature It has been observed that there has been an increase in the proportion of medical research papers that use the word "novel" in the title or abstract, and that this proportion can be accurately modeled by the function

$$p(x) = 0.001131e^{0.1268x},$$

where x is the number of years since 1970. ***Source: Nature.***

a. Find $p(40)$.

b. If this phenomenon continues, estimate the year in which every medical article will contain the word "novel" in its title or abstract.

c. Estimate the rate of increase in the proportion of medical papers using this word in the year 2010.

d. Explain some factors that may be contributing to researchers using this word.

52. Arctic Foxes The age/weight relationship of female Arctic foxes caught in Svalbard, Norway, can be estimated by the function

$$M(t) = 3102e^{-e^{-0.022(t-56)}},$$

where t is the age of the fox in days and $M(t)$ is the weight of the fox in grams. ***Source: Journal of Mammalogy.***

a. Estimate the weight of a female fox that is 200 days old.

b. Use $M(t)$ to estimate the largest size that a female fox can attain. (*Hint:* Find $\lim_{t\to\infty} M(t)$.)

c. Estimate the age of a female fox when it has reached 80% of its maximum weight.

d. Estimate the rate of change in weight of an Arctic fox that is 200 days old. (*Hint:* Recall that $D_t[e^{f(t)}] = f'(t)e^{f(t)}$.)

e. Use a graphing calculator to graph $M(t)$ and then describe the growth pattern.

f. Use the table function on a graphing calculator or a spreadsheet to develop a chart that shows the estimated weight and growth rate of female foxes for days 50, 100, 150, 200, 250, and 300.

53. Beef Cattle Researchers have compared two models that are used to predict the weight of beef cattle of various ages,

$$W_1(t) = 509.7(1 - 0.941e^{-0.00181t})$$

and

$$W_2(t) = 498.4(1 - 0.889e^{-0.00219t})^{1.25},$$

where $W_1(t)$ and $W_2(t)$ represent the weight (in kilograms) of a t-day-old beef cow. ***Source: Journal of Animal Science.***

a. What is the maximum weight predicted by each function for the average beef cow? Is this difference significant?

b. According to each function, find the age that the average beef cow reaches 90% of its maximum weight.

c. Find $W_1'(750)$ and $W_2'(750)$. Compare your results.

d. Graph the two functions on $[0, 2500]$ by $[0, 525]$ and comment on the differences in growth patterns for each of these functions.

e. Graph the derivative of these two functions on $[0, 2500]$ by $[0, 1]$ and comment on any differences you notice between these functions.

54. Cholesterol Researchers have found that the risk of coronary heart disease rises as blood cholesterol increases. This risk may be approximated by the function

$$R(c) = 3.19(1.006)^c, \quad 100 \le c \le 300,$$

where R is the risk in terms of coronary heart disease incidence per 1000 per year, and c is the cholesterol in mg/dL. Suppose a person's cholesterol is 180 mg/dL and going up at a rate of 15 mg/dL per year. At what rate is the person's risk of coronary heart disease going up? ***Source: Circulation***.

Social Sciences

55. Survival of Manuscripts Paleontologist John Cisne has demonstrated that the survival of ancient manuscripts can be modeled by the logistic equation. For example, the number of copies of the Venerable Bede's *De Temporum Ratione* was found to approach a limiting value over the five centuries after its publication in the year 725. Let $G(t)$ represent the proportion of manuscripts known to exist after t centuries out of the limiting value, so that $m = 1$. Cisne found that for Venerable Bede's *De Temporum Ratione,* $k = 3.5$ and $G_0 = 0.00369$. ***Source: Science.***

a. Find the growth function $G(t)$ for the proportion of copies of *De Temporum Ratione* found.

Find the proportion of manuscripts and their rate of growth after the following number of centuries.

b. 1 **c.** 2 **d.** 3

e. What happens to the rate of growth over time?

56. Habit Strength According to work by the psychologist C. L. Hull, the strength of a habit is a function of the number of times the habit is repeated. If N is the number of repetitions and $H(N)$ is the strength of the habit, then

$$H(N) = 1000(1 - e^{-kN}),$$

where k is a constant. Find $H'(N)$ if $k = 0.1$ and the number of times the habit is repeated is as follows.

a. 10 **b.** 100 **c.** 1000

d. Show that $H'(N)$ is always positive. What does this mean?

57. Online Learning The growth of the number of students taking at least one online course can be approximated by a logistic function with $k = 0.0440$, where t is the number of years since 2002. In 2002 (when $t = 0$), the number of students enrolled in at least one online course was 1.603 million. Assume that the number will level out at around 6.8 million students. ***Source: The Sloan Consortium***.

a. Find the growth function $G(t)$ for students enrolled in at least one online course.

Find the number of students enrolled in at least one online course and the rate of growth in the number in the following years.

b. 2004 **c.** 2006 **d.** 2010

e. What happens to the rate of growth over time?

Physical Sciences

58. Radioactive Decay The amount (in grams) of a sample of lead 214 present after t years is given by

$$A(t) = 500e^{-0.31t}.$$

Find the rate of change of the quantity present after each of the following years.

a. 4 **b.** 6 **c.** 10

d. What is happening to the rate of change of the amount present as the number of years increases?

e. Will the substance ever be gone completely?

59. Electricity In a series resistance-capacitance DC circuit, the instantaneous charge Q on the capacitor as a function of time (where $t = 0$ is the moment the circuit is energized by closing a switch) is given by the equation

$$Q(t) = CV(1 - e^{-t/RC}),$$

where C, V, and R are constants. Further, the instantaneous charging current I_C is the rate of change of charge on the capacitor, or $I_C = dQ/dt$. ***Source: Kevin Friedrich***.

a. Find the expression for I_C as a function of time.

b. If $C = 10^{-5}$ farads, $R = 10^7$ ohms, and $V = 10$ volts, what is the charging current after 200 seconds? (*Hint:* When placed into the function in part a the units can be combined into amps.)

60. Heat Index The heat index is a measure of how hot it really feels under different combinations of temperature and humidity. The heat index, in degrees Fahrenheit, can be approximated by

$$H(T) = T - 0.9971e^{0.02086T}[1 - e^{0.0445(D-57.2)}],$$

where the temperature T and dewpoint D are both expressed in degrees Fahrenheit. ***Source: American Meteorological Society***.

a. Assume the dewpoint is $D = 85°$ F. Find the function $H(T)$.

b. Using the function you found in part a, find the heat index when the temperature T is 80° F.

c. Find the rate of change of the heat index when $T = 80°$ F.

General Interest

61. Track and Field In 1958, L. Lucy developed a method for predicting the world record for any given year that a human could run a distance of 1 mile. His formula is given as follows:

$$t(n) = 218 + 31(0.933)^n,$$

where $t(n)$ is the world record (in seconds) for the mile run in year $1950 + n$. Thus, $n = 5$ corresponds to the year 1955. ***Source: Statistics in Sports***.

a. Find the estimate for the world record in the year 2010.

b. Calculate the instantaneous rate of change for the world record at the end of year 2010 and interpret.

c. Find $\lim_{n\to\infty} t(n)$ and interpret. How does this compare with the current world record?

62. Ballooning Suppose a person is going up in a hot air balloon. The surrounding air temperature in degrees Fahrenheit decreases with height according to the formula

$$T(h) = 80e^{-0.000065h},$$

where h is the height in feet. How fast is the temperature decreasing when the person is at a height of 1000 ft and rising at a height of 800 ft/hr?

63. The Gateway Arch The Gateway Arch in St. Louis, Missouri, is approximately 630 ft wide and 630 ft high. At first glance, the arch resembles a parabola, but its shape is actually known as a modified catenary. The height s (in feet) of the arch, measured from the ground to the middle of the arch, is approximated by the function

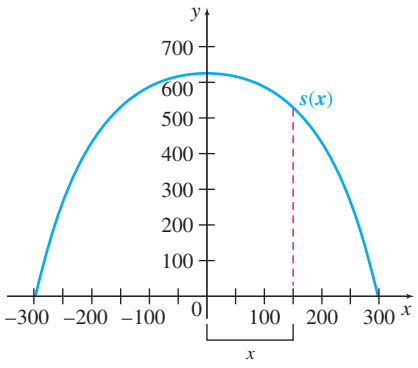
$$s(x) = 693.9 - 34.38(e^{0.01003x} + e^{-0.01003x}),$$

where $x = 0$ represents the center of the arch with $-299.2 \le x \le 299.2$. ***Source: Notices of the AMS.***

a. What is the height of the arch when $x = 0$?

b. What is the slope of the line tangent to the curve when $x = 0$? Does this make sense?

c. Find the rate of change of the height of the arc when $x = 150$ ft.



YOUR TURN ANSWERS

1. (a) $\dfrac{dy}{dx} = 3(\ln 4)4^{3x}$ **(b)** $\dfrac{dy}{dx} = 21x^2e^{7x^3+5}$

2. $\dfrac{dy}{dx} = 2e^{2x}(x^2 + 1)(x + 1)^2$

3. $f'(x) = \dfrac{2e^{-0.01x}}{(5 + 2e^{-0.01x})^2}$ **4.** -18.1 grams per year

12.5 Derivatives of Logarithmic Functions

APPLY IT **How does the average resale value of an automobile change with the age of the automobile?**

We will use the derivative to answer this question in Example 4.

Recall that in the section on Logarithmic Functions, we showed that the logarithmic function and the exponential function are inverses of each other. In the last section we showed that the derivative of a^x is $(\ln a)a^x$. We can use this information and the chain rule to find the derivative of $\log_a x$. We begin by solving the general logarithmic function for x.

$$f(x) = \log_a x$$
$$a^{f(x)} = x \qquad \text{Definition of the logarithm}$$

Now consider the left and right sides of the last equation as functions of x that are equal, so their derivatives with respect to x will also be equal. Notice in the first step that we need to use the chain rule when differentiating $a^{f(x)}$.

$$(\ln a)a^{f(x)}f'(x) = 1 \qquad \text{Derivative of the exponential function}$$
$$(\ln a)xf'(x) = 1 \qquad \text{Substitute } a^{f(x)} = x.$$

Finally, divide both sides of this equation by $(\ln a)x$ to get

$$f'(x) = \frac{1}{(\ln a)x}.$$

Derivative of $\log_a x$

$$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$$

(The derivative of a logarithmic function is the reciprocal of the product of the variable and the natural logarithm of the base.)

As with the exponential function, this formula becomes particularly simple when we let $a = e$, because of the fact that $\ln e = 1$.

Derivative of $\ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

This fact can be further justified geometrically. Notice what happens to the slope of the line $y = 2x + 4$ if the x-axis and y-axis are switched. That is, if we replace x with y and y with x, then the resulting line $x = 2y + 4$ or $y = x/2 - 2$ is a reflection of the line $y = 2x + 4$ across the line $y = x$, as seen in Figure 12. Furthermore, the slope of the new line is the reciprocal of the original line. In fact, the reciprocal property holds for all lines.

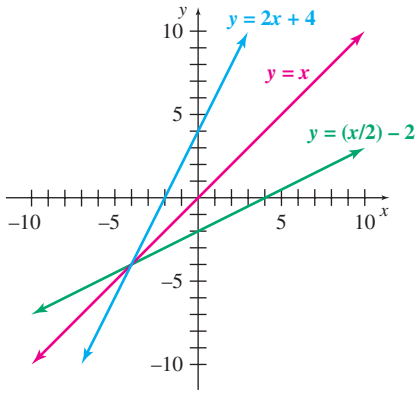


FIGURE 12

In the section on Logarithmic Functions, we showed that switching the x and y variables changes the exponential graph into a logarithmic graph, a defining property of functions that are inverses of each other. We also showed in the previous section that the slope of the tangent line of e^x at any point is e^x—that is, the y-coordinate itself. So, if we switch the x and y variables, the new slope of the tangent line will be $1/y$, except that it is no longer y, it is x. Thus, the slope of the tangent line of $y = \ln x$ must be $1/x$ and hence $D_x \ln x = 1/x$.

EXAMPLE 1 Derivatives of Logarithmic Functions

Find the derivative of each function.

(a) $f(x) = \ln 6x$

SOLUTION Use the properties of logarithms and the rules for derivatives.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\ln 6x) \\ &= \frac{d}{dx}(\ln 6 + \ln x) \\ &= \frac{d}{dx}(\ln 6) + \frac{d}{dx}(\ln x) = 0 + \frac{1}{x} = \frac{1}{x} \end{aligned}$$

(b) $y = \log x$

SOLUTION Recall that when the base is not specified, we assume that the logarithm is a common logarithm, which has a base of 10.

$$\frac{dy}{dx} = \frac{1}{(\ln 10)x}$$

YOUR TURN 1 Find the derivative of $f(x) = \log_3 x$.

TRY YOUR TURN 1

Applying the chain rule to the formulas for the derivative of logarithmic functions gives us

$$\frac{d}{dx}\log_a g(x) = \frac{1}{\ln a} \cdot \frac{g'(x)}{g(x)}$$

and

$$\frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}.$$

EXAMPLE 2 Derivatives of Logarithmic Functions

Find the derivative of each function.

(a) $f(x) = \ln(x^2 + 1)$

SOLUTION Here $g(x) = x^2 + 1$ and $g'(x) = 2x$. Thus,

$$f'(x) = \frac{g'(x)}{g(x)} = \frac{2x}{x^2 + 1}.$$

(b) $y = \log_2(3x^2 - 4x)$

SOLUTION

$$\frac{dy}{dx} = \frac{1}{\ln 2} \cdot \frac{6x - 4}{3x^2 - 4x}$$

$$= \frac{6x - 4}{(\ln 2)(3x^2 - 4x)}$$

YOUR TURN 2 Find the derivative of
(a) $y = \ln(2x^3 - 3)$,
(b) $f(x) = \log_4(5x + 3x^3)$.

TRY YOUR TURN 2

If $y = \ln(-x)$, where $x < 0$, the chain rule with $g(x) = -x$ and $g'(x) = -1$ gives

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{-1}{-x} = \frac{1}{x}.$$

The derivative of $y = \ln(-x)$ is the same as the derivative of $y = \ln x$. For this reason, these two results can be combined into one rule using the absolute value of x. A similar situation holds true for $y = \ln[g(x)]$ and $y = \ln[-g(x)]$, as well as for $y = \log_a[g(x)]$ and $y = \log_a[-g(x)]$. These results are summarized as follows.

Derivative of $\log_a|x|$, $\log_a|g(x)|$, $\ln|x|$, and $\ln|g(x)|$

$$\frac{d}{dx}[\log_a|x|] = \frac{1}{(\ln a)x} \qquad \frac{d}{dx}[\log_a|g(x)|] = \frac{1}{\ln a} \cdot \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \qquad \frac{d}{dx}[\ln|g(x)|] = \frac{g'(x)}{g(x)}$$

NOTE You need not memorize the previous four formulas. They are simply the result of the chain rule applied to the formula for the derivative of $y = \log_a x$, as well as the fact that when $\log_a x = \ln x$, so that $a = e$, then $\ln a = \ln e = 1$. An absolute value inside of a logarithm has no effect on the derivative, other than making the result valid for more values of x.

EXAMPLE 3 Derivatives of Logarithmic Functions

Find the derivative of each function.

(a) $y = \ln|5x|$

SOLUTION Let $g(x) = 5x$, so that $g'(x) = 5$. From the previous formula,

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{5}{5x} = \frac{1}{x}.$$

Notice that the derivative of $\ln|5x|$ is the same as the derivative of $\ln|x|$. Also notice that we would have found the exact same answer for the derivative of $y = \ln 5x$ (without the absolute value), but the result would not apply to negative values of x. Also, in Example 1, the derivative of $\ln 6x$ was the same as that for $\ln x$. This suggests that for any constant a,

$$\begin{aligned}\frac{d}{dx}\ln|ax| &= \frac{d}{dx}\ln|x| \\ &= \frac{1}{x}.\end{aligned}$$

Exercise 46 asks for a proof of this result.

(b) $f(x) = 3x \ln x^2$

SOLUTION This function is the product of the two functions $3x$ and $\ln x^2$, so use the product rule.

$$\begin{aligned}f'(x) &= (3x)\left[\frac{d}{dx}\ln x^2\right] + (\ln x^2)\left[\frac{d}{dx}3x\right] \\ &= 3x\left(\frac{2x}{x^2}\right) + (\ln x^2)(3) \\ &= 6 + 3\ln x^2\end{aligned}$$

By the power rule for logarithms,

$$\begin{aligned}f'(x) &= 6 + \ln(x^2)^3 \\ &= 6 + \ln x^6.\end{aligned}$$

Alternatively, write the answer as $f'(x) = 6 + 6\ln x$, except that this last form requires $x > 0$, while negative values of x are acceptable in $6 + \ln x^6$.

Another method would be to use a rule of logarithms to simplify the function to $f(x) = 3x \cdot 2\ln x = 6x \ln x$ and then to take the derivative.

(c) $s(t) = \dfrac{\log_8(t^{3/2} + 1)}{t}$

SOLUTION Use the quotient rule and the chain rule.

$$s'(t) = \frac{t \cdot \dfrac{1}{(t^{3/2}+1)\ln 8} \cdot \dfrac{3}{2}t^{1/2} - \log_8(t^{3/2}+1)\cdot 1}{t^2}$$

This expression can be simplified slightly by multiplying the numerator and the denominator by $2(t^{3/2} + 1)\ln 8$.

$$\begin{aligned}s'(t) &= \frac{t \cdot \dfrac{1}{(t^{3/2}+1)\ln 8} \cdot \dfrac{3}{2}t^{1/2} - \log_8(t^{3/2}+1)}{t^2} \cdot \frac{2(t^{3/2}+1)\ln 8}{2(t^{3/2}+1)\ln 8} \\ &= \frac{3t^{3/2} - 2(t^{3/2}+1)(\ln 8)\log_8(t^{3/2}+1)}{2t^2(t^{3/2}+1)\ln 8}\end{aligned}$$

YOUR TURN 3 Find the derivative of each function.

(a) $y = \ln|2x + 6|$

(b) $f(x) = x^2 \ln 3x$

(c) $s(t) = \dfrac{\ln(t^2 - 1)}{t + 1}$

TRY YOUR TURN 3

EXAMPLE 4 Automobile Resale Value

Based on projections from the *Kelly Blue Book*, the resale value of a 2010 Toyota Corolla 4-door sedan can be approximated by the following function

$$f(x) = 15{,}450 - 13{,}915 \log(t + 1),$$

where t is the number of years since 2010. Find and interpret $f(4)$ and $f'(4)$. ***Source: Kelly Blue Book.***

APPLY IT

SOLUTION Recognizing this function as a common (base 10) logarithm, we have

$$f(4) = 15{,}450 - 13{,}915 \log(4 + 1) \approx 5500.08.$$

The average resale value of a 2010 Toyota Corolla in 2014 would be approximately $5500.08.

The derivative of $f(t)$ is

$$f'(t) = \frac{-13{,}915}{(\ln 10)(t + 1)},$$

so $f'(4) \approx -1208.64$. In 2014, the average resale value of a 2010 Toyota Corolla is decreasing by $1208.64 per year.

12.5 EXERCISES

Find the derivative of each function.

1. $y = \ln(8x)$

2. $y = \ln(-4x)$

3. $y = \ln(8 - 3x)$

4. $y = \ln(1 + x^3)$

5. $y = \ln|4x^2 - 9x|$

6. $y = \ln|-8x^3 + 2x|$

7. $y = \ln\sqrt{x + 5}$

8. $y = \ln\sqrt{2x + 1}$

9. $y = \ln(x^4 + 5x^2)^{3/2}$

10. $y = \ln(5x^3 - 2x)^{3/2}$

11. $y = -5x \ln(3x + 2)$

12. $y = (3x + 7)\ln(2x - 1)$

13. $s = t^2 \ln|t|$

14. $y = x \ln|2 - x^2|$

15. $y = \dfrac{2\ln(x + 3)}{x^2}$

16. $v = \dfrac{\ln u}{u^3}$

17. $y = \dfrac{\ln x}{4x + 7}$

18. $y = \dfrac{-2 \ln x}{3x - 1}$

19. $y = \dfrac{3x^2}{\ln x}$

20. $y = \dfrac{x^3 - 1}{2 \ln x}$

21. $y = (\ln|x + 1|)^4$

22. $y = \sqrt{\ln|x - 3|}$

23. $y = \ln|\ln x|$

24. $y = (\ln 4)(\ln|3x|)$

25. $y = e^{x^2} \ln x$

26. $y = e^{2x-1} \ln(2x - 1)$

27. $y = \dfrac{e^x}{\ln x}$

28. $p(y) = \dfrac{\ln y}{e^y}$

29. $g(z) = (e^{2z} + \ln z)^3$

30. $s(t) = \sqrt{e^{-t} + \ln 2t}$

31. $y = \log(6x)$

32. $y = \log(4x - 3)$

33. $y = \log|1 - x|$

34. $y = \log|3x|$

35. $y = \log_5\sqrt{5x + 2}$

36. $y = \log_7\sqrt{4x - 3}$

37. $y = \log_3(x^2 + 2x)^{3/2}$

38. $y = \log_2(2x^2 - x)^{5/2}$

39. $w = \log_8(2^p - 1)$

40. $z = 10^y \log y$

41. $f(x) = e^{\sqrt{x}} \ln(\sqrt{x} + 5)$

42. $f(x) = \ln(xe^{\sqrt{x}} + 2)$

43. $f(t) = \dfrac{\ln(t^2 + 1) + t}{\ln(t^2 + 1) + 1}$

44. $f(t) = \dfrac{2t^{3/2}}{\ln(2t^{3/2} + 1)}$

45. Why do we use the absolute value of x or of $g(x)$ in the derivative formulas for the natural logarithm?

46. Prove $\dfrac{d}{dx}\ln|ax| = \dfrac{d}{dx}\ln|x|$ for any constant a.

47. A friend concludes that because $y = \ln 6x$ and $y = \ln x$ have the same derivative, namely $dy/dx = 1/x$, these two functions must be the same. Explain why this is incorrect.

48. Use a graphing calculator to sketch the graph of $y = [f(x + h) - f(x)]/h$ using $f(x) = \ln|x|$ and $h = 0.0001$. Compare it with the graph of $y = 1/x$ and discuss what you observe.

49. Using the fact that

$$\ln[u(x)v(x)] = \ln u(x) + \ln v(x),$$

use the chain rule and the formula for the derivative of $\ln x$ to derive the product rule. In other words, find $[u(x)v(x)]'$ without assuming the product rule.

50. Using the fact that

$$\ln \frac{u(x)}{v(x)} = \ln u(x) - \ln v(x),$$

use the chain rule and the formula for the derivative of $\ln x$ to derive the quotient rule. In other words, find $[u(x)/v(x)]'$ without assuming the quotient rule.

51. Use graphical differentiation to verify that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

52. Use the fact that $d \ln x/dx = 1/x$, as well as the change-of-base theorem for logarithms, to prove that

$$\frac{d \log_a x}{dx} = \frac{1}{x \ln a}.$$

53. Let

$$h(x) = u(x)^{v(x)}.$$

a. Using the fact that

$$\ln [u(x)^{v(x)}] = v(x) \ln u(x),$$

use the chain rule, the product rule, and the formula for the derivative of $\ln x$ to show that

$$\frac{d}{dx} \ln h(x) = \frac{v(x)u'(x)}{u(x)} + (\ln u(x))\, v'(x).$$

b. Use the result from part a and the fact that

$$\frac{d}{dx} \ln h(x) = \frac{h'(x)}{h(x)}$$

to show that

$$\frac{d}{dx} h(x) = u(x)^{v(x)} \left[\frac{v(x)u'(x)}{u(x)} + (\ln u(x))\, v'(x) \right].$$

The idea of taking the logarithm of a function before differentiating is known as logarithmic differentiation.

Use the ideas from Exercise 53 to find the derivative of each of the following functions.

54. $h(x) = x^x$ **55.** $h(x) = (x^2 + 1)^{5x}$

APPLICATIONS

Business and Economics

56. Profit If the total revenue received from the sale of x items is given by

$$R(x) = 30 \ln(2x + 1),$$

while the total cost to produce x items is $C(x) = x/2$, find the following.

a. The marginal revenue

b. The profit function $P(x)$

c. The marginal profit when $x = 60$

d. Interpret the results of part c.

57. Revenue Suppose the demand function for q units of a certain item is

$$p = D(q) = 100 + \frac{50}{\ln q}, \quad q > 1,$$

where p is in dollars.

a. Find the marginal revenue.

b. Approximate the revenue from one more unit when 8 units are sold.

c. How might a manager use the information from part b?

58. Profit If the cost function in dollars for q units of the item in Exercise 57 is $C(q) = 100q + 100$, find the following.

a. The marginal cost

b. The profit function $P(q)$

c. The approximate profit from one more unit when 8 units are sold

d. How might a manager use the information from part c?

59. Marginal Average Cost Suppose the cost in dollars to make x oboe reeds is given by

$$C(x) = 5 \log_2 x + 10.$$

Find the marginal average cost when the following numbers of reeds are sold.

a. 10 **b.** 20

Life Sciences

60. Body Surface Area There is a mathematical relationship between an infant's weight and total body surface area (BSA), given by

$$A(w) = 4.688 w^{0.8168 - 0.0154 \log_{10} w},$$

where w is the weight (in grams) and $A(w)$ is the BSA in square centimeters. ***Source: British Journal of Cancer.***

a. Find the BSA for an infant who weighs 4000 g.

b. Find $A'(4000)$ and interpret your answer.

c. Use a graphing calculator to graph $A(w)$ on $[2000, 10{,}000]$ by $[0, 6000]$.

61. Bologna Sausage Scientists have developed a model to predict the growth of bacteria in bologna sausage at 32°C. The number of bacteria is given by

$$\ln\left(\frac{N(t)}{N_0}\right) = 9.8901 e^{-e^{2.54197 - 0.2167t}},$$

where N_0 is the number of bacteria present at the beginning of the experiment and $N(t)$ is the number of bacteria present at time t (in hours). ***Source: Applied and Environmental Microbiology.***

a. Use the properties of logarithms to find an expression for $N(t)$. Assume that $N_0 = 1000$.

b. Use a graphing calculator to estimate the derivative of $N(t)$ when $t = 20$ and interpret.

c. Let $S(t) = \ln(N(t)/N_0)$. Graph $S(t)$ on $[0, 35]$ by $[0, 12]$.

d. Graph $N(t)$ on $[0, 35]$ by $[0, 20{,}000{,}000]$ and compare the graphs from parts c and d.

e. Find $\lim_{t\to\infty} S(t)$ and then use this limit to find $\lim_{t\to\infty} N(t)$.

62. Pronghorn Fawns The field metabolic rate (FMR), or the total energy expenditure per day in excess of growth, can be calculated for pronghorn fawns using Nagy's formula,

$$F(x) = 0.774 + 0.727 \log x,$$

where x is the mass (in grams) of the fawn and $F(x)$ is the energy expenditure (in kJ/day). ***Source: Animal Behavior.***

a. Determine the total energy expenditure per day in excess of growth for a pronghorn fawn that weighs 25,000 g.

b. Find $F'(25{,}000)$ and interpret the result.

c. Graph the function on $[5000, 30{,}000]$ by $[3, 5]$.

63. Fruit Flies A study of the relation between the rate of reproduction in *Drosophila* (fruit flies) bred in bottles and the density of the mated population found that the number of imagoes (sexually mature adults) per mated female per day (y) can be approximated by

$$\log y = 1.54 - 0.008x - 0.658 \log x,$$

where x is the mean density of the mated population (measured as flies per bottle) over a 16-day period. ***Source: Elements of Mathematical Biology.***

a. Show that the above equation is equivalent to

$$y = 34.7(1.0186)^{-x}x^{-0.658}.$$

b. Using your answer from part a, find the number of imagoes per mated female per day when the density is

i. 20 flies per bottle;

ii. 40 flies per bottle.

c. Using your answer from part a, find the rate of change in the number of imagoes per mated female per day with respect to the density when the density is

i. 20 flies per bottle;

ii. 40 flies per bottle.

64. Insect Mating Consider an experiment in which equal numbers of male and female insects of a certain species are permitted to intermingle. Assume that

$$M(t) = (0.1t + 1)\ln\sqrt{t}$$

represents the number of matings observed among the insects in an hour, where t is the temperature in degrees Celsius. (*Note:* The formula is an approximation at best and holds only for specific temperature intervals.)

a. Find the number of matings when the temperature is 15°C.

b. Find the number of matings when the temperature is 25°C.

c. Find the rate of change of the number of matings when the temperature is 15°C.

65. Population Growth Suppose that the population of a certain collection of rare Brazilian ants is given by

$$P(t) = (t + 100)\ln(t + 2),$$

where t represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

Social Sciences

66. Poverty The passage of the Social Security Amendments of 1965 resulted in the creation of the Medicare and Medicaid programs. Since then, the percent of persons 65 years and over with family income below the poverty level has declined. The percent can be approximated by the following function:

$$P(t) = 30.60 - 5.79 \ln t,$$

where t is the number of years since 1965. Find the percent of persons 65 years and over with family income below the poverty level and the rate of change in the following years. ***Source: U.S. Census.***

a. 1970 **b.** 1990 **c.** 2010

d. What happens to the rate of change over time?

Physical Sciences

67. Richter Scale The Richter scale provides a measure of the magnitude of an earthquake. In fact, the largest Richter number M ever recorded for an earthquake was 8.9 from the 1933 earthquake in Japan. The following formula shows a relationship between the amount of energy released and the Richter number.

$$M = \frac{2}{3}\log\frac{E}{0.007},$$

where E is measured in kilowatt-hours. ***Source: Mathematics Teacher.***

a. For the 1933 earthquake in Japan, what value of E gives a Richter number $M = 8.9$?

b. If the average household uses 247 kWh per month, how many months would the energy released by an earthquake of this magnitude power 10 million households?

c. Find the rate of change of the Richter number M with respect to energy when $E = 70{,}000$ kWh.

d. What happens to dM/dE as E increases?

General Interest

68. Street Crossing Consider a child waiting at a street corner for a gap in traffic that is large enough so that he can safely cross the street. A mathematical model for traffic shows that if the expected waiting time for the child is to be at most 1 minute, then the maximum traffic flow, in cars per hour, is given by

$$f(x) = \frac{29{,}000(2.322 - \log x)}{x},$$

where x is the width of the street in feet. Find the maximum traffic flow and the rate of change of the maximum traffic flow with respect to street width for the following values of the street width. *Source: An Introduction to Mathematical Modeling.*

a. 30 ft **b.** 40 ft

YOUR TURN ANSWERS

1. $f'(x) = \dfrac{1}{(\ln 3)x}$

2. (a) $\dfrac{dy}{dx} = \dfrac{6x^2}{2x^3 - 3}$

(b) $f'(x) = \dfrac{5 + 9x^2}{(\ln 4)(5x + 3x^3)}$

3. (a) $\dfrac{dy}{dx} = \dfrac{1}{x + 3}$

(b) $f'(x) = x + 2x \ln 3x$

(c) $s'(t) = \dfrac{2t - (t-1)\ln(t^2-1)}{(t-1)(t+1)^2}$

12 CHAPTER REVIEW

SUMMARY

In this chapter we used the definition of the derivative to develop techniques for finding derivatives of several types of functions. With the help of the rules that were developed, such as the power rule, product rule, quotient rule, and chain rule, we can now directly compute the derivative of a large variety of functions. In particular, we developed rules for finding derivatives of exponential and logarithmic functions. We also began to see the wide range of applications that these functions have in business, life sciences, social sciences, and the physical sciences. In the next chapter we will apply these techniques to study the behavior of certain functions, and we will learn that differentiation can be used to find maximum and minimum values of continuous functions.

Assume all indicated derivatives exist.

Constant Function If $f(x) = k$, where k is any real number, then $f'(x) = 0$.

Power Rule If $f(x) = x^n$, for any real number n, then $f'(x) = n \cdot x^{n-1}$.

Constant Times a Function Let k be a real number. Then the derivative of $y = k \cdot f(x)$ is $dy/dx = k \cdot f'(x)$.

Sum or Difference Rule If $y = u(x) \pm v(x)$, then $\dfrac{dy}{dx} = u'(x) \pm v'(x)$.

Product Rule If $f(x) = u(x) \cdot v(x)$, then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).$$

Quotient Rule If $f(x) = \dfrac{u(x)}{v(x)}$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

Chain Rule If y is a function of u, say $y = f(u)$, and if u is a function of x, say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Chain Rule (Alternative Form) Let $y = f[g(x)]$. Then $\dfrac{dy}{dx} = f'[g(x)] \cdot g'(x)$.

Exponential Functions

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = (\ln a)a^x$$

$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)}g'(x) \qquad \frac{d}{dx}(a^{g(x)}) = (\ln a)a^{g(x)}g'(x)$$

Logarithmic Functions

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a|x|) = \frac{1}{(\ln a)x}$$

$$\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)} \qquad \frac{d}{dx}(\log_a|g(x)|) = \frac{1}{\ln a}\cdot\frac{g'(x)}{g(x)}$$

KEY TERMS

12.1
marginal analysis

12.2
marginal average cost

12.3
composite function
composition
chain rule

12.4
logistic function

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The derivative of π^3 is $3\pi^2$.
2. The derivative of a sum is the sum of the derivatives.
3. The derivative of a product is the product of the derivatives.
4. The marginal cost function is the derivative of the cost function.
5. The chain rule is used to take the derivative of a product of functions.
6. The only function that is its own derivative is e^x.
7. The derivative of 10^x is $x10^{x-1}$.
8. The derivative of $\ln|x|$ is the same as the derivative of $\ln x$.
9. The derivative of $\ln kx$ is the same as the derivative of $\ln x$.
10. The derivative of $\log x$ is the same as the derivative of $\ln x$.

PRACTICE AND EXPLORATIONS

Use the rules for derivatives to find the derivative of each function defined as follows.

11. $y = 5x^3 - 7x^2 - 9x + \sqrt{5}$
12. $y = 7x^3 - 4x^2 - 5x + \sqrt{2}$
13. $y = 9x^{8/3}$
14. $y = -4x^{-3}$
15. $f(x) = 3x^{-4} + 6\sqrt{x}$
16. $f(x) = 19x^{-1} - 8\sqrt{x}$
17. $k(x) = \dfrac{3x}{4x + 7}$
18. $r(x) = \dfrac{-8x}{2x + 1}$
19. $y = \dfrac{x^2 - x + 1}{x - 1}$
20. $y = \dfrac{2x^3 - 5x^2}{x + 2}$
21. $f(x) = (3x^2 - 2)^4$
22. $k(x) = (5x^3 - 1)^6$
23. $y = \sqrt{2t^7 - 5}$
24. $y = -3\sqrt{8t^4 - 1}$
25. $y = 3x(2x + 1)^3$
26. $y = 4x^2(3x - 2)^5$
27. $r(t) = \dfrac{5t^2 - 7t}{(3t + 1)^3}$
28. $s(t) = \dfrac{t^3 - 2t}{(4t - 3)^4}$
29. $p(t) = t^2(t^2 + 1)^{5/2}$
30. $g(t) = t^3(t^4 + 5)^{7/2}$
31. $y = -6e^{2x}$
32. $y = 8e^{0.5x}$
33. $y = e^{-2x^3}$
34. $y = -4e^{x^2}$
35. $y = 5xe^{2x}$
36. $y = -7x^2e^{-3x}$
37. $y = \ln(2 + x^2)$
38. $y = \ln(5x + 3)$
39. $y = \dfrac{\ln|3x|}{x - 3}$
40. $y = \dfrac{\ln|2x - 1|}{x + 3}$
41. $y = \dfrac{xe^x}{\ln(x^2 - 1)}$
42. $y = \dfrac{(x^2 + 1)e^{2x}}{\ln x}$
43. $s = (t^2 + e^t)^2$
44. $q = (e^{2p+1} - 2)^4$
45. $y = 3 \cdot 10^{-x^2}$
46. $y = 10 \cdot 2^{\sqrt{x}}$
47. $g(z) = \log_2(z^3 + z + 1)$
48. $h(z) = \log(1 + e^z)$
49. $f(x) = e^{2x}\ln(xe^x + 1)$
50. $f(x) = \dfrac{e^{\sqrt{x}}}{\ln(\sqrt{x} + 1)}$

Consider the following table of values of the functions f and g and their derivatives at various points.

x	1	2	3	4
$f(x)$	3	4	2	1
$f'(x)$	−5	−6	−7	−11
$g(x)$	4	1	2	3
$g'(x)$	2/9	3/10	4/11	6/13

Find the following using the table.

51. **a.** $D_x(f[g(x)])$ at $x = 2$ **b.** $D_x(f[g(x)])$ at $x = 3$

52. **a.** $D_x(g[f(x)])$ at $x = 2$ **b.** $D_x(g[f(x)])$ at $x = 3$

Find the slope of the tangent line to the given curve at the given value of x. Find the equation of each tangent line.

53. $y = x^2 - 6x; \quad x = 2$

54. $y = 8 - x^2; \quad x = 1$

55. $y = \dfrac{3}{x - 1}; \quad x = -1$

56. $y = \dfrac{x}{x^2 - 1}; \quad x = 2$

57. $y = \sqrt{6x - 2}; \quad x = 3$

58. $y = -\sqrt{8x + 1}; \quad x = 3$

59. $y = e^x; \quad x = 0$

60. $y = xe^x; \quad x = 1$

61. $y = \ln x; \quad x = 1$

62. $y = x \ln x; \quad x = e$

63. Consider the graphs of the function $y = \sqrt{2x - 1}$ and the straight line $y = x + k$. Discuss the number of points of intersection versus the change in the value of k. ***Source: Japanese University Entrance Examination Problems in Mathematics.***

64. **a.** Verify that

$$\frac{d \ln f(x)}{dx} = \frac{f'(x)}{f(x)}.$$

This expression is called the *relative rate of change*. It expresses the rate of change of f relative to the size of f. Stephen B. Maurer denotes this expression by $\hat{f}$ and notes that economists commonly work with relative rates of change. ***Source: The College Mathematics Journal.***

b. Verify that

$$\widehat{fg} = \hat{f} + \hat{g}$$

Interpret this equation in terms of relative rates of change.

c. In his article, Maurer uses the result of part b to solve the following problem:

"Last year, the population grew by 1% and the average income per person grew by 2%. By what approximate percent did the national income grow?"

Explain why the result from part b implies that the answer to this question is approximately 3%.

65. Suppose that the student body in your college grows by 2% and the tuition goes up by 3%. Use the result from the previous exercise to calculate the approximate amount that the total tuition collected goes up, and compare this with the actual amount.

66. Why is e a convenient base for exponential and logarithmic functions?

APPLICATIONS

Business and Economics

Marginal Average Cost Find the marginal average cost function of each function defined as follows.

67. $C(x) = \sqrt{x + 1}$

68. $C(x) = \sqrt{3x + 2}$

69. $C(x) = (x^2 + 3)^3$

70. $C(x) = (4x + 3)^4$

71. $C(x) = 10 - e^{-x}$

72. $C(x) = \ln(x + 5)$

73. Sales The sales of a company are related to its expenditures on research by

$$S(x) = 1000 + 60\sqrt{x} + 12x,$$

where $S(x)$ gives sales in millions when x thousand dollars is spent on research. Find and interpret dS/dx if the following amounts are spent on research.

a. \$9000 **b.** \$16,000 **c.** \$25,000

d. As the amount spent on research increases, what happens to sales?

74. Profit Suppose that the profit (in hundreds of dollars) from selling x units of a product is given by

$$P(x) = \frac{x^2}{2x + 1}.$$

Find and interpret the marginal profit when the following numbers of units are sold.

a. 4 **b.** 12 **c.** 20

d. What is happening to the marginal profit as the number sold increases?

e. Find and interpret the marginal average profit when 4 units are sold.

75. Costs A company finds that its total costs are related to the amount spent on training programs by

$$T(x) = \frac{1000 + 60x}{4x + 5},$$

where $T(x)$ is costs in thousands of dollars when x hundred dollars are spent on training. Find and interpret $T'(x)$ if the following amounts are spent on training.

a. \$900 **b.** \$1900

c. Are costs per dollar spent on training always increasing or decreasing?

76. Compound Interest If a sum of \$1000 is deposited into an account that pays r% interest compounded quarterly, the balance after 12 years is given by

$$A = 1000\left(1 + \frac{r}{400}\right)^{48}.$$

Find and interpret $\dfrac{dA}{dr}$ when $r = 5$.

77. Continuous Compounding If a sum of \$1000 is deposited into an account that pays r% interest compounded continuously, the balance after 12 years is given by

$$A = 1000e^{12r/100}.$$

Find and interpret $\dfrac{dA}{dr}$ when $r = 5$.

78. Doubling Time If a sum of money is deposited into an account that pays $r\%$ interest compounded annually, the doubling time (in years) is given by

$$T = \frac{\ln 2}{\ln(1 + r/100)}.$$

Find and interpret dT/dr when $r = 5$.

79. U.S. Post Office The number (in billions) of pieces of mail handled by the U.S. Post Office each year from 1980 through 2009 can be approximated by

$$P(t) = -0.00132t^4 + 0.0665t^3 - 1.127t^2 + 11.581t + 105.655,$$

where t is the number of years since 1980. Find and interpret the rate of change in the volume of mail for the following years. ***Source: U.S. Postal Service***.

a. 1995 **b.** 2005

80. Elderly Employment After declining over the last century, the percentage of men aged 65 and older in the workforce has begun to rise in recent years, as shown by the following table. ***Source: U.S. Bureau of Labor Statistics***.

Year	Percent of Men 65 and Older in Workforce
1900	63.1
1920	55.6
1930	54.0
1940	41.8
1950	45.8
1960	33.1
1970	26.8
1980	19.0
1990	16.3
2000	17.7
2009	21.9

a. Using the regression feature on a graphing calculator, find a cubic and a quartic function that model this data, letting $t = 0$ correspond to the year 1900.

b. Using each of your answers to part a, find the rate that the percent of men aged 65 and older in the workforce was increasing in 2005.

c. Discuss which model from part a best describes the data, as well as which answer from part b best describes the rate that the percent of men aged 65 and older in the workforce was increasing in 2005.

d. Explore other functions that could be used to model the data, using the various regression features on a graphing calculator, and discuss to what extent any of them are useful descriptions of the data.

81. Value of the Dollar The U.S. dollar has been declining in value over the last century, except during the Great Depression, when it increased in value. The following table shows the number of dollars it took in various years to equal \$1 in 1913. ***Source: U.S. Bureau of Labor Statistics***.

Year	Number of Dollars It Took to Equal \$1 in 1913
1913	1.00
1920	2.02
1930	1.69
1940	1.41
1950	2.53
1960	2.99
1970	3.92
1980	8.32
1990	13.20
2000	17.39
2010	22.02

a. Using the regression feature on a graphing calculator, find a cubic and a quartic function that model this data, letting $t = 0$ correspond to the year 1900.

b. Using each of your answers to part a, find the rate that the number of dollars required to equal \$1 in 1913 was increasing in 2005.

c. Discuss which model from part a best describes the data, as well as which answer from part b best describes the rate that the number of dollars required to equal \$1 in 1913 was increasing in 2005.

d. Explore other functions that could be used to model the data, using the various regression features on a graphing calculator, and discuss to what extent any of them are useful descriptions of the data.

Life Sciences

82. Exponential Growth Suppose a population is growing exponentially with an annual growth constant $k = 0.05$. How fast is the population growing when it is 1,000,000? Use the derivative to calculate your answer, and then explain how the answer can be obtained without using the derivative.

83. Logistic Growth Suppose a population is growing logistically with $k = 5 \times 10^{-6}$, $m = 30{,}000$, and $G_0 = 2000$. Assume time is measured in years.

a. Find the growth function $G(t)$ for this population.

b. Find the population and rate of growth of the population after 6 years.

84. Fish The length of the monkeyface prickleback, a West Coast game fish, can be approximated by

$$L = 71.5(1 - e^{-0.1t})$$

and the weight by

$$W = 0.01289 \cdot L^{2.9},$$

where L is the length in centimeters, t is the age in years, and W is the weight in grams. ***Source: California Fish and Game***.

a. Find the approximate length of a 5-year-old monkeyface.

b. Find how fast the length of a 5-year-old monkeyface is growing.

c. Find the approximate weight of a 5-year-old monkeyface. (*Hint:* Use your answer from part a.)

d. Find the rate of change of the weight with respect to length for a 5-year-old monkeyface.

e. Using the chain rule and your answers to parts b and d, find how fast the weight of a 5-year-old monkeyface is growing.

85. Arctic Foxes The age/weight relationship of male Arctic foxes caught in Svalbard, Norway, can be estimated by the function

$$M(t) = 3583e^{-e^{-0.020(t-66)}},$$

where t is the age of the fox in days and $M(t)$ is the weight of the fox in grams. ***Source: Journal of Mammology***.

a. Estimate the weight of a male fox that is 250 days old.

b. Use $M(t)$ to estimate the largest size that a male fox can attain. (*Hint:* Find $\lim_{t\to\infty} M(t)$.)

c. Estimate the age of a male fox when it has reached 50% of its maximum weight.

d. Estimate the rate of change in weight of a male Arctic fox that is 250 days old. (*Hint:* Recall that $D_t e^{f(t)} = f'(t)e^{f(t)}$.)

e. Use a graphing calculator to graph $M(t)$ and then describe the growth pattern.

f. Use the table function on a graphing calculator or a spreadsheet to develop a chart that shows the estimated weight and growth rate of male foxes for days 50, 100, 150, 200, 250, and 300.

86. Hispanic Population In Section 10.4, Exercise 49, we found that the projected Asian population in the United States, in millions, can be approximated by

$$a(t) = 11.14(1.023)^t,$$

where t is the years since 2000. Find the instantaneous rate of change in the projected Asian population in the United States in each of the following years. ***Source: U.S. Census***.

a. 2005 **b.** 2025

Physical Sciences

87. Wind Energy In Section 10.4, Exercise 55, we found that the total world wind energy capacity (in megawatts) in recent years could be approximated by the function

$$C(t) = 19{,}231(1.2647)^t,$$

where t is the number of years since 2000. Find the rate of change in the energy capacity for the following years. ***Source: World Wind Energy Association***.

a. 2005 b. 2010 c. 2015

General Interest

88. Cats The distance from Lisa Wunderle's cat, Belmar, to a piece of string he is stalking is given in feet by

$$f(t) = \frac{8}{t+1} + \frac{20}{t^2+1},$$

where t is the time in seconds since he begins.

a. Find Belmar's average velocity between 1 second and 3 seconds.

b. Find Belmar's instantaneous velocity at 3 seconds.

89. Food Surplus In Section 10.4, Example 7, we found that the production of corn (in billions of bushels) in the United States since 1930 could be approximated by

$$p(x) = 1.757(1.0249)^{x-1930}$$

where x is the year. Find and interpret $p'(2000)$.

90. Dating a Language Over time, the number of original basic words in a language tends to decrease as words become obsolete or are replaced with new words. Linguists have used calculus to study this phenomenon and have developed a methodology for dating a language, called *glottochronology*. Experiments have indicated that a good estimate of the number of words that remain in use at a given time is given by

$$N(t) = N_0 e^{-0.217t},$$

where $N(t)$ is the number of words in a particular language, t is measured in the number of millennium, and N_0 is the original number of words in the language. ***Source: The UMAP Journal***.

a. In 1950, C. Feng and M. Swadesh established that of the original 210 basic ancient Chinese words from 950 A.D., 167 were still being used. Letting $t = 0$ correspond to 950, with $N_0 = 210$, find the number of words predicted to have been in use in 1950 A.D., and compare it with the actual number in use.

b. Estimate the number of words that will remain in the year 2050 ($t = 1.1$).

c. Find $N'(1.1)$ and interpret your answer.

91. Driving Fatalities A study by the National Highway Traffic Safety Administration found that driver fatalities rates were highest for the youngest and oldest drivers. The rates per 1000 licensed drivers for every 100 million miles may be approximated by the function

$$f(x) = k(x-49)^6 + 0.8,$$

where x is the driver's age in years and k is the constant 3.8×10^{-9}. Find and interpret the rate of change of the fatality rate when the driver is

a. 20 years old; **b.** 60 years old.

Source: National Highway Traffic Safety Administration.

EXTENDED APPLICATION

ELECTRIC POTENTIAL AND ELECTRIC FIELD

In physics, a major area of study is electricity, including such concepts as electric charge, electric force, and electric current. Two ideas that physicists use a great deal are electric potential and electric field. Electric potential is the same as voltage, such as for a battery. Electric field can be thought of in terms of a force field, which is often referred to in space movies as a deflector shield around a spaceship. An electric field causes an electric force to act on charged objects when they are in the electric field.

Both electric potential and electric field are produced by electric charges. It is important to physicists and electrical engineers to know what the electric field and the electric potential are near a charged object. Usually the problem involves finding the electric potential and taking the (negative) derivative of the electric potential to determine the electric field. More explicitly,

$$E = -\frac{dV}{dz}, \tag{1}$$

where V is the electric potential (voltage) near the charged object, z is the distance from the object, and E is the electric field.

Let's look at an example. Suppose we have a charged disk of radius R and we want to determine the electric potential and electric field at a distance z along the axis of the disk (Figure 13).

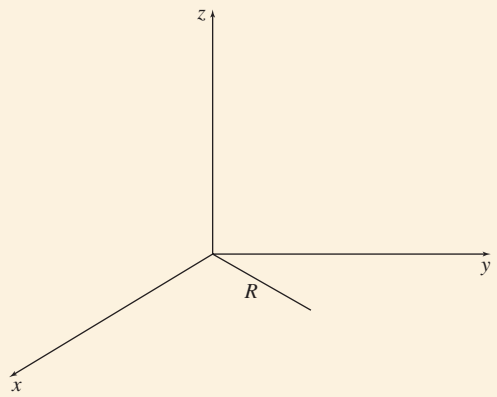


FIGURE 13

*For those interested in the constants k_1 and k_2:

$$k_1 = \frac{Q}{2\pi\varepsilon_0 R^2} \quad \text{and} \quad k_2 = \frac{1}{2}k_1R^2 = \frac{Q}{4\pi\varepsilon_0},$$

where Q is the charge, ε_0 is called the electric permittivity, and R is the radius of the disk.

Using some basic definitions and integral calculus (that you will learn about later in your calculus book), it can be shown that the electric potential on the axis is

$$V = k_1\left(\sqrt{z^2 + R^2} - z\right) \tag{2}$$

where k_1 is a constant.* To determine the electric field, we apply Equation (1) and find

$$E = -k_1\left(\frac{z}{\sqrt{z^2 + R^2}} - 1\right) \tag{3}$$

You will provide the details for this example by working through Exercise 1 at the end of this section.

Physicists often like to see if complicated expressions such as these can be simplified when certain conditions apply. An example would be to imagine that the location z is very far away from the disk. In that case, the disk doesn't look much like a disk anymore, but more like a point. In Equations (2) and (3), we see that z will be much larger than R. If you reach the topic of Taylor series in calculus, you will learn how these series can be used to approximate square roots, so that for large values of z the voltage is inversely proportional to z and looks like this:

$$V = \frac{k_2}{z}, \tag{4}$$

where k_2 is a constant. To determine the electric field, we apply Equation (1) again to find

$$E = \frac{k_2}{z^2}. \tag{5}$$

You will be asked to prove this result in Exercise 2.

Thus we see that the exact functions for V and E of the disk become much simpler at locations far away from the disk. The voltage is a reciprocal function and the electric field is called an inverse-square law. By the way, these functions are the same ones that would be used for a point charge, which is a charge that takes up very little space.

Now let's look at what happens when we are very close to the surface of the disk. You could imagine that an observer very close to the surface would see the disk as a large flat plane. If we apply the Taylor series once more to the exact function for the potential (Equation (2)), but this time with z much smaller than R, we find

$$V = k_1\left(R - z + \frac{z^2}{2R}\right). \tag{6}$$

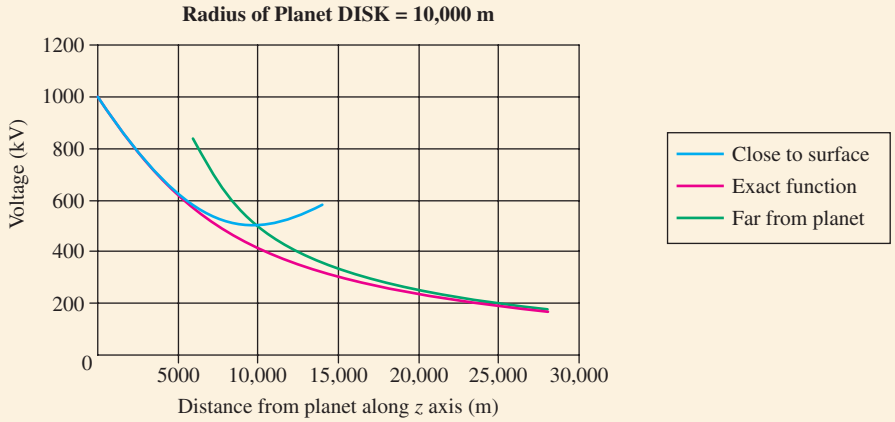


FIGURE 14

Notice that this is a quadratic function, or a parabola. Applying Equation (1) again, we see that the electric field is

$$E = k_1\left(1 - \frac{z}{R}\right). \tag{7}$$

This is just a linear function that increases as we approach the surface (which would be $z = 0$) and decreases the farther away from the surface we move. Again, the details of this calculation are left for you to do in Exercise 3.

Here is a fun way of making sense of the equations listed above. Suppose you are in a spaceship approaching the planet DISK on a very important mission. Planet DISK is noted for the fact that there is always a sizeable amount of charge on it. You are approaching the planet from very far away along its axis. In Figure 14 the three voltage functions (Equations (2), (4), and (6)) are plotted, and in Figure 15 the three electric field functions (Equations (3), (5), and (7)) are plotted. The graphs were generated using $R = 10{,}000$, $k_1 = 100$, and $k_2 = 5 \times 10^9$.

Notice that it looks like you can use the reciprocal function for the voltage (Equation (4)) and the inverse-square law function for the electric field (Equation (5)) when you are farther away than about 20,000 m because the exact functions and the approximate functions are almost exactly the same. Also, when you get close to the planet, say less than about 4000 m, you can use the quadratic function for the voltage (Equation (6)) and the linear function for the electric field (Equation (7)) because the exact functions and the approximate functions are nearly the same. This means that you should use the exact functions (Equations (2) and (3)) between about 20,000 m and about 4000 m because the other functions deviate substantially in that region.

There are many other examples that could be studied, but they involve other functions that you haven't covered yet, especially trigonometric functions. But the process is still the same: If you can determine the electric potential in the region around a charged object, then the electric field is found by taking the negative derivative of the electric potential.

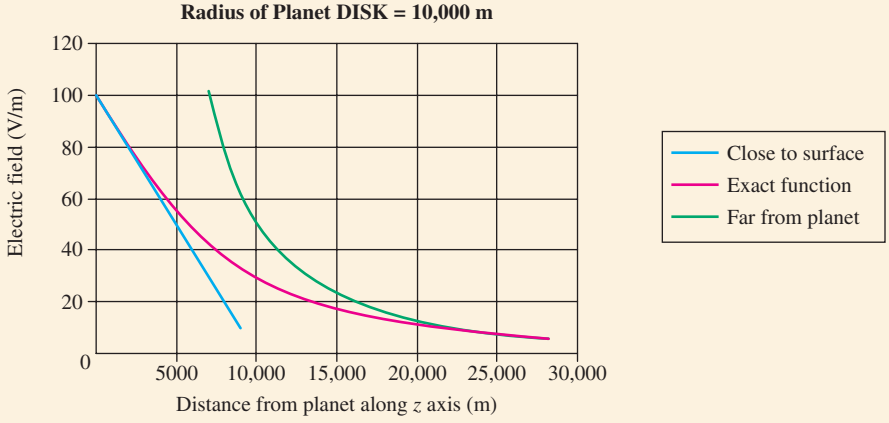


FIGURE 15

EXERCISES

1. Use Equation (1) to prove that the electric field of the disk (Equation (3)) is obtained from the voltage of the disk (Equation (2)). (*Hint:* It may help to write the square root in Equation (2) as a power.)

2. Apply Equation (1) to the voltage of a point charge (Equation (4)) to obtain the electric field of a point charge (Equation (5)).

3. Show that the electric field in Equation (7) results from the electric potential in Equation (6).

4. Sometimes for z very, very close to the disk, the third term in Equation (6) is so small that it can be dismissed. Show that the electric field is constant for this case.

5. Use a graphing calculator or Wolfram|Alpha (which can be found at www.wolframalpha.com) to recreate the graphs of the functions in Figures 14 and 15.

6. Use a spreadsheet to create a table of values for the functions displayed in Figures 14 and 15. Compare the three voltage functions and then compare the three electric field functions.

DIRECTIONS FOR GROUP PROJECT

Determine the electric potential and the electric field at various locations along the axis of a charged compact disc (CD). You will need to measure the radius of a typical CD and use the value for the electric permittivity (sometimes called the permittivity of free space), $\varepsilon_0 = 8.85 \times 10^{-12} C^2/Nm^2$. *To estimate the value for the charge on the CD, assume that a typical CD has about one mole of atoms (*6.0×10^{23}*, which is Avogadro's number) and that one out of every billion of these atoms loses an electron. The charge is found by multiplying* 10^{-9} *(one billionth) by Avogadro's number and by the charge of one electron (or proton), which is* 1.6×10^{-19} *C. With this information you can calculate the constants* k_1 *and* k_2. *Use appropriate graphing software, such as Microsoft Excel, to plot all three of the voltage functions (Equations (2), (4), and (6)) on one graph and all three electric field functions (Equations (3), (5), and (7)) on one graph.*

13 Graphs and the Derivative

13.1 Increasing and Decreasing Functions

13.2 Relative Extrema

13.3 Higher Derivatives, Concavity, and the Second Derivative Test

13.4 Curve Sketching

Chapter 13 Review

Extended Application: A Drug Concentration Model for Orally Administered Medications

Derivatives provide useful information about the behavior of functions and the shapes of their graphs. The first derivative describes the rate of increase or decrease, while the second derivative indicates how the rate of increase or decrease is changing. In an exercise at the end of this chapter, we will see what changes in the sign of the second derivative tell us about the shape of the graph that shows a weightlifter's performance as a function of age.

The graph in Figure 1 shows the relationship between the number of sleep-related accidents and traffic density during a 24-hour period. The blue line indicates the hourly distribution of sleep-related accidents. The green line indicates the hourly distribution of traffic density. The red line indicates the relative risk of sleep-related accidents. For example, the relative risk graph shows us that a person is nearly seven times as likely to have an accident at 4:00 A.M. than at 10:00 P.M. *Source: Sleep.*

Given a graph like the one in Figure 1, we can often locate maximum and minimum values simply by looking at the graph. It is difficult to get *exact* values or *exact* locations of maxima and minima from a graph, however, and many functions are difficult to graph without the aid of technology. In Chapter 10, we saw how to find exact maximum and minimum values for quadratic functions by identifying the vertex. A more general approach is to use the derivative of a function to determine precise maximum and minimum values of the function. The procedure for doing this is described in this chapter, which begins with a discussion of increasing and decreasing functions.

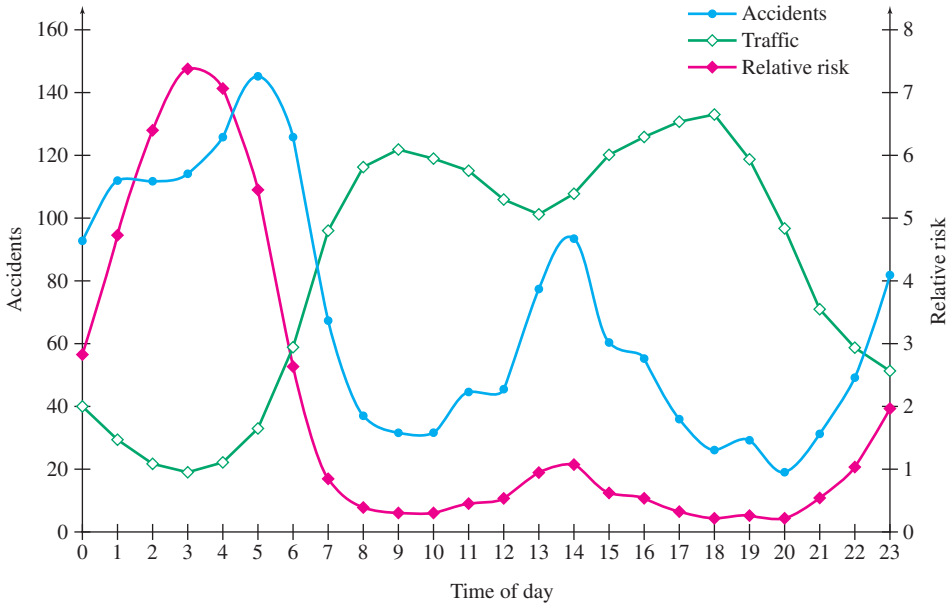


FIGURE 1

13.1 Increasing and Decreasing Functions

APPLY IT **How long is it profitable to increase production?**

We will answer this question in Example 5 after further investigating increasing and decreasing functions.

A function is *increasing* if the graph goes *up* from left to right and *decreasing* if its graph goes *down* from left to right. Examples of increasing functions are shown in Figures 2(a)–(c), and examples of decreasing functions in Figures 2(d)–(f).

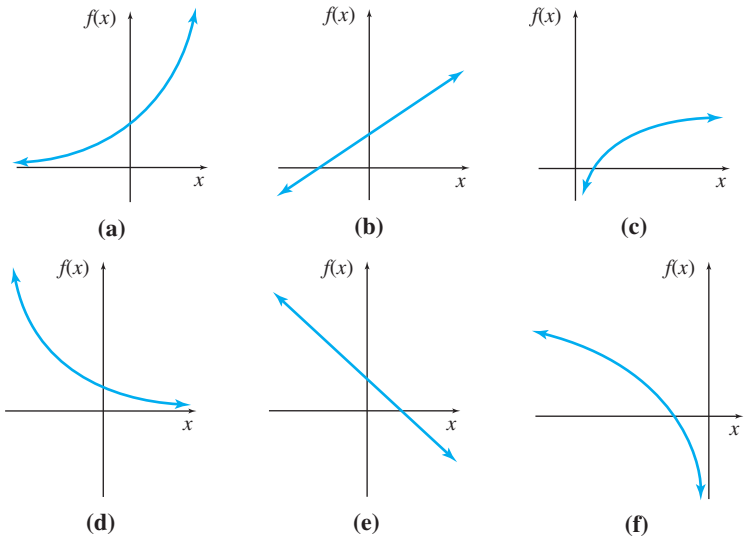


FIGURE 2

Increasing and Decreasing Functions

Let f be a function defined on some interval. Then for any two numbers x_1 and x_2 in the interval, f is **increasing** on the interval if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2,$$

and f is **decreasing** on the interval if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2.$$

EXAMPLE 1 Increasing and Decreasing

Where is the function graphed in Figure 3 increasing? Where is it decreasing?

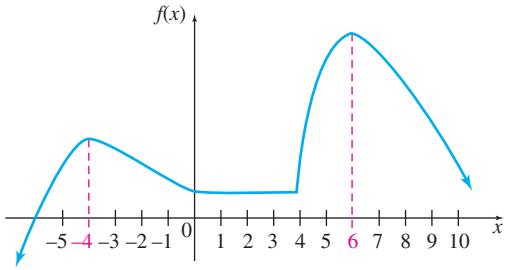
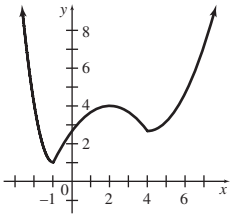


FIGURE 3

SOLUTION Moving from left to right, the function is increasing for x-values up to -4, then decreasing for x-values from -4 to 0, constant (neither increasing nor decreasing) for x-values from 0 to 4, increasing for x-values from 4 to 6, and decreasing for all x-values larger than 6. In interval notation, the function is increasing on $(-\infty, -4)$ and $(4, 6)$, decreasing on $(-4, 0)$ and $(6, \infty)$, and constant on $(0, 4)$. TRY YOUR TURN 1

YOUR TURN 1 Find where the function is increasing and decreasing.



How can we tell from the equation that defines a function where the graph increases and where it decreases? The derivative can be used to answer this question. Remember that the derivative of a function at a point gives the slope of the line tangent to the function at that point. Recall also that a line with a positive slope rises from left to right and a line with a negative slope falls from left to right.

The graph of a typical function, f, is shown in Figure 4. Think of the graph of f as a roller coaster track moving from left to right along the graph. Now, picture one of the cars on the roller coaster. As shown in Figure 5, when the car is on level ground or parallel to level ground, its floor is horizontal, but as the car moves up the slope, its floor tilts upward. When the car reaches a peak, its floor is again horizontal, but it then begins to tilt downward (very steeply) as the car rolls downhill. The floor of the car as it moves from left to right along the track represents the tangent line at each point. Using this analogy, we can see that the slope of the tangent line will be *positive* when the car travels uphill and f is *increasing*, and the slope of the tangent line will be *negative* when the car travels downhill and f is *decreasing*. (In this case it is also true that the slope of the tangent line will be zero at "peaks" and "valleys.")

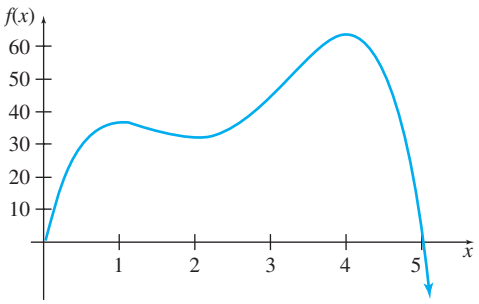


FIGURE 4

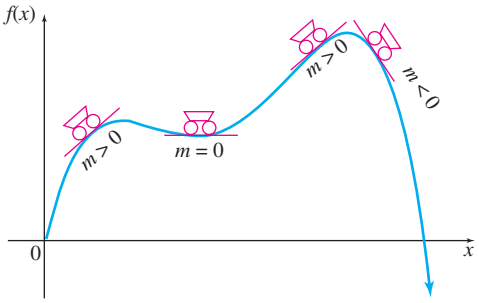


FIGURE 5

Thus, on intervals where $f'(x) > 0$, the function $f(x)$ will increase, and on intervals where $f'(x) < 0$, the function $f(x)$ will decrease. We can determine where $f(x)$ peaks by finding the intervals on which it increases and decreases.

Our discussion suggests the following test.

Test for Intervals Where $f(x)$ is Increasing and Decreasing

Suppose a function f has a derivative at each point in an open interval; then
if $f'(x) > 0$ for each x in the interval, f is *increasing* on the interval; ↗
if $f'(x) < 0$ for each x in the interval, f is *decreasing* on the interval; ↘
if $f'(x) = 0$ for each x in the interval, f is *constant* on the interval. →

NOTE
The third condition must hold for an entire open interval, not a single point. It would not be correct to say that because $f'(x) = 0$ at a point, then $f(x)$ is constant at that point.

The derivative $f'(x)$ can change signs from positive to negative (or negative to positive) at points where $f'(x) = 0$ and at points where $f'(x)$ does not exist. The values of x where this occurs are called *critical numbers*.

Critical Numbers

The **critical numbers** for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist. A **critical point** is a point whose x-coordinate is the critical number c and whose y-coordinate is $f(c)$.

It is shown in more advanced classes that if the critical numbers of a function are used to determine open intervals on a number line, then the sign of the derivative at any point in an interval will be the same as the sign of the derivative at any other point in the interval. This suggests that the test for increasing and decreasing functions be applied as follows (assuming that no open intervals exist where the function is constant).

FOR REVIEW

The method for finding where a function is increasing and decreasing is similar to the method introduced in Section R.5 for solving quadratic inequalities.

Applying the Test

1. Locate the critical numbers for f on a number line, as well as any points where f is undefined. These points determine several open intervals.
2. Choose a value of x in each of the intervals determined in Step 1. Use these values to decide whether $f'(x) > 0$ or $f'(x) < 0$ in that interval.
3. Use the test on the previous page to decide whether f is increasing or decreasing on the interval.

EXAMPLE 2 Increasing and Decreasing

Find the intervals in which the following function is increasing or decreasing. Locate all points where the tangent line is horizontal. Graph the function.

$$f(x) = x^3 + 3x^2 - 9x + 4$$

FOR REVIEW

In this chapter you will need all of the rules for derivatives you learned in the previous chapter. If any of these are still unclear, go over the Derivative Summary at the end of that chapter and practice some of the Review Exercises before proceeding.

SOLUTION Here $f'(x) = 3x^2 + 6x - 9$. To find the critical numbers, set this derivative equal to 0 and solve the resulting equation by factoring.

$$\begin{aligned} 3x^2 + 6x - 9 &= 0 \\ 3(x^2 + 2x - 3) &= 0 \\ 3(x + 3)(x - 1) &= 0 \\ x = -3 \quad &\text{or} \quad x = 1 \end{aligned}$$

The tangent line is horizontal at $x = -3$ or $x = 1$. Since there are no values of x where $f'(x)$ fails to exist, the only critical numbers are -3 and 1. To determine where the function is increasing or decreasing, locate -3 and 1 on a number line, as in Figure 6. (Be sure to place the values on the number line in numerical order.) These points determine three intervals: $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$.

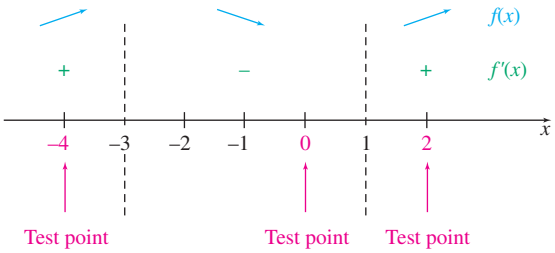


FIGURE 6

Now choose any value of x in the interval $(-\infty, -3)$. Choosing $x = -4$ and evaluating $f'(-4)$ using the factored form of $f'(x)$ gives

$$f'(-4) = 3(-4 + 3)(-4 - 1) = 3(-1)(-5) = 15,$$

which is positive. You could also substitute $x = -4$ in the unfactored form of $f'(x)$, but using the factored form makes it easier to see whether the result is positive or negative, depending upon whether you have an even or an odd number of negative factors. Since one value of x in this interval makes $f'(x) > 0$, all values will do so, and therefore, f is increasing on $(-\infty, -3)$. Selecting 0 from the middle interval gives $f'(0) = -9$, so f is decreasing on $(-3, 1)$. Finally, choosing 2 in the right-hand region gives $f'(2) = 15$, with f increasing on $(1, \infty)$. The arrows in each interval in Figure 6 indicate where f is increasing or decreasing.

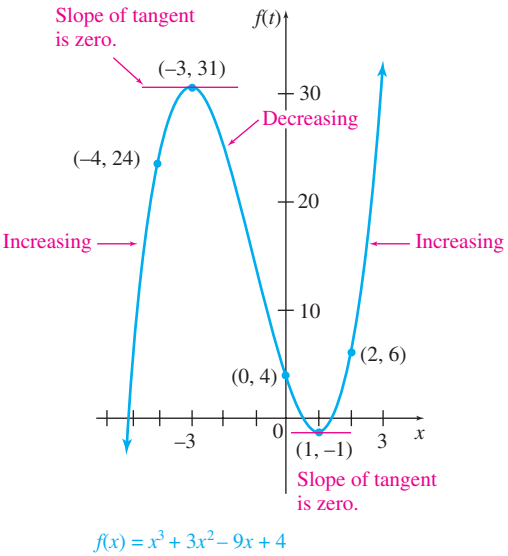


FIGURE 7

YOUR TURN 2
Find the intervals in which $f(x) = -x^3 - 2x^2 + 15x + 10$ is increasing or decreasing. Graph the function.

We now have an additional tool for graphing functions: the test for determining where a function is increasing or decreasing. (Other tools are discussed in the next few sections.) To graph the function, plot a point at each of the critical numbers by finding $f(-3) = 31$ and $f(1) = -1$. Also plot points for $x = -4$, 0, and 2, the test values of each interval. Use these points along with the information about where the function is increasing and decreasing to get the graph in Figure 7. TRY YOUR TURN 2

CAUTION Be careful to use $f(x)$, not $f'(x)$, to find the y-value of the points to plot.

Recall critical numbers are numbers c in the domain of f for which $f'(c) = 0$ or $f'(x)$ does not exist. In Example 2, there are no critical values, c, where $f'(c)$ fails to exist. The next example illustrates the case where a function has a critical number at c because the derivative does not exist at c.

EXAMPLE 3 Increasing and Decreasing

Find the critical numbers and decide where f is increasing and decreasing if $f(x) = (x - 1)^{2/3}$.

SOLUTION We find $f'(x)$ first, using the power rule and the chain rule.

$$f'(x) = \frac{2}{3}(x - 1)^{-1/3}(1) = \frac{2}{3(x - 1)^{1/3}}$$

To find the critical numbers, we first find any values of x that make $f'(x) = 0$, but here $f'(x)$ is never 0. Next, we find any values of x where $f'(x)$ fails to exist. This occurs whenever the denominator of $f'(x)$ is 0, so set the denominator equal to 0 and solve.

$$\begin{aligned} 3(x - 1)^{1/3} &= 0 && \text{Divide by 3.} \\ [(x - 1)^{1/3}]^3 &= 0^3 && \text{Raise both sides to the 3rd power.} \\ x - 1 &= 0 \\ x &= 1 \end{aligned}$$

Since $f'(1)$ does not exist but $f(1)$ is defined, $x = 1$ is a critical number, the only critical number. This point divides the number line into two intervals: $(-\infty,1)$ and $(1,\infty)$. Draw a number line for f', and use a test point in each of the intervals to find where f is increasing and decreasing.

$$f'(0) = \frac{2}{3(0-1)^{1/3}} = \frac{2}{-3} = -\frac{2}{3}$$

$$f'(2) = \frac{2}{3(2-1)^{1/3}} = \frac{2}{3}$$

Since f is defined for all x, these results show that f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. The graph of f is shown in Figure 8. TRY YOUR TURN 3

YOUR TURN 3 Find where f is increasing and decreasing if $f(x) = (2x + 4)^{2/5}$. Graph the function.

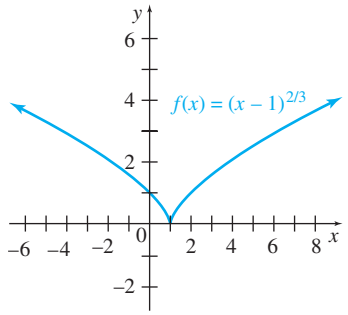


FIGURE 8

In Example 3, we found a critical number where $f'(x)$ failed to exist. This occurred when the denominator of $f'(x)$ was zero. Be on the alert for such values of x. Also be alert for values of x that would make the expression under a square root, or other even root, negative. For example, if $f(x) = \sqrt{x}$, then $f'(x) = 1/(2\sqrt{x})$. Notice that $f'(x)$ does not exist for $x \le 0$, but the values of $x < 0$ are not critical numbers because those values of x are not in the domain of f. The function $f(x) = \sqrt{x}$ does have a critical point at $x = 0$, because 0 is in the domain of f.

Sometimes a function may not have any critical numbers, but we are still able to determine where the function is increasing and decreasing, as shown in the next example.

EXAMPLE 4 Increasing and Decreasing (No Critical Numbers)

Find the intervals for which the following function increases and decreases. Graph the function.

$$f(x) = \frac{x-1}{x+1}$$

SOLUTION Notice that the function f is undefined when $x = -1$, so -1 is not in the domain of f. To determine any critical numbers, first use the quotient rule to find $f'(x)$.

$$f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$$

$$= \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

This derivative is never 0, but it fails to exist at $x = -1$, where the function is undefined. Since -1 is not in the domain of f, there are no critical numbers for f.

We can still apply the first derivative test, however, to find where f is increasing and decreasing. The number -1 (where f is undefined) divides the number line into two intervals: $(-\infty,-1)$ and $(-1,\infty)$. Draw a number line for f', and use a test point in each of these intervals to find that $f'(x) > 0$ for all x except -1. (This can also be determined by observing that $f'(x)$ is the quotient of 2, which is positive, and $(x+1)^2$, which is always positive or 0.) This means that the function f is increasing on both $(-\infty,-1)$ and $(-1,\infty)$.

To graph the function, we find any asymptotes. Since the value $x = -1$ makes the denominator 0 but not the numerator, the line $x = -1$ is a vertical asymptote. To find the horizontal asymptote, we find

$$\lim_{x\to\infty} \frac{x-1}{x+1} = \lim_{x\to\infty} \frac{1-1/x}{1+1/x} \qquad \text{Divide numerator and denominator by } x.$$

$$= 1.$$

We get the same limit as x approaches $-\infty$, so the graph has the line $y = 1$ as a horizontal asymptote. Using this information, as well as the x-intercept (1, 0) and the y-intercept $(0,-1)$, gives the graph in Figure 9. TRY YOUR TURN 4

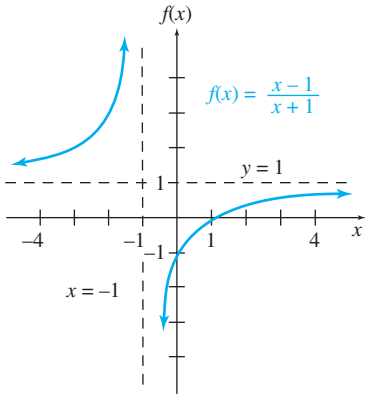


FIGURE 9

YOUR TURN 4 Find where f is increasing and decreasing if $f(x) = \dfrac{-2x}{x+2}$. Graph the function.

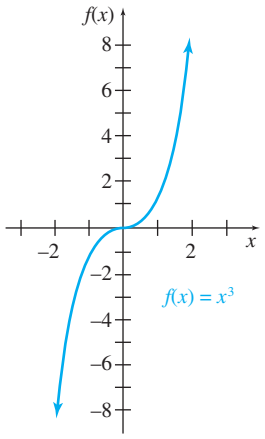


FIGURE 10

CAUTION It is important to note that the reverse of the test for increasing and decreasing functions is not true—it is possible for a function to be increasing on an interval even though the derivative is not positive at every point in the interval. A good example is given by $f(x) = x^3$, which is increasing on every interval, even though $f'(x) = 0$ when $x = 0$. See Figure 10.

Similarly, it is incorrect to assume that the sign of the derivative in regions separated by critical numbers must alternate between + and −. If this were always so, it would lead to a simple rule for finding the sign of the derivative: just check one test point, and then make the other regions alternate in sign. But this is not true if one of the factors in the derivative is raised to an even power. In the function $f(x) = x^3$ just considered, $f'(x) = 3x^2$ is positive on both sides of the critical number $x = 0$.

TECHNOLOGY NOTE

A graphing calculator can be used to find the derivative of a function at a particular x-value. The screen in Figure 11 supports our results in Example 2 for the test values, -4 and 2. (Notice that the calculator screen does not show the entire command.) The results are not exact because the calculator uses a numerical method to approximate the derivative at the given x-value.

Some graphing calculators can find where a function changes from increasing to decreasing by finding a maximum or minimum. The calculator windows in Figure 12 show this feature for the function in Example 2. Note that these, too, are approximations. This concept will be explored further in the next section.

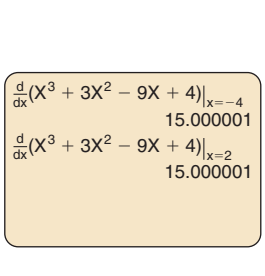


FIGURE 11

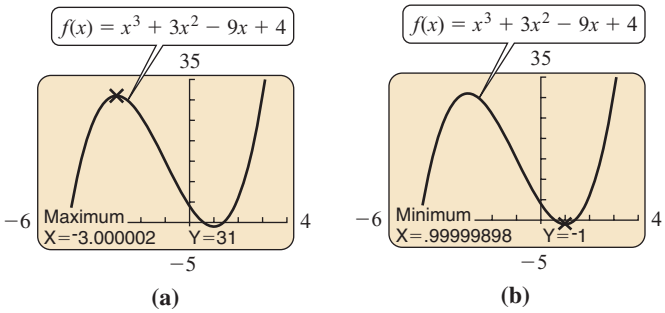


FIGURE 12

Knowing the intervals where a function is increasing or decreasing can be important in applications, as shown by the next examples.

EXAMPLE 5 Profit Analysis

A company selling computers finds that the cost per computer decreases linearly with the number sold monthly, decreasing from $1000 when none are sold to $800 when 1000 are sold. Thus, the average cost function has a y-intercept of 1000 and a slope of $-200/1000 = -0.2$, so it is given by the formula

$$\overline{C}(x) = 1000 - 0.2x, \quad 0 \le x \le 1000,$$

where x is the number of computers sold monthly. Since $\overline{C}(x) = C(x)/x$, the cost function is given by

$$\begin{aligned} C(x) = x\overline{C}(x) &= x(1000 - 0.2x) \\ &= 1000x - 0.2x^2, \quad 0 \le x \le 1000. \end{aligned}$$

Suppose the revenue function can be approximated by

$$R(x) = 0.0008x^3 - 2.4x^2 + 2400x, \quad 0 \le x \le 1000.$$

Determine any intervals on which the profit function is increasing.

APPLY IT

SOLUTION First find the profit function $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (0.0008x^3 - 2.4x^2 + 2400x) - (1000x - 0.2x^2) \\ &= 0.0008x^3 - 2.2x^2 + 1400x \end{aligned}$$

To find any intervals where this function is increasing, set $P'(x) = 0$.

$$P'(x) = 0.0024x^2 - 4.4x + 1400 = 0$$

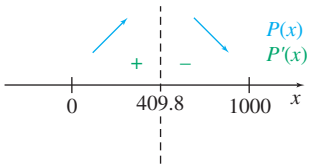


FIGURE 13

Solving this with the quadratic formula gives the approximate solutions $x = 409.8$ and $x = 1423.6$. The latter number is outside of the domain. Use $x = 409.8$ to determine two intervals on a number line, as shown in Figure 13. Choose $x = 0$ and $x = 1000$ as test points.

$$P'(0) = 0.0024(0^2) - 4.4(0) + 1400 = 1400$$
$$P'(1000) = 0.0024(1000^2) - 4.4(1000) + 1400 = -600$$

This means that when no computers are sold monthly, the profit is going up at a rate of \$1400 per computer. When 1000 computers are sold monthly, the profit is going down at a rate of \$600 per computer. The test points show that the function increases on $(0, 409.8)$ and decreases on $(409.8, 1000)$. See Figure 13. Thus, the profit is increasing when 409 computers or fewer are sold, and decreasing when 410 or more are sold, as shown in Figure 14.

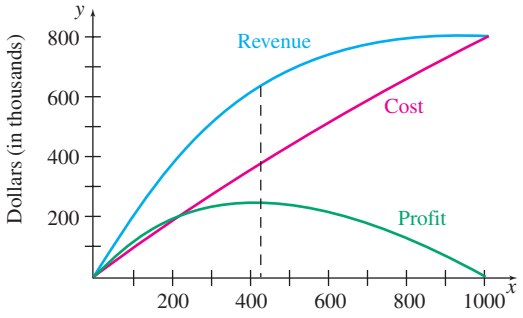


FIGURE 14

As the graph in Figure 14 shows, the profit will increase as long as the revenue function increases faster than the cost function. That is, increasing production will produce more profit as long as the marginal revenue is greater than the marginal cost.

EXAMPLE 6 Recollection of Facts

In the exercises in the previous chapter, the function

$$f(t) = \frac{90t}{99t - 90}$$

gave the number of facts recalled after t hours for $t > 10/11$. Find the intervals in which $f(t)$ is increasing or decreasing.

SOLUTION First use the quotient rule to find the derivative, $f'(t)$.

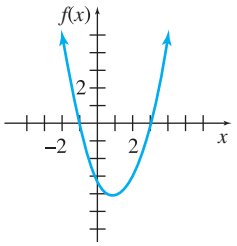
$$\begin{aligned} f'(t) &= \frac{(99t - 90)(90) - 90t(99)}{(99t - 90)^2} \\ &= \frac{8910t - 8100 - 8910t}{(99t - 90)^2} = \frac{-8100}{(99t - 90)^2} \end{aligned}$$

Since $(99t - 90)^2$ is positive everywhere in the domain of the function and since the numerator is a negative constant, $f'(t) < 0$ for all t in the domain of $f(t)$. Thus $f(t)$ always decreases and, as expected, the number of words recalled decreases steadily over time.

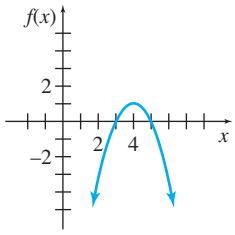
13.1 EXERCISES

Find the open intervals where the functions graphed as follows are (a) increasing, or (b) decreasing.

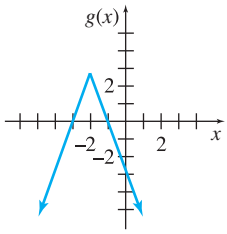
1.



2.



3.



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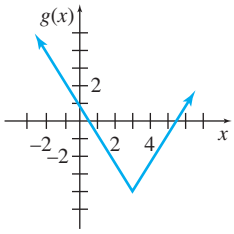
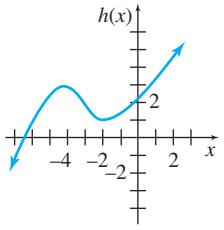
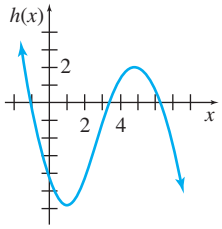


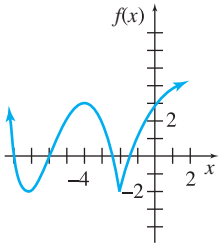
5.



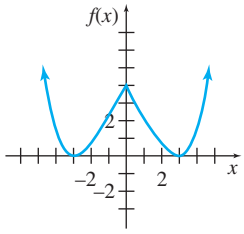
6.



7.



8.



For each of the exercises listed below, suppose that the function that is graphed is not $f(x)$, but $f'(x)$. Find the open intervals where $f(x)$ is (a) increasing or (b) decreasing.

9. Exercise 1

10. Exercise 2

11. Exercise 7

12. Exercise 8

For each function, find (a) the critical numbers; (b) the open intervals where the function is increasing; and (c) the open intervals where it is decreasing.

13. $y = 2.3 + 3.4x - 1.2x^2$

14. $y = 1.1 - 0.3x - 0.3x^2$

15. $f(x) = \frac{2}{3}x^3 - x^2 - 24x - 4$

16. $f(x) = \frac{2}{3}x^3 - x^2 - 4x + 2$

17. $f(x) = 4x^3 - 15x^2 - 72x + 5$

18. $f(x) = 4x^3 - 9x^2 - 30x + 6$

19. $f(x) = x^4 + 4x^3 + 4x^2 + 1$

20. $f(x) = 3x^4 + 8x^3 - 18x^2 + 5$

21. $y = -3x + 6$

22. $y = 6x - 9$

23. $f(x) = \frac{x+2}{x+1}$

24. $f(x) = \frac{x+3}{x-4}$

25. $y = \sqrt{x^2+1}$

26. $y = x\sqrt{9-x^2}$

27. $f(x) = x^{2/3}$

28. $f(x) = (x+1)^{4/5}$

29. $y = x - 4\ln(3x - 9)$

30. $f(x) = \ln\frac{5x^2+4}{x^2+1}$

31. $f(x) = xe^{-3x}$

32. $f(x) = xe^{x^2-3x}$

33. $f(x) = x^2 2^{-x}$

34. $f(x) = x2^{-x^2}$

35. $y = x^{2/3} - x^{5/3}$

36. $y = x^{1/3} + x^{4/3}$

37. A friend looks at the graph of $y = x^2$ and observes that if you start at the origin, the graph increases whether you go to the right or the left, so the graph is increasing everywhere. Explain why this reasoning is incorrect.

38. Use the techniques of this chapter to find the vertex and intervals where f is increasing and decreasing, given

$$f(x) = ax^2 + bx + c,$$

where we assume $a > 0$. Verify that this agrees with what we found in Chapter 10.

39. Repeat Exercise 38 under the assumption $a < 0$.

40. Where is the function defined by $f(x) = e^x$ increasing? Decreasing? Where is the tangent line horizontal?

41. Repeat Exercise 40 with the function defined by $f(x) = \ln x$.

42. a. For the function in Exercise 15, find the average of the critical numbers.

b. For the function in Exercise 15, use a graphing calculator to find the roots of the function, and then find the average of those roots.

c. Compare your answers to parts a and b. What do you notice?

d. Repeat part a for the function in Exercise 17.

e. Repeat part b for the function in Exercise 17.

f. Compare your answers to parts d and e. What do you notice?

It can be shown that the average of the roots of a polynomial (including the complex roots, if there are any) and the critical numbers of a polynomial (including complex roots of $f'(x) = 0$, if there are any) are always equal. ***Source: The Mathematics Teacher.***

For each of the following functions, use a graphing calculator to find the open intervals where $f(x)$ is (a) increasing, or (b) decreasing.

43. $f(x) = e^{0.001x} - \ln x$

44. $f(x) = \ln(x^2+1) - x^{0.3}$

APPLICATIONS

Business and Economics

45. Housing Starts A county realty group estimates that the number of housing starts per year over the next three years will be

$$H(r) = \frac{300}{1 + 0.03r^2},$$

where r is the mortgage rate (in percent).

a. Where is $H(r)$ increasing?

b. Where is $H(r)$ decreasing?

46. Cost Suppose the total cost $C(x)$ (in dollars) to manufacture a quantity x of weed killer (in hundreds of liters) is given by

$$C(x) = x^3 - 2x^2 + 8x + 50.$$

a. Where is $C(x)$ decreasing?

b. Where is $C(x)$ increasing?

47. Profit A manufacturer sells video games with the following cost and revenue functions (in dollars), where x is the number of games sold, for $0 \le x \le 3300$.

$$C(x) = 0.32x^2 - 0.00004x^3$$
$$R(x) = 0.848x^2 - 0.0002x^3$$

Determine the interval(s) on which the profit function is increasing.

48. Profit A manufacturer of CD players has determined that the profit $P(x)$ (in thousands of dollars) is related to the quantity x of CD players produced (in hundreds) per month by

$$P(x) = -(x - 4)e^x - 4, \qquad 0 < x \le 3.9.$$

a. At what production levels is the profit increasing?

b. At what levels is it decreasing?

49. Social Security Assets The projected year-end assets in the Social Security trust funds, in trillions of dollars, where t represents the number of years since 2000, can be approximated by

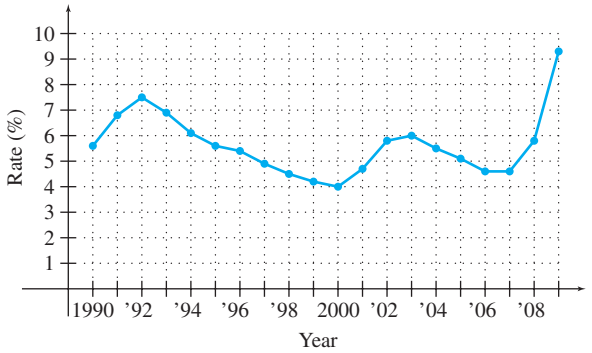
$$A(t) = 0.0000329t^3 - 0.00450t^2 + 0.0613t + 2.34,$$

where $0 \le t \le 50$. ***Source: Social Security Administration***.

a. Where is $A(t)$ increasing?

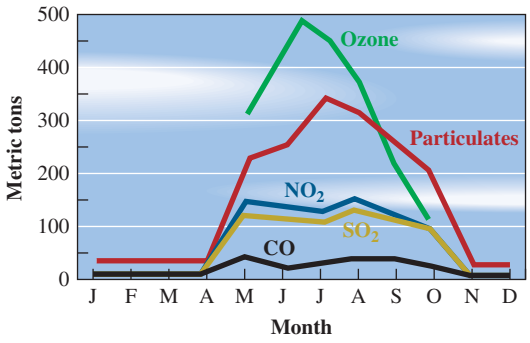
b. Where is $A(t)$ decreasing?

50. Unemployment The annual unemployment rates of the U.S. civilian noninstitutional population for 1990–2009 are shown in the graph. When is the function increasing? Decreasing? Constant? ***Source: Bureau of Labor Statistics***.



Life Sciences

51. Air Pollution The graph shows the amount of air pollution removed by trees in the Chicago urban region for each month of the year. From the graph we see, for example, that the ozone level starting in May increases up to June, and then abruptly decreases. ***Source: National Arbor Day Foundation***.



a. Are these curves the graphs of functions?

b. Look at the graph for particulates. Where is the function increasing? Decreasing? Constant?

c. On what intervals do all four lower graphs indicate that the corresponding functions are constant? Why do you think the functions are constant on those intervals?

52. Spread of Infection The number of people $P(t)$ (in hundreds) infected t days after an epidemic begins is approximated by

$$P(t) = \frac{10 \ln(0.19t + 1)}{0.19t + 1}.$$

When will the number of people infected start to decline?

53. Alcohol Concentration In Exercise 55 in the section on Polynomial and Rational Functions, we gave the function defined by

$$A(x) = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$$

as the approximate blood alcohol concentration in a 170-lb woman x hours after drinking 2 oz of alcohol on an empty stomach, for x in the interval $[0, 5]$. ***Source: Medicolegal Aspects of Alcohol Determination in Biological Specimens***.

a. On what time intervals is the alcohol concentration increasing?

b. On what intervals is it decreasing?

54. Drug Concentration The percent of concentration of a drug in the bloodstream x hours after the drug is administered is given by

$$K(x) = \frac{4x}{3x^2 + 27}.$$

a. On what time intervals is the concentration of the drug increasing?

b. On what intervals is it decreasing?

55. Drug Concentration Suppose a certain drug is administered to a patient, with the percent of concentration of the drug in the bloodstream t hours later given by

$$K(t) = \frac{5t}{t^2 + 1}.$$

a. On what time intervals is the concentration of the drug increasing?

b. On what intervals is it decreasing?

56. Cardiology The aortic pressure-diameter relation in a particular patient who underwent cardiac catheterization can be modeled by the polynomial

$$D(p) = 0.000002p^3 - 0.0008p^2 + 0.1141p + 16.683,$$
$$55 \le p \le 130,$$

where $D(p)$ is the aortic diameter (in millimeters) and p is the aortic pressure (in mmHg). Determine where this function is increasing and where it is decreasing within the interval given above. ***Source: Circulation***.

57. Thermic Effect of Food The metabolic rate of a person who has just eaten a meal tends to go up and then, after some time has passed, returns to a resting metabolic rate. This phenomenon is known as the thermic effect of food. Researchers have indicated that the thermic effect of food for one particular person is

$$F(t) = -10.28 + 175.9te^{-t/1.3},$$

where $F(t)$ is the thermic effect of food (in kJ/hr) and t is the number of hours that have elapsed since eating a meal. ***Source: American Journal of Clinical Nutrition***.

a. Find $F'(t)$.

b. Determine where this function is increasing and where it is decreasing. Interpret your answers.

58. Holstein Dairy Cattle Researchers have developed the following function that can be used to accurately predict the weight of Holstein cows (females) of various ages:

$$W_1(t) = 619(1 - 0.905e^{-0.002t})^{1.2386},$$

where $W_1(t)$ is the weight of the Holstein cow (in kilograms) that is t days old. Where is this function increasing? ***Source: Canadian Journal of Animal Science***.

Social Sciences

59. Population The standard normal probability function is used to describe many different populations. Its graph is the well-known normal curve. This function is defined by

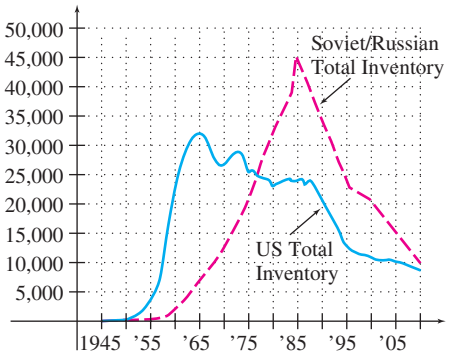
$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Give the intervals where the function is increasing and decreasing.

60. Nuclear Arsenals The figure shows estimated totals of nuclear weapons inventory for the United States and the Soviet Union (and its successor states) from 1945 to 2010. ***Source: Federation of American Scientists***.

a. On what intervals were the total inventories of both countries increasing?

b. On what intervals were the total inventories of both countries decreasing?



General Interest

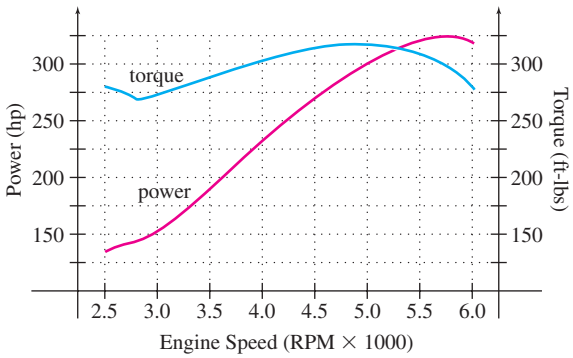
61. Sports Cars The following graph shows the horsepower and torque as a function of the engine speed for a 1964 Ford Mustang. ***Source: Online with Fuel Line Exhaust***.

a. On what intervals is the power increasing with engine speed?

b. On what intervals is the power decreasing with engine speed?

c. On what intervals is the torque increasing with engine speed?

d. On what intervals is the torque decreasing with engine speed?



62. Automobile Mileage As a mathematics professor loads more weight in the back of his Subaru, the mileage goes down. Let x be the amount of weight (in pounds) that he adds, and let $y = f(x)$ be the mileage (in mpg).

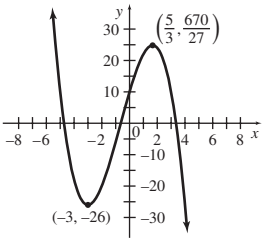
a. Is $f'(x)$ positive or negative? Explain.

b. What are the units of $f'(x)$?

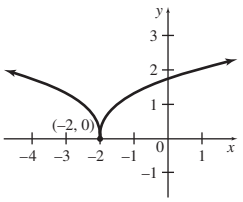
YOUR TURN ANSWERS

1. Increasing on $(-1, 2)$ and $(4, \infty)$. Decreasing on $(-\infty, -1)$ and $(2, 4)$.

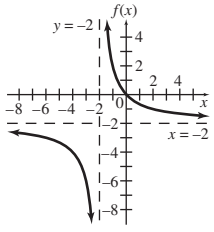
2. Increasing on $(-3, 5/3)$. Decreasing on $(-\infty, -3)$ and $(5/3, \infty)$



3. Increasing on $(-2, \infty)$ and decreasing on $(-\infty, -2)$.



4. Never increasing. Decreasing on $(-\infty, -2)$ and $(-2, \infty)$.



13.2 Relative Extrema

APPLY IT **In a 30-second commercial, when is the best time to present the sales message?**

We will answer this question in Example 1 by investigating the idea of a relative maximum.

As we have seen throughout this text, the graph of a function may have peaks and valleys. It is important in many applications to determine where these points occur. For example, if the function represents the profit of a company, these peaks and valleys indicate maximum profits and losses. When the function is given as an equation, we can use the derivative to determine these points, as shown in the first example.

EXAMPLE 1 Maximizing Viewer's Attention

Suppose that the manufacturer of a diet soft drink is disappointed by sales after airing a new series of 30-second television commercials. The company's market research analysts hypothesize that the problem lies in the timing of the commercial's message, Drink Sparkling Light. Either it comes too early in the commercial, before the viewer has become involved; or it comes too late, after the viewer's attention has faded. After extensive experimentation, the research group finds that the percent of full attention that a viewer devotes to a commercial is a function of time (in seconds) since the commercial began, where

$$\text{Viewer's attention} = f(t) = -\frac{3}{20}t^2 + 6t + 20, \quad 0 \le t \le 30.$$

When is the best time to present the commercial's sales message?

APPLY IT **SOLUTION** Clearly, the message should be delivered when the viewer's attention is at a maximum. To find this time, find $f'(t)$.

$$f'(t) = -\frac{3}{10}t + 6 = -0.3t + 6$$

The derivative $f'(t)$ is greater than 0 when

$$-0.3t + 6 > 0,$$
$$-3t > -60, \text{ or}$$
$$t < 20.$$

Similarly, $f'(t) < 0$ when $-0.3t + 6 < 0$, or $t > 20$. Thus, attention increases for the first 20 seconds and decreases for the last 10 seconds. The message should appear about 20 seconds into the commercial. At that time the viewer will devote $f(20) = 80\%$ of his attention to the commercial.

The maximum level of viewer attention (80%) in Example 1 is a *relative maximum*, defined as follows.

Relative Maximum or Minimum

Let c be a number in the domain of a function f. Then $f(c)$ is a **relative** (or **local**) **maximum** for f if there exists an open interval (a, b) containing c such that

$$f(x) \le f(c)$$

for all x in (a, b).

Likewise, $f(c)$ is a **relative** (or **local**) **minimum** for f if there exists an open interval (a, b) containing c such that

$$f(x) \ge f(c)$$

for all x in (a, b). (*continued*)

A function has a **relative** (or **local**) **extremum** (plural: **extrema**) at c if it has either a relative maximum or a relative minimum there.

If c is an endpoint of the domain of f, we only consider x in the half-open interval that is in the domain.*

The intuitive idea is that a relative maximum is the greatest value of the function in some region right around the point, although there may be greater values elsewhere. For example, the highest value of the Dow Jones industrial average this week is a relative maximum, although the Dow may have reached a higher value earlier this year. Similarly, a relative minimum is the least value of a function in some region around the point.

A simple way to view these concepts is that a relative maximum is a peak, and a relative minimum is the bottom of a valley, although either a relative minimum or maximum can also occur at the endpoint of the domain.

NOTE
Recall from Section 10.3 on Polynomials and Rational Functions that a relative extremum that is not an endpoint is also referred to as a turning point.

EXAMPLE 2 Relative Extrema

Identify the x-values of all points where the graph in Figure 15 has relative extrema.

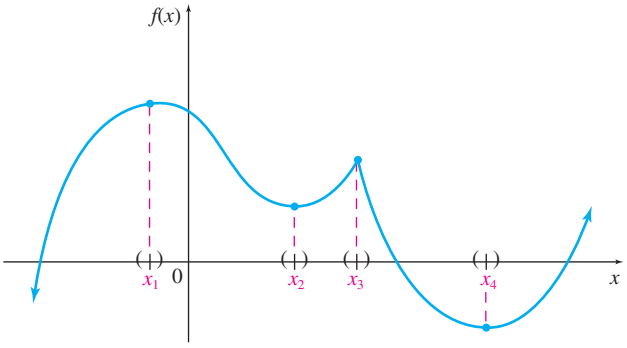
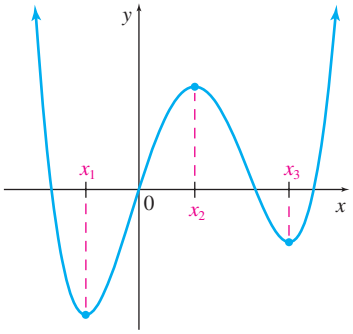


FIGURE 15

SOLUTION The parentheses around x_1 show an open interval containing x_1 such that $f(x) \le f(x_1)$, so there is a relative maximum of $f(x_1)$ at $x = x_1$. Notice that many other open intervals would work just as well. Similar intervals around x_2, x_3, and x_4 can be used to find a relative maximum of $f(x_3)$ at $x = x_3$ and relative minima of $f(x_2)$ at $x = x_2$ and $f(x_4)$ at $x = x_4$. TRY YOUR TURN 1

YOUR TURN 1 Identify the x-values of all points where the graph has relative extrema.



The function graphed in Figure 16 has relative maxima when $x = x_1$ or $x = x_3$ and relative minima when $x = x_2$ or $x = x_4$. The tangent lines at the points having x-values x_1 and

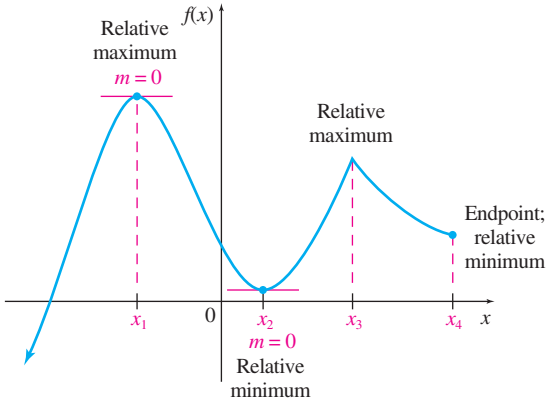


FIGURE 16

*There is disagreement on calling an endpoint a maximum or minimum. We define it this way because this is an applied calculus book, and in an application it would be considered a maximum or minimum value of the function.

x_2 are shown in the figure. Both tangent lines are horizontal and have slope 0. There is no single tangent line at the point where $x = x_3$.

Since the derivative of a function gives the slope of a line tangent to the graph of the function, to find relative extrema we first identify all critical numbers and endpoints. A relative extremum *may* exist at a critical number. (A rough sketch of the graph of the function near a critical number often is enough to tell whether an extremum has been found.) These facts about extrema are summarized below.

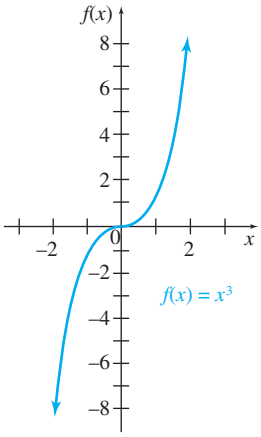


FIGURE 17

If a function f has a relative extremum at c, then c is a critical number or c is an endpoint of the domain.

CAUTION Be very careful not to get this result backward. It does *not* say that a function has relative extrema at all critical numbers of the function. For example, Figure 17 shows the graph of $f(x) = x^3$. The derivative, $f'(x) = 3x^2$, is 0 when $x = 0$, so that 0 is a critical number for that function. However, as suggested by the graph of Figure 17, $f(x) = x^3$ has neither a relative maximum nor a relative minimum at $x = 0$ (or anywhere else, for that matter). A critical number is a candidate for the location of a relative extremum, but only a candidate.

First Derivative Test

Suppose all critical numbers have been found for some function f. How is it possible to tell from the equation of the function whether these critical numbers produce relative maxima, relative minima, or neither? One way is suggested by the graph in Figure 18.

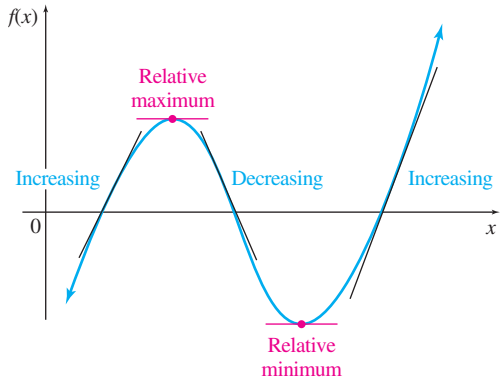


FIGURE 18

As shown in Figure 18, on the left of a relative maximum the tangent lines to the graph of a function have positive slopes, indicating that the function is increasing. At the relative maximum, the tangent line is horizontal. On the right of the relative maximum the tangent lines have negative slopes, indicating that the function is decreasing. Around a relative minimum the opposite occurs. As shown by the tangent lines in Figure 18, the function is decreasing on the left of the relative minimum, has a horizontal tangent at the minimum, and is increasing on the right of the minimum.

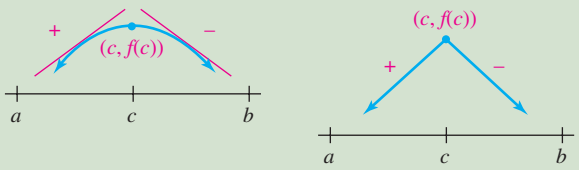
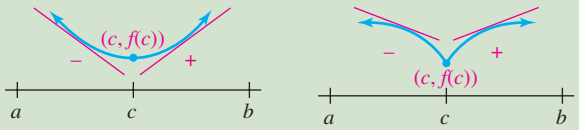
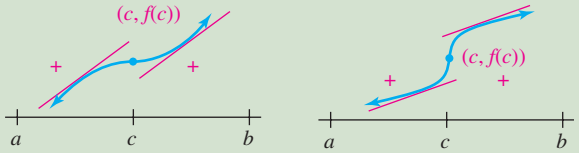
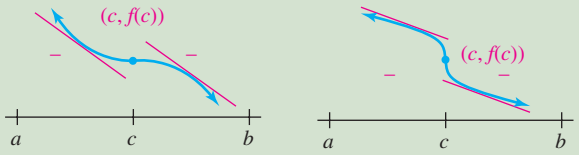
Putting this together with the methods from Section 1 for identifying intervals where a function is increasing or decreasing gives the following **first derivative test** for locating relative extrema.

First Derivative Test

Let c be a critical number for a function f. Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c, and that c is the only critical number for f in (a, b).

1. $f(c)$ is a relative maximum of f if the derivative $f'(x)$ is positive in the interval (a, c) and negative in the interval (c, b).
2. $f(c)$ is a relative minimum of f if the derivative $f'(x)$ is negative in the interval (a, c) and positive in the interval (c, b).

The sketches in the following table show how the first derivative test works. Assume the same conditions on a, b, and c for the table as those given for the first derivative test.

Relative Extrema			
$f(x)$ has:	Sign of f' in (a, c)	Sign of f' in (c, b)	Sketches
Relative maximum	+	−	
Relative minimum	−	+	
No relative extrema	+	+	
No relative extrema	−	−	

EXAMPLE 3 Relative Extrema

Find all relative extrema for the following functions, as well as where each function is increasing and decreasing.

(a) $f(x) = 2x^3 - 3x^2 - 72x + 15$

Method I
First Derivative Test

SOLUTION

The derivative is $f'(x) = 6x^2 - 6x - 72$. There are no points where $f'(x)$ fails to exist, so the only critical numbers will be found where the derivative equals 0. Setting the derivative equal to 0 gives

$$\begin{aligned} 6x^2 - 6x - 72 &= 0 \\ 6(x^2 - x - 12) &= 0 \\ 6(x - 4)(x + 3) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad x + 3 = 0 \\ x = 4 \quad &\text{or} \quad x = -3. \end{aligned}$$

As in the previous section, the critical numbers 4 and -3 are used to determine the three intervals $(-\infty, -3)$, $(-3, 4)$, and $(4, \infty)$ shown on the number line in Figure 19.

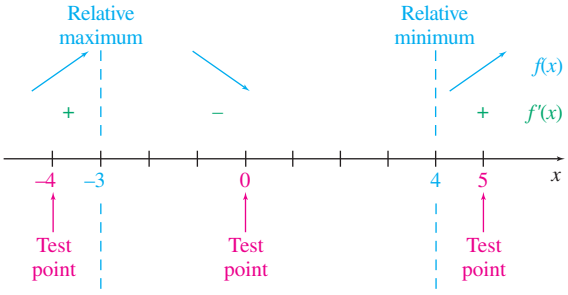


FIGURE 19

Any number from each of the three intervals can be used as a test point to find the sign of f' in each interval. Using -4, 0, and 5 gives the following information.

$$f'(-4) = 6(-8)(-1) > 0$$
$$f'(0) = 6(-4)(3) < 0$$
$$f'(5) = 6(1)(8) > 0$$

Thus, the derivative is positive on $(-\infty, -3)$, negative on $(-3, 4)$, and positive on $(4, \infty)$. By Part 1 of the first derivative test, this means that the function has a relative maximum of $f(-3) = 150$ when $x = -3$; by Part 2, f has a relative minimum of $f(4) = -193$ when $x = 4$. The function is increasing on $(-\infty, -3)$ and $(4, \infty)$ and decreasing on $(-3, 4)$. The graph is shown in Figure 20. TRY YOUR TURN 2

YOUR TURN 2 Find all relative extrema of $f(x) = -x^3 - 2x^2 + 15x + 10$.

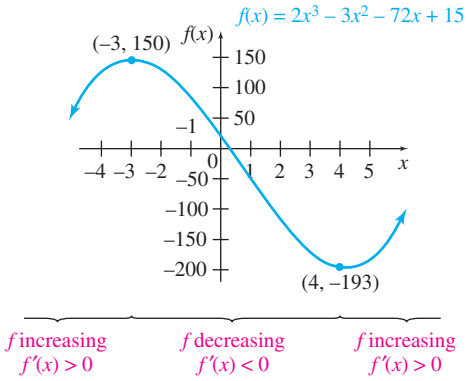


FIGURE 20

Method 2 Graphing Calculator

Many graphing calculators can locate a relative extremum when supplied with an interval containing the extremum. For example, after graphing the function $f(x) = 2x^3 - 3x^2 - 72x + 15$ on a TI-84 Plus, we selected "maximum" from the CALC menu and entered a left bound of -4 and a right bound of 0. The calculator asks for an initial guess, but in this example it doesn't matter what we enter. The result of this process, as well as a similar process for finding the relative minimum, is shown in Figure 21.

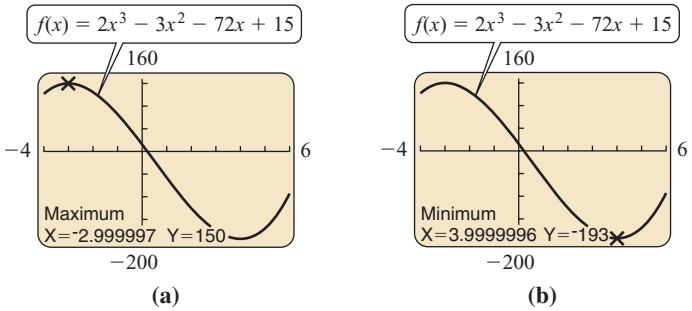


FIGURE 21

Another way to verify the extrema with a graphing calculator is to graph $y = f'(x)$ and find where the graph crosses the x-axis. Figure 22 shows the result of this approach for finding the relative minimum of the previous function.

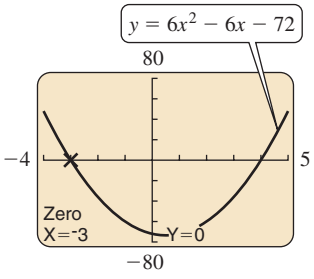


FIGURE 22

(b) $f(x) = 6x^{2/3} - 4x$

SOLUTION Find $f'(x)$.

$$f'(x) = 4x^{-1/3} - 4 = \frac{4}{x^{1/3}} - 4$$

The derivative fails to exist when $x = 0$, but the function itself is defined when $x = 0$, making 0 a critical number for f. To find other critical numbers, set $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{4}{x^{1/3}} - 4 &= 0 \\ \frac{4}{x^{1/3}} &= 4 \\ 4 &= 4x^{1/3} && \text{Multiply both sides by } x^{1/3}. \\ 1 &= x^{1/3} && \text{Divide both sides by } 4. \\ 1 &= x && \text{Cube both sides.} \end{aligned}$$

The critical numbers 0 and 1 are used to locate the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ on a number line as in Figure 23. Evaluating $f'(x)$ at the test points -1, $1/2$, and 2 and using the first derivative test shows that f has a relative maximum at $x = 1$; the value of this relative maximum is $f(1) = 2$. Also, f has a relative minimum at $x = 0$; this relative minimum is $f(0) = 0$. The function is increasing on $(0, 1)$ and decreasing on $(-\infty, 0)$ and $(1, \infty)$. Notice that the graph, shown in Figure 24, has a sharp point at the critical number where the derivative does not exist. In the last section of this chapter we will show how to verify other features of the graph. TRY YOUR TURN 3

YOUR TURN 3 Find all relative extrema of $f(x) = x^{2/3} - x^{5/3}$.

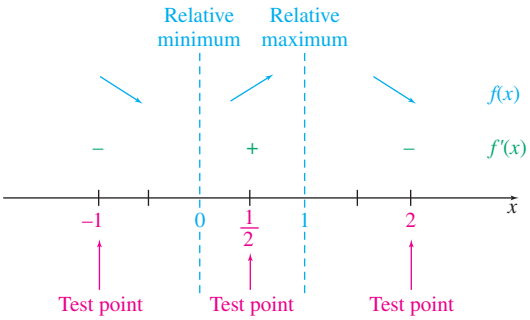


FIGURE 23

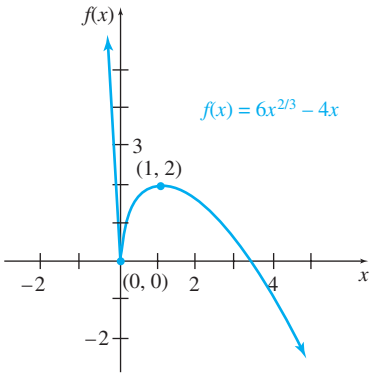


FIGURE 24

(c) $f(x) = xe^{2-x^2}$

SOLUTION The derivative, found by using the product rule and the chain rule, is

$$\begin{aligned} f'(x) &= x(-2x)e^{2-x^2} + e^{2-x^2} \\ &= e^{2-x^2}(-2x^2 + 1). \end{aligned}$$

This expression exists for all x in the domain of f. Since e^{2-x^2} is always positive, the derivative is 0 when

$$\begin{aligned} -2x^2 + 1 &= 0 \\ 1 &= 2x^2 \\ \frac{1}{2} &= x^2 \\ x &= \pm\sqrt{1/2} \\ x &= \pm\frac{1}{\sqrt{2}} \approx \pm 0.707. \end{aligned}$$

FOR REVIEW

Recall that $e^x > 0$ for all x, so there can never be a solution to $e^{g(x)} = 0$ for any function $g(x)$.

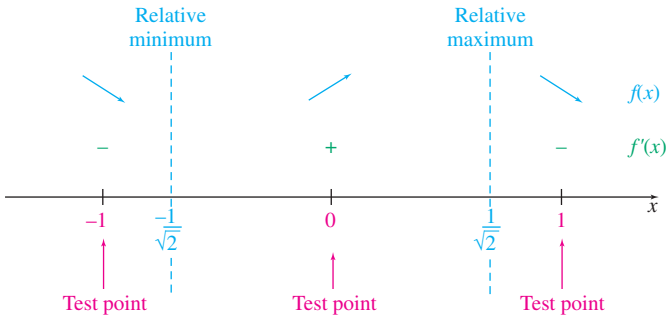


FIGURE 25

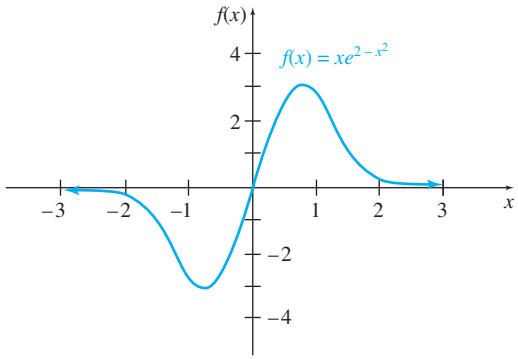


FIGURE 26

There are two critical points, $-1/\sqrt{2}$ and $1/\sqrt{2}$. Using test points of -1, 0, and 1 gives the results shown in Figure 25.

The function has a relative minimum at $-1/\sqrt{2}$ of $f(-1/\sqrt{2}) \approx -3.17$ and a relative maximum at $1/\sqrt{2}$ of $f(1/\sqrt{2}) \approx 3.17$. It is decreasing on the interval $(-\infty, -1/\sqrt{2})$, increasing on the interval $(-1/\sqrt{2}, 1/\sqrt{2})$, and decreasing on the interval $(1/\sqrt{2}, \infty)$. The graph is shown in Figure 26. TRY YOUR TURN 4

YOUR TURN 4 Find all relative extrema of $f(x) = x^2e^x$.

CAUTION A critical number must be in the domain of the function. For example, the derivative of $f(x) = x/(x-4)$ is $f'(x) = -4/(x-4)^2$, which fails to exist when $x = 4$. But $f(4)$ does not exist, so 4 is not a critical number, and the function has no relative extrema.

As mentioned at the beginning of this section, finding the maximum or minimum value of a quantity is important in applications of mathematics. The final example gives a further illustration.

EXAMPLE 4 Bicycle Sales

A small company manufactures and sells bicycles. The production manager has determined that the cost and demand functions for q $(q \geq 0)$ bicycles per week are

$$C(q) = 10 + 5q + \frac{1}{60}q^3 \quad \text{and} \quad p = D(q) = 90 - q,$$

where p is the price per bicycle.

(a) Find the maximum weekly revenue.

SOLUTION The revenue each week is given by

$$R(q) = qp = q(90 - q) = 90q - q^2.$$

To maximize $R(q) = 90q - q^2$, find $R'(q)$. Then find the critical numbers.

$$\begin{aligned} R'(q) &= 90 - 2q = 0 \\ 90 &= 2q \\ q &= 45 \end{aligned}$$

Since $R'(q)$ exists for all q, 45 is the only critical number. To verify that $q = 45$ will produce a *maximum*, evaluate the derivative on both sides of $q = 45$.

$$R'(40) = 10 \quad \text{and} \quad R'(50) = -10$$

This shows that $R(q)$ is increasing up to $q = 45$, then decreasing, so there is a maximum value at $q = 45$ of $R(45) = 2025$. The maximum revenue will be \$2025 and will occur when 45 bicycles are produced and sold each week.

(b) Find the maximum weekly profit.

SOLUTION Since profit equals revenue minus cost, the profit is given by

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= (90q - q^2) - \left(10 + 5q + \frac{1}{60}q^3\right) \\ &= -\frac{1}{60}q^3 - q^2 + 85q - 10. \end{aligned}$$

Find the derivative and set it equal to 0 to find the critical numbers. (The derivative exists for all q.)

$$P'(q) = -\frac{1}{20}q^2 - 2q + 85 = 0$$

Solving this equation by the quadratic formula gives the solutions $q \approx 25.8$ and $q \approx -65.8$. Since q cannot be negative, the only critical number of concern is 25.8. Determine whether $q = 25.8$ produces a maximum by testing a value on either side of 25.8 in $P'(q)$.

$$P'(0) = 85 \qquad \text{and} \qquad P'(40) = -75$$

These results show that $P(q)$ increases to $q = 25.8$ and then decreases. Since q must be an integer, evaluate $P(q)$ at $q = 25$ and $q = 26$. Since $P(25) = 1229.58$ and $P(26) = 1231.07$, the maximum value occurs when $q = 26$. Thus, the maximum profit of \$1231.07 occurs when 26 bicycles are produced and sold each week. Notice that this is not the same as the number that should be produced to yield maximum revenue.

(c) Find the price the company should charge to realize maximum profit.

SOLUTION As shown in part (b), 26 bicycles per week should be produced and sold to get the maximum profit of \$1231.07 per week. Since the price is given by

$$p = 90 - q,$$

if $q = 26$, then $p = 64$. The manager should charge \$64 per bicycle and produce and sell 26 bicycles per week to get the maximum profit of \$1231.07 per week. Figure 27 shows the graphs of the functions used in this example. Notice that the slopes of the revenue and cost functions are the same at the point where the maximum profit occurs. Why is this true?

YOUR TURN 5 Find the maximum weekly profit and the price a company should charge to realize maximum profit if $C(q) = 100 + 10q$ and $p = D(q) = 50 - 2q$.

TRY YOUR TURN 5

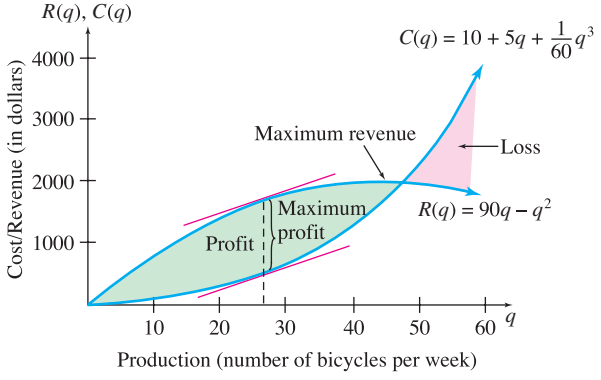


FIGURE 27

CAUTION Be careful to give the y-value of the point where an extremum occurs. Although we solve the equation $f'(x) = 0$ for x to find the extremum, the maximum or minimum value of the function is the corresponding y-value. Thus, in Example 4(a), we found that at $q = 45$, the maximum weekly revenue is \$2025 (not \$45).

The examples in this section involving the maximization of a quadratic function, such as the advertising example and the bicycle revenue example, could be solved by the methods described in Chapter 10 on Nonlinear Functions. But those involving more complicated functions, such as the bicycle profit example, are difficult to analyze without the tools of calculus.

Finding extrema for realistic problems requires an accurate mathematical model of the problem. In particular, it is important to be aware of restrictions on the values of the variables. For example, if $T(x)$ closely approximates the number of items that can be manufactured daily on a production line when x is the number of employees on the line, x must certainly be restricted to the positive integers or perhaps to a few common fractional values. (We can imagine half-time workers, but not 1/49-time workers.)

On the other hand, to apply the tools of calculus to obtain an extremum for some function, the function must be defined and be meaningful at every real number in some interval. Because of this, the answer obtained from a mathematical model might be a number that is not feasible in the actual problem.

Usually, the requirement that a continuous function be used, rather than one that can take on only certain selected values, is of theoretical interest only. In most cases, the methods of calculus give acceptable results as long as the assumptions of continuity and differentiability are not totally unreasonable. If they lead to the conclusion, say, that $80\sqrt{2}$ workers should be hired, it is usually only necessary to investigate acceptable values close to $80\sqrt{2}$. This was done in Example 4.

13.2 EXERCISES

Find the locations and values of all relative extrema for the functions with graphs as follows. Compare with Exercises 1–8 in the preceding section.

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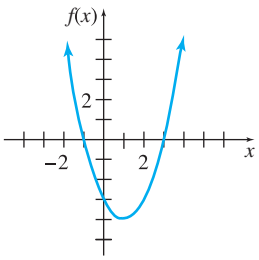

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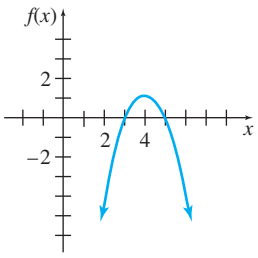

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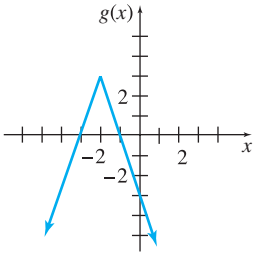

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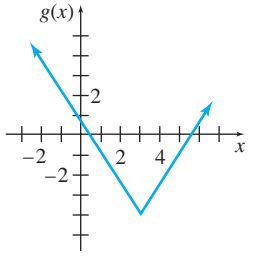

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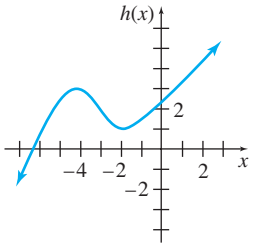

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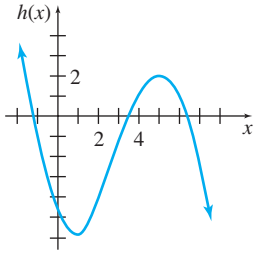

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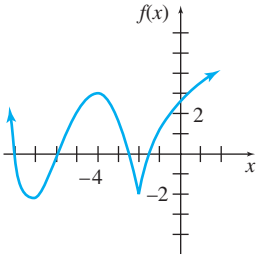

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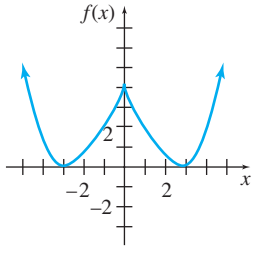


For each of the exercises listed below, suppose that the function that is graphed is not $f(x)$ but $f'(x)$. Find the locations of all relative extrema, and tell whether each extremum is a relative maximum or minimum.

9. Exercise 1 **10.** Exercise 2

11. Exercise 7 **12.** Exercise 8

Find the x-value of all points where the functions defined as follows have any relative extrema. Find the value(s) of any relative extrema.

13. $f(x) = x^2 - 10x + 33$ **14.** $f(x) = x^2 + 8x + 5$

15. $f(x) = x^3 + 6x^2 + 9x - 8$

16. $f(x) = x^3 + 3x^2 - 24x + 2$

17. $f(x) = -\frac{4}{3}x^3 - \frac{21}{2}x^2 - 5x + 8$

18. $f(x) = -\frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x - 4$

19. $f(x) = x^4 - 18x^2 - 4$ **20.** $f(x) = x^4 - 8x^2 + 9$

21. $f(x) = 3 - (8 + 3x)^{2/3}$ **22.** $f(x) = \frac{(5 - 9x)^{2/3}}{7} + 1$

23. $f(x) = 2x + 3x^{2/3}$ **24.** $f(x) = 3x^{5/3} - 15x^{2/3}$

25. $f(x) = x - \frac{1}{x}$ **26.** $f(x) = x^2 + \frac{1}{x}$

27. $f(x) = \frac{x^2 - 2x + 1}{x - 3}$ **28.** $f(x) = \frac{x^2 - 6x + 9}{x + 2}$

29. $f(x) = x^2e^x - 3$ **30.** $f(x) = 3xe^x + 2$

31. $f(x) = 2x + \ln x$ **32.** $f(x) = \frac{x^2}{\ln x}$

33. $f(x) = \frac{2^x}{x}$ **34.** $f(x) = x + 8^{-x}$

Use the derivative to find the vertex of each parabola.

35. $y = -2x^2 + 12x - 5$ **36.** $y = ax^2 + bx + c$

Graph each function on a graphing calculator, and then use the graph to find all relative extrema (to three decimal places). Then confirm your answer by finding the derivative and using the calculator to solve the equation $f'(x) = 0$.

37. $f(x) = x^5 - x^4 + 4x^3 - 30x^2 + 5x + 6$

38. $f(x) = -x^5 - x^4 + 2x^3 - 25x^2 + 9x + 12$

39. Graph $f(x) = 2|x + 1| + 4|x - 5| - 20$ with a graphing calculator in the window $[-10, 10]$ by $[-15, 30]$. Use the graph and the function to determine the x-values of all extrema.

40. Consider the function

$$g(x) = \frac{1}{x^{12}} - 2\left(\frac{1000}{x}\right)^6.$$

Source: Mathematics Teacher.

a. Using a graphing calculator, try to find any local minima, or tell why finding a local minimum is difficult for this function.

b. Find any local minima using the techniques of calculus.

c. Based on your results in parts a and b, describe circumstances under which relative extrema are easier to find using the techniques of calculus than using a graphing calculator.

APPLICATIONS

Business and Economics

Profit In Exercises 41–44, find (a) the number, q, of units that produces maximum profit; (b) the price, p, per unit that produces maximum profit; and (c) the maximum profit, P.

41. $C(q) = 80 + 18q; \quad p = 70 - 2q$

42. $C(q) = 25q + 5000; \quad p = 90 - 0.02q$

43. $C(q) = 100 + 20qe^{-0.01q}; \quad p = 40e^{-0.01q}$

44. $C(q) = 21.047q + 3; \quad p = 50 - 5\ln(q + 10)$

45. Power On August 8, 2007, the power used in New York state (in thousands of megawatts) could be approximated by the function

$$P(t) = -0.005846t^3 + 0.1614t^2 - 0.4910t + 20.47,$$

where t is the number of hours since midnight, for $0 \le t \le 24$. Find any relative extrema for power usage, as well as when they occurred. ***Source: Current Energy.***

46. Profit The total profit $P(x)$ (in thousands of dollars) from the sale of x units of a certain prescription drug is given by

$$P(x) = \ln(-x^3 + 3x^2 + 72x + 1)$$

for x in $[0, 10]$.

a. Find the number of units that should be sold in order to maximize the total profit.

b. What is the maximum profit?

47. Revenue The demand equation for telephones at one store is

$$p = D(q) = 200e^{-0.1q},$$

where p is the price (in dollars) and q is the quantity of telephones sold per week. Find the values of q and p that maximize revenue.

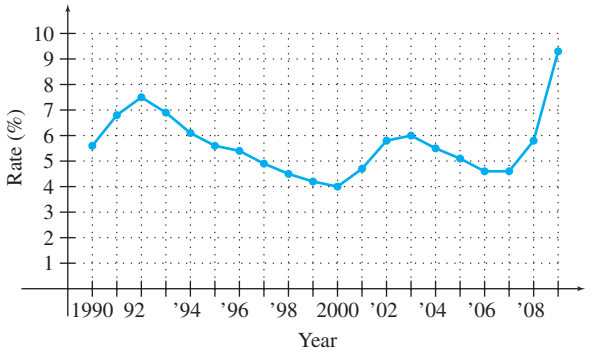
48. Revenue The demand equation for one type of computer networking system is

$$p = D(q) = 500qe^{-0.0016q^2},$$

where p is the price (in dollars) and q is the quantity of servers sold per month. Find the values of q and p that maximize revenue.

49. Cost Suppose that the cost function for a product is given by $C(x) = 0.002x^3 + 9x + 6912$. Find the production level (i.e., value of x) that will produce the minimum average cost per unit $\overline{C}(x)$.

50. Unemployment The annual unemployment rates of the U.S. civilian noninstitutional population for 1990–2009 are shown in the graph. Identify the years where relative extrema occur, and estimate the unemployment rate at each of these years. *Source: Bureau of Labor Statistics*.



Life Sciences

51. Activity Level In the summer the activity level of a certain type of lizard varies according to the time of day. A biologist has determined that the activity level is given by the function

$$a(t) = 0.008t^3 - 0.288t^2 + 2.304t + 7,$$

where t is the number of hours after 12 noon. When is the activity level highest? When is it lowest?

52. Milk Consumption The average individual daily milk consumption for herds of Charolais, Angus, and Hereford calves can be described by the function

$$M(t) = 6.281t^{0.242}e^{-0.025t}, \qquad 1 \le t \le 26,$$

where $M(t)$ is the milk consumption (in kilograms) and t is the age of the calf (in weeks). *Source: Animal Production*.

a. Find the time in which the maximum daily consumption occurs and the maximum daily consumption.

b. If the general formula for this model is given by

$$M(t) = at^be^{-ct},$$

find the time where the maximum consumption occurs and the maximum consumption. (*Hint:* Express your answer in terms of *a, b,* and *c.*)

53. Alaskan Moose The mathematical relationship between the age of a captive female moose and its mass can be described by the function

$$M(t) = 369(0.93)^t t^{0.36}, \quad t \le 12,$$

where $M(t)$ is the mass of the moose (in kilograms) and t is the age (in years) of the moose. Find the age at which the mass of a female moose is maximized. What is the maximum mass? *Source: Journal of Wildlife Management*.

54. Thermic Effect of Food As we saw in the last section, the metabolic rate after a person eats a meal tends to go up and then, after some time has passed, returns to a resting metabolic rate. This phenomenon is known as the thermic effect of food and can be described for a particular individual as

$$F(t) = -10.28 + 175.9te^{-t/1.3},$$

where $F(t)$ is the thermic effect of food (in kJ/hr), and t is the number of hours that have elapsed since eating a meal. Find the time after the meal when the thermic effect of the food is maximized. *Source: American Journal of Clinical Nutrition*.

Social Sciences

55. Attitude Change Social psychologists have found that as the discrepancy between the views of a speaker and those of an audience increases, the attitude change in the audience also increases to a point but decreases when the discrepancy becomes too large, particularly if the communicator is viewed by the audience as having low credibility. Suppose that the degree of change can be approximated by the function

$$D(x) = -x^4 + 8x^3 + 80x^2,$$

where x is the discrepancy between the views of the speaker and those of the audience, as measured by scores on a questionnaire. Find the amount of discrepancy the speaker should aim for to maximize the attitude change in the audience. *Source: Journal of Personality and Social Psychology*.

56. Film Length A group of researchers found that people prefer training films of moderate length; shorter films contain too little information, while longer films are boring. For a training film on the care of exotic birds, the researchers determined that the ratings people gave for the film could be approximated by

$$R(t) = \frac{20t}{t^2 + 100},$$

where t is the length of the film (in minutes). Find the film length that received the highest rating.

Physical Sciences

57. Height After a great deal of experimentation, two Atlantic Institute of Technology senior physics majors determined that when a bottle of French champagne is shaken several times, held upright, and uncorked, its cork travels according to

$$s(t) = -16t^2 + 40t + 3,$$

where s is its height (in feet) above the ground t seconds after being released.

a. How high will it go?

b. How long is it in the air?

YOUR TURN ANSWERS

1. Relative maximum of $f(x_2)$ at $x = x_2$; relative minima of $f(x_1)$ at $x = x_1$ and $f(x_3)$ at $x = x_3$.
2. Relative maximum of $f(5/3) = 670/27 \approx 24.8$ at $x = 5/3$ and relative minimum of $f(-3) = -26$ at $x = -3$.
3. Relative maximum of $f\left(\frac{2}{5}\right) = \frac{3}{5}\left(\frac{2}{5}\right)^{2/3} \approx 0.3257$ at $x = 2/5$ and relative minimum of $f(0) = 0$ at $x = 0$.
4. Relative maximum of $f(-2) = 4e^{-2} \approx 0.5413$ at $x = -2$ and relative minimum of $f(0) = 0$ at $x = 0$.
5. Maximum weekly profit is \$100 when $q = 10$ and the company should charge \$30 per item.

13.3 Higher Derivatives, Concavity, and the Second Derivative Test

APPLY IT **Just because the price of a stock is increasing, does that alone make it a good investment?**

We will address this question in Example 1.

In the first section of this chapter, we used the derivative to determine intervals where a function is increasing or decreasing. For example, if the function represents the price of a stock, we can use the derivative to determine when the price is increasing. In addition, it would be important for us to know how the *rate of increase* is changing. We can determine how the *rate of increase* (or the *rate of decrease*) is changing by determining the rate of change of the derivative of the function. In other words, we can find the derivative of the derivative, called the **second derivative**, as shown in the following example.

EXAMPLE 1 Stock Prices

Suppose a friend is trying to get you to invest in the stock of a young company. The following function represents the price $P(t)$ of the company's stock since it became available two years ago:

$$P(t) = 17 + t^{1/2},$$

where t is the number of months since the stock became available. He claims that the price of the stock is always increasing and that you will make a fortune on it. Verify his claims. Is the price of the stock increasing? How fast? How much will you make if you invest now?

APPLY IT **SOLUTION** The derivative of $P(t)$,

$$P'(t) = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}},$$

is always positive because $\sqrt{t}$ is positive for $t > 0$. This means that the price function $P(t)$ is always increasing. But *how fast* is it increasing?

The derivative $P'(t) = 1/(2\sqrt{t})$ tells how fast the price is increasing at any number of months, t, since the stock became available. For example, when $t = 1$ month, $P'(t) = 1/2$, and the price is increasing at 1/2 dollar, or 50 cents, per month. When $t = 4$ months, $P'(t) = 1/4$; the stock is increasing at 25 cents per month. By the time you buy in at $t = 24$ months, the price is increasing at 10 cents per month, and the *rate of increase* looks as though it will continue to decrease.

In general, the rate of increase in P' is given by the derivative of $P'(t)$, called the second derivative and denoted by $P''(t)$. Since $P'(t) = (1/2)t^{-1/2}$,

$$P''(t) = -\frac{1}{4}t^{-3/2} = -\frac{1}{4\sqrt{t^3}}.$$

$P''(t)$ is negative for $t > 0$ and, therefore, confirms the suspicion that the *rate* of increase in price does indeed decrease for all $t > 0$. The price of the company's stock will not drop, but the amount of return will certainly not be the fortune your friend predicts.

If you invest now, at $t = 24$ months, the price would be \$21.90. A year later, it would be worth \$23 a share. If you were rich enough to buy 100 shares for \$21.90 each, the total investment would be worth \$2300 in a year. The increase of \$110 is about 5% of the investment. The only investors to make a lot of money on this stock would be those who bought early, when the rate of increase was much greater.

As mentioned earlier, the second derivative of a function f, written f'', gives the rate of change of the *derivative* of f. Before continuing to discuss applications of the second derivative, we need to introduce some additional terminology and notation.

Higher Derivatives

If a function f has a derivative f', then the derivative of f', if it exists, is the second derivative of f, written f''. The derivative of f'', if it exists, is called the **third derivative** of f, and so on. By continuing this process, we can find **fourth derivatives** and other higher derivatives. For example, if $f(x) = x^4 + 2x^3 + 3x^2 - 5x + 7$, then

$$f'(x) = 4x^3 + 6x^2 + 6x - 5, \quad \text{First derivative of } f$$
$$f''(x) = 12x^2 + 12x + 6, \quad \text{Second derivative of } f$$
$$f'''(x) = 24x + 12, \quad \text{Third derivative of } f$$

and

$$f^{(4)}(x) = 24. \quad \text{Fourth derivative of } f$$

Notation for Higher Derivatives

The second derivative of $y = f(x)$ can be written using any of the following notations:

$$f''(x), \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad D_x^2[f(x)].$$

The third derivative can be written in a similar way. For $n \geq 4$, the nth derivative is written $f^{(n)}(x)$.

CAUTION Notice the difference in notation between $f^{(4)}(x)$, which indicates the fourth derivative of $f(x)$, and $f^4(x)$, which indicates $f(x)$ raised to the fourth power.

EXAMPLE 2 Second Derivative

Let $f(x) = x^3 + 6x^2 - 9x + 8$.

(a) Find $f''(x)$.

SOLUTION To find the second derivative of $f(x)$, find the first derivative, and then take its derivative.

$$f'(x) = 3x^2 + 12x - 9$$
$$f''(x) = 6x + 12$$

(b) Find $f''(0)$.

SOLUTION Since $f''(x) = 6x + 12$,

$$f''(0) = 6(0) + 12 = 12.$$

YOUR TURN 1 Find $f''(1)$ if $f(x) = 5x^4 - 4x^3 + 3x$.

TRY YOUR TURN 1

EXAMPLE 3 Second Derivative

Find the second derivative for the functions defined as follows.

(a) $f(x) = (x^2 - 1)^2$

SOLUTION Here, using the chain rule,

$$f'(x) = 2(x^2 - 1)(2x) = 4x(x^2 - 1).$$

Use the product rule to find $f''(x)$.

$$f''(x) = 4x(2x) + (x^2 - 1)(4)$$
$$= 8x^2 + 4x^2 - 4$$
$$= 12x^2 - 4$$

(b) $g(x) = 4x(\ln x)$

SOLUTION Use the product rule.

$$g'(x) = 4x \cdot \frac{1}{x} + (\ln x) \cdot 4 = 4 + 4(\ln x)$$

$$g''(x) = 0 + 4 \cdot \frac{1}{x} = \frac{4}{x}$$

(c) $h(x) = \dfrac{x}{e^x}$

SOLUTION Here, we need the quotient rule.

$$h'(x) = \frac{e^x - xe^x}{(e^x)^2} = \frac{e^x(1 - x)}{(e^x)^2} = \frac{1 - x}{e^x}$$

$$h''(x) = \frac{e^x(-1) - (1 - x)e^x}{(e^x)^2} = \frac{e^x(-1 - 1 + x)}{(e^x)^2} = \frac{-2 + x}{e^x}$$

YOUR TURN 2 Find the second derivative for
(a) $f(x) = (x^3 + 1)^2$
(b) $g(x) = xe^x$
(c) $h(x) = \dfrac{\ln x}{x}$.

TRY YOUR TURN 2

Earlier, we saw that the first derivative of a function represents the rate of change of the function. The second derivative, then, represents the rate of change of the first derivative. If a function describes the position of a vehicle (along a straight line) at time t, then the first derivative gives the velocity of the vehicle. That is, if $y = s(t)$ describes the position (along a straight line) of the vehicle at time t, then $v(t) = s'(t)$ gives the velocity at time t.

We also saw that *velocity* is the rate of change of distance with respect to time. Recall, the difference between velocity and speed is that velocity may be positive or negative, whereas speed is always positive. A negative velocity indicates travel in a negative direction (backing up) with regard to the starting point; positive velocity indicates travel in the positive direction (going forward) from the starting point.

The instantaneous rate of change of velocity is called **acceleration**. Since instantaneous rate of change is the same as the derivative, acceleration is the derivative of velocity. Thus if $a(t)$ represents the acceleration at time t, then

$$a(t) = \frac{d}{dt}v(t) = s''(t).$$

If the velocity is positive and the acceleration is positive, the velocity is increasing, so the vehicle is speeding up. If the velocity is positive and the acceleration is negative, the vehicle is slowing down. A negative velocity and a positive acceleration mean the vehicle is backing up and slowing down. If both the velocity and acceleration are negative, the vehicle is speeding up in the negative direction.

EXAMPLE 4 Velocity and Acceleration

Suppose a car is moving in a straight line, with its position from a starting point (in feet) at time t (in seconds) given by

$$s(t) = t^3 - 2t^2 - 7t + 9.$$

Find the following.

(a) The velocity at any time t

SOLUTION The velocity is given by

$$v(t) = s'(t) = 3t^2 - 4t - 7$$

feet per second.

(b) The acceleration at any time t

SOLUTION Acceleration is given by

$$a(t) = v'(t) = s''(t) = 6t - 4$$

feet per second per second.

(c) The time intervals (for $t \geq 0$) when the car is going forward or backing up

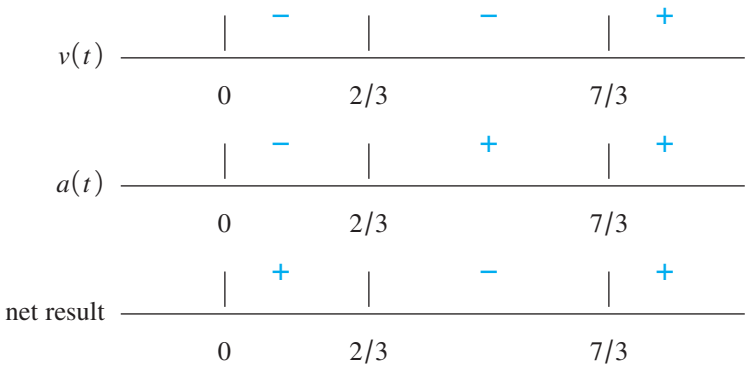
SOLUTION We first find when the velocity is 0, that is, when the car is stopped.

$$\begin{aligned} v(t) &= 3t^2 - 4t - 7 = 0 \\ (3t - 7)(t + 1) &= 0 \\ t = 7/3 \quad &\text{or} \quad t = -1 \end{aligned}$$

We are interested in $t \geq 0$. Choose a value of t in each of the intervals $(0, 7/3)$ and $(7/3, \infty)$ to see that the velocity is negative in $(0, 7/3)$ and positive in $(7/3, \infty)$. The car is backing up for the first 7/3 seconds, then going forward.

(d) The time intervals (for $t \geq 0$) when the car is speeding up or slowing down

SOLUTION The car will speed up when the velocity and acceleration are the same sign and slow down when they have opposite signs. Here, the acceleration is positive when $6t - 4 > 0$, that is, $t > 2/3$ seconds, and negative for $t < 2/3$ seconds. Since the velocity is negative in $(0, 7/3)$ and positive in $(7/3, \infty)$, the car is speeding up for $0 < t < 2/3$ seconds, slowing down for $2/3 < t < 7/3$ seconds, and speeding up again for $t > 7/3$ seconds. See the sign graphs.

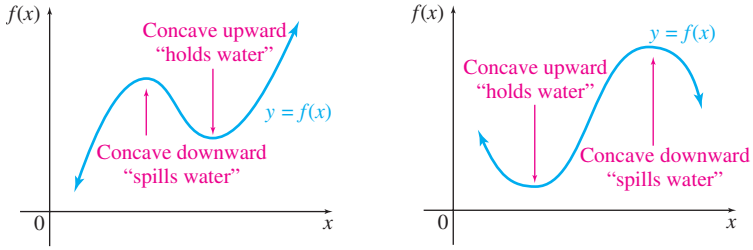


YOUR TURN 3 Find the velocity and acceleration of the car if the distance (in feet) is given by $s(t) = t^3 - 3t^2 - 24t + 10$, at time t (in seconds). When is the car going forward or backing up? When is the car speeding up or slowing down?

TRY YOUR TURN 3

Concavity of a Graph

The first derivative has been used to show where a function is increasing or decreasing and where the extrema occur. The second derivative gives the rate of change of the first derivative; it indicates *how fast* the function is increasing or decreasing. The rate of change of the derivative (the second derivative) affects the *shape* of the graph. Intuitively, we say that a graph is *concave upward* on an interval if it "holds water" and *concave downward* if it "spills water." See Figure 28.



(a) **(b)**

FIGURE 28

More precisely, a function is **concave upward** on an interval (a, b) if the graph of the function lies above its tangent line at each point of (a, b). A function is **concave downward** on (a, b) if the graph of the function lies below its tangent line at each point of (a, b). A point where a graph changes **concavity** is called an **inflection point**. See Figure 29.

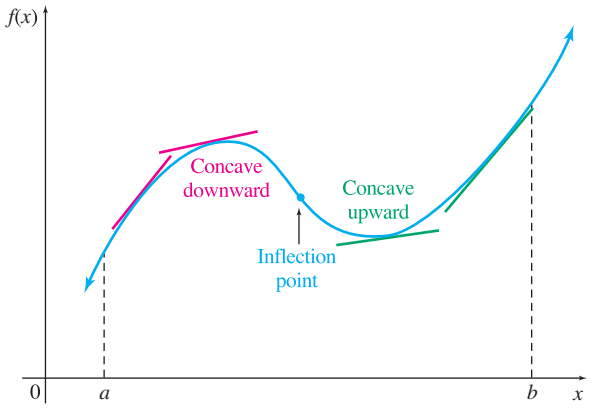


FIGURE 29

Users of soft contact lenses recognize concavity as the way to tell if a lens is inside out. As Figure 30 shows, a correct contact lens has a profile that is entirely concave upward. The profile of an inside-out lens has inflection points near the edges, where the profile begins to turn concave downward very slightly.

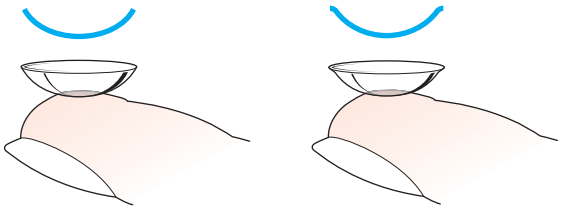


FIGURE 30

Just as a function can be either increasing or decreasing on an interval, it can be either concave upward or concave downward on an interval. Examples of various combinations are shown in Figure 31.

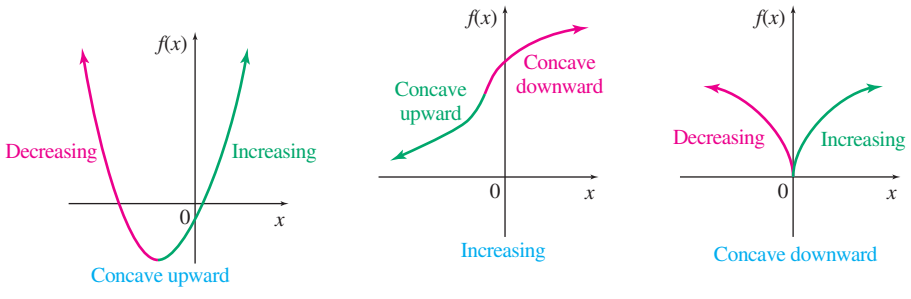


FIGURE 31

Figure 32 shows two functions that are concave upward on an interval (a, b) Several tangent lines are also shown. In Figure 32(a), the slopes of the tangent lines (moving from left to right) are first negative, then 0, and then positive. In Figure 32(b), the slopes are all positive, but they get larger.

In both cases, the slopes are *increasing*. The slope at a point on a curve is given by the derivative. Since a function is increasing if its derivative is positive, its slope is increasing if the derivative of the slope function is positive. Since the derivative of a derivative is the second derivative, a function is concave upward on an interval if its second derivative is positive at each point of the interval.

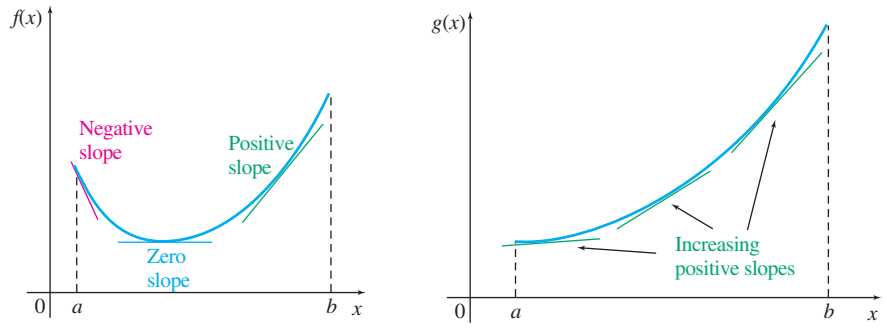


FIGURE 32

A similar result is suggested by Figure 33 for functions whose graphs are concave downward. In both graphs, the slopes of the tangent lines are *decreasing* as we move from left to right. Since a function is decreasing if its derivative is negative, a function is concave downward on an interval if its second derivative is negative at each point of the interval. These observations suggest the following test.

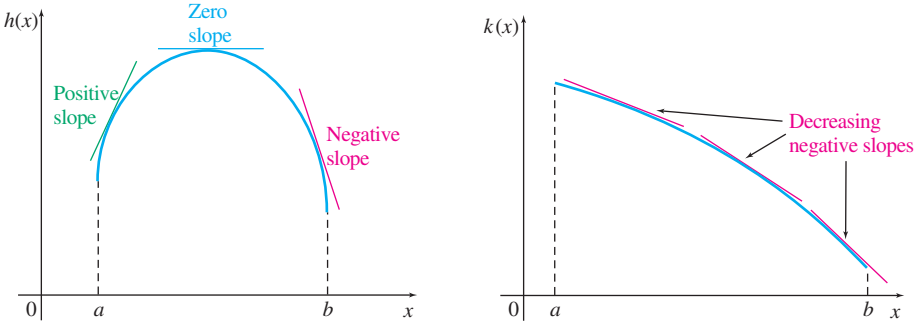


FIGURE 33

Test for Concavity

Let f be a function with derivatives f' and f'' existing at all points in an interval (a, b). Then f is concave upward on (a, b) if $f''(x) > 0$ for all x in (a, b) and concave downward on (a, b) if $f''(x) < 0$ for all x in (a, b).

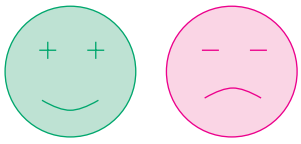


FIGURE 34

An easy way to remember this test is by the faces shown in Figure 34. When the second derivative is positive at a point $(+\ +)$, the graph is concave upward $(\smile)$. When the second derivative is negative at a point $(-\ -)$, the graph is concave downward $(\frown)$.

EXAMPLE 5 Concavity

Find all intervals where $f(x) = x^4 - 8x^3 + 18x^2$ is concave upward or downward, and find all inflection points.

SOLUTION The first derivative is $f'(x) = 4x^3 - 24x^2 + 36x$, and the second derivative is $f''(x) = 12x^2 - 48x + 36$. We factor $f''(x)$ as $12(x - 1)(x - 3)$, and then create a number line for $f''(x)$ as we did in the previous two sections for $f'(x)$.

We see from Figure 35 that $f''(x) > 0$ on the intervals $(-\infty, 1)$ and $(3, \infty)$, so f is concave upward on these intervals. Also, $f''(x) < 0$ on the interval $(1, 3)$, so f is concave downward on this interval.

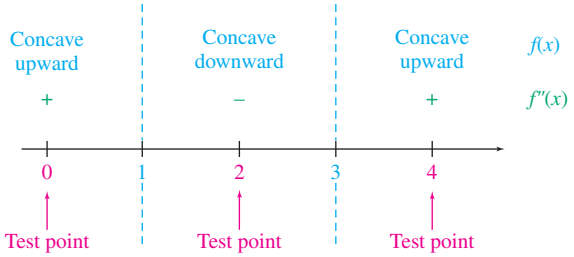


FIGURE 35

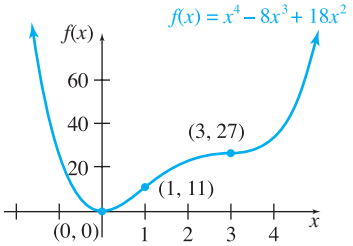


FIGURE 36

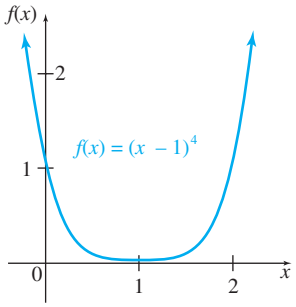
Finally, we have inflection points where f'' changes sign, namely, at $x = 1$ and $x = 3$. Since $f(1) = 11$ and $f(3) = 27$, the inflection points are $(1, 11)$ and $(3, 27)$.

Although we were only seeking information about concavity and inflection points in this example, it is also worth noting that $f'(x) = 4x^3 - 24x^2 + 36x = 4x(x - 3)^2$, which has roots at $x = 0$ and $x = 3$. Verify that there is a relative minimum at (0, 0), but that (3, 27) is neither a relative minimum nor a relative maximum. The function is graphed in Figure 36. TRY YOUR TURN 4

YOUR TURN 4 Find all intervals where $f(x) = x^5 - 30x^3$ is concave upward or downward, and find all inflection points.

Example 5 suggests the following result.

At an inflection point for a function f, the second derivative is 0 or does not exist.



Second derivative is 0 at $x = 1$, but $(1, f(1))$ is not an inflection point.

FIGURE 37

CAUTION

1. Be careful with the previous statement. Finding a value of x where $f''(x) = 0$ does not mean that an inflection point has been located. For example, if $f(x) = (x - 1)^4$, then $f''(x) = 12(x - 1)^2$, which is 0 at $x = 1$. The graph of $f(x) = (x - 1)^4$ is always concave upward, however, so it has no inflection point. See Figure 37.
2. Note that the concavity of a function might change not only at a point where $f''(x) = 0$ but also where $f''(x)$ does not exist. For example, this happens at $x = 0$ for $f(x) = x^{1/3}$.

TECHNOLOGY NOTE

Most graphing calculators do not have a feature for finding inflection points. Nevertheless, a graphing calculator sketch can be useful for verifying that your calculations for finding inflection points and intervals where the function is concave up or down are correct.

Second Derivative Test

The idea of concavity can often be used to decide whether a given critical number produces a relative maximum or a relative minimum. This test, an alternative to the first derivative test, is based on the fact that a curve with a horizontal tangent at a point c and concave downward on an open interval containing c also has a relative maximum at c. A relative minimum occurs when a graph has a horizontal tangent at a point d and is concave upward on an open interval containing d. See Figure 38.

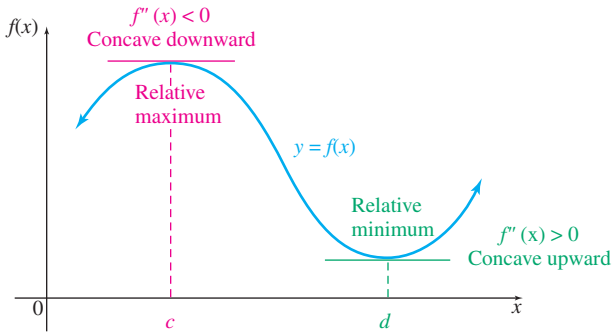


FIGURE 38

A function f is concave upward on an interval if $f''(x) > 0$ for all x in the interval, while f is concave downward on an interval if $f''(x) < 0$ for all x in the interval. These ideas lead to the **second derivative test** for relative extrema.

Second Derivative Test

Let f'' exist on some open interval containing c, (except possibly at c itself) and let $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or $f''(c)$ does not exist, then the test gives no information about extrema, so use the first derivative test.

NOTE
In Case 3 of the second derivative test (when $f''(c) = 0$ or does not exist), observe that if $f''(x)$ changes sign at c, there is an inflection point at $x = c$.

EXAMPLE 6 Second Derivative Test

Find all relative extrema for

$$f(x) = 4x^3 + 7x^2 - 10x + 8.$$

SOLUTION First, find the points where the derivative is 0. Here $f'(x) = 12x^2 + 14x - 10$. Solve the equation $f'(x) = 0$ to get

$$\begin{aligned} 12x^2 + 14x - 10 &= 0 \\ 2(6x^2 + 7x - 5) &= 0 \\ 2(3x + 5)(2x - 1) &= 0 \\ 3x + 5 = 0 \quad \text{or} \quad & 2x - 1 = 0 \\ 3x = -5 \qquad\quad & 2x = 1 \\ x = -\frac{5}{3} \qquad\quad & x = \frac{1}{2}. \end{aligned}$$

Now use the second derivative test. The second derivative is $f''(x) = 24x + 14$. Evaluate $f''(x)$ first at $-5/3$, getting

$$f''\left(-\frac{5}{3}\right) = 24\left(-\frac{5}{3}\right) + 14 = -40 + 14 = -26 < 0,$$

so that by Part 2 of the second derivative test, $-5/3$ leads to a relative maximum of $f(-5/3) = 691/27$. Also, when $x = 1/2$,

$$f''\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}\right) + 14 = 12 + 14 = 26 > 0,$$

with $1/2$ leading to a relative minimum of $f(1/2) = 21/4$.

YOUR TURN 5
Find all relative extrema of $f(x) = -2x^3 + 3x^2 + 72x$.

TRY YOUR TURN 5

CAUTION The second derivative test works only for those critical numbers c that make $f'(c) = 0$. This test does not work for critical numbers c for which $f'(c)$ does not exist (since $f''(c)$ would not exist either). Also, the second derivative test does not work for critical numbers c that make $f''(c) = 0$. In both of these cases, use the first derivative test.

The *law of diminishing returns* in economics is related to the idea of concavity. The function graphed in Figure 39 gives the output y from a given input x. If the input were advertising costs for some product, for example, the output might be the corresponding revenue from sales.

The graph in Figure 39 shows an inflection point at $(c, f(c))$. For $x < c$, the graph is concave upward, so the rate of change of the slope is increasing. This indicates that the output y is increasing at a faster rate with each additional dollar spent. When $x > c$, however, the graph is concave downward, the rate of change of the slope is decreasing, and the increase in y is smaller with each additional dollar spent. Thus, further input beyond c dollars produces diminishing returns. The inflection point at $(c, f(c))$ is called the **point of diminishing returns**. Beyond this point there is a smaller and smaller return for each dollar invested.

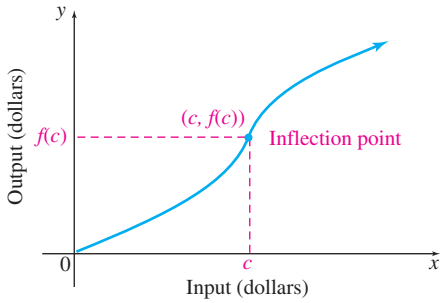


FIGURE 39

As another example of diminishing returns from agriculture, with a fixed amount of land, machinery, fertilizer, and so on, adding workers increases production a lot at first, then less and less with each additional worker.

EXAMPLE 7 Point of Diminishing Returns

The revenue $R(x)$ generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x) = \frac{1}{15{,}000}(600x^2 - x^3), \quad 0 \le x \le 600,$$

where x and $R(x)$ are in thousands of dollars. Is there a point of diminishing returns for this function? If so, what is it?

SOLUTION Since a point of diminishing returns occurs at an inflection point, look for an x-value that makes $R''(x) = 0$. Write the function as

$$R(x) = \frac{600}{15{,}000}x^2 - \frac{1}{15{,}000}x^3 = \frac{1}{25}x^2 - \frac{1}{15{,}000}x^3.$$

Now find $R'(x)$ and then $R''(x)$.

$$R'(x) = \frac{2x}{25} - \frac{3x^2}{15{,}000} = \frac{2}{25}x - \frac{1}{5000}x^2$$

$$R''(x) = \frac{2}{25} - \frac{1}{2500}x$$

Set $R''(x)$ equal to 0 and solve for x.

$$\frac{2}{25} - \frac{1}{2500}x = 0$$

$$-\frac{1}{2500}x = -\frac{2}{25}$$

$$x = \frac{5000}{25} = 200$$

Test a number in the interval $(0, 200)$ to see that $R''(x)$ is positive there. Then test a number in the interval $(200, 600)$ to find $R''(x)$ negative in that interval. Since the sign of $R''(x)$ changes from positive to negative at $x = 200$, the graph changes from concave upward to concave downward at that point, and there is a point of diminishing returns at the inflection point $(200, 1066\frac{2}{3})$. Investments in advertising beyond \$200,000 return less and less for each dollar invested. Verify that $R'(200) = 8$. This means that when \$200,000 is invested, another \$1000 invested returns approximately \$8000 in additional revenue. Thus it may still be economically sound to invest in advertising beyond the point of diminishing returns.

13.3 EXERCISES

Find $f''(x)$ for each function. Then find $f''(0)$ and $f''(2)$.

1. $f(x) = 5x^3 - 7x^2 + 4x + 3$

2. $f(x) = 4x^3 + 5x^2 + 6x - 7$

3. $f(x) = 4x^4 - 3x^3 - 2x^2 + 6$

4. $f(x) = -x^4 + 7x^3 - \dfrac{x^2}{2}$

5. $f(x) = 3x^2 - 4x + 8$

6. $f(x) = 8x^2 + 6x + 5$

7. $f(x) = \dfrac{x^2}{1 + x}$

8. $f(x) = \dfrac{-x}{1 - x^2}$

9. $f(x) = \sqrt{x^2 + 4}$

10. $f(x) = \sqrt{2x^2 + 9}$

11. $f(x) = 32x^{3/4}$

12. $f(x) = -6x^{1/3}$

13. $f(x) = 5e^{-x^2}$

14. $f(x) = 0.5e^{x^2}$

15. $f(x) = \dfrac{\ln x}{4x}$

16. $f(x) = \ln x + \dfrac{1}{x}$

Find $f'''(x)$, the third derivative of f, and $f^{(4)}(x)$, the fourth derivative of f, for each function.

17. $f(x) = 7x^4 + 6x^3 + 5x^2 + 4x + 3$

18. $f(x) = -2x^4 + 7x^3 + 4x^2 + x$

19. $f(x) = 5x^5 - 3x^4 + 2x^3 + 7x^2 + 4$

20. $f(x) = 2x^5 + 3x^4 - 5x^3 + 9x - 2$

21. $f(x) = \dfrac{x - 1}{x + 2}$

22. $f(x) = \dfrac{x + 1}{x}$

23. $f(x) = \dfrac{3x}{x - 2}$

24. $f(x) = \dfrac{x}{2x + 1}$

25. Let $f(x) = \ln x$.

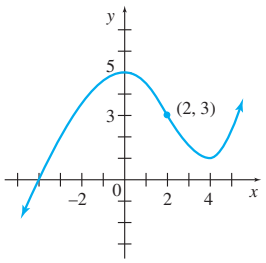
a. Compute $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, and $f^{(5)}(x)$.

b. Guess a formula for $f^{(n)}(x)$, where n is any positive integer.

26. For $f(x) = e^x$, find $f''(x)$ and $f'''(x)$. What is the nth derivative of f with respect to x?

In Exercises 27–48, find the open intervals where the functions are concave upward or concave downward. Find any inflection points.

27.



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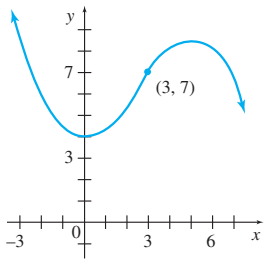

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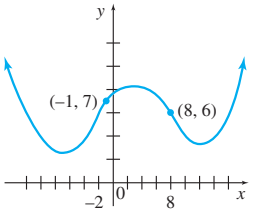

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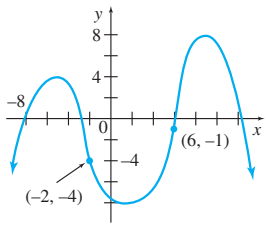

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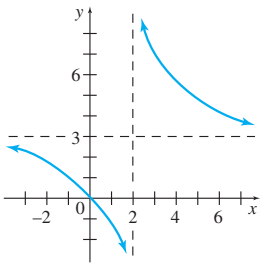

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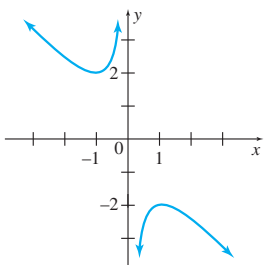


33. $f(x) = x^2 + 10x - 9$

34. $f(x) = 8 - 6x - x^2$

35. $f(x) = -2x^3 + 9x^2 + 168x - 3$

36. $f(x) = -x^3 - 12x^2 - 45x + 2$

37. $f(x) = \dfrac{3}{x - 5}$

38. $f(x) = \dfrac{-2}{x + 1}$

39. $f(x) = x(x + 5)^2$

40. $f(x) = -x(x - 3)^2$

41. $f(x) = 18x - 18e^{-x}$

42. $f(x) = 2e^{-x^2}$

43. $f(x) = x^{8/3} - 4x^{5/3}$

44. $f(x) = x^{7/3} + 56x^{4/3}$

45. $f(x) = \ln(x^2 + 1)$

46. $f(x) = x^2 + 8 \ln |x + 1|$

47. $f(x) = x^2 \log |x|$

48. $f(x) = 5^{-x^2}$

For each of the exercises listed below, suppose that the function that is graphed is not $f(x)$, but $f'(x)$. Find the open intervals where the original function is concave upward or concave downward, and find the location of any inflection points.

49. Exercise 27

50. Exercise 28

51. Exercise 29

52. Exercise 30

53. Give an example of a function $f(x)$ such that $f'(0) = 0$ but $f''(0)$ does not exist. Is there a relative minimum or maximum or an inflection point at $x = 0$?

54. a. Graph the two functions $f(x) = x^{7/3}$ and $g(x) = x^{5/3}$ on the window $[-2, 2]$ by $[-2, 2]$.

b. Verify that both f and g have an inflection point at $(0, 0)$.

c. How is the value of $f''(0)$ different from $g''(0)$?

d. Based on what you have seen so far in this exercise, is it always possible to tell the difference between a point where the second derivative is 0 or undefined based on the graph? Explain.

55. Describe the slope of the tangent line to the graph of $f(x) = e^x$ for the following.

a. $x \to -\infty$ **b.** $x \to 0$

56. What is true about the slope of the tangent line to the graph of $f(x) = \ln x$ as $x \to \infty$? As $x \to 0$?

Find any critical numbers for f in Exercises 57–64 and then use the second derivative test to decide whether the critical numbers lead to relative maxima or relative minima. If $f''(c) = 0$ or $f''(c)$ does not exist for a critical number c, then the second derivative test gives no information. In this case, use the first derivative test instead.

57. $f(x) = -x^2 - 10x - 25$

58. $f(x) = x^2 - 12x + 36$

59. $f(x) = 3x^3 - 3x^2 + 1$

60. $f(x) = 2x^3 - 4x^2 + 2$

61. $f(x) = (x + 3)^4$

62. $f(x) = x^3$

63. $f(x) = x^{7/3} + x^{4/3}$

64. $f(x) = x^{8/3} + x^{5/3}$

Sometimes the derivative of a function is known, but not the function. We will see more of this later in the book. For each function f' defined in Exercises 65–68, find $f''(x)$, then use a graphing calculator to graph f' and f'' in the indicated window. Use the graph to do the following.

a. Give the (approximate) x-values where f has a maximum or minimum.

b. By considering the sign of $f'(x)$, give the (approximate) intervals where $f(x)$ is increasing and decreasing.

c. Give the (approximate) x-values of any inflection points.

d. By considering the sign of $f''(x)$, give the intervals where f is concave upward or concave downward.

65. $f'(x) = x^3 - 6x^2 + 7x + 4$; $[-5, 5]$ by $[-5, 15]$

66. $f'(x) = 10x^2(x - 1)(5x - 3)$; $[-1, 1.5]$ by $[-20, 20]$

67. $f'(x) = \dfrac{1 - x^2}{(x^2 + 1)^2}$; $[-3, 3]$ by $[-1.5, 1.5]$

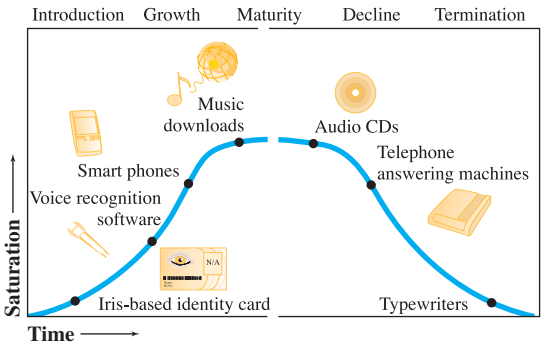
68. $f'(x) = x^2 + x \ln x$; $[0, 1]$ by $[-2, 2]$

69. Suppose a friend makes the following argument. A function f is increasing and concave downward. Therefore, f' is positive and decreasing, so it eventually becomes 0 and then negative, at which point f decreases. Show that your friend is wrong by giving an example of a function that is always increasing and concave downward.

APPLICATIONS

Business and Economics

70. Product Life Cycle The accompanying figure shows the *product life cycle* graph, with typical products marked on it. It illustrates the fact that a new product is often purchased at a faster and faster rate as people become familiar with it. In time, saturation is reached and the purchase rate stays constant until the product is made obsolete by newer products, after which it is purchased less and less. ***Source: tutor2u***.



a. Which products on the left side of the graph are closest to the left-hand inflection point? What does the inflection point mean here?

b. Which product on the right side of the graph is closest to the right-hand inflection point? What does the inflection point mean here?

c. Discuss where portable Blu-ray players, iPads, and other new technologies should be placed on the graph.

71. Social Security Assets As seen in the first section of this chapter, the projected year-end assets in the Social Security trust funds, in trillions of dollars, where t represents the number of years since 2000, can be approximated by

$$A(t) = 0.0000329t^3 - 0.00450t^2 + 0.0613t + 2.34,$$

where $0 \le t \le 50$. Find the value of t when Social Security assets will decrease most rapidly. Approximately when does this occur? ***Source: Social Security Administration***.

Point of Diminishing Returns **In Exercises 72–75, find the point of diminishing returns (x, y) for the given functions, where $R(x)$, represents revenue (in thousands of dollars) and x represents the amount spent on advertising (in thousands of dollars).**

72. $R(x) = 10{,}000 - x^3 + 42x^2 + 800x, \quad 0 \le x \le 20$

73. $R(x) = \frac{4}{27}(-x^3 + 66x^2 + 1050x - 400), \quad 0 \le x \le 25$

74. $R(x) = -0.3x^3 + x^2 + 11.4x, \quad 0 \le x \le 6$

75. $R(x) = -0.6x^3 + 3.7x^2 + 5x, \quad 0 \le x \le 6$

76. Risk Aversion In economics, an index of *absolute risk aversion* is defined as

$$I(M) = \frac{-U''(M)}{U'(M)},$$

where M measures how much of a commodity is owned and $U(M)$ is a *utility function,* which measures the ability of quantity M of a commodity to satisfy a consumer's wants. Find $I(M)$ for $U(M) = \sqrt{M}$ and for $U(M) = M^{2/3}$, and determine which indicates a greater aversion to risk.

77. Demand Function The authors of an article in an economics journal state that if $D(q)$ is the demand function, then the inequality

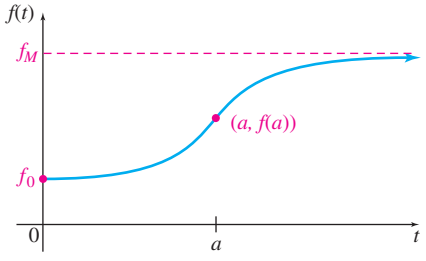
$$qD''(q) + D'(q) < 0$$

is equivalent to saying that the marginal revenue declines more quickly than does the price. Prove that this equivalence is true. ***Source: Bell Journal of Economics***.

Life Sciences

78. Population Growth When a hardy new species is introduced into an area, the population often increases as shown. Explain the significance of the following function values on the graph.

a. f_0 **b.** $f(a)$ **c.** f_M



79. Bacteria Population Assume that the number of bacteria $R(t)$ (in millions) present in a certain culture at time t (in hours) is given by

$$R(t) = t^2(t - 18) + 96t + 1000.$$

a. At what time before 8 hours will the population be maximized?

b. Find the maximum population.

80. Ozone Depletion According to an article in *The New York Times,* "Government scientists reported last week that they had detected a slowdown in the rate at which chemicals that deplete the earth's protective ozone layer are accumulating in the atmosphere." Letting $c(t)$ be the amount of ozone-depleting chemicals at time t, what does this statement tell you about $c(t)$, $c'(t)$, and $c''(t)$? ***Source: The New York Times***.

81. Drug Concentration The percent of concentration of a certain drug in the bloodstream x hours after the drug is administered is given by

$$K(x) = \frac{3x}{x^2 + 4}.$$

For example, after 1 hour the concentration is given by

$$K(1) = \frac{3(1)}{1^2 + 4} = \frac{3}{5}\% = 0.6\% = 0.006.$$

a. Find the time at which concentration is a maximum.

b. Find the maximum concentration.

82. Drug Concentration The percent of concentration of a drug in the bloodstream x hours after the drug is administered is given by

$$K(x) = \frac{4x}{3x^2 + 27}.$$

a. Find the time at which the concentration is a maximum.

b. Find the maximum concentration.

The next two exercises are a continuation of exercises first given in the section on Derivatives of Exponential Functions. Find the inflection point of the graph of each logistic function. This is the point at which the growth rate begins to decline.

83. Insect Growth The growth function for a population of beetles is given by

$$G(t) = \frac{10{,}000}{1 + 49e^{-0.1t}}.$$

84. Clam Population Growth The population of a bed of clams is described by

$$G(t) = \frac{5200}{1 + 12e^{-0.52t}}.$$

Hints for Exercises 85 and 86: Leave B, c, and k as constants until you are ready to calculate your final answer.

85. Clam Growth Researchers used a version of the Gompertz curve to model the growth of razor clams during the first seven years of the clams' lives with the equation

$$L(t) = Be^{-ce^{-kt}},$$

where $L(t)$ gives the length (in centimeters) after t years, $B = 14.3032$, $c = 7.267963$, and $k = 0.670840$. Find the inflection point and describe what it signifies. ***Source: Journal of Experimental Biology***.

86. Breast Cancer Growth Researchers used a version of the Gompertz curve to model the growth of breast cancer tumors with the equation

$$N(t) = e^{c(1-e^{-kt})},$$

where $N(t)$ is the number of cancer cells after t days, $c = 27.3$, and $k = 0.011$. Find the inflection point and describe what it signifies. ***Source: Cancer Research***.

87. Popcorn Researchers have determined that the amount of moisture present in a kernel of popcorn affects the volume of the popped corn and can be modeled for certain sizes of kernels by the function

$$v(x) = -35.98 + 12.09x - 0.4450x^2,$$

where x is moisture content (%, wet basis) and $v(x)$ is the expansion volume (in cm^3/gram). Describe the concavity of this function. ***Source: Cereal Chemistry***.

88. Alligator Teeth Researchers have developed a mathematical model that can be used to estimate the number of teeth $N(t)$ at time t (days of incubation) for *Alligator mississippiensis*, where

$$N(t) = 71.8e^{-8.96e^{-0.0685t}}.$$

Find the inflection point and describe its importance to this research. ***Source: Journal of Theoretical Biology***.

Social Sciences

89. Crime In 1995, the rate of violent crimes in New York City continued to decrease, but at a slower rate than in previous years. Letting $f(t)$ be the rate of violent crime as a function of time, what does this tell you about $f(t)$, $f'(t)$, and $f''(t)$? ***Source: The New York Times***.

Physical Sciences

90. Chemical Reaction An autocatalytic chemical reaction is one in which the product being formed causes the rate of formation to increase. The rate of a certain autocatalytic reaction is given by

$$V(x) = 12x(100 - x),$$

where x is the quantity of the product present and 100 represents the quantity of chemical present initially. For what value of x is the rate of the reaction a maximum?

91. Velocity and Acceleration When an object is dropped straight down, the distance (in feet) that it travels in t seconds is given by

$$s(t) = -16t^2.$$

Find the velocity at each of the following times.

a. After 3 seconds **b.** After 5 seconds

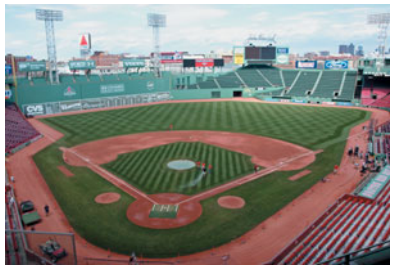
c. After 8 seconds

d. Find the acceleration. (The answer here is a constant—the acceleration due to the influence of gravity alone near the surface of Earth.)

92. Baseball Roger Clemens, ace pitcher for many major league teams, including the Boston Red Sox, is standing on top of the 37-ft-high "Green Monster" left-field wall in Boston's Fenway Park, to which he has returned for a visit. We have asked him to fire his famous 95 mph (140 ft per second) fastball straight up. The position equation, which gives the height of the ball at any time t, in seconds, is given by $s(t) = -16t^2 + 140t + 37$. Find the following. ***Source: Frederick Russell***.

a. The maximum height of the ball

b. The time and velocity when the ball hits the ground



93. Height of a Ball If a cannonball is shot directly upward with a velocity of 256 ft per second, its height above the ground after t seconds is given by $s(t) = 256t - 16t^2$. Find the velocity and the acceleration after t seconds. What is the maximum height the cannonball reaches? When does it hit the ground?

94. Velocity and Acceleration of a Car A car rolls down a hill. Its distance (in feet) from its starting point is given by $s(t) = 1.5t^2 + 4t$, where t is in seconds.

a. How far will the car move in 10 seconds?

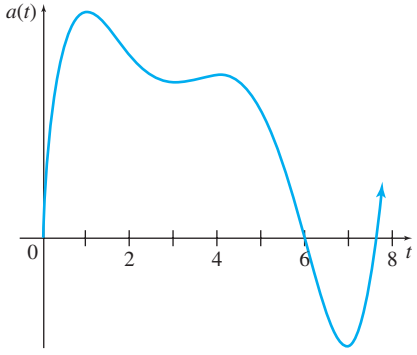
b. What is the velocity at 5 seconds? At 10 seconds?

c. How can you tell from $v(t)$ that the car will not stop?

d. What is the acceleration at 5 seconds? At 10 seconds?

e. What is happening to the velocity and the acceleration as t increases?

95. Velocity and Acceleration A car is moving along a straight stretch of road. The acceleration of the car is given by the graph shown. Assume that the velocity of the car is always positive. At what time was the car moving most rapidly? Explain. ***Source: Larry Taylor***.



YOUR TURN ANSWERS

1. 36

2. **(a)** $f''(x) = 30x^4 + 12x$ **(b)** $g''(x) = 2e^x + xe^x$ **(c)** $h''(x) = \dfrac{-3 + 2\ln x}{x^3}$

3. $v(t) = 3t^2 - 6t - 24$ and $a(t) = 6t - 6$. Car backs up for the first 4 seconds and then goes forward. It speeds up for $0 < t < 1$, slows down for $1 < t < 4$, and then speeds up for $t > 4$.

4. Concave up on $(-3, 0)$ and $(3, \infty)$; concave down on $(-\infty, -3)$ and $(0, 3)$; inflection points are $(-3, 567)$, $(0, 0)$, and $(3, -567)$

5. Relative maximum of $f(4) = 208$ at $x = 4$ and relative minimum of $f(-3) = -135$ at $x = -3$.

13.4 Curve Sketching

APPLY IT **How can we use differentiation to help us sketch the graph of a function, and describe its behavior?**

In the following examples, the test for concavity, the test for increasing and decreasing functions, and the concept of limits at infinity will help us sketch the graphs and describe the behavior of a variety of functions. This process, called **curve sketching**, has decreased somewhat in importance in recent years due to the widespread use of graphing calculators. We believe, however, that this topic is worth studying for the following reasons.

For one thing, a graphing calculator picture can be misleading, particularly if important points lie outside the viewing window. Even if all important features are within the viewing windows, there is still the problem that the calculator plots and connects points and misses what goes on between those points. As an example of the difficulty in choosing an appropriate window without a knowledge of calculus, see Exercise 40 in the second section of this chapter.

Furthermore, curve sketching may be the best way to learn the material in the previous three sections. You may feel confident that you understand what increasing and concave upward mean, but using those concepts in a graph will put your understanding to the test.

Curve sketching may be done with the following steps.

Curve Sketching

To sketch the graph of a function f:

1. Consider the domain of the function, and note any restrictions. (That is, avoid dividing by 0, taking a square root of a negative number, or taking the logarithm of 0 or a negative number.)
2. Find the y-intercept (if it exists) by substituting $x = 0$ into $f(x)$. Find any x-intercepts by solving $f(x) = 0$ if this is not too difficult.
3. **a.** If f is a rational function, find any vertical asymptotes by investigating where the denominator is 0, and find any horizontal asymptotes by finding the limits as $x \to \infty$ and $x \to -\infty$.

   b. If f is an exponential function, find any horizontal asymptotes; if f is a logarithmic function, find any vertical asymptotes.
4. Investigate symmetry. If $f(-x) = f(x)$, the function is even, so the graph is symmetric about the y-axis. If $f(-x) = -f(x)$, the function is odd, so the graph is symmetric about the origin.
5. Find $f'(x)$. Locate any critical points by solving the equation $f'(x) = 0$ and determining where $f'(x)$ does not exist, but $f(x)$ does. Find any relative extrema and determine where f is increasing or decreasing.
6. Find $f''(x)$. Locate potential inflection points by solving the equation $f''(x) = 0$ and determining where $f''(x)$ does not exist. Determine where f is concave upward or concave downward.
7. Plot the intercepts, the critical points, the inflection points, the asymptotes, and other points as needed. Take advantage of any symmetry found in Step 4.
8. Connect the points with a smooth curve using the correct concavity, being careful not to connect points where the function is not defined.
9. Check your graph using a graphing calculator. If the picture looks very different from what you've drawn, see in what ways the picture differs and use that information to help find your mistake.

There are four possible combinations for a function to be increasing or decreasing and concave up or concave down, as shown in the following table.

	Concavity Summary	
$f''(x)$ \ $f'(x)$	+ (Function Is Increasing)	− (Function Is Decreasing)
+ (function is concave up)		
− (function is concave down)		

EXAMPLE 1 Polynomial Function Graph

Graph $f(x) = 2x^3 - 3x^2 - 12x + 1$.

SOLUTION The domain is $(-\infty, \infty)$. The y-intercept is located at $y = f(0) = 1$. Finding the x-intercepts requires solving the equation $f(x) = 0$. But this is a third-degree equation; since we have not covered a procedure for solving such equations, we will skip this step. This is neither a rational nor an exponential function, so we also skip step 3. Observe that $f(-x) = 2(-x)^3 - 3(-x)^2 - 12(-x) + 1 = -2x^3 - 3x^2 + 12x + 1$, which is neither $f(x)$ nor $-f(x)$, so there is no symmetry about the y-axis or origin.

To find the intervals where the function is increasing or decreasing, find the first derivative.

$$f'(x) = 6x^2 - 6x - 12$$

This derivative is 0 when

$$6(x^2 - x - 2) = 0$$
$$6(x - 2)(x + 1) = 0$$
$$x = 2 \quad \text{or} \quad x = -1.$$

These critical numbers divide the number line in Figure 40 into three regions. Testing a number from each region in $f'(x)$ shows that f is increasing on $(-\infty, -1)$ and $(2, \infty)$ and decreasing on $(-1, 2)$. This is shown with the arrows in Figure 41. By the first derivative test, f has a relative maximum when $x = -1$ and a relative minimum when $x = 2$. The relative maximum is $f(-1) = 8$, while the relative minimum is $f(2) = -19$.

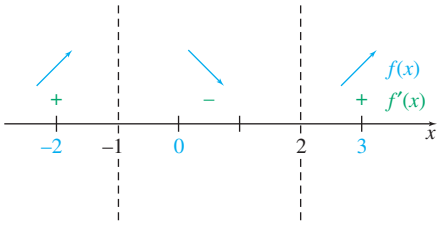


FIGURE 40

Now use the second derivative to find the intervals where the function is concave upward or downward. Here

$$f''(x) = 12x - 6,$$

which is 0 when $x = 1/2$. Testing a point with x less than $1/2$, and one with x greater than $1/2$, shows that f is concave downward on $(-\infty, 1/2)$ and concave upward on $(1/2, \infty)$.

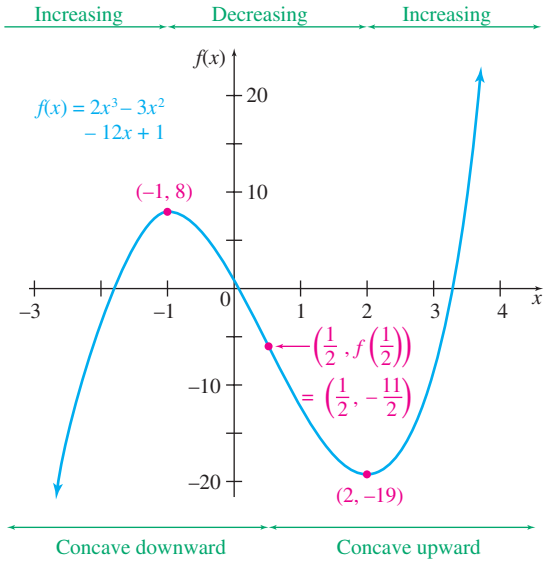


FIGURE 41

The graph has an inflection point at $(1/2, f(1/2))$, or $(1/2, -11/2)$. This information is summarized in the following table.

Graph Summary

Interval	$(-\infty, -1)$	$(-1, 1/2)$	$(1/2, 2)$	$(2, \infty)$
Sign of f′	+	−	−	+
Sign of f″	−	−	+	+
f Increasing or Decreasing	Increasing	Decreasing	Decreasing	Increasing
Concavity of f	Downward	Downward	Upward	Upward
Shape of Graph	⌒	⌒	⌒	⌒

Use this information and the critical points to get the graph shown in Figure 41. Notice that the graph appears to be symmetric about its inflection point. It can be shown that is always true for third-degree polynomials. In other words, if you put your pencil point at the inflection point and then spin the book 180° about the pencil point, the graph will appear to be unchanged.

YOUR TURN 1 Graph $f(x) = -x^3 + 3x^2 + 9x - 10$.

TRY YOUR TURN 1

TECHNOLOGY NOTE

A graphing calculator picture of the function in Figure 41 on the arbitrarily chosen window $[-3, 3]$ by $[-7, 7]$ gives a misleading picture, as Figure 42(a) shows. Knowing where the turning points lie tells us that a better window would be $[-3, 4]$ by $[-20, 20]$, with the results shown in Figure 42(b).

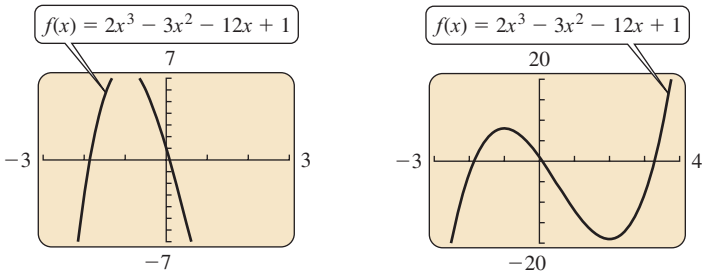


FIGURE 42

EXAMPLE 2 Rational Function Graph

Graph $f(x) = x + \frac{1}{x}$.

SOLUTION Notice that $x = 0$ is not in the domain of the function, so there is no y-intercept. To find the x-intercept, solve $f(x) = 0$.

$$x + \frac{1}{x} = 0$$
$$x = -\frac{1}{x}$$
$$x^2 = -1$$

Since x^2 is always positive, there is also no x-intercept.

The function is a rational function, but it is not written in the usual form of one polynomial over another. By getting a common denominator and adding the fractions, it can be rewritten in that form:

$$f(x) = x + \frac{1}{x} = \frac{x^2 + 1}{x}.$$

FOR REVIEW

Asymptotes were discussed in Section 10.3 on Polynomial and Rational Functions. You may wish to refer back to that section to review. To review limits, refer to Section 11.1 in the chapter titled The Derivative.

Because $x = 0$ makes the denominator (but not the numerator) 0, the line $x = 0$ is a vertical asymptote. To find any horizontal asymptotes, we investigate

$$\lim_{x\to\infty} \frac{x^2 + 1}{x} = \lim_{x\to\infty} \left(\frac{x^2}{x} + \frac{1}{x}\right) = \lim_{x\to\infty} \left(x + \frac{1}{x}\right).$$

The second term, $1/x$, approaches 0 as $x \to \infty$, but the first term, x, becomes infinitely large, so the limit does not exist. Verify that $\lim_{x\to-\infty} f(x)$ also does not exist, so there are no horizontal asymptotes.

Observe that as x gets very large, the second term $(1/x)$ in $f(x)$ gets very small, so $f(x) = x + (1/x) \approx x$. The graph gets closer and closer to the straight line $y = x$ as x becomes larger and larger. This is what is known as an **oblique asymptote**.

Observe that

$$f(-x) = (-x) + \frac{1}{-x} = -\left(x + \frac{1}{x}\right) = -f(x),$$

so the graph is symmetric about the origin. This means that the left side of the graph can be found by rotating the right side 180° about the origin.

Here $f'(x) = 1 - (1/x^2)$, which is 0 when

$$\frac{1}{x^2} = 1$$
$$x^2 = 1$$
$$x^2 - 1 = (x - 1)(x + 1) = 0$$
$$x = 1 \quad \text{or} \quad x = -1.$$

The derivative fails to exist at 0, where the vertical asymptote is located. Evaluating $f'(x)$ in each of the regions determined by the critical numbers and the asymptote shows that f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 0)$ and $(0, 1)$. See Figure 43(a). By the first derivative test, f has a relative maximum of $y = f(-1) = -2$ when $x = -1$, and a relative minimum of $y = f(1) = 2$ when $x = 1$.

The second derivative is

$$f''(x) = \frac{2}{x^3},$$

which is never equal to 0 and does not exist when $x = 0$. (The function itself also does not exist at 0.) Because of this, there may be a change of concavity, but not an inflection point,

YOUR TURN 2 Graph $f(x) = 4x + \frac{1}{x}$.

when $x = 0$. The second derivative is negative when x is negative, making f concave downward on $(-\infty, 0)$. Also, $f''(x) > 0$ when $x > 0$, making f concave upward on $(0, \infty)$. See Figure 43(b).

Use this information, the asymptotes, and the critical points to get the graph shown in Figure 44. TRY YOUR TURN 2

Graph Summary

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'	+	−	−	+
Sign of f''	−	−	+	+
f Increasing or Decreasing	Increasing	Decreasing	Decreasing	Increasing
Concavity of f	Downward	Downward	Upward	Upward
Shape of Graph				

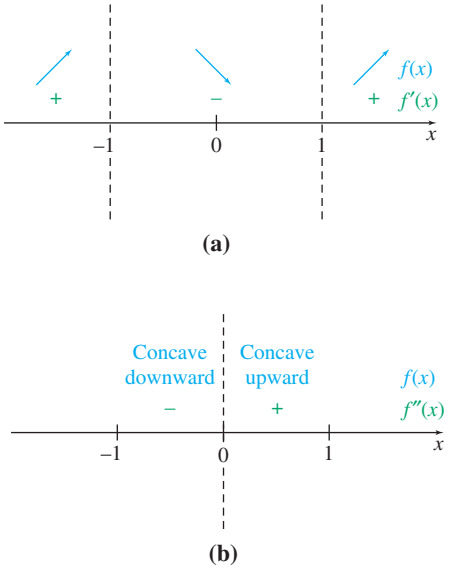


FIGURE 43

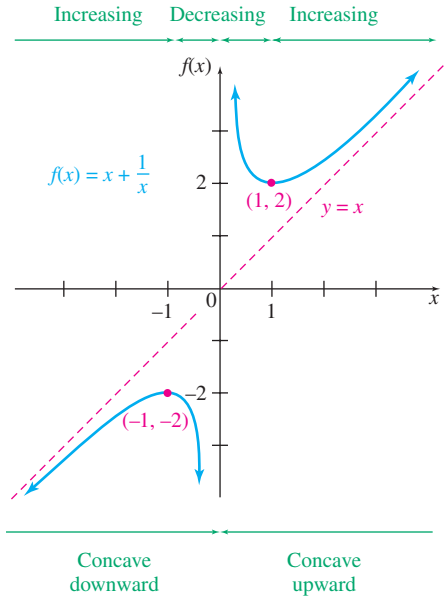


FIGURE 44

EXAMPLE 3 Rational Function Graph

Graph $f(x) = \dfrac{3x^2}{x^2 + 5}$.

SOLUTION The y-intercept is located at $y = f(0) = 0$. Verify that this is also the only x-intercept. There is no vertical asymptote, because $x^2 + 5 \neq 0$ for any value of x. Find any horizontal asymptote by calculating $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. First, divide both the numerator and the denominator of $f(x)$ by x^2.

$$\lim_{x\to\infty} \frac{3x^2}{x^2 + 5} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}}{\frac{x^2}{x^2} + \frac{5}{x^2}} = \frac{3}{1 + 0} = 3$$

Verify that the limit of $f(x)$ as $x \to -\infty$ is also 3. Thus, the horizontal asymptote is $y = 3$.

Observe that

$$f(-x) = \frac{3(-x)^2}{(-x)^2 + 5} = \frac{3x^2}{x^2 + 5} = f(x),$$

so the graph is symmetric about the y-axis. This means that the left side of the graph is the mirror image of the right side.

We now compute $f'(x)$:

$$f'(x) = \frac{(x^2 + 5)(6x) - (3x^2)(2x)}{(x^2 + 5)^2}.$$

Notice that $6x$ can be factored out of each term in the numerator:

$$f'(x) = \frac{(6x)[(x^2 + 5) - x^2]}{(x^2 + 5)^2}$$

$$= \frac{(6x)(5)}{(x^2 + 5)^2} = \frac{30x}{(x^2 + 5)^2}.$$

From the numerator, $x = 0$ is a critical number. The denominator is always positive. (Why?) Evaluating $f'(x)$ in each of the regions determined by $x = 0$ shows that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. By the first derivative test, f has a relative minimum when $x = 0$.

The second derivative is

$$f''(x) = \frac{(x^2 + 5)^2(30) - (30x)(2)(x^2 + 5)(2x)}{(x^2 + 5)^4}.$$

Factor $30(x^2 + 5)$ out of the numerator:

$$f''(x) = \frac{30(x^2 + 5)[(x^2 + 5) - (x)(2)(2x)]}{(x^2 + 5)^4}.$$

Divide a factor of $(x^2 + 5)$ out of the numerator and denominator, and simplify the numerator:

$$f''(x) = \frac{30[(x^2 + 5) - (x)(2)(2x)]}{(x^2 + 5)^3}$$

$$= \frac{30[(x^2 + 5) - (4x^2)]}{(x^2 + 5)^3}$$

$$= \frac{30(5 - 3x^2)}{(x^2 + 5)^3}.$$

The numerator of $f''(x)$ is 0 when $x = \pm\sqrt{5/3} \approx \pm 1.29$. Testing a point in each of the three intervals defined by these points shows that f is concave downward on $(-\infty, -1.29)$ and $(1.29, \infty)$, and concave upward on $(-1.29, 1.29)$. The graph has inflection points at $(\pm\sqrt{5/3}, f(\pm\sqrt{5/3})) \approx (\pm 1.29, 0.75)$.

Use this information, the asymptote, the critical point, and the inflection points to get the graph shown in Figure 45. TRY YOUR TURN 3

YOUR TURN 3

Graph $f(x) = \dfrac{4x^2}{x^2 + 4}$.

Graph Summary				
Interval	$(-\infty, -1.29)$	$(-1.29, 0)$	$(0, 1.29)$	$(1.29, \infty)$
Sign of f'	$-$	$-$	$+$	$+$
Sign of f''	$-$	$+$	$+$	$-$
f Increasing or Decreasing	Decreasing	Decreasing	Increasing	Increasing
Concavity of f	Downward	Upward	Upward	Downward
Shape of Graph				

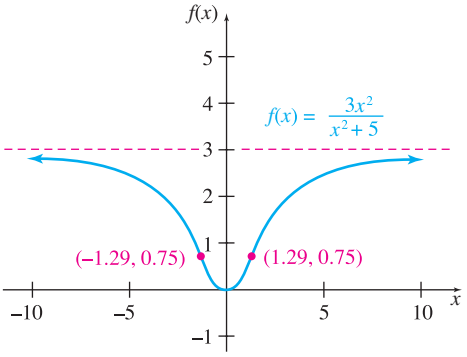


FIGURE 45

EXAMPLE 4 Graph with Logarithm

Graph $f(x) = \dfrac{\ln x}{x^2}$.

SOLUTION The domain is $x > 0$, so there is no y-intercept. The x-intercept is 1, because $\ln 1 = 0$. We know that $y = \ln x$ has a vertical asymptote at $x = 0$, because $\lim_{x \to 0^+} \ln x = -\infty$. Dividing by x^2 when x is small makes $(\ln x)/x^2$ even more negative than $\ln x$. Therefore, $(\ln x)/x^2$ has a vertical asymptote at $x = 0$ as well. The first derivative is

$$f'(x) = \frac{x^2 \cdot \dfrac{1}{x} - 2x \ln x}{(x^2)^2} = \frac{x(1 - 2 \ln x)}{x^4} = \frac{1 - 2 \ln x}{x^3}$$

by the quotient rule. Setting the numerator equal to 0 and solving for x gives

$$\begin{aligned} 1 - 2 \ln x &= 0 \\ 1 &= 2 \ln x \\ \ln x &= 0.5 \\ x &= e^{0.5} \approx 1.65. \end{aligned}$$

Since $f'(1)$ is positive and $f'(2)$ is negative, f increases on $(0, 1.65)$ then decreases on $(1.65, \infty)$, with a maximum value of $f(1.65) \approx 0.18$.

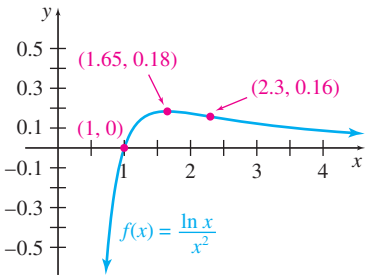


FIGURE 46

YOUR TURN 4 Graph $f(x) = (x + 2)e^{-x}$. (Recall $\lim_{x\to\infty} x^n e^{-x} = 0$)

To find any inflection points, we set $f''(x) = 0$.

$$f''(x) = \frac{x^3\left(-2 \cdot \frac{1}{x}\right) - (1 - 2\ln x) \cdot 3x^2}{(x^3)^2}$$

$$= \frac{-2x^2 - 3x^2 + 6x^2 \ln x}{x^6} = \frac{-5 + 6\ln x}{x^4}$$

$$\frac{-5 + 6\ln x}{x^4} = 0$$

$$-5 + 6\ln x = 0 \qquad \text{Set the numerator equal to 0.}$$

$$6\ln x = 5 \qquad \text{Add 5 to both sides.}$$

$$\ln x = 5/6 \qquad \text{Divide both sides by 6.}$$

$$x = e^{5/6} \approx 2.3 \qquad e^{\ln x} = x.$$

There is an inflection point at $(2.3, f(2.3)) \approx (2.3, 0.16)$. Verify that $f''(1)$ is negative and $f''(3)$ is positive, so the graph is concave downward on $(1, 2.3)$ and upward on $(2.3, \infty)$. This information is summarized in the following table and could be used to sketch the graph. A graph of the function is shown in Figure 46. TRY YOUR TURN 4

	Graph Summary		
Interval	**(0, 1.65)**	**(1.65, 2.3)**	**(2.3, ∞)**
Sign of f′	+	−	−
Sign of f″	−	−	+
f Increasing or Decreasing	Increasing	Decreasing	Decreasing
Concavity of f	Downward	Downward	Upward
Shape of Graph	⌒	⌒	⌒

As we saw earlier, a graphing calculator, when used with care, can be helpful in studying the behavior of functions. This section has illustrated that calculus is also a great help. The techniques of calculus show where the important points of a function, such as the relative extrema and the inflection points, are located. Furthermore, they tell how the function behaves between and beyond the points that are graphed, something a graphing calculator cannot always do.

13.4 EXERCISES

1. By sketching a graph of the function or by investigating values of the function near 0, find $\lim_{x\to 0} x \ln |x|$. (This result will be useful in Exercise 21.)

2. Describe how you would find the equation of the horizontal asymptote for the graph of

$$f(x) = \frac{3x^2 - 2x}{2x^2 + 5}.$$

Graph each function, considering the domain, critical points, symmetry, regions where the function is increasing or decreasing, inflection points, regions where the function is concave upward or concave downward, intercepts where possible, and asymptotes where applicable. (*Hint*: In Exercise 21, use the result of Exercise 1. In Exercises 25–27, recall from Exercise 66 in the section on Limits that $\lim_{x\to\infty} x^n e^{-x} = 0$.)

3. $f(x) = -2x^3 - 9x^2 + 108x - 10$

4. $f(x) = x^3 - \frac{15}{2}x^2 - 18x - 1$

5. $f(x) = -3x^3 + 6x^2 - 4x - 1$

6. $f(x) = x^3 - 6x^2 + 12x - 11$

7. $f(x) = x^4 - 24x^2 + 80$

8. $f(x) = -x^4 + 6x^2$

9. $f(x) = x^4 - 4x^3$

10. $f(x) = x^5 - 15x^3$

11. $f(x) = 2x + \dfrac{10}{x}$

12. $f(x) = 16x + \dfrac{1}{x^2}$

13. $f(x) = \dfrac{-x + 4}{x + 2}$

14. $f(x) = \dfrac{3x}{x - 2}$

15. $f(x) = \dfrac{1}{x^2 + 4x + 3}$

16. $f(x) = \dfrac{-8}{x^2 - 6x - 7}$

17. $f(x) = \dfrac{x}{x^2 + 1}$

18. $f(x) = \dfrac{1}{x^2 + 4}$

19. $f(x) = \dfrac{1}{x^2 - 9}$

20. $f(x) = \dfrac{-2x}{x^2 - 4}$

21. $f(x) = x \ln |x|$

22. $f(x) = x - \ln |x|$

23. $f(x) = \dfrac{\ln x}{x}$

24. $f(x) = \dfrac{\ln x^2}{x^2}$

25. $f(x) = xe^{-x}$

26. $f(x) = x^2e^{-x}$

27. $f(x) = (x - 1)e^{-x}$

28. $f(x) = e^x + e^{-x}$

29. $f(x) = x^{2/3} - x^{5/3}$

30. $f(x) = x^{1/3} + x^{4/3}$

31. The default window on many calculators is $[-10, 10]$ by $[-10, 10]$. For the odd exercises between 3 and 15, tell which would give a poor representation in this window. (*Note:* Your answers may differ from ours, depending on what you consider "poor.")

32. Repeat Exercise 31 for the even exercises between 4 and 16.

33. Repeat Exercise 31 for the odd exercises between 17 and 29.

34. Repeat Exercise 31 for the even exercises between 18 and 30.

In Exercises 35–39, sketch the graph of a single function that has all of the properties listed.

35. a. Continuous and differentiable everywhere except at $x = 1$, where it has a vertical asymptote

b. $f'(x) < 0$ everywhere it is defined

c. A horizontal asymptote at $y = 2$

d. $f''(x) < 0$ on $(-\infty, 1)$ and $(2, 4)$

e. $f''(x) > 0$ on $(1, 2)$ and $(4, \infty)$

36. a. Continuous for all real numbers

b. $f'(x) < 0$ on $(-\infty, -6)$ and $(1, 3)$

c. $f'(x) > 0$ on $(-6, 1)$ and $(3, \infty)$

d. $f''(x) > 0$ on $(-\infty, -6)$ and $(3, \infty)$

e. $f''(x) < 0$ on $(-6, 3)$

f. A y-intercept at $(0, 2)$

37. a. Continuous and differentiable for all real numbers

b. $f'(x) > 0$ on $(-\infty, -3)$ and $(1, 4)$

c. $f'(x) < 0$ on $(-3, 1)$ and $(4, \infty)$

d. $f''(x) < 0$ on $(-\infty, -1)$ and $(2, \infty)$

e. $f''(x) > 0$ on $(-1, 2)$

f. $f'(-3) = f'(4) = 0$

g. $f''(x) = 0$ at $(-1, 3)$ and $(2, 4)$

38. a. Continuous for all real numbers

b. $f'(x) > 0$ on $(-\infty, -2)$ and $(0, 3)$

c. $f'(x) < 0$ on $(-2, 0)$ and $(3, \infty)$

d. $f''(x) < 0$ on $(-\infty, 0)$ and $(0, 5)$

e. $f''(x) > 0$ on $(5, \infty)$

f. $f'(-2) = f'(3) = 0$

g. $f'(0)$ doesn't exist

h. Differentiable everywhere except at $x = 0$

i. An inflection point at $(5, 1)$

39. a. Continuous for all real numbers

b. Differentiable everywhere except at $x = 4$

c. $f(1) = 5$

d. $f'(1) = 0$ and $f'(3) = 0$

e. $f'(x) > 0$ on $(-\infty, 1)$ and $(4, \infty)$

f. $f'(x) < 0$ on $(1, 3)$ and $(3, 4)$

g. $\lim\limits_{x \to 4^-} f'(x) = -\infty$ and $\lim\limits_{x \to 4^+} f'(x) = \infty$

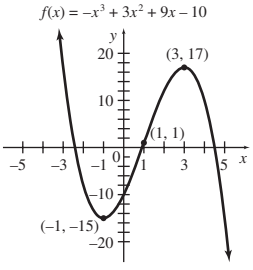
h. $f''(x) > 0$ on $(2, 3)$

i. $f''(x) < 0$ on $(-\infty, 2)$, $(3, 4)$, and $(4, \infty)$

40. On many calculators, graphs of rational functions produce lines at vertical asymptotes. For example, graphing $y = (x - 1)/(x + 1)$ on the window $[-4.9, 4.9]$ by $[-4.9, 4.9]$ produces such a line at $x = -1$ on the TI-84 Plus and TI-89. But with the window $[-4.7, 4.7]$ by $[-4.7, 4.7]$ on a TI-84 Plus; or $[-7.9, 7.9]$ by $[-7.9, 7.9]$ on a TI-89, the spurious line does not appear. Experiment with this function on your calculator, trying different windows, and try to figure out an explanation for this phenomenon. (*Hint:* Consider the number of pixels on the calculator screen.)

YOUR TURN ANSWERS

1.



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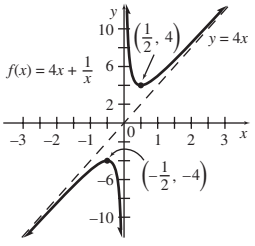

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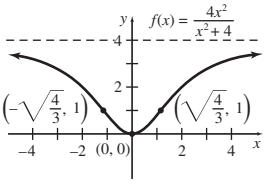

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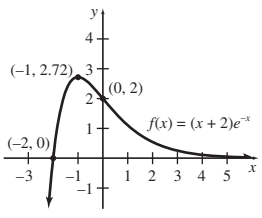


13 CHAPTER REVIEW

SUMMARY

In this chapter we have explored various concepts related to the graph of a function:

- increasing and decreasing,
- critical numbers (numbers c in the domain of f for which $f'(x) = 0$ or $f'(x)$ does not exist),
- critical points (whose x-coordinate is a critical number c and whose y-coordinate is $f(c)$),
- relative maxima and minima (together known as relative extrema),
- concavity, and
- inflection points (where the concavity changes).

The first and second derivative tests provide ways to locate relative extrema. The last section brings all these concepts together. Also, we investigated two applications of the second derivative:

- acceleration (the second derivative of the position function), and
- the point of diminishing returns (an inflection point on an input/output graph).

Test for Increasing/Decreasing

On any open interval,
if $f'(x) > 0$, then f is increasing;
if $f'(x) < 0$, then f is decreasing;
if $f'(x) = 0$, then f is constant.

First Derivative Test

If c is a critical number for f on the open interval (a, b), f is continuous on (a, b), and f is differentiable on (a, b) (except possibly at c), then

1. $f(c)$ is a relative maximum if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b);
2. $f(c)$ is a relative minimum if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b).

Test for Concavity

On any open interval,
if $f''(x) > 0$, then f is concave upward;
if $f''(x) < 0$, then f is concave downward.

Second Derivative Test

Suppose f'' exists on an open interval containing c and $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or $f''(c)$ does not exist, then the test gives no information about extrema, so use the first derivative test.

Curve Sketching

To sketch the graph of a function f:

1. Consider the domain of the function, and note any restrictions. (That is, avoid dividing by 0, taking a square root of a negative number, or taking the logarithm of 0 or a negative number.)
2. Find the y-intercept (if it exists) by substituting $x = 0$ into $f(x)$. Find any x-intercepts by solving $f(x) = 0$ if this is not too difficult.
3. **a.** If f is a rational function, find any vertical asymptotes by investigating where the denominator is 0, and find any horizontal asymptotes by finding the limits as $x \to \infty$ and $x \to -\infty$.
   b. If f is an exponential function, find any horizontal asymptotes; if f is a logarithmic function, find any vertical asymptotes.
4. Investigate symmetry. If $f(-x) = f(x)$, the function is even, so the graph is symmetric about the y-axis. If $f(-x) = -f(x)$, the function is odd, so the graph is symmetric about the origin.
5. Find $f'(x)$. Locate any critical points by solving the equation $f'(x) = 0$ and determining where $f'(x)$ does not exist, but $f(x)$ does. Find any relative extrema and determine where f is increasing or decreasing.
6. Find $f''(x)$. Locate potential inflection points by solving the equation $f''(x) = 0$ and determining where $f''(x)$ does not exist. Determine where f is concave upward or concave downward.
7. Plot the intercepts, the critical points, the inflection points, the asymptotes, and other points as needed. Take advantage of any symmetry found in Step 4.

8. Connect the points with a smooth curve using the correct concavity, being careful not to connect points where the function is not defined.
9. Check your graph using a graphing calculator. If the picture looks very different from what you've drawn, see in what ways the picture differs and use that information to help find your mistake.

KEY TERMS

13.1
increasing function
decreasing function
critical number
critical point

13.2
relative (or local) maximum
relative (or local) minimum
relative (or local) extremum
first derivative test

13.3
second derivative
third derivative
fourth derivative
acceleration
concave upward and downward
concavity
inflection point
second derivative test
point of diminishing returns

13.4
curve sketching
oblique asymptote

REVIEW EXERCISES

CONCEPT CHECK

For Exercises 1–12 determine whether each of the following statements is true or false, and explain why.

1. A critical number c is a number in the domain of a function f for which $f'(c) = 0$ or $f'(c)$ does not exist.
2. If $f'(x) > 0$ on an interval, the function is positive on that interval.
3. If c is a critical number, then the function must have a relative maximum or minimum at c.
4. If f is continuous on (a, b), $f'(x) < 0$ on (a, c), and $f'(x) > 0$ on (c, b), then f has a relative minimum at c.
5. If $f'(c)$ exists, $f''(c)$ also exists.
6. The acceleration is the second derivative of the position function.
7. If $f''(x) > 0$ on an interval, the function is increasing on that interval.
8. If $f''(c) = 0$, the function has an inflection point at c.
9. If $f''(c) = 0$, the function does not have a relative maximum or minimum at c.
10. Every rational function has either a vertical or a horizontal asymptote.
11. If an odd function has a y-intercept, it must pass through the origin.
12. If $f'(c) = 0$, where c is a value in interval (a, b), then f is a constant on the interval (a, b).

PRACTICE AND EXPLORATIONS

13. When given the equation for a function, how can you determine where it is increasing and where it is decreasing?
14. When given the equation for a function, how can you determine where the relative extrema are located? Give two ways to test whether a relative extremum is a minimum or a maximum.
15. Does a relative maximum of a function always have the largest y-value in the domain of the function? Explain your answer.
16. What information about a graph can be found from the second derivative?

Find the open intervals where f is increasing or decreasing.

17. $f(x) = x^2 + 9x + 8$
18. $f(x) = -2x^2 + 7x + 14$
19. $f(x) = -x^3 + 2x^2 + 15x + 16$
20. $f(x) = 4x^3 + 8x^2 - 16x + 11$
21. $f(x) = \dfrac{16}{9 - 3x}$
22. $f(x) = \dfrac{15}{2x + 7}$
23. $f(x) = \ln|x^2 - 1|$
24. $f(x) = 8xe^{-4x}$

Find the locations and values of all relative maxima and minima.

25. $f(x) = -x^2 + 4x - 8$
26. $f(x) = x^2 - 6x + 4$
27. $f(x) = 2x^2 - 8x + 1$
28. $f(x) = -3x^2 + 2x - 5$
29. $f(x) = 2x^3 + 3x^2 - 36x + 20$
30. $f(x) = 2x^3 + 3x^2 - 12x + 5$
31. $f(x) = \dfrac{xe^x}{x - 1}$
32. $f(x) = \dfrac{\ln(3x)}{2x^2}$

Find the second derivative of each function, and then find $f''(1)$ and $f''(-3)$.

33. $f(x) = 3x^4 - 5x^2 - 11x$
34. $f(x) = 9x^3 + \dfrac{1}{x}$

35. $f(x) = \dfrac{4x + 2}{3x - 6}$ **36.** $f(x) = \dfrac{1 - 2x}{4x + 5}$

37. $f(t) = \sqrt{t^2 + 1}$ **38.** $f(t) = -\sqrt{5 - t^2}$

Graph each function, considering the domain, critical points, symmetry, regions where the function is increasing or decreasing, inflection points, regions where the function is concave up or concave down, intercepts where possible, and asymptotes where applicable.

39. $f(x) = -2x^3 - \dfrac{1}{2}x^2 + x - 3$

40. $f(x) = -\dfrac{4}{3}x^3 + x^2 + 30x - 7$

41. $f(x) = x^4 - \dfrac{4}{3}x^3 - 4x^2 + 1$

42. $f(x) = -\dfrac{2}{3}x^3 + \dfrac{9}{2}x^2 + 5x + 1$

43. $f(x) = \dfrac{x - 1}{2x + 1}$ **44.** $f(x) = \dfrac{2x - 5}{x + 3}$

45. $f(x) = -4x^3 - x^2 + 4x + 5$

46. $f(x) = x^3 + \dfrac{5}{2}x^2 - 2x - 3$

47. $f(x) = x^4 + 2x^2$ **48.** $f(x) = 6x^3 - x^4$

49. $f(x) = \dfrac{x^2 + 4}{x}$ **50.** $f(x) = x + \dfrac{8}{x}$

51. $f(x) = \dfrac{2x}{3 - x}$ **52.** $f(x) = \dfrac{-4x}{1 + 2x}$

53. $f(x) = xe^{2x}$ **54.** $f(x) = x^2e^{2x}$

55. $f(x) = \ln(x^2 + 4)$ **56.** $f(x) = x^2 \ln x$

57. $f(x) = 4x^{1/3} + x^{4/3}$ **58.** $f(x) = 5x^{2/3} + x^{5/3}$

In Exercises 59 and 60, sketch the graph of a single function that has all of the properties listed.

59. a. Continuous everywhere except at $x = -4$, where there is a vertical asymptote

b. A y-intercept at $y = -2$

c. x-intercepts at $x = -3$, 1, and 4

d. $f'(x) < 0$ on $(-\infty, -5)$, $(-4, -1)$, and $(2, \infty)$

e. $f'(x) > 0$ on $(-5, -4)$ and $(-1, 2)$

f. $f''(x) > 0$ on $(-\infty, -4)$ and $(-4, -3)$

g. $f''(x) < 0$ on $(-3, -1)$ and $(-1, \infty)$

h. Differentiable everywhere except at $x = -4$ and $x = -1$

60. a. Continuous and differentiable everywhere except at $x = -3$, where it has a vertical asymptote

b. A horizontal asymptote at $y = 1$

c. An x-intercept at $x = -2$

d. A y-intercept at $y = 4$

e. $f'(x) > 0$ on the intervals $(-\infty, -3)$ and $(-3, 2)$

f. $f'(x) < 0$ on the interval $(2, \infty)$

g. $f''(x) > 0$ on the intervals $(-\infty, -3)$ and $(4, \infty)$

h. $f''(x) < 0$ on the interval $(-3, 4)$

i. $f'(2) = 0$

j. An inflection point at $(4, 3)$

APPLICATIONS

Business and Economics

Stock Prices **In Exercises 61 and 62, $P(t)$ is the price of a certain stock at time t during a particular day.**

61. a. If the price of the stock is falling faster and faster, are $P'(t)$ and $P''(t)$ positive or negative?

b. Explain your answer.

62. a. When the stock reaches its highest price of the day, are $P'(t)$ and $P''(t)$ positive, zero, or negative?

b. Explain your answer.

63. Cat Brushes The cost function to produce q electric cat brushes is given by $C(q) = -10q^2 + 250q$. The demand equation is given by $p = -q^2 - 3q + 299$, where p is the price in dollars.

a. Find and simplify the profit function.

b. Find the number of brushes that will produce the maximum profit.

c. Find the price that produces the maximum profit.

d. Find the maximum profit.

e. Find the point of diminishing returns for the profit function.

64. Gasoline Prices In 2008, the price of gasoline in the United States spiked and then dropped. The average monthly price (in cents per gallon) of unleaded regular gasoline for 2008 can be approximated by the function

$$p(t) = -0.614t^3 + 6.25t^2 + 1.94t + 297, \quad \text{for } 0 < t < 12,$$

where t is in months and $t = 1$ corresponds to January 2008. ***Source: U.S. Energy Information Administration.***

a. Determine the interval(s) on which the price is increasing.

b. Determine the interval(s) on which the price is decreasing.

c. Find any relative extrema for the price of gasoline, as well as when they occurred.

Life Sciences

65. Weightlifting An abstract for an article states, "We tentatively conclude that Olympic weightlifting ability in trained subjects undergoes a nonlinear decline with age, in which the second derivative of the performance versus age curve repeatedly changes sign." ***Source: Medicine and Science in Sports and Exercise.***

a. What does this quote tell you about the first derivative of the performance versus age curve?

b. Describe what you know about the performance versus age curve based on the information in the quote.

66. Scaling Laws Many biological variables depend on body mass, with a functional relationship of the form

$$Y = Y_0 M^b,$$

where M represents body mass, b is a multiple of $1/4$, and Y_0 is a constant. For example, when Y represents metabolic rate, $b = 3/4$. When Y represents heartbeat, $b = -1/4$. When Y represents life span, $b = 1/4$. ***Source: Science***.

a. Determine which of metabolic rate, heartbeat, and life span are increasing or decreasing functions of mass. Also determine which have graphs that are concave upward and which have graphs that are concave downward.

b. Verify that all functions of the form given above satisfy the equation

$$\frac{dY}{dM} = \frac{b}{M}Y.$$

This means that the rate of change of Y is proportional to Y and inversely proportional to body mass.

67. Thoroughbred Horses The association between velocity during exercise and blood lactate concentration after submaximal 800-m exercise of thoroughbred racehorses on sand and grass tracks has been studied. The lactate-velocity relationship can be described by the functions.

$$l_1(v) = 0.08e^{0.33v} \quad \text{and}$$
$$l_2(v) = -0.87v^2 + 28.17v - 211.41,$$

where $l_1(v)$ and $l_2(v)$ are the lactate concentrations (in mmol/L) and v is the velocity (in m/sec) of the horse during workout on sand and grass tracks, respectively. Sketch the graph of both functions for $13 \le v \le 17$. ***Source: The Veterinary Journal***.

68. Neuron Communications In the FitzHugh-Nagumo model of how neurons communicate, the rate of change of the electric potential v with respect to time is given as a function of v by $f(v) = v(a - v)(v - 1)$, where a is a positive constant. Sketch a graph of this function when $a = 0.25$ and $0 \le v \le 1$. ***Source: Mathematical Biology***.

69. Fruit Flies The number of imagoes (sexually mature adult fruit flies) per mated female per day (y) can be approximated by

$$y = 34.7(1.0186)^{-x}x^{-0.658},$$

where x is the mean density of the mated population (measured as flies per bottle) over a 16-day period. Sketch the graph of the function. ***Source: Elements of Mathematical Biology***.

70. Blood Volume A formula proposed by Hurley for the red cell volume (RCV) in milliliters for males is

$$RCV = 1486S^2 - 4106S + 4514,$$

where S is the surface area (in square meters). A formula given by Pearson et al., is

$$RCV = 1486S - 825.$$

Source: Journal of Nuclear Medicine and British Journal of Haematology.

a. For the value of S which the RCV values given by the two formulas are closest, find the rate of change of RCV with respect to S for both formulas. What does this number represent?

b. The formula for plasma volume for males given by Hurley is

$$PV = 995e^{0.6085S},$$

while the formula given by Pearson et al., is

$$PV = 1578S,$$

where PV is measured in milliliters and S in square meters. Find the value of S for which the PV values given by the two formulas are the closest. Then find the value of PV that each formula gives for this value of S.

c. For the value of S found in part b, find the rate of change of PV with respect to S for both formulas. What does this number represent?

d. Notice in parts a and c that both formulas give the same instantaneous rate of change at the value of S for which the function values are closest. Prove that if two functions f and g are differentiable and never cross but are closest together when $x = x_0$, then $f'(x_0) = g'(x_0)$.

Social Sciences

71. Learning Researchers used a version of the Gompertz curve to model the rate that children learn with the equation

$$y(t) = A^{c^t},$$

where $y(t)$ is the portion of children of age t years passing a certain mental test, $A = 0.3982 \times 10^{-291}$, and $c = 0.4252$. Find the inflection point and describe what it signifies. (*Hint:* Leave A and c as constants until you are ready to calculate your final answer. If A is too small for your calculator to handle, use common logarithms and properties of logarithms to calculate $(\log A)/(\log e)$.) ***Source: School and Society***.

72. Population Under the scenario that the fertility rate in the European Union (EU) remains at 1.8 until 2020, when it rises to replacement level, the predicted population (in millions) of the 15 member countries of the EU can be approximated over the next century by

$$P(t) = 325 + 7.475(t + 10)e^{-(t+10)/20},$$

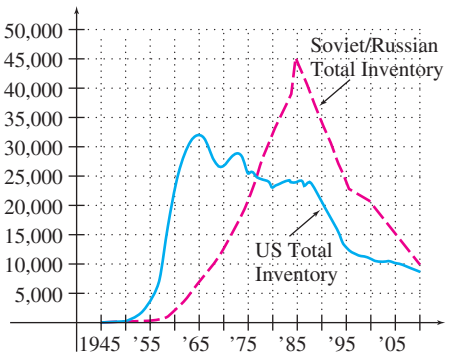
where t is the number of years since 2000. ***Source: Science***.

a. In what year is the population predicted to be largest? What is the population predicted to be in that year?

b. In what year is the population declining most rapidly?

c. What is the population approaching as time goes on?

73. Nuclear Weapons The graph shows the total inventory of nuclear weapons held by the United States and by the Soviet



Union and its successor states from 1945 to 2010. (See Exercise 60 in the first section of this chapter.) ***Source: Federation of American Scientists***.

a. In what years was the U.S. total inventory of weapons at a relative maximum?

b. When the U.S. total inventory of weapons was at the largest relative maximum, is the graph for the Soviet stockpile concave up or concave down? What does this mean?

Physical Sciences

74. Velocity and Acceleration A projectile is shot straight up with an initial velocity of 512 ft per second. Its height above the ground after t seconds is given by $s(t) = 512t - 16t^2$.

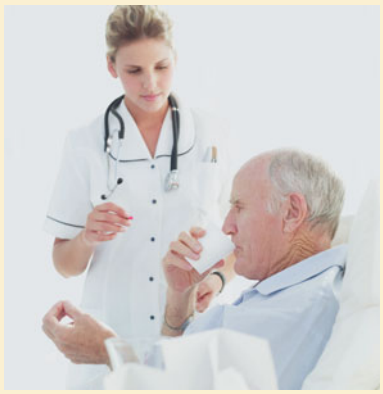
a. Find the velocity and acceleration after t seconds.

b. What is the maximum height attained?

c. When does the projectile hit the ground and with what velocity?

EXTENDED APPLICATION

A DRUG CONCENTRATION MODEL FOR ORALLY ADMINISTERED MEDICATIONS



Finding a range for the concentration of a drug in the bloodstream that is both safe and effective is one of the primary goals in pharmaceutical research and development. This range is called the *therapeutic window*. When determining the proper dosage (both the size of the dose and the frequency of administration), it is important to understand the behavior of the drug once it enters the body. Using data gathered during research we can create a mathematical model that predicts the concentration of the drug in the bloodstream at any given time.

We will look at two examples that explore a mathematical model for the concentration of a particular drug in the bloodstream. We will find the maximum and minimum concentrations of the drug given the size of the dose and the frequency of administration. We will then determine what dose should be administered to maintain concentrations within a given therapeutic window.

The drug tolbutamide is used for the management of mild to moderately severe type 2 diabetes. Suppose a 1000-mg dose of this drug is taken every 12 hours for three days. The concentration of the drug in the bloodstream, t hours after the initial dose is taken, is shown in Figure 47.

Looking at the graph, you can see that after a few doses have been administered, the maximum values of the concentration function begin to level off. The function also becomes periodic, repeating itself between every dose. At this point, the concentration is said to be at steady-state. The time it takes to reach steady-state depends on the elimination half-life of the drug (the time it takes for half the dose to be eliminated from the body). The elimination half-life of the drug used for this function is about 7 hours. Generally speaking, we say that steady-state is reached after about 5 half-lives.

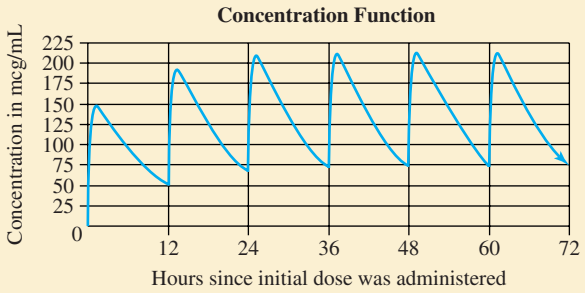


FIGURE 47

We will define the *steady-state concentration function*, $C_{ss}(t)$, to be the concentration of drug in the bloodstream t hours after a dose has been administered once steady-state has been reached.

The steady-state concentration function can be written as the difference of two exponential decay functions, or

$$C_{ss}(t) = c_1e^{kt} - c_2e^{k_at}.$$

The constants c_1 and c_2 are influenced by several factors, including the size of the dose and how widely the particular drug disperses through the body. The constants k_a and k are decay constants reflecting the rate at which the drug is being absorbed into the bloodstream and eliminated from the bloodstream, respectively.

Consider the following steady-state concentration function:

$$C_{ss}(t) = 0.2473De^{-0.1t} - 0.1728De^{-2.8t}\ \text{mcg/mL}$$

where D is the size of the dose (in milligrams) administered every 12 hours. The concentration is given in micrograms per milliliter.

If a single dose is 1000 mg, then the concentration of drug in the bloodstream is

$$C_{ss}(t) = 247.3e^{-0.1t} - 172.8e^{-2.8t}\ \text{mcg/mL}.$$

The graph of $C_{ss}(t)$ is given below in Figure 48.

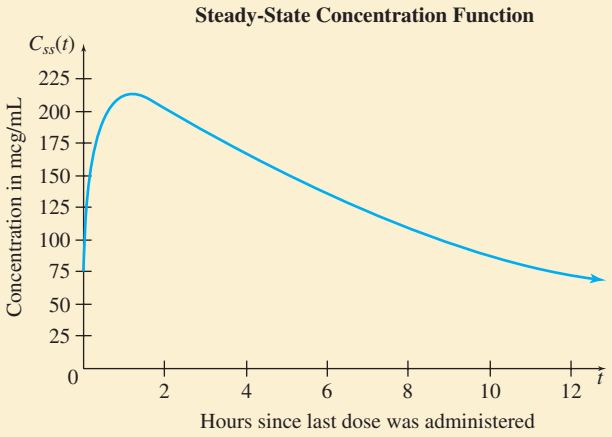


FIGURE 48

EXAMPLE 1 Drug Concentration

Find the maximum and minimum concentrations for the steady-state concentration function

$$C_{ss}(t) = 247.3e^{-0.1t} - 172.8e^{-2.8t} \text{ mcg/mL}.$$

SOLUTION The maximum concentration occurs when $C'_{ss}(t) = 0$.

Calculating the derivative, we get:

$$\begin{aligned} C'_{ss}(t) &= 247.3(-0.1)e^{-0.1t} - 172.8(-2.8)e^{-2.8t} \\ &= -24.73e^{-0.1t} + 483.84e^{-2.8t}. \end{aligned}$$

If we factor out $e^{-0.1t}$, we can find where the derivative is equal to zero.

$$C'_{ss}(t) = e^{-0.1t}(-24.73 + 483.84e^{-2.7t}) = 0$$

$C'_{ss}(t) = 0$ when

$$-24.73 + 483.84e^{-2.7t} = 0.$$

Solving this equation for t, we get

$$t = \frac{\ln\left(\frac{24.73}{483.84}\right)}{-2.7} \approx 1.1 \text{ hours}.$$

Therefore, the maximum concentration is

$$C_{ss}(1.1) = 247.3e^{-0.1(1.1)} - 172.8e^{-2.8(1.1)} \approx 214 \text{ mcg/mL}.$$

Looking at the graph of $C_{ss}(t)$ in Figure 48, you can see that the minimum concentration occurs at the endpoints (when $t = 0$ and $t = 12$; immediately after a dose is administered and immediately before a next dose is to be administered, respectively).

Therefore, the minimum concentration is

$$\begin{aligned} C_{ss}(0) &= 247.3e^{-0.1(0)} - 172.8e^{-2.8(0)} = 247.3 - 172.8 \\ &= 74.5 \text{ mcg/mL}. \end{aligned}$$

Verify that $C_{ss}(12)$ gives the same value.

If the therapeutic window for this drug is 70–240 mcg/mL, then, once steady-state has been reached, the concentration remains safe and effective as long as treatment continues.

Suppose, however, that a new study found that this drug is effective only if the concentration remains between 100 and 400 mcg/mL. How could you adjust the dose so that the maximum and minimum steady-state concentrations fall within this range?

EXAMPLE 2 Therapeutic Window

Find a range for the size of doses such that the steady-state concentration remains within the therapeutic window of 100 to 400 mcg/mL.

SOLUTION Recall that the steady-state concentration function is

$$C_{ss}(t) = 0.2473De^{-0.1t} - 0.1728De^{-2.8t} \text{ mcg/mL},$$

where D is the size of the dose given (in milligrams) every 12 hours.

From Example 1, we found that the minimum concentration occurs when $t = 0$. Therefore, we want the minimum concentration, $C_{ss}(0)$, to be greater than or equal to 100 mcg/mL.

$$C_{ss}(0) = 0.2473De^{-0.1(0)} - 0.1728De^{-2.8(0)} \geq 100$$

or

$$0.2473D - 0.1728D \geq 100$$

Solving for D, we get

$$0.0745D \geq 100$$

$$D \geq 1342 \text{ mg}.$$

In Example 1, we also found that the maximum concentration occurs when $t = 1.1$ hours. If we change the size of the dose, the maximum concentration will change; however, the time it takes to reach the maximum concentration does not change. Can you see why this is true?

Since the maximum concentration occurs when $t = 1.1$, we want $C_{ss}(1.1)$, the maximum concentration, to be less than or equal to 400 mcg/mL.

$$C_{ss}(1.1) = 0.2473De^{-0.1(1.1)} - 0.1728De^{-2.8(1.1)} \leq 400$$

or

$$0.2215D - 0.0079D \leq 400.$$

Solving for D, we get

$$0.2136D \leq 400$$

$$D \leq 1873 \text{ mg}.$$

Therefore, if the dose is between 1342 mg and 1873 mg, the steady-state concentration remains within the new therapeutic window.

EXERCISES

Use the following information to answer Exercises 1–3.
A certain drug is given to a patient every 12 hours. The steady-state concentration function is given by

$$C_{ss}(t) = 1.99De^{-0.14t} - 1.62De^{-2.08t} \ mcg/mL,$$

where D is the size of the dose in milligrams.

1. If a 500-mg dose is given every 12 hours, find the maximum and minimum steady-state concentrations.
2. If the dose is increased to 1500 mg every 12 hours, find the maximum and minimum steady-state concentrations.
3. What dose should be given every 12 hours to maintain a steady-state concentration between 80 and 400 mcg/mL?

DIRECTIONS FOR GROUP PROJECT

Because of declining health, many elderly people rely on prescription medications to stabilize or improve their medical condition. Your group has been assigned the task of developing a brochure to be made available at senior citizens' centers and physicians' offices that describes drug concentrations in the body for orally administered medications. The brochure should summarize the facts presented in this extended application but at a level that is understandable to a typical layperson. The brochure should be designed to look professional with a marketing flair.

14 Applications of the Derivative

14.1 Absolute Extrema

14.2 Applications of Extrema

14.3 Further Business Applications: Economic Lot Size; Economic Order Quantity; Elasticity of Demand

14.4 Implicit Differentiation

14.5 Related Rates

14.6 Differentials: Linear Approximation

Chapter 14 Review

Extended Application: A Total Cost Model for a Training Program

When several variables are related by a single equation, their rates of change are also related. For example, the height and horizontal distance of a kite are related to the length of the string holding the kite. In an exercise in Section 5 we differentiate this relationship to discover how fast the kite flier must let out the string to maintain the kite at a constant height and constant horizontal speed.

The previous chapter included examples in which we used the derivative to find the maximum or minimum value of a function. This problem is ubiquitous; consider the efforts people expend trying to maximize their income, or to minimize their costs or the time required to complete a task. In this chapter we will treat the topic of optimization in greater depth.

The derivative is applicable in far wider circumstances, however. In roughly 500 B.C., Heraclitus said, "Nothing endures but change," and his observation has relevance here. If change is continuous, rather than in sudden jumps, the derivative can be used to describe the rate of change. This explains why calculus has been applied to so many fields.

14.1 Absolute Extrema

APPLY IT **During a 10-year period, when did the U.S. dollar reach a minimum exchange rate with the Canadian dollar, and how much was the U.S. dollar worth then?**

We will answer this question in Example 3.

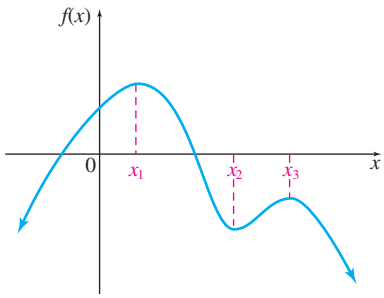


FIGURE 1

A function may have more than one relative maximum. It may be important, however, in some cases to determine if one function value is larger than any other. In other cases, we may want to know whether one function value is smaller than any other. For example, in Figure 1, $f(x_1) \geq f(x)$ for all x in the domain. There is no function value that is smaller than all others, however, because $f(x) \to -\infty$ as $x \to \infty$ or as $x \to -\infty$.

The largest possible value of a function is called the *absolute maximum* and the smallest possible value of a function is called the *absolute minimum*. As Figure 1 shows, one or both of these may not exist on the domain of the function, $(-\infty, \infty)$ here. Absolute extrema often coincide with relative extrema, as with $f(x_1)$ in Figure 1. Although a function may have several relative maxima or relative minima, it never has more than one *absolute maximum* or *absolute minimum,* although the absolute maximum or minimum might occur at more than one value of x.

Absolute Maximum or Minimum

Let f be a function defined on some interval. Let c be a number in the interval. Then $f(c)$ is the **absolute maximum** of f on the interval if

$$f(x) \leq f(c)$$

for every x in the interval, and $f(c)$ is the **absolute minimum** of f on the interval if

$$f(x) \geq f(c)$$

for every x in the interval.

A function has an **absolute extremum** (plural: **extrema**) at c if it has either an absolute maximum or an absolute minimum there.

CAUTION Notice that, just like a relative extremum, an absolute extremum is a y-value, not an x-value.

Now look at Figure 2, which shows three functions defined on closed intervals. In each case there is an absolute maximum value and an absolute minimum value. These absolute extrema may occur at the endpoints or at relative extrema. As the graphs in Figure 2 show, an absolute extremum is either the largest or the smallest function value occurring on a closed interval, while a relative extremum is the largest or smallest function value in some (perhaps small) open interval.

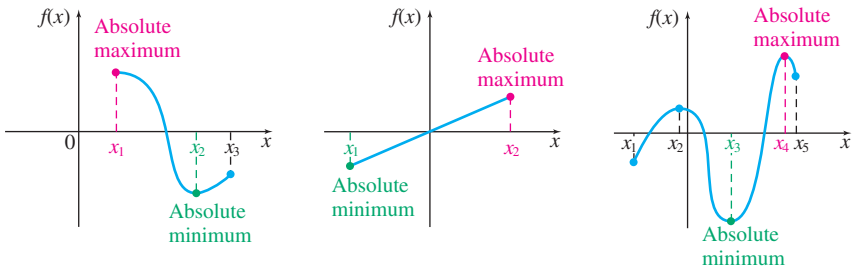
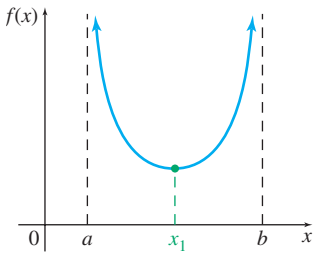


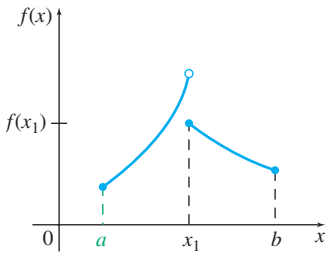
FIGURE 2

Although a function can have only one absolute minimum value and only one absolute maximum value, it can have many points where these values occur. (Note that the absolute maximum value and absolute minimum value are numbers, not points.) As an extreme example, consider the function $f(x) = 2$. The absolute minimum value of this function is clearly 2, as is the absolute maximum value. Both the absolute minimum and the absolute maximum occur at every real number x.

One of the main reasons for the importance of absolute extrema is given by the **extreme value theorem** (which is proved in more advanced courses).



(a)



(b)

FIGURE 3

Extreme Value Theorem

A function f that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum on the interval.

A continuous function on an open interval may or may not have an absolute maximum or minimum. For example, the function in Figure 3(a) has an absolute minimum on the interval (a, b) at x_1, but it does not have an absolute maximum. Instead, it becomes arbitrarily large as x approaches a or b. Also, a discontinuous function on a closed interval may or may not have an absolute minimum or maximum. The function in Figure 3(b) has an absolute minimum at $x = a$, yet it has no absolute maximum. It may appear at first to have an absolute maximum at x_1, but notice that $f(x_1)$ has a smaller value than f at values of x less than x_1.

The extreme value theorem guarantees the existence of absolute extrema for a continuous function on a closed interval. To find these extrema, use the following steps.

Finding Absolute Extrema

To find absolute extrema for a function f continuous on a closed interval $[a, b]$:

1. Find all critical numbers for f in (a, b).
2. Evaluate f for all critical numbers in (a, b).
3. Evaluate f for the endpoints a and b of the interval $[a, b]$.
4. The largest value found in Step 2 or 3 is the absolute maximum for f on $[a, b]$, and the smallest value found is the absolute minimum for f on $[a, b]$.

EXAMPLE 1 Absolute Extrema

Find the absolute extrema of the function

$$f(x) = x^{8/3} - 16x^{2/3}$$

on the interval $[-1, 8]$.

FOR REVIEW

Recall from Section R.6 that an exponential expression can be simplified by factoring out the smallest power of the variable. The expression $\frac{8}{3}x^{5/3} - \frac{32}{3}x^{-1/3}$ has a common factor of $\frac{8}{3}x^{-1/3}$.

SOLUTION First look for critical numbers in the interval $(-1, 8)$.

$$\begin{aligned} f'(x) &= \frac{8}{3}x^{5/3} - \frac{32}{3}x^{-1/3} \\ &= \frac{8}{3}x^{-1/3}(x^2 - 4) \qquad \text{Factor.} \\ &= \frac{8}{3}\left(\frac{x^2 - 4}{x^{1/3}}\right) \end{aligned}$$

Set $f'(x) = 0$ and solve for x. Notice that $f'(x) = 0$ at $x = 2$ and $x = -2$, but -2 is not in the interval $(-1, 8)$, so we ignore it. The derivative is undefined at $x = 0$, but the function is defined there, so 0 is also a critical number.

Evaluate the function at the critical numbers and the endpoints.

x-Value	Extrema Candidates Value of Function	
-1	-15	
0	0	
2	-19.05	← Absolute minimum
8	192	← Absolute maximum

YOUR TURN 1 Find the absolute extrema of the function $f(x) = 3x^{2/3} - 3x^{5/3}$ on the interval $[0, 8]$.

The absolute maximum, 192, occurs when $x = 8$, and the absolute minimum, approximately -19.05, occurs when $x = 2$. A graph of f is shown in Figure 4.

TRY YOUR TURN 1

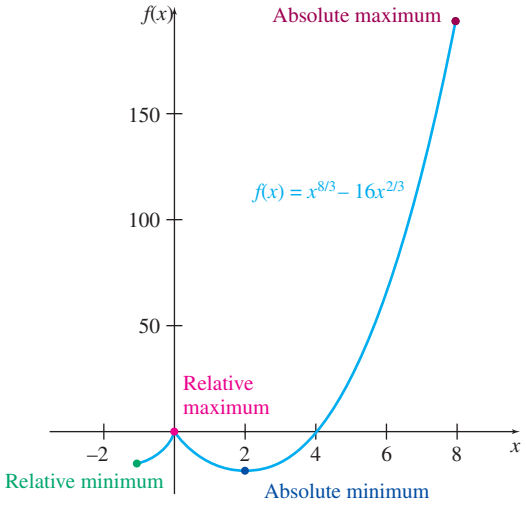


FIGURE 4

TECHNOLOGY NOTE

In Example 1, a graphing calculator that gives the maximum and minimum values of a function on an interval, such as the `fMax` or `fMin` feature of the TI-84 Plus, could replace the table. Alternatively, we could first graph the function on the given interval and then select the feature that gives the maximum or minimum value of the graph of the function instead of completing the table.

EXAMPLE 2 Absolute Extrema

Find the locations and values of the absolute extrema, if they exist, for the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 2.$$

SOLUTION In this example, the extreme value theorem does not apply since the domain is an open interval, $(-\infty, \infty)$, which has no endpoints. Begin as before by finding any critical numbers.

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 24x = 0 \\ 12x(x^2 - x - 2) &= 0 \\ 12x(x+1)(x-2) &= 0 \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 2$$

There are no values of x where $f'(x)$ does not exist. Evaluate the function at the critical numbers.

Extrema Candidates	
x-Value	Value of Function
-1	-3
0	2
2	-30 ← Absolute minimum

For an open interval, rather than evaluating the function at the endpoints, we evaluate the limit of the function when the endpoints are approached. Because the positive x^4-term dominates the other terms as x becomes large,

$$\lim_{x \to \infty} (3x^4 - 4x^3 - 12x^2 + 2) = \infty.$$

The limit is also ∞ as x approaches $-\infty$. Since the function can be made arbitrarily large, it has no absolute maximum. The absolute minimum, -30, occurs at $x = 2$. This result can be confirmed with a graphing calculator, as shown in Figure 5. TRY YOUR TURN 2

YOUR TURN 2 Find the locations and values of the absolute extrema, if they exist, for the function $f(x) = -x^4 - 4x^3 + 8x^2 + 20$.

In many of the applied extrema problems in the next section, a continuous function on an open interval has just one critical number. In that case, we can use the following theorem, which also applies to closed intervals.

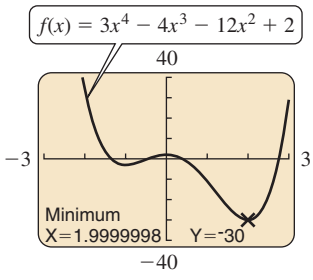


FIGURE 5

Critical Point Theorem

Suppose a function f is continuous on an interval I and that f has exactly one critical number in the interval I, located at $x = c$.

If f has a relative maximum at $x = c$, then this relative maximum is the absolute maximum of f on the interval I.

If f has a relative minimum at $x = c$, then this relative minimum is the absolute minimum of f on the interval I.

The critical point theorem is of no help in the previous two examples because they each had more than one critical point on the interval under consideration. But the theorem could be useful for some of Exercises 31–38 at the end of this section, and we will make good use of it in the next section.

EXAMPLE 3 U.S. and Canadian Dollar Exchange

The U.S. and Canadian exchange rate changes daily. The value of the U.S. dollar (in Canadian dollars) between 2000 and 2010 can be approximated by the function

$$f(t) = 0.00316t^3 - 0.0471t^2 + 0.114t + 1.47$$

where t is the number of years since 2000. Based on this approximation, in what year during this period did the value of the U.S. dollar reach its absolute minimum? What is the minimum value of the dollar during this period? ***Source: The Federal Reserve***.

APPLY IT

SOLUTION The function is defined on the interval [0, 10]. We first look for critical numbers in this interval. Here $f'(t) = 0.00948t^2 - 0.0942t + 0.114$. We set this derivative equal to 0 and use the quadratic formula to solve for t.

$$0.00948t^2 - 0.0942t + 0.114 = 0$$

$$t = \frac{0.0942 \pm \sqrt{(-0.0942)^2 - 4(0.00948)(0.114)}}{2(0.00948)}$$

$$t = 1.41 \quad \text{or} \quad t = 8.53$$

Both values are in the interval [0,10]. Now evaluate the function at the critical numbers and the endpoints 0 and 10.

	Extrema Candidates	
t-Value	Value of Function	
0	1.47	
1.41	1.55	
8.53	0.977	← Absolute minimum
10	1.06	

About 8.53 years after 2000, that is, around the middle of 2008, the U.S. dollar was worth about \$0.98 Canadian, which was an absolute minimum during this period. It is also worth noting the absolute maximum value of the U.S. dollar, which was about \$1.55 Canadian, occurred approximately 1.41 years after 2000, or around May of 2001.

Graphical Optimization

Figure 6 shows the production output for a family owned business that produces landscape mulch. As the number of workers (measured in hours of labor) varies, the total production of mulch also varies. In Section 11.5, Example 1, we saw that maximum production occurs with 8 workers (corresponding to 320 hours of labor). A manager, however, may want to know how many hours of labor to use in order to maximize the output per hour of labor. For any point on the curve, the y-coordinate measures the output and the x-coordinate measures the hours of labor, so the y-coordinate divided by the x-coordinate gives the output per hour of labor. This quotient is also the slope of the line through the origin and the point on the curve. Therefore, to maximize the output per hour of labor, we need to find where this slope is greatest. As shown in Figure 6, this occurs when approximately 270 hours of labor are used. Notice that this is also where the line from the origin to the curve is tangent to the curve. Another way of looking at this is to say that the point on the curve where the tangent line passes through the origin is the point that maximizes the output per hour of labor.

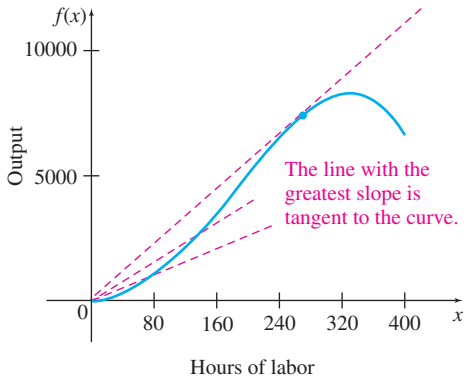


FIGURE 6

We can show that, in general, when $y = f(x)$ represents the output as a function of input, the maximum output per unit input occurs when the line from the origin to a point on the graph of the function is tangent to the function. Our goal is to maximize

$$g(x) = \frac{\textbf{output}}{\textbf{input}} = \frac{f(x)}{x}.$$

Taking the derivative and setting it equal to 0 gives

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = 0$$

$$xf'(x) = f(x)$$

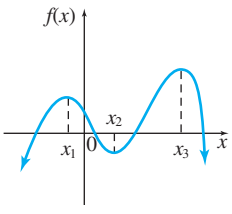
$$f'(x) = \frac{f(x)}{x}.$$

Notice that $f'(x)$ gives the slope of the tangent line at the point, and $f(x)/x$ gives the slope of the line from the origin to the point. When these are equal, as in Figure 6, the output per input is maximized. In other examples, the point on the curve where the tangent line passes through the origin gives a minimum. For a life science example of this, see Exercise 54 in Section 11.4 on the Definition of the Derivative.

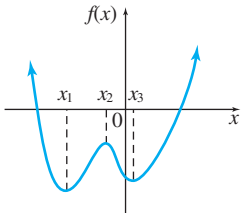
14.1 EXERCISES

Find the locations of any absolute extrema for the functions with graphs as follows.

1.



2.



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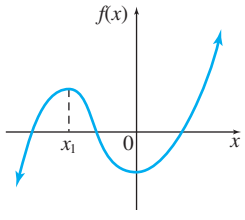

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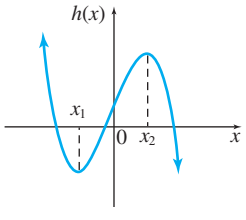

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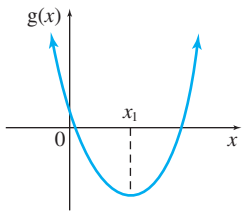

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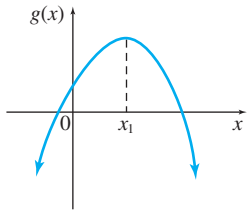

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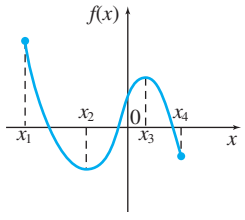

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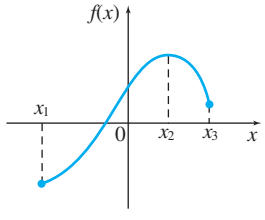


9. What is the difference between a relative extremum and an absolute extremum?

10. Can a relative extremum be an absolute extremum? Is a relative extremum necessarily an absolute extremum?

Find the absolute extrema if they exist, as well as all values of x where they occur, for each function, and specified domain. If you have one, use a graphing calculator to verify your answers.

11. $f(x) = x^3 - 6x^2 + 9x - 8; \quad [0, 5]$

12. $f(x) = x^3 - 3x^2 - 24x + 5; \quad [-3, 6]$

13. $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x + 1; \quad [-5, 2]$

14. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 3; \quad [-4, 4]$

15. $f(x) = x^4 - 18x^2 + 1; \quad [-4, 4]$

16. $f(x) = x^4 - 32x^2 - 7; \quad [-5, 6]$

17. $f(x) = \frac{1 - x}{3 + x}; \quad [0, 3]$

18. $f(x) = \frac{8 + x}{8 - x}; \quad [4, 6]$

19. $f(x) = \frac{x - 1}{x^2 + 1}; \quad [1, 5]$

20. $f(x) = \frac{x}{x^2 + 2}; \quad [0, 4]$

21. $f(x) = (x^2 - 4)^{1/3}; \quad [-2, 3]$

22. $f(x) = (x^2 - 16)^{2/3}; \quad [-5, 8]$

23. $f(x) = 5x^{2/3} + 2x^{5/3}; \quad [-2, 1]$

24. $f(x) = x + 3x^{2/3}; \quad [-10, 1]$

25. $f(x) = x^2 - 8\ln x; \quad [1, 4]$

26. $f(x) = \frac{\ln x}{x^2}; \quad [1, 4]$

27. $f(x) = x + e^{-3x}; \quad [-1, 3]$

28. $f(x) = x^2 e^{-0.5x}; \quad [2, 5]$

Graph each function on the indicated domain, and use the capabilities of your calculator to find the location and value of the absolute extrema.

29. $f(x) = \frac{-5x^4 + 2x^3 + 3x^2 + 9}{x^4 - x^3 + x^2 + 7}; \quad [-1, 1]$

30. $f(x) = \frac{x^3 + 2x + 5}{x^4 + 3x^3 + 10}; \quad [-3, 0]$

Find the absolute extrema if they exist, as well as all values of x where they occur.

31. $f(x) = 2x + \frac{8}{x^2} + 1, \quad x > 0$

32. $f(x) = 12 - x - \frac{9}{x}, \quad x > 0$

33. $f(x) = -3x^4 + 8x^3 + 18x^2 + 2$

34. $f(x) = x^4 - 4x^3 + 4x^2 + 1$

35. $f(x) = \frac{x - 1}{x^2 + 2x + 6}$

36. $f(x) = \frac{x}{x^2 + 1}$

37. $f(x) = \frac{\ln x}{x^3}$

38. $f(x) = x\ln x$

39. Find the absolute maximum and minimum of $f(x) = 2x - 3x^{2/3}$ (**a**) on the interval $[-1, 0.5]$; (**b**) on the interval $[0.5, 2]$.

40. Let $f(x) = e^{-2x}$. For $x > 0$, let $P(x)$ be the perimeter of the rectangle with vertices $(0, 0)$, $(x, 0)$, $(x, f(x))$ and $(0, f(x))$. Which of the following statements is true? ***Source: Society of Actuaries***.

a. The function P has an absolute minimum but not an absolute maximum on the interval $(0, \infty)$.

b. The function P has an absolute maximum but not an absolute minimum on the interval $(0, \infty)$.

c. The function P has both an absolute minimum and an absolute maximum on the interval $(0, \infty)$.

d. The function P has neither an absolute maximum nor an absolute minimum on the interval $(0, \infty)$, but the graph of the function P does have an inflection point with positive x-coordinate.

e. The function P has neither an absolute maximum nor an absolute minimum on the interval $(0, \infty)$, and the graph of the function P does not have an inflection point with positive x-coordinate.

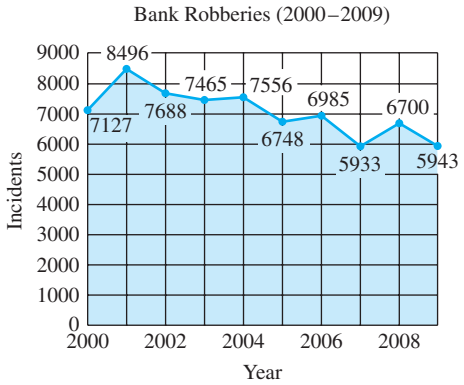
APPLICATIONS

Business and Economics

41. Bank Robberies The number of bank robberies in the United States for the years 2000–2009 is given in the following figure. Consider the closed interval [2000, 2009]. ***Source: FBI.***

a. Give all relative maxima and minima and when they occur on the interval.

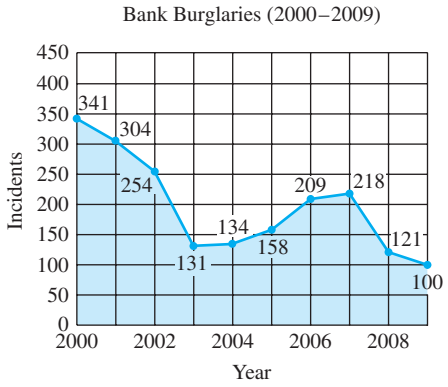
b. Give the absolute maxima and minima and when they occur on the interval. Interpret your results.



42. Bank Burglaries The number of bank burglaries (entry into or theft from a bank during nonbusiness hours) in the United States for the years 2000–2009 is given in the figure on the following page. Consider the closed interval [2000, 2009]. ***Source: FBI.***

a. Give all relative maxima and minima and when they occur on the interval.

b. Give the absolute maxima and minima and when they occur on the interval. Interpret your results.



43. Profit The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred thousand automobile tires is approximated by

$$P(x) = -x^3 + 9x^2 + 120x - 400, \quad x \geq 5.$$

Find the number of hundred thousands of tires that must be sold to maximize profit. Find the maximum profit.

44. Profit A company has found that its weekly profit from the sale of x units of an auto part is given by

$$P(x) = -0.02x^3 + 600x - 20{,}000.$$

Production bottlenecks limit the number of units that can be made per week to no more than 150, while a long-term contract requires that at least 50 units be made each week. Find the maximum possible weekly profit that the firm can make.

Average Cost Find the minimum value of the average cost for the given cost function on the given intervals.

45. $C(x) = x^3 + 37x + 250$ on the following intervals.

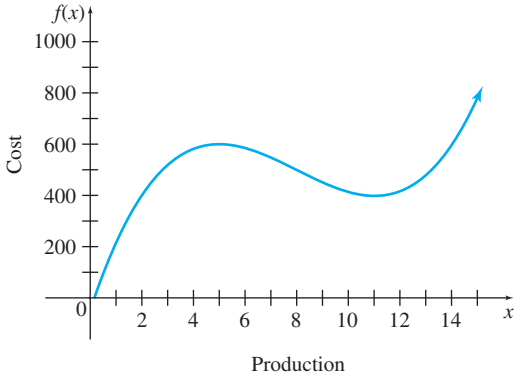
a. $1 \leq x \leq 10$ **b.** $10 \leq x \leq 20$

46. $C(x) = 81x^2 + 17x + 324$ on the following intervals.

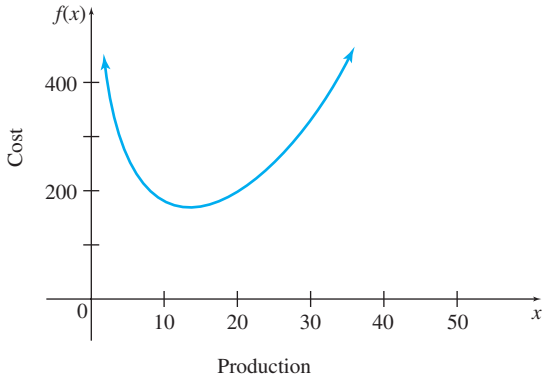
a. $1 \leq x \leq 10$ **b.** $10 \leq x \leq 20$

Cost Each graph gives the cost as a function of production level. Use the method of graphical optimization to estimate the production level that results in the minimum cost per item produced.

47.

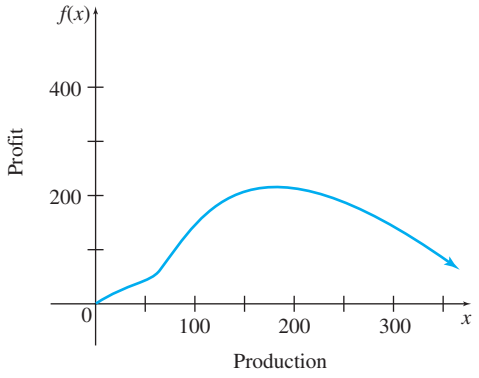


48.

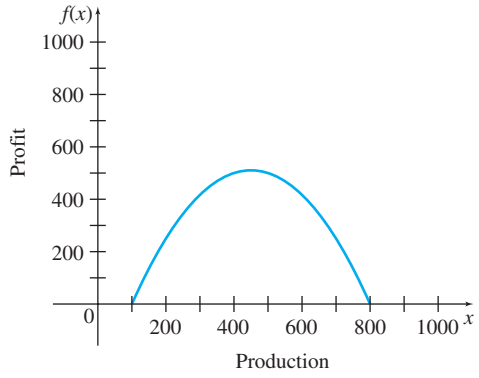


Profit Each graph gives the profit as a function of production level. Use graphical optimization to estimate the production level that gives the maximum profit per item produced.

49.



50.



Life Sciences

51. Pollution A marshy region used for agricultural drainage has become contaminated with selenium. It has been determined that flushing the area with clean water will reduce the selenium for a while, but it will then begin to build up again. A biologist has found that the percent of selenium in the soil x months after the flushing begins is given by

$$f(x) = \frac{x^2 + 36}{2x}, \quad 1 \leq x \leq 12.$$

When will the selenium be reduced to a minimum? What is the minimum percent?

52. Salmon Spawning The number of salmon swimming upstream to spawn is approximated by

$$S(x) = -x^3 + 3x^2 + 360x + 5000, \quad 6 \le x \le 20,$$

where x represents the temperature of the water in degrees Celsius. Find the water temperature that produces the maximum number of salmon swimming upstream.

53. Molars Researchers have determined that the crown length of first molars in fetuses is related to the postconception age of the tooth as

$$L(t) = -0.01t^2 + 0.788t - 7.048,$$

where $L(t)$ is the crown length (in millimeters) of the molar t weeks after conception. Find the maximum length of the crown of first molars during weeks 22 through 28. ***Source: American Journal of Physical Anthropology***.

54. Fungal Growth Because of the time that many people spend indoors, there is a concern about the health risk of being exposed to harmful fungi that thrive in buildings. The risk appears to increase in damp environments. Researchers have discovered that by controlling both the temperature and the relative humidity in a building, the growth of the fungus *A. versicolor* can be limited. The relationship between temperature and relative humidity, which limits growth, can be described by

$$R(T) = -0.00007T^3 + 0.0401T^2 - 1.6572T + 97.086,$$
$$15 \le T \le 46,$$

where $R(T)$ is the relative humidity (in percent) and T is the temperature (in degrees Celsius). Find the temperature at which the relative humidity is minimized. ***Source: Applied and Environmental Microbiology***.

Physical Sciences

55. Gasoline Mileage From information given in a recent business publication, we constructed the mathematical model

$$M(x) = -\frac{1}{45}x^2 + 2x - 20, \quad 30 \le x \le 65,$$

to represent the miles per gallon used by a certain car at a speed of x mph. Find the absolute maximum miles per gallon and the absolute minimum and the speeds at which they occur.

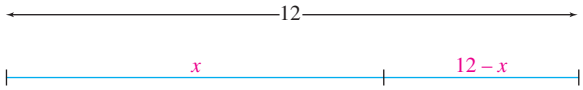
56. Gasoline Mileage For a certain sports utility vehicle,

$$M(x) = -0.015x^2 + 1.31x - 7.3, \quad 30 \le x \le 60,$$

represents the miles per gallon obtained at a speed of x mph. Find the absolute maximum miles per gallon and the absolute minimum and the speeds at which they occur.

General Interest

Area A piece of wire 12 ft long is cut into two pieces. (See the figure.) One piece is made into a circle and the other piece is made into a square. Let the piece of length x be formed into a circle. We allow x to equal 0 or 12, so all the wire may be used for the square or for the circle.



$$\text{Radius of circle} = \frac{x}{2\pi} \qquad \text{Area of circle} = \pi\left(\frac{x}{2\pi}\right)^2$$

$$\text{Side of square} = \frac{12 - x}{4} \qquad \text{Area of square} = \left(\frac{12 - x}{4}\right)^2$$

57. Where should the cut be made in order to minimize the sum of the areas enclosed by both figures?

58. Where should the cut be made in order to make the sum of the areas maximum? (*Hint*: Remember to use the endpoints of a domain when looking for absolute maxima and minima.)

59. For the solution to Exercise 57, show that the side of the square equals the diameter of the circle, that is, that the circle can be inscribed in the square.*

60. Information Content Suppose dots and dashes are transmitted over a telegraph line so that dots occur a fraction p of the time (where $0 < p < 1$) and dashes occur a fraction $1 - p$ of the time. The *information content* of the telegraph line is given by $I(p)$, where

$$I(p) = -p \ln p - (1 - p)\ln(1 - p).$$

a. Show that $I'(p) = -\ln p + \ln(1 - p)$.

b. Set $I'(p) = 0$ and find the value of p that maximizes the information content.

c. How might the result in part b be used?

YOUR TURN ANSWERS

1. Absolute maximum of about 0.977 occurs when $x = 2/5$ and absolute minimum of -84 when $x = 8$.
2. Absolute maximum, 148, occurs at $x = -4$. No absolute minimum.

*For a generalization of this phenomenon, see Cade, Pat and Russell A. Gordon, "An Apothem Apparently Appears," *The College Mathematics Journal,* Vol. 36, No. 1, Jan. 2005, pp. 52–55.

14.2 Applications of Extrema

APPLY IT **How should boxes and cans be designed to minimize the material needed to construct them or to maximize the volume?**

In Examples 3 and 4 we will use the techniques of calculus to find an answer to these questions.

In this section we give several examples showing applications of calculus to maximum and minimum problems. To solve these examples, go through the following steps.

Solving an Applied Extrema Problem

1. Read the problem carefully. Make sure you understand what is given and what is unknown.
2. If possible, sketch a diagram. Label the various parts.
3. Decide on the variable that must be maximized or minimized. Express that variable as a function of *one* other variable.
4. Find the domain of the function.
5. Find the critical points for the function from Step 3.
6. If the domain is a closed interval, evaluate the function at the endpoints and at each critical number to see which yields the absolute maximum or minimum. If the domain is an open interval, apply the critical point theorem when there is only one critical number. If there is more than one critical number, evaluate the function at the critical numbers and find the limit as the endpoints of the interval are approached to determine if an absolute maximum or minimum exists at one of the critical points.

CAUTION Do not skip Step 6 in the preceding box. If a problem asks you to maximize a quantity and you find a critical point at Step 5, do not automatically assume the maximum occurs there, for it may occur at an endpoint, as in Exercise 58 of the previous section, or it may not exist at all.

An infamous case of such an error occurred in a 1945 study of "flying wing" aircraft designs similar to the Stealth bomber. In seeking to maximize the range of the aircraft (how far it can fly on a tank of fuel), the study's authors found that a critical point occurred when almost all of the volume of the plane was in the wing. They claimed that this critical point was a maximum. But another engineer later found that this critical point, in fact, *minimized* the range of the aircraft! ***Source: Science***.

EXAMPLE 1 Maximization

Find two nonnegative numbers x and y for which $2x + y = 30$, such that xy^2 is maximized.

SOLUTION Step 1, reading and understanding the problem, is up to you. Step 2 does not apply in this example; there is nothing to draw. We proceed to Step 3, in which we decide what is to be maximized and assign a variable to that quantity. Here, xy^2 is to be maximized, so let

$$M = xy^2.$$

According to Step 3, we must express M in terms of just *one* variable, which can be done using the equation $2x + y = 30$ by solving for either x or y. Solving for y gives

$$2x + y = 30$$
$$y = 30 - 2x.$$

Substitute for y in the expression for M to get

$$\begin{aligned} M &= x(30 - 2x)^2 \\ &= x(900 - 120x + 4x^2) \\ &= 900x - 120x^2 + 4x^3. \end{aligned}$$

We are now ready for Step 4, when we find the domain of the function. Because of the nonnegativity requirement, x must be at least 0. Since y must also be at least 0, we require $30 - 2x \geq 0$, so $x \leq 15$. Thus x is confined to the interval $[0, 15]$.

Moving on to Step 5, we find the critical points for M by finding dM/dx, and then solving the equation $dM/dx = 0$ for x.

$$\frac{dM}{dx} = 900 - 240x + 12x^2 = 0$$

$$12(75 - 20x + x^2) = 0 \qquad \text{Factor out the 12.}$$

$$12(5 - x)(15 - x) = 0 \qquad \text{Factor the quadratic.}$$

$$x = 5 \quad \text{or} \quad x = 15$$

Extrema Candidates		
x	M	
0	0	
5	2000	← Maximum
15	0	

YOUR TURN 1 Find two nonnegative numbers x and y for which $x + 3y = 30$, such that x^2y is maximized.

Finally, at Step 6, we find M for the critical numbers $x = 5$ and $x = 15$, as well as for $x = 0$, an endpoint of the domain. The other endpoint, $x = 15$, has already been included as a critical number. We see in the table that the maximum value of the function occurs when $x = 5$. Since $y = 30 - 2x = 30 - 2(5) = 20$, the values that maximize xy^2 are $x = 5$ and $y = 20$. TRY YOUR TURN 1

NOTE A critical point is only a candidate for an absolute maximum or minimum. The absolute maximum or minimum might occur at a different critical point or at an endpoint.

EXAMPLE 2 Minimizing Time

A math professor participating in the sport of orienteering must get to a specific tree in the woods as fast as possible. He can get there by traveling east along the trail for 300 m and then north through the woods for 800 m. He can run 160 m per minute along the trail but only 70 m per minute through the woods. Running directly through the woods toward the tree minimizes the distance, but he will be going slowly the whole time. He could instead run 300 m along the trail before entering the woods, maximizing the total distance but minimizing the time in the woods. Perhaps the fastest route is a combination, as shown in Figure 7. Find the path that will get him to the tree in the minimum time.

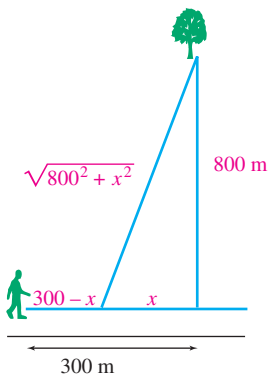


FIGURE 7

SOLUTION As in Example 1, the first step is to read and understand the problem. If the statement of the problem is not clear to you, go back and reread it until you understand it before moving on.

We have already started Step 2 by providing Figure 7. Let x be the distance shown in Figure 7, so the distance he runs on the trail is $300 - x$. By the Pythagorean theorem, the distance he runs through the woods is $\sqrt{800^2 + x^2}$.

The first part of Step 3 is noting that we are trying to minimize the total amount of time, which is the sum of the time on the trail and the time through the woods. We must express this time as a function of x. Since time = distance/speed, the total time is

$$T(x) = \frac{300 - x}{160} + \frac{\sqrt{800^2 + x^2}}{70}.$$

To complete Step 4, notice in this equation that $0 \leq x \leq 300$.

We now move to Step 5, in which we find the critical points by calculating the derivative and setting it equal to 0. Since $\sqrt{800^2 + x^2} = (800^2 + x^2)^{1/2}$,

$$T'(x) = -\frac{1}{160} + \frac{1}{70}\left(\frac{1}{2}\right)(800^2 + x^2)^{-1/2}(2x) = 0.$$

$$\frac{x}{70\sqrt{800^2 + x^2}} = \frac{1}{160}$$

$$16x = 7\sqrt{800^2 + x^2} \qquad \text{Cross multiply and divide by 10.}$$

$$256x^2 = 49(800^2 + x^2) = (49 \cdot 800^2) + 49x^2 \qquad \text{Square both sides.}$$

$$207x^2 = 49 \cdot 800^2 \qquad \text{Subtract } 49x^2 \text{ from both sides.}$$

$$x^2 = \frac{49 \cdot 800^2}{207}$$

$$x = \frac{7 \cdot 800}{\sqrt{207}} \approx 389 \qquad \text{Take the square root of both sides}$$

Since 389 is not in the interval $[0, 300]$, the minimum time must occur at one of the endpoints.

We now complete Step 6 by creating a table with $T(x)$ evaluated at the endpoints. We see from the table that the time is minimized when $x = 300$, that is, when the professor heads straight for the tree.

Extrema Candidates		
x	$T(x)$	
0	13.30	
300	12.21	← Minimum

YOUR TURN 2 Suppose the professor in Example 2 can only run 40 m per minute through the woods. Find the path that will get him to the tree in the minimum time.

TRY YOUR TURN 2

EXAMPLE 3 Maximizing Volume

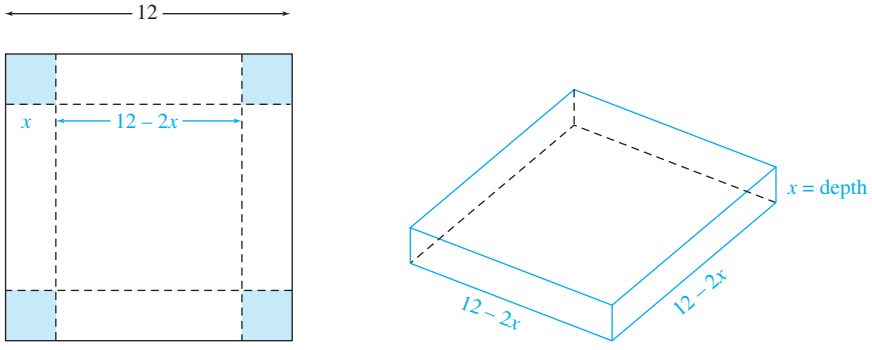
An open box is to be made by cutting a square from each corner of a 12-in. by 12-in. piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume?

APPLY IT

SOLUTION Let x represent the length of a side of the square that is cut from each corner, as shown in Figure 8(a). The width of the box is $12 - 2x$, with the length also $12 - 2x$. As shown in Figure 8(b), the depth of the box will be x inches. The volume of the box is given by the product of the length, width, and height. In this example, the volume, $V(x)$, depends on x:

$$V(x) = x(12 - 2x)(12 - 2x) = 144x - 48x^2 + 4x^3.$$

Clearly, $0 \le x$, and since neither the length nor the width can be negative, $0 \le 12 - 2x$, so $x \le 6$. Thus, the domain of V is the interval $[0, 6]$.



(a) (b)

FIGURE 8

Extrema Candidates		
x	$V(x)$	
0	0	
2	128	← Maximum
6	0	

YOUR TURN 3 Repeat Example 3 using an 8-m by 8-m piece of metal.

The derivative is $V'(x) = 144 - 96x + 12x^2$. Set this derivative equal to 0.

$$\begin{aligned} 12x^2 - 96x + 144 &= 0 \\ 12(x^2 - 8x + 12) &= 0 \\ 12(x-2)(x-6) &= 0 \\ x - 2 = 0 \quad \text{or} \quad & x - 6 = 0 \\ x = 2 \quad \text{or} \quad & x = 6 \end{aligned}$$

Find $V(x)$ for x equal to 0, 2, and 6 to find the depth that will maximize the volume. The table indicates that the box will have maximum volume when $x = 2$ and that the maximum volume will be 128 in^3. TRY YOUR TURN 3

EXAMPLE 4 Minimizing Area

A company wants to manufacture cylindrical aluminum cans with a volume of 1000 cm^3 (1 liter). What should the radius and height of the can be to minimize the amount of aluminum used?

APPLY IT

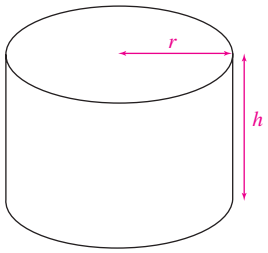


FIGURE 9

SOLUTION The two variables in this problem are the radius and the height of the can, which we shall label r and h, as in Figure 9. Minimizing the amount of aluminum used requires minimizing the surface area of the can, which we will designate S. The surface area consists of a top and a bottom, each of which is a circle with an area πr^2, plus the side. If the side were sliced vertically and unrolled, it would form a rectangle with height h and width equal to the circumference of the can, which is $2\pi r$. Thus the surface area is given by

$$S = 2\pi r^2 + 2\pi rh.$$

The right side of the equation involves two variables. We need to get a function of a single variable. We can do this by using the information about the volume of the can:

$$V = \pi r^2 h = 1000.$$

(Here we have used the formula for the volume of a cylinder.) Solve this for h:

$$h = \frac{1000}{\pi r^2}.$$

(Solving for r would have involved a square root and a more complicated function.)

We now substitute this expression for h into the equation for S to get

$$S = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}.$$

There are no restrictions on r other than that it be a positive number, so the domain of S is $(0, \infty)$.

Find the critical points for S by finding dS/dr, then solving the equation $dS/dr = 0$ for r.

$$\begin{aligned} \frac{dS}{dr} = 4\pi r - \frac{2000}{r^2} &= 0 \\ 4\pi r^3 &= 2000 \\ r^3 &= \frac{500}{\pi} \end{aligned}$$

Take the cube root of both sides to get

$$r = \left(\frac{500}{\pi}\right)^{1/3} \approx 5.419$$

centimeters. Substitute this expression into the equation for h to get

$$h = \frac{1000}{\pi 5.419^2} \approx 10.84$$

centimeters. Notice that the height of the can is twice its radius.

There are several ways to carry out Step 6 to verify that we have found the minimum. Because there is only one critical number, the critical point theorem applies.

Method 1 Critical Point Theorem with First Derivative Test

Verify that when $r < 5.419$, then $dS/dr < 0$, and when $r > 5.419$, then $dS/dr > 0$. Since the function is decreasing before 5.419 and increasing after 5.419, there must be a relative minimum at $r = 5.419$ cm. By the critical point theorem, there is an absolute minimum there.

Method 2 Critical Point Theorem with Second Derivative Test

We could also use the critical point theorem with the second derivative test.

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3}$$

Notice that for positive r, the second derivative is always positive, so there is a relative minimum at $r = 5.419$ cm. By the critical point theorem, there is an absolute minimum there.

Method 3 Limits at Endpoints

We could also find the limit as the endpoints are approached.

$$\lim_{r \to 0} S = \lim_{r \to \infty} S = \infty$$

The surface area becomes arbitrarily large as r approaches the endpoints of the domain, so the absolute minimum surface area must be at the critical point.

YOUR TURN 4 Repeat Example 4 if the volume is to be 500 cm^3.

The graphing calculator screen in Figure 10 confirms that there is an absolute minimum at $r = 5.419$ cm. TRY YOUR TURN 4

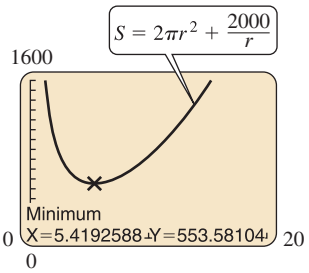


FIGURE 10

Notice that if the previous example had asked for the height and radius that maximize the amount of aluminum used, the problem would have no answer. There is no maximum for a function that can be made arbitrarily large.

Maximum Sustainable Harvest

For most living things, reproduction is *seasonal*—it can take place only at selected times of the year. Large whales, for example, reproduce every two years during a relatively short time span of about two months. Shown on the time axis in Figure 11 are the reproductive periods. Let S = number of adults present during the reproductive period and let R = number of adults that return the next season to reproduce. ***Source: Mathematics for the Biosciences.***

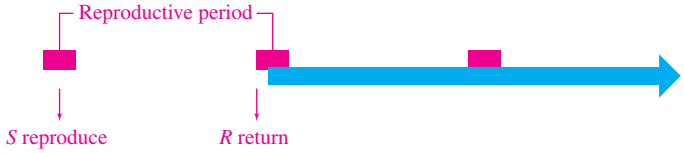


FIGURE 11

If we find a relationship between R and S, $R = f(S)$, then we have formed a **spawner-recruit** function or **parent-progeny** function. These functions are notoriously hard to develop because of the difficulty of obtaining accurate counts and because of the many hypotheses that can be made about the life stages. We will simply suppose that the function f takes various forms.

If $R > S$, we can presumably harvest

$$H = R - S = f(S) - S$$

individuals, leaving S to reproduce. Next season, $R = f(S)$ will return and the harvesting process can be repeated, as shown in Figure 12 on the next page.

Let S_0 be the number of spawners that will allow as large a harvest as possible without threatening the population with extinction. Then $H(S_0)$ is called the **maximum sustainable harvest**.

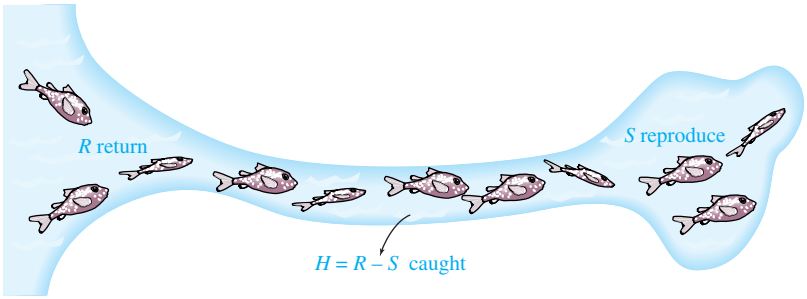


FIGURE 12

EXAMPLE 5 Maximum Sustainable Harvest

Suppose the spawner-recruit function for Idaho rabbits is $f(S) = 2.17\sqrt{S}\ln(S+1)$, where S is measured in thousands of rabbits. Find S_0 and the maximum sustainable harvest, $H(S_0)$.

SOLUTION S_0 is the value of S that maximizes H. Since

$$\begin{aligned} H(S) &= f(S) - S \\ &= 2.17\sqrt{S}\ln(S+1) - S, \end{aligned}$$

$$H'(S) = 2.17\left(\frac{\ln(S+1)}{2\sqrt{S}} + \frac{\sqrt{S}}{S+1}\right) - 1.$$

Now we want to set this derivative equal to 0 and solve for S.

$$0 = 2.17\left(\frac{\ln(S+1)}{2\sqrt{S}} + \frac{\sqrt{S}}{S+1}\right) - 1.$$

This equation cannot be solved analytically, so we will graph $H'(S)$ with a graphing calculator and find any S-values where $H'(S)$ is 0. (An alternative approach is to use the equation solver some graphing calculators have.) The graph with the value where $H'(S)$ is 0 is shown in Figure 13.

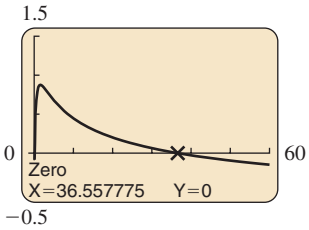


FIGURE 13

From the graph we see that $H'(S) = 0$ when $S = 36.557775$, so the number of rabbits needed to sustain the population is about 36,600. A graph of H will show that this is a maximum. From the graph, using the capability of the calculator, we find that the harvest is $H(36.557775) \approx 11.015504$. These results indicate that after one reproductive season, a population of 36,600 rabbits will have increased to 47,600. Of these, 11,000 may be harvested, leaving 36,600 to regenerate the population. Any harvest larger than 11,000 will threaten the future of the rabbit population, while a harvest smaller than 11,000 will allow the population to grow larger each season. Thus 11,000 is the maximum sustainable harvest for this population.

14.2 EXERCISES

In Exercises 1–4, use the steps shown in Exercise 1 to find nonnegative numbers x and y that satisfy the given requirements. Give the optimum value of the indicated expression.

1. $x + y = 180$ and the product $P = xy$ is as large as possible.

a. Solve $x + y = 180$ for y.

b. Substitute the result from part a into $P = xy$, the equation for the variable that is to be maximized.

c. Find the domain of the function P found in part b.

d. Find dP/dx. Solve the equation $dP/dx = 0$.

e. Evaluate P at any solutions found in part d, as well as at the endpoints of the domain found in part c.

f. Give the maximum value of P, as well as the two numbers x and y whose product is that value.

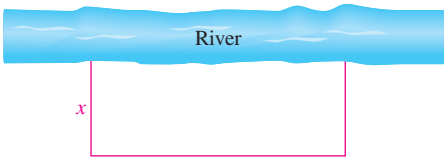
2. The sum of x and y is 140 and the sum of the squares of x and y is minimized.
3. $x + y = 90$ and x^2y is maximized.
4. $x + y = 105$ and xy^2 is maximized.

APPLICATIONS

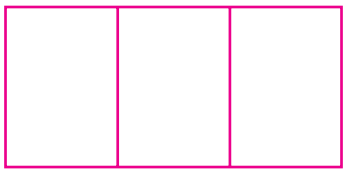
Business and Economics

Average Cost **In Exercises 5 and 6, determine the average cost function $\overline{C}(x) = C(x)/x$. To find where the average cost is smallest, first calculate $\overline{C}'(x)$, the derivative of the average cost function. Then use a graphing calculator to find where the derivative is 0. Check your work by finding the minimum from the graph of the function $\overline{C}(x)$.**

5. $C(x) = \frac{1}{2}x^3 + 2x^2 - 3x + 35$
6. $C(x) = 10 + 20x^{1/2} + 16x^{3/2}$
7. **Revenue** If the price charged for a candy bar is $p(x)$ cents, then x thousand candy bars will be sold in a certain city, where
$$p(x) = 160 - \frac{x}{10}.$$
   a. Find an expression for the total revenue from the sale of x thousand candy bars.
   b. Find the value of x that leads to maximum revenue.
   c. Find the maximum revenue.
8. **Revenue** The sale of compact disks of "lesser" performers is very sensitive to price. If a CD manufacturer charges $p(x)$ dollars per CD, where
$$p(x) = 12 - \frac{x}{8},$$
   then x thousand CDs will be sold.
   a. Find an expression for the total revenue from the sale of x thousand CDs.
   b. Find the value of x that leads to maximum revenue.
   c. Find the maximum revenue.
9. **Area** A campground owner has 1400 m of fencing. He wants to enclose a rectangular field bordering a river, with no fencing needed along the river. (See the sketch.) Let x represent the width of the field.



   a. Write an expression for the length of the field.
   b. Find the area of the field (area = length × width).
   c. Find the value of x leading to the maximum area.
   d. Find the maximum area.
10. **Area** Find the dimensions of the rectangular field of maximum area that can be made from 300 m of fencing material. (This fence has four sides.)
11. **Area** An ecologist is conducting a research project on breeding pheasants in captivity. She first must construct suitable pens. She wants a rectangular area with two additional fences across its width, as shown in the sketch. Find the maximum area she can enclose with 3600 m of fencing.

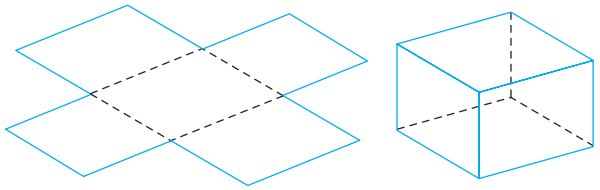


12. **Area** A farmer is constructing a rectangular pen with one additional fence across its width. Find the maximum area that can be enclosed with 2400 m of fencing.
13. **Cost with Fixed Area** A fence must be built in a large field to enclose a rectangular area of 25,600 m^2. One side of the area is bounded by an existing fence; no fence is needed there. Material for the fence costs \$3 per meter for the two ends and \$1.50 per meter for the side opposite the existing fence. Find the cost of the least expensive fence.
14. **Cost with Fixed Area** A fence must be built to enclose a rectangular area of 20,000 ft^2. Fencing material costs \$2.50 per foot for the two sides facing north and south and \$3.20 per foot for the other two sides. Find the cost of the least expensive fence.
15. **Revenue** A local club is arranging a charter flight to Hawaii. The cost of the trip is \$1600 each for 90 passengers, with a refund of \$10 per passenger for each passenger in excess of 90.
   a. Find the number of passengers that will maximize the revenue received from the flight.
   b. Find the maximum revenue.
16. **Profit** In planning a restaurant, it is estimated that a profit of \$8 per seat will be made if the number of seats is no more than 50, inclusive. On the other hand, the profit on each seat will decrease by 10¢ for each seat above 50.
   a. Find the number of seats that will produce the maximum profit.
   b. What is the maximum profit?
17. **Timing Income** A local group of scouts has been collecting aluminum cans for recycling. The group has already collected 12,000 lb of cans, for which they could currently receive \$7.50 per hundred pounds. The group can continue to collect cans at the rate of 400 lb per day. However, a glut in the aluminum market has caused the recycling company to announce that it will lower its price, starting immediately, by \$0.15 per hundred pounds per day. The scouts can make only one trip to the recycling center. Find the best time for the trip. What total income will be received?
18. **Pricing** Decide what you would do if your assistant presented the following contract for your signature:

    Your firm offers to deliver 250 tables to a dealer, at \$160 per table, and to reduce the price per table on the entire order by 50¢ for each additional table over 250.

    Find the dollar total involved in the largest possible transaction between the manufacturer and the dealer; then find the smallest possible dollar amount.

19. Packaging Design A television manufacturing firm needs to design an open-topped box with a square base. The box must hold 32 in^3. Find the dimensions of the box that can be built with the minimum amount of materials. (See the figure.)



20. Packaging Design A company wishes to manufacture a box with a volume of 36 ft^3 that is open on top and is twice as long as it is wide. Find the dimensions of the box produced from the minimum amount of material.

21. Container Design An open box will be made by cutting a square from each corner of a 3-ft by 8-ft piece of cardboard and then folding up the sides. What size square should be cut from each corner in order to produce a box of maximum volume?

22. Container Design Consider the problem of cutting corners out of a rectangle and folding up the sides to make a box. Specific examples of this problem are discussed in Example 3 and Exercise 21.

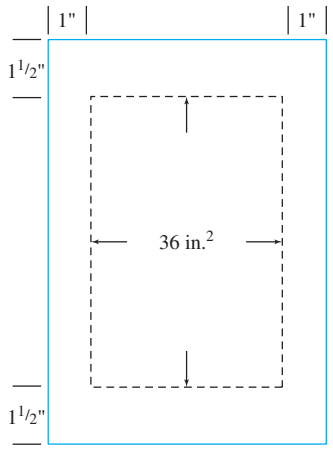
a. In the solution to Example 3, compare the area of the base of the box with the area of the walls.

b. Repeat part a for the solution to Exercise 21.

c. Make a conjecture about the area of the base compared with the area of the walls for the box with the maximum volume.

23. Packaging Cost A closed box with a square base is to have a volume of 16,000 cm^3. The material for the top and bottom of the box costs \$3 per square centimeter, while the material for the sides costs \$1.50 per square centimeter. Find the dimensions of the box that will lead to the minimum total cost. What is the minimum total cost?

24. Use of Materials A mathematics book is to contain 36 in^2 of printed matter per page, with margins of 1 in. along the sides and $1\frac{1}{2}$ in. along the top and bottom. Find the dimensions of the page that will require the minimum amount of paper. (See the figure.)



25. Can Design

a. For the can problem in Example 4, the minimum surface area required that the height be twice the radius. Show that this is true for a can of arbitrary volume V.

b. Do many cans in grocery stores have a height that is twice the radius? If not, discuss why this may be so.

26. Container Design Your company needs to design cylindrical metal containers with a volume of 16 cubic feet. The top and bottom will be made of a sturdy material that costs \$2 per square foot, while the material for the sides costs \$1 per square foot. Find the radius, height, and cost of the least expensive container.

In Exercises 27–29, use a graphing calculator to determine where the derivative is equal to zero.

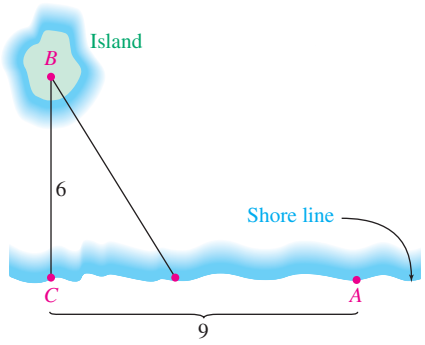
27. Can Design Modify the can problem in Example 4 so the cost must be minimized. Assume that aluminum costs 3¢ per square centimeter, and that there is an additional cost of 2¢ per cm times the perimeter of the top, and a similar cost for the bottom, to seal the top and bottom of the can to the side.

28. Can Design In this modification of the can problem in Example 4, the cost must be minimized. Assume that aluminum costs 3¢ per square centimeter, and that there is an additional cost of 1¢ per cm times the height of the can to make a vertical seam on the side.

29. Can Design This problem is a combination of Exercises 27 and 28. We will again minimize the cost of the can, assuming that aluminum costs 3¢ per square centimeter. In addition, there is a cost of 2¢ per cm to seal the top and bottom of the can to the side, plus 1¢ per cm to make a vertical seam.

30. Packaging Design A cylindrical box will be tied up with ribbon as shown in the figure. The longest piece of ribbon available is 130 cm long, and 10 cm of that are required for the bow. Find the radius and height of the box with the largest possible volume.

31. Cost A company wishes to run a utility cable from point A on the shore (see the figure on the next page) to an installation at point B on the island. The island is 6 miles from the shore. It costs \$400 per mile to run the cable on land and \$500 per mile underwater. Assume that the cable starts at A and runs along the shoreline, then angles and runs underwater to the island. Find the point at which the line should begin to angle in order to yield the minimum total cost.



32. Cost Repeat Exercise 31, but make point A 7 miles from point C.

Life Sciences

33. Disease Epidemiologists have found a new communicable disease running rampant in College Station, Texas. They estimate that t days after the disease is first observed in the community, the percent of the population infected by the disease is approximated by

$$p(t) = \frac{20t^3 - t^4}{1000}$$

for $0 \le t \le 20$.

a. After how many days is the percent of the population infected a maximum?

b. What is the maximum percent of the population infected?

34. Disease Another disease hits the chronically ill town of College Station, Texas. This time the percent of the population infected by the disease t days after it hits town is approximated by $p(t) = 10te^{-t/8}$ for $0 \le t \le 40$.

a. After how many days is the percent of the population infected a maximum?

b. What is the maximum percent of the population infected?

Maximum Sustainable Harvest Find the maximum sustainable harvest in Exercises 35 and 36. See Example 5.

35. $f(S) = 12S^{0.25}$ **36.** $f(S) = \dfrac{25S}{S + 2}$

37. Pollution A lake polluted by bacteria is treated with an antibacterial chemical. After t days, the number N of bacteria per milliliter of water is approximated by

$$N(t) = 20\left(\frac{t}{12} - \ln\left(\frac{t}{12}\right)\right) + 30$$

for $1 \le t \le 15$.

a. When during this time will the number of bacteria be a minimum?

b. What is the minimum number of bacteria during this time?

c. When during this time will the number of bacteria be a maximum?

d. What is the maximum number of bacteria during this time?

38. Maximum Sustainable Harvest The population of salmon next year is given by $f(S) = Se^{r(1-S/P)}$, where S is this year's salmon population, P is the equilibrium population, and r is a constant that depends upon how fast the population grows. The number of salmon that can be fished next year while keeping the population the same is $H(S) = f(S) - S$. The maximum value of $H(S)$ is the maximum sustainable harvest. ***Source: Journal of the Fisheries Research Board of Canada.***

a. Show that the maximum sustainable harvest occurs when $f'(S) = 1$. (*Hint:* To maximize, set $H'(S) = 0$.)

b. Let the value of S found in part a be denoted by S_0. Show that the maximum sustainable harvest is given by

$$S_0\left(\frac{1}{1 - rS_0/P} - 1\right).$$

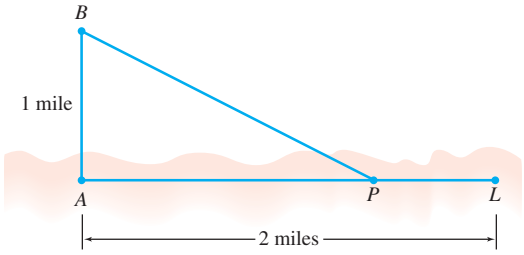
(*Hint:* Set $f'(S_0) = 1$ and solve for $e^{r(1-S_0/P)}$. Then find $H(S_0)$ and substitute the expression for $e^{r(1-S_0/P)}$.)

Maximum Sustainable Harvest In Exercises 39 and 40, refer to Exercise 38. Find $f'(S_0)$ and solve the equation $f'(S_0) = 1$, using a calculator to find the intersection of the graphs of $f'(S_0)$ and $y = 1$.

39. Find the maximum sustainable harvest if $r = 0.1$ and $P = 100$.

40. Find the maximum sustainable harvest if $r = 0.4$ and $P = 500$.

41. Pigeon Flight Homing pigeons avoid flying over large bodies of water, preferring to fly around them instead. (One possible explanation is the fact that extra energy is required to fly over water because air pressure drops over water in the daytime.) Assume that a pigeon released from a boat 1 mile from the shore of a lake (point B in the figure) flies first to point P on the shore and then along the straight edge of the lake to reach its home at L. If L is 2 miles from point A, the point on the shore closest to the boat, and if a pigeon needs $4/3$ as much energy per mile to fly over water as over land, find the location of point P, which minimizes energy used.



42. Pigeon Flight Repeat Exercise 41, but assume a pigeon needs $10/9$ as much energy to fly over water as over land.

43. Harvesting Cod A recent article described the population $f(S)$ of cod in the North Sea next year as a function of this year's population S (in thousands of tons) by various mathematical models.

$$\text{Shepherd:} \quad f(S) = \frac{aS}{1 + (S/b)^c};$$

$$\text{Ricker:} \quad f(S) = aSe^{-bS};$$

$$\text{Beverton–Holt:} \quad f(S) = \frac{aS}{1 + (S/b)},$$

where a, b, and c are constants. ***Source: Nature.***

a. Find a replacement of variables in the Ricker model above that will make it the same as another form of the Ricker model described in Exercise 38 of this section, $f(S) = Se^{r(1-S/P)}$.

b. Find $f'(S)$ for all three models.

c. Find $f'(0)$ for all three models. From your answer, describe in words the geometric meaning of the constant a.

d. The values of a, b, and c reported in the article for the Shepherd model are 3.026, 248.72, and 3.24, respectively. Find the value of this year's population that maximizes next year's population using the Shepherd model.

e. The values of a and b reported in the article for the Ricker model are 4.151 and 0.0039, respectively. Find the value of this year's population that maximizes next year's population using the Ricker model.

f. Explain why, for the Beverton-Holt model, there is no value of this year's population that maximizes next year's population.

44. Bird Migration Suppose a migrating bird flies at a velocity v, and suppose the amount of time the bird can fly depends on its velocity according to the function $T(v)$. ***Source: A Concrete Approach to Mathematical Modelling.***

a. If E is the bird's initial energy, then the bird's effective power is given by kE/T, where k is the fraction of the power that can be converted into mechanical energy. According to principles of aerodynamics,

$$\frac{kE}{T} = aSv^3 + I,$$

where a is a constant, S is the wind speed, and I is the induced power, or rate of working against gravity. Using this result and the fact that distance is velocity multiplied by time, show that the distance that the bird can fly is given by

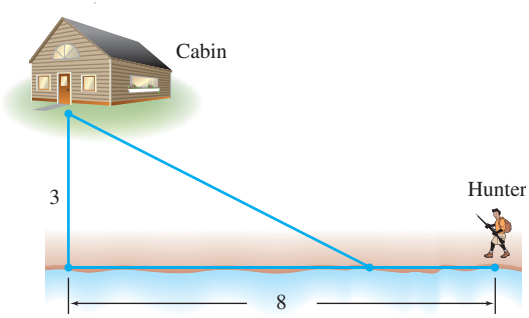
$$D(v) = \frac{kEv}{aSv^3 + I}.$$

b. Show that the migrating bird can fly a maximum distance by flying at a velocity

$$v = \left(\frac{I}{2aS}\right)^{1/3}.$$

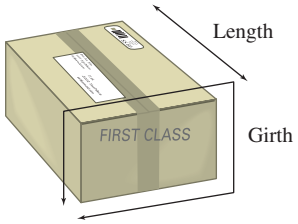
General Interest

45. Travel Time A hunter is at a point along a river bank. He wants to get to his cabin, located 3 miles north and 8 miles west. (See the figure.) He can travel 5 mph along the river but only 2 mph on this very rocky land. How far upriver should he go in order to reach the cabin in minimum time?



46. Travel Time Repeat Exercise 45, but assume the cabin is 19 miles north and 8 miles west.

47. Postal Regulations The U.S. Postal Service stipulates that any boxes sent through the mail must have a length plus girth totaling no more than 108 in. (See the figure.) Find the dimensions of the box with maximum volume that can be sent through the U.S. mail, assuming that the width and the height of the box are equal. ***Source: U.S. Postal Service.***



YOUR TURN ANSWERS

1. $x = 20$ and $y = 10/3$
2. Go 93 m along the trail and then head into the woods.
3. Box will have maximum volume when $x = 4/3$ m and the maximum volume is $1024/27$ m^3.
4. Radius is 4.3 cm and height is 8.6 cm.

14.3 Further Business Applications: Economic Lot Size; Economic Order Quantity; Elasticity of Demand

APPLY IT **How many batches of primer should a paint company produce per year to minimize its costs while meeting its customers' demand?**
We will answer this question in Example 1 using the concept of economic lot size.

In this section we introduce three common business applications of calculus. The first two, *economic lot size* and *economic order quantity*, are related. A manufacturer

must determine the production lot (or batch) size that will result in minimum production and storage costs, while a purchaser must decide what quantity of an item to order in an effort to minimize reordering and storage costs. The third application, *elasticity of demand*, deals with the sensitivity of demand for a product to changes in the price of the product.

Economic Lot Size

Suppose that a company manufactures a constant number of units of a product per year and that the product can be manufactured in several batches of equal size throughout the year. On the one hand, if the company were to manufacture one large batch every year, it would minimize setup costs but incur high warehouse costs. On the other hand, if it were to make many small batches, this would increase setup costs. Calculus can be used to find the number that should be manufactured in each batch in order to minimize the total cost. This number is called the **economic lot size**.

Figure 14 below shows several possibilities for a product having an annual demand of 12,000 units. The top graph shows the results if all 12,000 units are made in one batch per year. In this case an average of 6000 units will be held in a warehouse. If 3000 units are made in each batch, four batches will be made at equal time intervals during the year, and the average number of units in the warehouse falls to only 1500. If 1000 units are made in each of twelve batches, an average of 500 units will be in the warehouse.

The variable in our discussion of economic lot size will be

q = number of units in each batch.

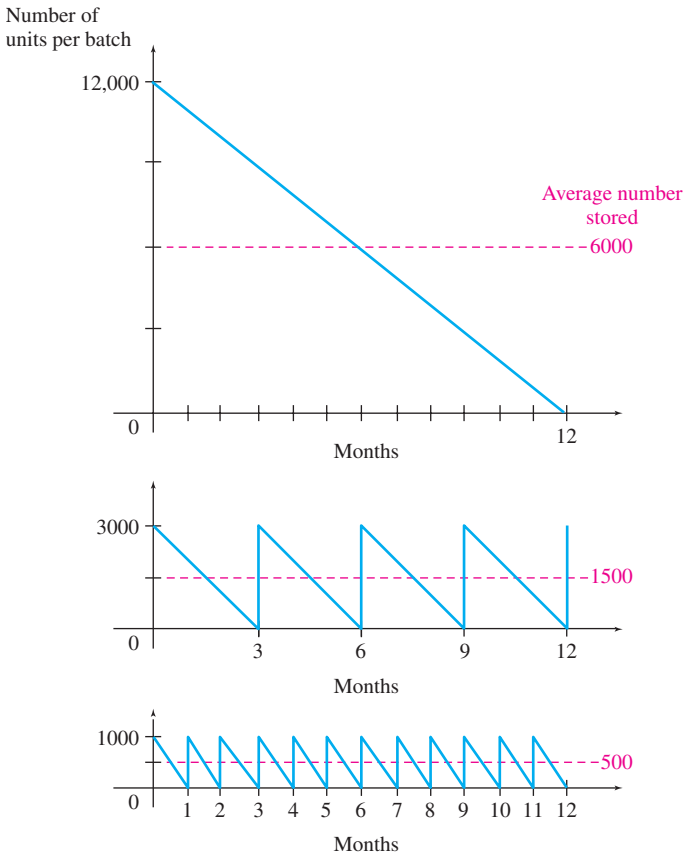


FIGURE 14

In addition, we have the following constants:

k = cost of storing one unit of the product for one year;
f = fixed setup cost to manufacture the product;
g = cost of manufacturing a single unit of the product;
M = total number of units produced annually.

The company has two types of costs: a cost associated with manufacturing the item and a cost associated with storing the finished product. Because q units are produced in each batch, and each batch has a fixed cost f and a variable cost g per unit, the manufacturing cost per batch is

$$f + gq.$$

The number of units produced in a year is M, so the number of batches per year must be M/q. Therefore, the total annual manufacturing cost is

$$(f + gq)\frac{M}{q} = \frac{fM}{q} + gM. \tag{1}$$

Since demand is constant, the inventory goes down linearly from q to 0, as in Figure 14, with an average inventory of $q/2$ units per year. The cost of storing one unit of the product for a year is k, so the total storage cost is

$$k\left(\frac{q}{2}\right) = \frac{kq}{2}. \tag{2}$$

The total production cost is the sum of the manufacturing and storage costs, or the sum of Equations (1) and (2). If $T(q)$ is the total cost of producing M units in batches of size q,

$$T(q) = \frac{fM}{q} + gM + \frac{kq}{2}.$$

In words, we have found that the total cost is equal to

$$\left(\text{fixed cost} + \frac{\text{cost}}{\text{unit}} \times \frac{\#\text{ units}}{\text{batch}}\right)\frac{\#\text{ batches}}{\text{year}} + \text{storage cost} \times \#\text{ units in storage.}$$

Since the only constraint on q is that it be a positive number, the domain of T is $(0, \infty)$. To find the value of q that will minimize $T(q)$, remember that f, g, k, and M are constants and find $T'(q)$.

$$T'(q) = \frac{-fM}{q^2} + \frac{k}{2}$$

Set this derivative equal to 0.

$$\frac{-fM}{q^2} + \frac{k}{2} = 0$$

$$\frac{k}{2} = \frac{fM}{q^2} \qquad \text{Add } \frac{fM}{q^2} \text{ to both sides.}$$

$$q^2\frac{k}{2} = fM \qquad \text{Multiply both sides by } q^2.$$

$$q^2 = \frac{2fM}{k} \qquad \text{Multiply both sides by } 2/k.$$

$$q = \sqrt{\frac{2fM}{k}} \qquad \text{Take square root of both sides.} \tag{3}$$

The critical point theorem can be used to show that $\sqrt{(2fM)/k}$ is the economic lot size that minimizes total production costs. (See Exercise 1.)

This application is referred to as the *inventory problem* and is treated in more detail in management science courses. Please note that Equation (3) was derived under very specific assumptions. If the assumptions are changed slightly, a different conclusion might be reached, and it would not necessarily be valid to use Equation (3).

In some examples Equation (3) may not give an integer value, in which case we must investigate the next integer smaller than q and the next integer larger to see which gives the minimum cost.

EXAMPLE 1 Lot Size

APPLY IT

A paint company has a steady annual demand for 24,500 cans of automobile primer. The comptroller for the company says that it costs \$2 to store one can of paint for 1 year and \$500 to set up the plant for the production of the primer. Find the number of cans of primer that should be produced in each batch, as well as the number of batches per year, in order to minimize total production costs.

YOUR TURN 1 Suppose the annual demand in Example 1 is only 18,000 cans, the setup cost is \$750, and the storage cost is \$3 per can. Find the number of cans that should be produced in each batch and the number of batches per year to minimize production costs.

SOLUTION Use Equation (3), with $k = 2$, $M = 24{,}500$, and $f = 500$.

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(500)(24{,}500)}{2}} = \sqrt{12{,}250{,}000} = 3500$$

The company should make 3500 cans of primer in each batch to minimize production costs. The number of batches per year is $M/q = 24{,}500/3500 = 7$.

TRY YOUR TURN 1

Economic Order Quantity

We can extend our previous discussion to the problem of reordering an item that is used at a constant rate throughout the year. Here, the company using a product must decide how often to order and how many units to request each time an order is placed; that is, it must identify the **economic order quantity**. In this case, the variable is

q = number of units to order each time.

We also have the following constants:

k = cost of storing one unit for one year

f = fixed cost to place an order

M = total units needed per year

The goal is to minimize the total cost of ordering over a year's time, where

Total cost = Storage cost + Reorder cost.

Again assume an average inventory of $q/2$, so the yearly storage cost is $kq/2$. The number of orders placed annually is M/q. The reorder cost is the product of this quantity and the cost per order, f. Thus, the reorder cost is fM/q, and the total cost is

$$T(q) = \frac{fM}{q} + \frac{kq}{2}.$$

This is almost the same formula we derived for the inventory problem, which also had a constant term gM. Since a constant does not affect the derivative, Equation (3) is also valid for the economic order quantity problem. As before, the number of orders placed annually is M/q. This illustrates how two different applications might have the same mathematical structure, so a solution to one applies to both.

EXAMPLE 2 Order Quantity

A large pharmacy has an annual need for 480 units of a certain antibiotic. It costs \$3 to store one unit for one year. The fixed cost of placing an order (clerical time, mailing, and so on) amounts to \$31. Find the number of units to order each time, and how many times a year the antibiotic should be ordered.

YOUR TURN 2 Suppose the annual need in Example 2 is 320 units, the fixed cost amounts to \$30, and the storage cost is \$2 per unit. Find the number of units to order each time and how many times a year to order to minimize cost.

SOLUTION Here $k = 3$, $M = 480$, and $f = 31$. We have

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(31)(480)}{3}} = \sqrt{9920} \approx 99.6$$

$T(99) = 298.803$ and $T(100) = 298.800$, so ordering 100 units of the drug each time minimizes the annual cost. The drug should be ordered $M/q = 480/100 = 4.8$ times a year, or about once every $2\frac{1}{2}$ months. TRY YOUR TURN 2

Elasticity of Demand

Anyone who sells a product or service is concerned with how a change in price affects demand. The sensitivity of demand to changes in price varies with different items. Luxury items tend to be more sensitive to price than essentials. For items such as milk, heating fuel, and light bulbs, relatively small percentage changes in price will not change the demand for the item much, so long as the price is not far from its normal range. For cars, home loans, jewelry, and concert tickets, however, small percentage changes in price can have a significant effect on demand.

FOR REVIEW

Recall from Chapter 1 that the Greek letter Δ, pronounced *delta*, is used in mathematics to mean "change."

One way to measure the sensitivity of demand to changes in price is by the relative change—the ratio of percent change in demand to percent change in price. If q represents the quantity demanded and p the price, this ratio can be written as

$$\frac{\Delta q/q}{\Delta p/p},$$

where Δq represents the change in q and Δp represents the change in p. This ratio is always negative, because q and p are positive, while Δq and Δp have opposite signs. (An *increase* in price causes a *decrease* in demand.) If the absolute value of this ratio is large, it suggests that a relatively small increase in price causes a relatively large drop (decrease) in demand.

This ratio can be rewritten as

$$\frac{\Delta q/q}{\Delta p/p} = \frac{\Delta q}{q} \cdot \frac{p}{\Delta p} = \frac{p}{q} \cdot \frac{\Delta q}{\Delta p}.$$

Suppose $q = f(p)$. (Note that this is the inverse of the way our demand functions have been expressed so far; previously we had $p = D(q)$.) Then $\Delta q = f(p + \Delta p) - f(p)$, and

$$\frac{\Delta q}{\Delta p} = \frac{f(p + \Delta p) - f(p)}{\Delta p}.$$

As $\Delta p \to 0$, this quotient becomes

$$\lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} = \lim_{\Delta p \to 0} \frac{f(p + \Delta p) - f(p)}{\Delta p} = \frac{dq}{dp},$$

and

$$\lim_{\Delta p \to 0} \frac{p}{q} \cdot \frac{\Delta q}{\Delta p} = \frac{p}{q} \cdot \frac{dq}{dp}.$$

The negative of this last quantity is called the *elasticity of demand* (E) and measures the instantaneous responsiveness of demand to price. *

Elasticity of Demand

Let $q = f(p)$, where q is demand at a price p. The **elasticity of demand** is

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}.$$

Demand is inelastic if $E < 1$.

Demand is elastic if $E > 1$.

Demand has unit elasticity if $E = 1$.

*Economists often define elasticity as the negative of our definition.

For example, E has been estimated at 0.6 for physician services and at 2.3 for restaurant meals. The demand for medical care is much less responsive to price changes than is the demand for nonessential commodities, such as restaurant meals. ***Source: Economics: Private and Public Choice***.

If $E < 1$, the relative change in demand is less than the relative change in price, and the demand is called *inelastic*. If $E > 1$, the relative change in demand is greater than the relative change in price, and the demand is called *elastic*. In other words, inelastic means that a small change in price has little effect on demand, while elastic means that a small change in price has more effect on demand. When $E = 1$, the percentage changes in price and demand are relatively equal and the demand is said to have **unit elasticity**.

EXAMPLE 3 Elasticity

Terrence Wales described the demand for distilled spirits as

$$q = f(p) = -0.00375p + 7.87,$$

where p represents the retail price of a case of liquor in dollars per case. Here q represents the average number of cases purchased per year by a consumer. Calculate and interpret the elasticity of demand when $p = \$118.30$ per case. ***Source: The American Economic Review***.

SOLUTION From $q = -0.00375p + 7.87$, we determine $dq/dp = -0.00375$. Now we find E.

$$\begin{aligned} E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ &= -\frac{p}{-0.00375p + 7.87}(-0.00375) \\ &= \frac{0.00375p}{-0.00375p + 7.87} \end{aligned}$$

Let $p = 118.30$ to get

$$E = \frac{0.00375(118.30)}{-0.00375(118.30) + 7.87} \approx 0.0597.$$

Since $0.0597 < 1$, the demand is inelastic, and a percentage change in price will result in a smaller percentage change in demand. Thus an increase in price will increase revenue. For example, a 10% increase in price will cause an approximate decrease in demand of $(0.0597)(0.10) = 0.00597$ or about 0.6%. TRY YOUR TURN 3

YOUR TURN 3 Suppose the demand equation for a given commodity is $q = 24{,}000 - 3p^2$. Calculate and interpret E when $p = \$50$.

EXAMPLE 4 Elasticity

The demand for beer was modeled by Hogarty and Elzinga with the function given by $q = f(p) = 1/p$. The price was expressed in dollars per can of beer, and the quantity sold in cans per day per adult. Calculate and interpret the elasticity of demand. ***Source: The Review of Economics and Statistics***.

SOLUTION Since $q = 1/p$,

$$\frac{dq}{dp} = \frac{-1}{p^2}, \quad \text{and}$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p}{1/p} \cdot \frac{-1}{p^2} = 1.$$

Here, the elasticity is 1, unit elasticity, at every (positive) price. As we will see shortly, this means that revenues remain constant when the price changes. TRY YOUR TURN 4

YOUR TURN 4 Suppose the demand equation for a given product is $q = 200e^{-0.4p}$. Calculate and interpret E when $p = \$100$.

Elasticity can be related to the total revenue, R, by considering the derivative of R. Since revenue is given by price times sales (demand),

$$R = pq.$$

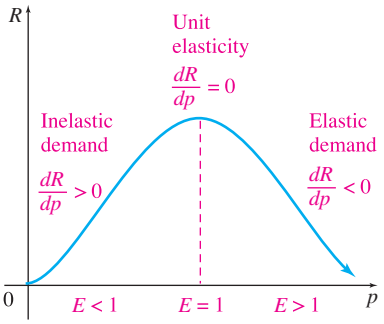


FIGURE 15

Differentiate with respect to p using the product rule.

$$\begin{aligned}
\frac{dR}{dp} &= p \cdot \frac{dq}{dp} + q \cdot 1 \\
&= \frac{q}{q} \cdot p \cdot \frac{dq}{dp} + q && \text{Multiply by } \frac{q}{q} \text{ (or 1).} \\
&= q\left(\frac{p}{q} \cdot \frac{dq}{dp}\right) + q \\
&= q(-E) + q && -E = \frac{p}{q} \cdot \frac{dq}{dp}. \\
&= q(-E + 1) && \text{Factor.} \\
&= q(1 - E)
\end{aligned}$$

Total revenue R is increasing, optimized, or decreasing depending on whether $dR/dp > 0$, $dR/dp = 0$, or $dR/dp < 0$. These three situations correspond to $E < 1$, $E = 1$, or $E > 1$. See Figure 15.

In summary, total revenue is related to elasticity as follows.

Revenue and Elasticity

1. If the demand is inelastic, total revenue increases as price increases.
2. If the demand is elastic, total revenue decreases as price increases.
3. Total revenue is maximized at the price where demand has unit elasticity.

EXAMPLE 5 Elasticity

Assume that the demand for a product is $q = 216 - 2p^2$, where p is the price in dollars.

(a) Find the price intervals where demand is elastic and where demand is inelastic.

SOLUTION Since $q = 216 - 2p^2$, $dq/dp = -4p$, and

$$\begin{aligned}
E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\
&= -\frac{p}{216 - 2p^2}(-4p) \\
&= \frac{4p^2}{216 - 2p^2}.
\end{aligned}$$

To decide where $E < 1$ or $E > 1$, solve the corresponding *equation*.

$$\begin{aligned}
E &= 1 \\
\frac{4p^2}{216 - 2p^2} &= 1 && E = \frac{4p^2}{216 - 2p^2} \\
4p^2 &= 216 - 2p^2 && \text{Multiply both sides by } 216 - 2p^2. \\
6p^2 &= 216 && \text{Add } 2p^2 \text{ to both sides.} \\
p^2 &= 36 && \text{Divide both sides by 6.} \\
p &= 6 && \text{Take square root of both sides.}
\end{aligned}$$

Substitute a test number on either side of 6 in the expression for E to see which values make $E < 1$ and which make $E > 1$.

$$\text{Let } p = 1\text{: } E = \frac{4(1)^2}{216 - 2(1)^2} = \frac{4}{214} < 1.$$

$$\text{Let } p = 10\text{: } E = \frac{4(10)^2}{216 - 2(10)^2} = \frac{400}{216 - 200} > 1.$$

YOUR TURN 5 Suppose the demand function for a given product is $q = 3600 - 3p^2$. Find the price intervals where demand is elastic and where demand is inelastic. What price results in maximum revenue? What is the maximum revenue?

Demand is inelastic when $E < 1$. This occurs when $p < 6$. Demand is elastic when $E > 1$; that is, when $p > 6$.

(b) What price results in the maximum revenue? What is the maximum revenue?

SOLUTION Total revenue is maximized at the price where demand has unit elasticity. As we saw in part (a), this occurs when the price is set at \$6 per item. The demand at this price is $q = 216 - 2(6)^2 = 144$. The maximum revenue is then $pq = 6 \cdot 144 = \$864$.

TRY YOUR TURN 5

14.3 EXERCISES

1. In the discussion of economic lot size, use the critical point theorem to show that $\sqrt{(2fM)/k}$ is the economic lot size that minimizes total production costs.
2. Why do you think that the cost g does not appear in the equation for q [Equation (3)]?
3. Choose the correct answer. ***Source: American Institute of Certified Public Accountants***.

   The economic order quantity formula assumes that

   a. Purchase costs per unit differ due to quantity discounts.

   b. Costs of placing an order vary with quantity ordered.

   c. Periodic demand for the goods is known.

   d. Erratic usage rates are cushioned by safety stocks.
4. Describe elasticity of demand in your own words.
5. A Giffen good is a product for which the demand function is increasing. Economists debate whether such goods actually exist. What is true about the elasticity of a Giffen good? ***Source: investopedia.com***.
6. What must be true about the demand function if $E = 0$?
7. Suppose that a demand function is linear—that is, $q = m - np$ for $0 \le p \le m/n$, where m and n are positive constants. Show that $E = 1$ at the midpoint of the demand curve on the interval $0 \le p \le m/n$; that is, at $p = m/(2n)$.
8. Suppose the demand function is of the form $q = Cp^{-k}$, where C and k are positive constants.

   a. Find the elasticity E.

   b. If $0 < k < 1$, what does your answer from part a say about how prices should be set to maximize the revenue?

   c. If $k > 1$, what does your answer from part a say about how prices should be set to maximize the revenue?

   d. If $k = 1$, what does your answer from part a tell you about setting prices to maximize revenue?

   e. Based on your answers above, is a demand function of the form $q = Cp^{-k}$ realistic? Explain your answer.

APPLICATIONS

Business and Economics

9. **Lot Size** Suppose 100,000 lamps are to be manufactured annually. It costs \$1 to store a lamp for 1 year, and it costs \$500 to set up the factory to produce a batch of lamps. Find the number of lamps to produce in each batch.
10. **Lot Size** A manufacturer has a steady annual demand for 13,950 cases of sugar. It costs \$9 to store 1 case for 1 year, \$31 in setup cost to produce each batch, and \$16 to produce each case. Find the number of cases per batch that should be produced.
11. **Lot Size** Find the number of batches of lamps that should be manufactured annually in Exercise 9.
12. **Lot Size** Find the number of batches of sugar that should be manufactured annually in Exercise 10.
13. **Order Quantity** A bookstore has an annual demand for 100,000 copies of a best-selling book. It costs \$0.50 to store 1 copy for 1 year, and it costs \$60 to place an order. Find the optimum number of copies per order.
14. **Order Quantity** A restaurant has an annual demand for 900 bottles of a California wine. It costs \$1 to store 1 bottle for 1 year, and it costs \$5 to place a reorder. Find the optimum number of bottles per order.
15. **Lot Size** Suppose that in the inventory problem, the storage cost depends on the maximum inventory size, rather than the average. This would be more realistic if, for example, the company had to build a warehouse large enough to hold the maximum inventory, and the cost of storage was the same no matter how full or empty the warehouse was. Show that in this case the number of units that should be ordered or manufactured to minimize the total cost is

$$q = \sqrt{\frac{fM}{k}}.$$

16. **Lot Size** A book publisher wants to know how many times a year a print run should be scheduled. Suppose it costs \$1000 to set up the printing process, and the subsequent cost per book is so low it can be ignored. Suppose further that the annual warehouse cost is \$6 times the maximum number of books stored. Assuming 5000 copies of the book are needed per year, how many books should be printed in each print run? (See Exercise 15.)
17. **Lot Size** Suppose that in the inventory problem, the storage cost is a combination of the cost described in the text and the cost described in Exercise 15. In other words, suppose there is an annual cost, k_1, for storing a single unit, plus an annual cost per unit, k_2, that must be paid for each unit up to the maximum number of units stored. Show that the number of units that

should be ordered or manufactured to minimize the total cost in this case is

$$q = \sqrt{\frac{2fM}{k_1 + 2k_2}}.$$

18. Lot Size Every year, Corinna Paolucci sells 30,000 cases of her Famous Spaghetti Sauce. It costs her \$1 per year in electricity to store a case, plus she must pay annual warehouse fees of \$2 per case for the maximum number of cases she will store. If it costs her \$750 to set up a production run, plus \$8 per case to manufacture a single case, how many production runs should she have each year to minimize her total costs? (See Exercise 17.)

Elasticity For each of the following demand functions, find (a) E, and (b) values of q (if any) at which total revenue is maximized.

19. $q = 50 - \frac{p}{4}$

20. $q = 25{,}000 - 50p$

21. $q = 37{,}500 - 5p^2$

22. $q = 48{,}000 - 10p^2$

23. $p = 400e^{-0.2q}$

24. $q = 10 - \ln p$

Elasticity Find the elasticity of demand (E) for the given demand function at the indicated values of p. Is the demand elastic, inelastic, or neither at the indicated values? Interpret your results.

25. $q = 400 - 0.2p^2$

a. $p = \$20$ **b.** $p = \$40$

26. $q = 300 - 2p$

a. $p = \$100$ **b.** $p = \$50$

27. Elasticity of Crude Oil The short-term demand for crude oil in the United States in 2008 can be approximated by

$$q = f(p) = 2{,}431{,}129p^{-0.06},$$

where p represents the price of crude oil in dollars per barrel and q represents the per capita consumption of crude oil. Calculate and interpret the elasticity of demand when the price is \$40 per barrel. ***Source: 2003 OPEC Review***.

28. Elasticity of Rice The demand for rice in Japan for a particular year was estimated by the general function

$$q = f(p) = Ap^{-0.13},$$

where p represents the price of a unit of rice, A represents a constant that can be calculated uniquely for a particular year, and q represents the annual per capita rice demand. Calculate and interpret the elasticity of demand. ***Source: Agricultural Economics***.

29. Elasticity of Software In 2008, the Valve Corporation, a software entertainment company, ran a holiday sale on its popular Steam software program. Using data collected from the sale, it is possible to estimate the demand corresponding to various discounts in the price of the software. Assuming that the original price was \$40, the demand for the software can be estimated by the function

$$q = 3{,}751{,}000p^{-2.826},$$

where p is the price and q is the demand. Calculate and interpret the elasticity of demand. ***Source: codinghorror.com***.

30. Elasticity The price of beef in the United States has been found to depend on the demand (measured by per capita consumption) according to the equation

$$q = \frac{342.5}{p^{0.5314}}.$$

Find the elasticity. Is the demand for beef elastic or inelastic? ***Source: SAS Institute Inc***.

31. Elasticity A study of the demand for air travel in Australia found that the demand for discount air travel from Sydney to Melbourne (in revenue passenger kilometre per capita, the product of the number of passengers travelling on a route and the distance of the route, divided by the populations of the host cities) depends on the airfare according to the equation

$$q = 55.2 - 0.022p.$$

Source: International Journal of Transport Economics.

a. Find the elasticity when the price is \$166.10, the average discount airfare deflated by the consumer price index to 1989–1990 prices, according to the authors of the study.

b. Is the demand for airfare elastic or inelastic at this price?

c. Find the price that maximizes revenue.

32. Elasticity The price along the West Coast of the United States for Japanese spruce logs (in dollars per cubic meter) based on the demand (in thousands of cubic meters per day) has been approximated by

$$p = 0.604q^2 - 20.16q + 263.067.$$

Source: Kamiak Econometrics.

a. Find the elasticity when the demand is 11 thousand cubic meters a day. (*Hint:* To find dq/dp when p is expressed in terms of q, you may use the fact that

$$\frac{dq}{dp} = \frac{1}{dp/dq}.$$

Review the explanation of the derivative of the natural logarithm to see why this is true.)

b. Is the demand for spruce logs elastic or inelastic?

c. What happens to the elasticity as q approaches 16.6887? Discuss the limitations of this model for the price as a function of the demand.

33. Elasticity A geometric interpretation of elasticity is as follows. Consider the tangent line to the demand curve $q = f(p)$ at the point $P_0 = (p_0, q_0)$. Let the point where the tangent line intersects the p-axis be called A, and the point where it intersects the q-axis be called B. Let P_0A and P_0B be the distances from P_0 to A and to B, respectively. Calculate the ratio P_0B/P_0A in terms of p_0, q_0, and $f'(p_0)$, and show that this ratio equals the elasticity. ***Source: The AMATYC Review***.

YOUR TURN ANSWERS

1. 6 batches per year with 3000 cans per batch
2. Order 98 units about every 3.675 months.
3. $E = 0.909$, the demand is inelastic.
4. $E = 40$, the demand is elastic.
5. Demand is inelastic when $p < 20$ and demand is elastic when $p > 20$. The maximum revenue is \$48,000 when $p = \$20$.

14.4 Implicit Differentiation

APPLY IT **How does demand for a certain commodity change with respect to price?**
We will answer this question in Example 4.

In almost all of the examples and applications so far, all functions have been defined in the form

$$y = f(x),$$

with y given **explicitly** in terms of x, or as an **explicit function** of x. For example,

$$y = 3x - 2, \qquad y = x^2 + x + 6, \qquad \text{and} \qquad y = -x^3 + 2$$

are all explicit functions of x. The equation $4xy - 3x = 6$ can be expressed as an explicit function of x by solving for y. This gives

$$\begin{aligned} 4xy - 3x &= 6 \\ 4xy &= 3x + 6 \\ y &= \frac{3x + 6}{4x}. \end{aligned}$$

On the other hand, some equations in x and y cannot be readily solved for y, and some equations cannot be solved for y at all. For example, while it would be possible (but tedious) to use the quadratic formula to solve for y in the equation $y^2 + 2yx + 4x^2 = 0$, it is not possible to solve for y in the equation $y^5 + 8y^3 + 6y^2x^2 + 2yx^3 + 6 = 0$. In equations such as these last two, y is said to be given **implicitly** in terms of x.

In such cases, it may still be possible to find the derivative dy/dx by a process called **implicit differentiation**. In doing so, we assume that there exists some function or functions f, which we may or may not be able to find, such that $y = f(x)$ and dy/dx exists. It is useful to use dy/dx here rather than $f'(x)$ to make it clear which variable is independent and which is dependent.

EXAMPLE 1 Implicit Differentiation

Find dy/dx if $3xy + 4y^2 = 10$.

SOLUTION Differentiate with respect to x on both sides of the equation.

$$3xy + 4y^2 = 10$$

$$\frac{d}{dx}(3xy + 4y^2) = \frac{d}{dx}(10) \tag{1}$$

FOR REVIEW

In Chapter 1, we pointed out that when y is given as a function of x, x is the independent variable and y is the dependent variable. We later defined the derivative dy/dx when y is a function of x. In an equation such as $3xy + 4y^2 = 10$, either variable can be considered the independent variable. If a problem asks for dy/dx, consider x the independent variable; if it asks for dy/dx, consider y the independent variable. A similar rule holds when other variables are used.

Now differentiate each term on the left side of the equation. Think of $3xy$ as the product $(3x)(y)$ and use the product rule and the chain rule. Since

$$\frac{d}{dx}(3x) = 3 \quad \text{and} \quad \frac{d}{dx}(y) = \frac{dy}{dx},$$

the derivative of $(3x)(y)$ is

$$(3x)\frac{dy}{dx} + (y)3 = 3x\frac{dy}{dx} + 3y.$$

To differentiate the second term, $4y^2$, use the chain rule, since y is assumed to be some function of x.

$$\frac{d}{dx}(4y^2) = \overbrace{4(2y^1)\frac{dy}{dx}}^{\text{Derivative of } y^2} = 8y\frac{dy}{dx}$$

On the right side of Equation (1), the derivative of 10 is 0. Taking the indicated derivatives in Equation (1) term by term gives

$$3x\frac{dy}{dx} + 3y + 8y\frac{dy}{dx} = 0.$$

Now solve this result for dy/dx.

$$(3x + 8y)\frac{dy}{dx} = -3y$$

$$\frac{dy}{dx} = \frac{-3y}{3x + 8y}$$

YOUR TURN 1 Find dy/dx if $x^2 + y^2 = xy$.

TRY YOUR TURN 1

NOTE Because we are treating y as a function of x, notice that each time an expression has y in it, we use the chain rule.

EXAMPLE 2 Implicit Differentiation

Find dy/dx for $x + \sqrt{x}\sqrt{y} = y^2$.

SOLUTION Take the derivative on both sides with respect to x.

$$\frac{d}{dx}(x + \sqrt{x}\sqrt{y}) = \frac{d}{dx}(y^2)$$

Since $\sqrt{x} \cdot \sqrt{y} = x^{1/2} \cdot y^{1/2}$, use the product rule and the chain rule as follows.

$$\overbrace{1}^{\text{Derivative of } x} + \overbrace{x^{1/2}\left(\frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx}\right) + y^{1/2}\left(\frac{1}{2}x^{-1/2}\right)}^{\text{Derivative of } x^{1/2}y^{1/2}} = \overbrace{2y\frac{dy}{dx}}^{\text{Derivative of } y^2}$$

$$1 + \frac{x^{1/2}}{2y^{1/2}} \cdot \frac{dy}{dx} + \frac{y^{1/2}}{2x^{1/2}} = 2y\frac{dy}{dx}$$

Multiply both sides by $2x^{1/2} \cdot y^{1/2}$.

$$2x^{1/2} \cdot y^{1/2} + x\frac{dy}{dx} + y = 4x^{1/2} \cdot y^{3/2} \cdot \frac{dy}{dx}$$

Combine terms and solve for dy/dx.

$$2x^{1/2} \cdot y^{1/2} + y = (4x^{1/2} \cdot y^{3/2} - x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x^{1/2} \cdot y^{1/2} + y}{4x^{1/2} \cdot y^{3/2} - x}$$

YOUR TURN 2 Find dy/dx for $xe^y + x^2 = \ln y$.

TRY YOUR TURN 2

EXAMPLE 3 Tangent Line

The graph of $x^3 + y^3 = 9xy$, shown in Figure 16, is a *folium of Descartes*.* Find the equation of the tangent line at the point $(2, 4)$, shown in Figure 16.

SOLUTION Since this is not the graph of a function, y is not a function of x, and dy/dx is not defined. But if we restrict the curve to the vicinity of $(2, 4)$, as shown in Figure 17,

*Information on this curve and others is available on the Famous Curves section of the MacTutor History of Mathematics Archive website at www-history.mcs.st-and.ac.uk/~history. See Exercises 33–36 for more curves.

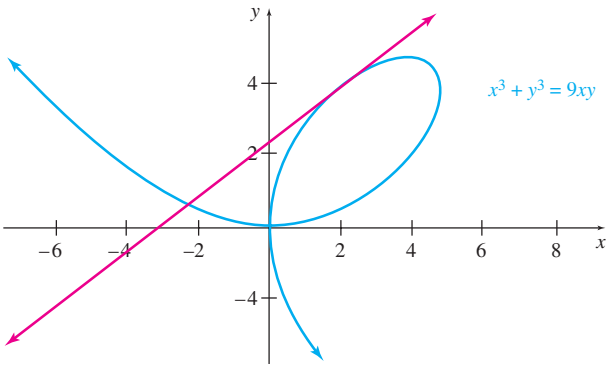


FIGURE 16

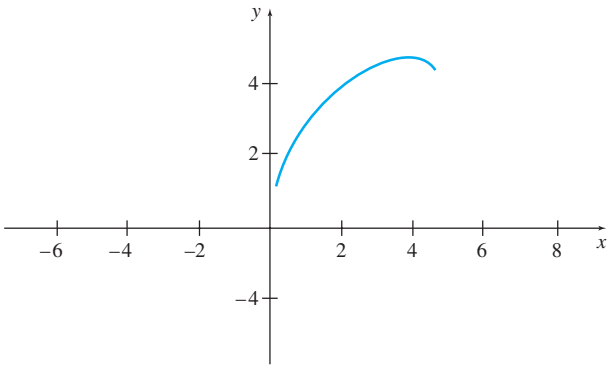


FIGURE 17

the curve does represent the graph of a function, and we can calculate dy/dx by implicit differentiation.

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 9x\frac{dy}{dx} + 9y \qquad \text{Chain rule and product rule}$$

$$3y^2 \cdot \frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2 \qquad \text{Move all } dy/dx \text{ terms to the same side of the equation.}$$

$$\frac{dy}{dx}(3y^2 - 9x) = 9y - 3x^2 \qquad \text{Factor.}$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$

$$= \frac{3(3y - x^2)}{3(y^2 - 3x)} = \frac{3y - x^2}{y^2 - 3x}$$

To find the slope of the tangent line at the point $(2, 4)$, let $x = 2$ and $y = 4$. The slope is

$$m = \frac{3y - x^2}{y^2 - 3x} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The equation of the tangent line is then found by using the point-slope form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{4}{5}(x - 2)$$

$$y - 4 = \frac{4}{5}x - \frac{8}{5}$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

The tangent line is graphed in Figure 16.

YOUR TURN 3 The graph of $y^4 - x^4 - y^2 + x^2 = 0$ is called the *devil's curve*. Find the equation of the tangent line at the point (1, 1).

TRY YOUR TURN 3

NOTE In Example 3, we could have substituted $x = 2$ and $y = 4$ immediately after taking the derivative implicitly. You may find that such a substitution makes solving the equation for dy/dx easier.

The steps used in implicit differentiation can be summarized as follows.

Implicit Differentiation

To find dy/dx for an equation containing x and y:

1. Differentiate on both sides of the equation with respect to x, keeping in mind that y is assumed to be a function of x.
2. Using algebra, place all terms with dy/dx on one side of the equals sign and all terms without dy/dx on the other side.
3. Factor out dy/dx, and then solve for dy/dx.

When an applied problem involves an equation that is not given in explicit form, implicit differentiation can be used to locate maxima and minima or to find rates of change.

EXAMPLE 4 Demand

The demand function for a certain commodity is given by

$$p = \frac{500{,}000}{2q^3 + 400q + 5000},$$

where p is the price in dollars and q is the demand in hundreds of units. Find the rate of change of demand with respect to price when $q = 100$ (that is, find dq/dp).

APPLY IT

SOLUTION Since we don't have q as a function of p, we will use implicit differentiation. Differentiate both sides with respect to p using the power rule (with a power of -1) and the chain rule.

$$1 = \frac{-500{,}000}{(2q^3 + 400q + 5000)^2}\left(6q^2\frac{dq}{dp} + 400\frac{dq}{dp}\right)$$

$$= \frac{-500{,}000}{(2q^3 + 400q + 5000)^2}(6q^2 + 400)\frac{dq}{dp}$$

Now substitute $q = 100$.

$$1 = \frac{-500{,}000}{(2 \cdot 100^3 + 400 \cdot 100 + 5000)^2}(6 \cdot 100^2 + 400)\frac{dq}{dp}$$

$$= \frac{-500{,}000 \cdot 60{,}400}{2{,}045{,}000^2} \cdot \frac{dq}{dp}$$

Therefore,

$$\frac{dq}{dp} = -\frac{2{,}045{,}000^2}{500{,}000 \cdot 60{,}400} \approx -138.$$

This means that when demand (q) is 100 hundreds, or 10,000, demand is decreasing at the rate of 139 hundred, or 13,900, units per dollar change in price. TRY YOUR TURN 4

YOUR TURN 4 Find the rate of change of demand with respect to price when $q = 200$ if the demand function is given by $p = \dfrac{100{,}000}{q^2 + 100q}$.

14.4 EXERCISES

Find dy/dx by implicit differentiation for the following.

1. $6x^2 + 5y^2 = 36$
2. $7x^2 - 4y^2 = 24$
3. $8x^2 - 10xy + 3y^2 = 26$
4. $7x^2 = 5y^2 + 4xy + 1$
5. $5x^3 = 3y^2 + 4y$
6. $3x^3 - 8y^2 = 10y$
7. $3x^2 = \dfrac{2 - y}{2 + y}$
8. $2y^2 = \dfrac{5 + x}{5 - x}$
9. $2\sqrt{x} + 4\sqrt{y} = 5y$
10. $4\sqrt{x} - 8\sqrt{y} = 6y^{3/2}$
11. $x^4y^3 + 4x^{3/2} = 6y^{3/2} + 5$
12. $(xy)^{4/3} + x^{1/3} = y^6 + 1$
13. $e^{x^2y} = 5x + 4y + 2$
14. $x^2e^y + y = x^3$
15. $x + \ln y = x^2y^3$
16. $y \ln x + 2 = x^{3/2}y^{5/2}$

Find the equation of the tangent line at the given point on each curve.

17. $x^2 + y^2 = 25; \quad (-3, 4)$
18. $x^2 + y^2 = 100; \quad (8, -6)$
19. $x^2y^2 = 1; \quad (-1, 1)$
20. $x^2y^3 = 8; \quad (-1, 2)$
21. $2y^2 - \sqrt{x} = 4; \quad (16, 2)$
22. $y + \dfrac{\sqrt{x}}{y} = 3; \quad (4, 2)$
23. $e^{x^2+y^2} = xe^{5y} - y^2e^{5x/2}; \quad (2, 1)$
24. $2xe^{xy} = e^{x^3} + ye^{x^2}; \quad (1, 1)$
25. $\ln(x + y) = x^3y^2 + \ln(x^2 + 2) - 4; \quad (1, 2)$
26. $\ln(x^2 + y^2) = \ln 5x + \dfrac{y}{x} - 2; \quad (1, 2)$

In Exercises 27–32, find the equation of the tangent line at the given value of x on each curve.

27. $y^3 + xy - y = 8x^4; \quad x = 1$
28. $y^3 + 2x^2y - 8y = x^3 + 19; \quad x = 2$

29. $y^3 + xy^2 + 1 = x + 2y^2; \quad x = 2$

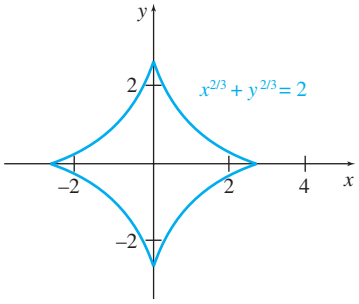
30. $y^4(1 - x) + xy = 2; \quad x = 1$

31. $2y^3(x - 3) + x\sqrt{y} = 3; \quad x = 3$

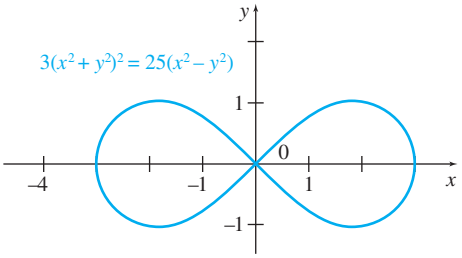
32. $\frac{y}{18}(x^2 - 64) + x^{2/3}y^{1/3} = 12; \quad x = 8$

Information on curves in Exercises 33–36, as well as many other curves, is available on the Famous Curves section of the MacTutor History of Mathematics Archive website at www-history.mcs.st-and.ac.uk/~history.

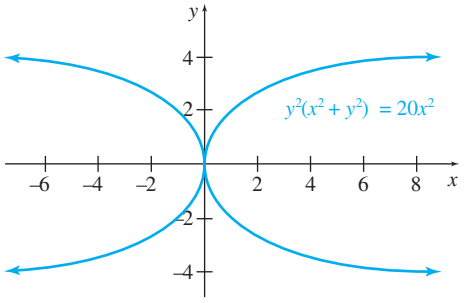
33. The graph of $x^{2/3} + y^{2/3} = 2$, shown in the figure, is an *astroid*. Find the equation of the tangent line at the point $(1, 1)$.



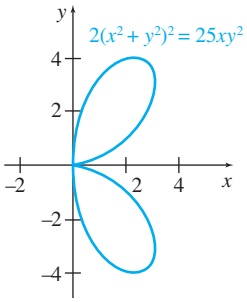
34. The graph of $3(x^2 + y^2)^2 = 25(x^2 - y^2)$, shown in the figure, is a *lemniscate of Bernoulli*. Find the equation of the tangent line at the point $(2, 1)$.



35. The graph of $y^2(x^2 + y^2) = 20x^2$, shown in the figure, is a *kappa curve*. Find the equation of the tangent line at the point $(1, 2)$.



36. The graph of $2(x^2 + y^2)^2 = 25xy^2$, shown in the figure, is a *double folium*. Find the equation of the tangent line at the point $(2, 1)$.



37. The graph of $x^2 + y^2 = 100$ is a circle having center at the origin and radius 10.

a. Write the equations of the tangent lines at the points where $x = 6$.

b. Graph the circle and the tangent lines.

38. Much has been written recently about elliptic curves because of their role in Andrew Wiles's 1995 proof of Fermat's Last Theorem. An elliptic curve can be written in the form

$$y^2 = x^3 + ax + b,$$

where a and b are constants, and the cubic function on the right has distinct roots. Find dy/dx for this curve.

39. Let $\sqrt{u} + \sqrt{2v + 1} = 5$. Find each derivative.

a. $\frac{du}{dv}$ **b.** $\frac{dv}{du}$

c. Based on your answers to parts a and b, what do you notice about the relationship between du/dv and dv/du?

40. Let $e^{u^2 - v} - v = 1$. Find each derivative.

a. $\frac{du}{dv}$ **b.** $\frac{dv}{du}$

c. Based on your answers to parts a and b, what do you notice about the relationship between du/dv and dv/du?

41. Suppose $x^2 + y^2 + 1 = 0$. Use implicit differentiation to find dy/dx. Then explain why the result you got is meaningless. (*Hint*: Can $x^2 + y^2 + 1$ equal 0?)

APPLICATIONS

Business and Economics

42. Demand The demand equation for a certain product is $2p^2 + q^2 = 1600$, where p is the price per unit in dollars and q is the number of units demanded.

a. Find and interpret dq/dp.

b. Find and interpret dp/dq.

43. Cost and Revenue For a certain product, cost C and revenue R are given as follows, where x is the number of units sold (in hundreds)

$$\text{Cost: } C^2 = x^2 + 100\sqrt{x} + 50$$
$$\text{Revenue: } 900(x - 5)^2 + 25R^2 = 22{,}500$$

a. Find and interpret the marginal cost dC/dx at $x = 5$.

b. Find and interpret the marginal revenue dR/dx at $x = 5$.

44. Elasticity of Demand Researchers found the demand for milk in Mexico for a particular year can be estimated by the implicit equation

$$\ln q = C - 0.678 \ln p,$$

where p represents the price of a unit of fluid milk and C represents a constant that can be calculated uniquely for a particular year. Here q represents the annual per capita fluid milk demand. ***Source: Agricultural Economics***.

a. Use implicit differentiation to calculate and interpret the elasticity of demand. (Recall from the previous section that elasticity of demand is $E = -(p/q) \cdot dq/dp$.)

b. Solve the equation for q, then calculate the elasticity of demand.

45. Elasticity of Demand Researchers found the demand for cheese in Mexico for a particular year can be estimated by the implicit equation

$$\ln q = D - 0.44 \ln p,$$

where p represents the price of a unit of cheese and D represents a constant that can be calculated uniquely for a particular year. Here q represents the annual per capita cheese demand. ***Source: Agricultural Economics.***

a. Use implicit differentiation to calculate and interpret the elasticity of demand. (Recall from the previous section that elasticity of demand is $E = -(p/q) \cdot dq/dp$.)

b. Solve the equation for q, then calculate the elasticity of demand.

Life Sciences

46. Respiratory Rate Researchers have found a correlation between respiratory rate and body mass in the first three years of life. This correlation can be expressed by the function

$$\log R(w) = 1.83 - 0.43 \log(w),$$

where w is the body weight (in kilograms) and $R(w)$ is the respiratory rate (in breaths per minute). ***Source: Archives of Disease in Children***.

a. Find $R'(w)$ using implicit differentiation.

b. Find $R'(w)$ by first solving the equation for $R(w)$.

c. Discuss the two procedures. Is there a situation when you would want to use one method over another?

47. Biochemical Reaction A simple biochemical reaction with three molecules has solutions that oscillate toward a steady state when positive constants a and b are below the curve $b - a = (b + a)^3$. Find the largest possible value of a for which the reaction has solutions that oscillate toward a steady state. (*Hint:* Find where $da/db = 0$. Derive values for $a + b$ and $a - b$, and then solve the equations in two unknowns.) ***Source: Mathematical Biology***.

48. Species The relationship between the number of species in a genus (x) and the number of genera (y) comprising x species is given by

$$xy^a = k,$$

where a and k are constants. Find dy/dx. ***Source: Elements of Mathematical Biology***.

Physical Sciences

Velocity **The position of a particle at time t is given by s. Find the velocity ds/dt.**

49. $s^3 - 4st + 2t^3 - 5t = 0$

50. $2s^2 + \sqrt{st} - 4 = 3t$

YOUR TURN ANSWERS

1. $dy/dx = (y - 2x)/(2y - x)$

2. $\dfrac{dy}{dx} = \dfrac{ye^y + 2xy}{1 - xye^y}$

3. $y = x$

4. $dq/dp = -72$

14.5 Related Rates

APPLY IT **When a skier's blood vessels contract because of the cold, how fast is the velocity of blood changing?**

We use related rates to answer this question in Example 6 of this section.

It is common for variables to be functions of time; for example, sales of an item may depend on the season of the year, or a population of animals may be increasing at a certain rate several months after being introduced into an area. Time is often present implicitly in a mathematical model, meaning that derivatives with respect to time must be found by the method of implicit differentiation discussed in the previous section.

We start with a simple algebraic example.

EXAMPLE 1 Related Rates

Suppose that x and y are both functions of t, which can be considered to represent time, and that x and y are related by the equation

$$xy^2 + y = x^2 + 17.$$

Suppose further that when $x = 2$ and $y = 3$, then $dx/dt = 13$. Find the value of dy/dt at that moment.

SOLUTION We start by taking the derivative of the relationship, using the product and chain rules. Keep in mind that both x and y are functions of t. The result is

$$x\left(2y\frac{dy}{dt}\right) + y^2\frac{dx}{dt} + \frac{dy}{dt} = 2x\frac{dx}{dt}.$$

Now substitute $x = 2$, $y = 3$, and $dx/dt = 13$ to get

$$2\left(6\frac{dy}{dt}\right) + 9(13) + \frac{dy}{dt} = 4(13),$$

$$12\frac{dy}{dt} + 117 + \frac{dy}{dt} = 52.$$

Solve this last equation for dy/dt to get

$$13\frac{dy}{dt} = -65,$$

$$\frac{dy}{dt} = -5.$$

YOUR TURN 1 Suppose x and y are both functions of t and $x^3 + 2xy + y^2 = 1$. If $x = 1$, $y = -2$, and $dx/dt = 6$, then find dy/dt.

TRY YOUR TURN 1

Our next example is typical of the word problems involving related rates.

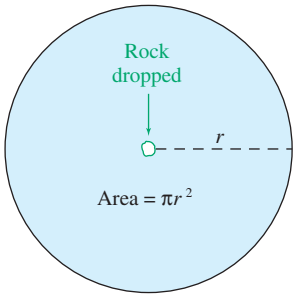


FIGURE 18

EXAMPLE 2 Area

A small rock is dropped into a lake. Circular ripples spread over the surface of the water, with the radius of each circle increasing at the rate of 3/2 ft per second. Find the rate of change of the area inside the circle formed by a ripple at the instant the radius is 4 ft.

SOLUTION As shown in Figure 18, the area A and the radius r are related by

$$A = \pi r^2.$$

Both A and r are functions of the time t in seconds. Take the derivative of both sides with respect to time.

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \tag{1}$$

Since the radius is increasing at the rate of 3/2 ft per second,

$$\frac{dr}{dt} = \frac{3}{2}.$$

The rate of change of area at the instant $r = 4$ is given by dA/dt evaluated at $r = 4$. Substituting into Equation (1) gives

$$\frac{dA}{dt} = 2\pi \cdot 4 \cdot \frac{3}{2}$$

$$\frac{dA}{dt} = 12\pi \approx 37.7 \text{ ft}^2 \text{ per second.}$$

In Example 2, the derivatives (or rates of change) dA/dt and dr/dt are related by Equation (1); for this reason they are called **related rates**. As suggested by Example 2, four basic steps are involved in solving problems about related rates.

Solving a Related Rate Problem

1. Identify all given quantities, as well as the quantities to be found. Draw a sketch when possible.
2. Write an equation relating the variables of the problem.
3. Use implicit differentiation to find the derivative of both sides of the equation in Step 2 with respect to time.
4. Solve for the derivative giving the unknown rate of change and substitute the given values.

CAUTION

1. Differentiate *first*, and *then* substitute values for the variables. If the substitutions were performed first, differentiating would not lead to useful results.
2. Some students confuse related rates problems with applied extrema problems, perhaps because they are both word problems. There is an easy way to tell the difference. In applied extrema problems, you are always trying to maximize or minimize something, that is, make it as large or as small as possible. In related rate problems, you are trying to find how fast something is changing; time is always the independent variable.

EXAMPLE 3 Sliding Ladder

A 50-ft ladder is placed against a large building. The base of the ladder is resting on an oil spill, and it slips (to the right in Figure 19) at the rate of 3 ft per minute. Find the rate of change of the height of the top of the ladder above the ground at the instant when the base of the ladder is 30 ft from the base of the building.

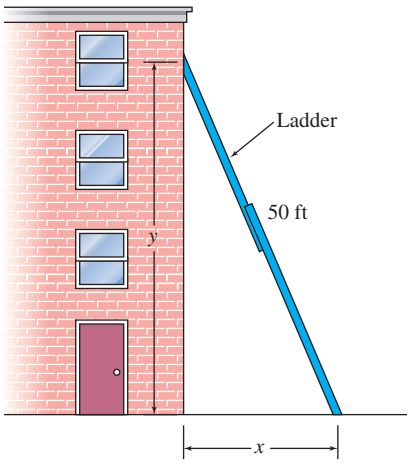


FIGURE 19

SOLUTION Starting with Step 1, let y be the height of the top of the ladder above the ground, and let x be the distance of the base of the ladder from the base of the building. We are trying to find dy/dt when $x = 30$. To perform Step 2, use the Pythagorean theorem to write

$$x^2 + y^2 = 50^2. \tag{2}$$

Both x and y are functions of time t (in minutes) after the moment that the ladder starts slipping. According to Step 3, take the derivative of both sides of Equation (2) with respect to time, getting

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(50^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0. \tag{3}$$

To complete Step 4, we need to find the values of x, y, and dx/dt. Once we find these, we can substitute them into Equation (3) to find dy/dt.

Since the base is sliding at the rate of 3 ft per minute,

$$\frac{dx}{dt} = 3.$$

Also, the base of the ladder is 30 ft from the base of the building, so $x = 30$. Use this to find y.

$$\begin{aligned} 50^2 &= 30^2 + y^2 \\ 2500 &= 900 + y^2 \\ 1600 &= y^2 \\ y &= 40 \end{aligned}$$

In summary, $y = 40$ when $x = 30$. Also, the rate of change of x over time t is $dx/dt = 3$. Substituting these values into Equation (3) to find the rate of change of y over time gives

$$2(30)(3) + 2(40)\frac{dy}{dt} = 0$$

$$180 + 80\frac{dy}{dt} = 0$$

$$80\frac{dy}{dt} = -180$$

$$\frac{dy}{dt} = \frac{-180}{80} = \frac{-9}{4} = -2.25.$$

At the instant when the base of the ladder is 30 ft from the base of the building, the top of the ladder is sliding down the building at the rate of 2.25 ft per minute. (The minus sign shows that the ladder is sliding *down*, so the distance y is *decreasing*.)*

TRY YOUR TURN 2

YOUR TURN 2 A 25-ft ladder is placed against a building. The base of the ladder is slipping away from the building at a rate of 3 ft per minute. Find the rate at which the top of the ladder is sliding down the building when the bottom of the ladder is 7 ft from the base of the building.

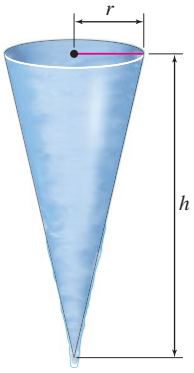


FIGURE 20

EXAMPLE 4 Icicle

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 cm per hour, while the length is increasing at a rate of 0.8 cm per hour. If the icicle is currently 4 cm in radius and 20 cm long, is the volume of the icicle increasing or decreasing, and at what rate?

SOLUTION For this problem we need the formula for the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h, \tag{4}$$

where r is the radius of the cone and h is the height of the cone, which in this case is the length of the icicle, as in Figure 20.

In this problem, V, r, and h are functions of the time t in hours. Taking the derivative of both sides of Equation (4) with respect to time yields

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2\frac{dh}{dt} + (h)(2r)\frac{dr}{dt}\right]. \tag{5}$$

Since the radius is decreasing at a rate of 0.2 cm per hour and the length is increasing at a rate of 0.8 cm per hour,

$$\frac{dr}{dt} = -0.2 \quad \text{and} \quad \frac{dh}{dt} = 0.8.$$

Substituting these, as well as $r = 4$ and $h = 20$, into Equation (5) yields

$$\frac{dV}{dt} = \frac{1}{3}\pi[4^2(0.8) + (20)(8)(-0.2)]$$

$$= \frac{1}{3}\pi(-19.2) \approx -20.$$

Because the sign of dV/dt is negative, the volume of the icicle is decreasing at a rate of 20 cm^3 per hour.

TRY YOUR TURN 3

YOUR TURN 3 Suppose that in Example 4 the volume of the icicle is decreasing at a rate of 10 cm^3 per hour and the radius is decreasing at a rate of 0.4 cm per hour. Find the rate of change of the length of the icicle when the radius is 4 cm and the length is 20 cm.

*The model in Example 3 breaks down as the top of the ladder nears the ground. As y approaches 0, dy/dt becomes infinitely large. In reality, the ladder loses contact with the wall before y reaches 0.

EXAMPLE 5 Revenue

A company is increasing production of peanuts at the rate of 50 cases per day. All cases produced can be sold. The daily demand function is given by

$$p = 50 - \frac{q}{200},$$

where q is the number of units produced (and sold) and p is price in dollars. Find the rate of change of revenue with respect to time (in days) when the daily production is 200 units.

SOLUTION The revenue function,

$$R = qp = q\left(50 - \frac{q}{200}\right) = 50q - \frac{q^2}{200},$$

relates R and q. The rate of change of q over time (in days) is $dq/dt = 50$. The rate of change of revenue over time, dR/dt, is to be found when $q = 200$. Differentiate both sides of the equation

$$R = 50q - \frac{q^2}{200}$$

with respect to t.

$$\frac{dR}{dt} = 50\frac{dq}{dt} - \frac{1}{100}q\frac{dq}{dt} = \left(50 - \frac{1}{100}q\right)\frac{dq}{dt}$$

Now substitute the known values for q and dq/dt.

$$\frac{dR}{dt} = \left[50 - \frac{1}{100}(200)\right](50) = 2400$$

Thus revenue is increasing at the rate of \$2400 per day. TRY YOUR TURN 4

YOUR TURN 4 Repeat Example 5 using the daily demand function given by

$$p = 2000 - \frac{q^2}{100}.$$

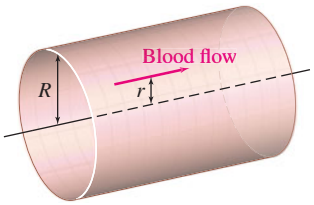


FIGURE 21

EXAMPLE 6 Blood Flow

Blood flows faster the closer it is to the center of a blood vessel. According to Poiseuille's laws, the velocity V of blood is given by

$$V = k(R^2 - r^2),$$

where R is the radius of the blood vessel, r is the distance of a layer of blood flow from the center of the vessel, and k is a constant, assumed here to equal 375. See Figure 21. Suppose a skier's blood vessel has radius $R = 0.08$ mm and that cold weather is causing the vessel to contract at a rate of $dR/dt = -0.01$ mm per minute. How fast is the velocity of blood changing?

APPLY IT

SOLUTION Find dV/dt. Treat r as a constant. Assume the given units are compatible.

$$V = 375(R^2 - r^2)$$

$$\frac{dV}{dt} = 375\left(2R\frac{dR}{dt} - 0\right) \quad r \text{ is a constant.}$$

$$\frac{dV}{dt} = 750R\frac{dR}{dt}$$

Here $R = 0.08$ and $dR/dt = -0.01$, so

$$\frac{dV}{dt} = 750(0.08)(-0.01) = -0.6.$$

That is, the velocity of the blood is decreasing at a rate of -0.6 mm per minute each minute. The minus sign indicates that this is a deceleration (negative acceleration), since it represents a negative rate of change of velocity.

14.5 EXERCISES

Assume x and y are functions of t. Evaluate dy/dt for each of the following.

1. $y^2 - 8x^3 = -55;\ \dfrac{dx}{dt} = -4,\ x = 2,\ y = 3$

2. $8y^3 + x^2 = 1;\ \dfrac{dx}{dt} = 2,\ x = 3,\ y = -1$

3. $2xy - 5x + 3y^3 = -51;\ \dfrac{dx}{dt} = -6,\ x = 3,\ y = -2$

4. $4x^3 - 6xy^2 + 3y^2 = 228;\ \dfrac{dx}{dt} = 3,\ x = -3,\ y = 4$

5. $\dfrac{x^2 + y}{x - y} = 9;\ \dfrac{dx}{dt} = 2,\ x = 4,\ y = 2$

6. $\dfrac{y^3 - 4x^2}{x^3 + 2y} = \dfrac{44}{31};\ \dfrac{dx}{dt} = 5,\ x = -3,\ y = -2$

7. $xe^y = 2 - \ln 2 + \ln x;\ \dfrac{dx}{dt} = 6,\ x = 2,\ y = 0$

8. $y \ln x + xe^y = 1;\ \dfrac{dx}{dt} = 5,\ x = 1,\ y = 0$

APPLICATIONS

Business and Economics

9. Cost A manufacturer of handcrafted wine racks has determined that the cost to produce x units per month is given by $C = 0.2x^2 + 10{,}000$. How fast is cost per month changing when production is changing at the rate of 12 units per month and the production level is 80 units?

10. Cost/Revenue The manufacturer in Exercise 9 has found that the cost C and revenue R (in dollars) in one month are related by the equation

$$C = \frac{R^2}{450{,}000} + 12{,}000.$$

Find the rate of change of revenue with respect to time when the cost is changing by \$15 per month and the monthly revenue is \$25,000.

11. Revenue/Cost/Profit Given the revenue and cost functions $R = 50x - 0.4x^2$ and $C = 5x + 15$ (in dollars), where x is the daily production (and sales), find the following when 40 units are produced daily and the rate of change of production is 10 units per day.

a. The rate of change of revenue with respect to time

b. The rate of change of cost with respect to time

c. The rate of change of profit with respect to time

12. Revenue/Cost/Profit Repeat Exercise 11, given that 80 units are produced daily and the rate of change of production is 12 units per day.

13. Demand The demand function for a certain product is determined by the fact that the product of the price and the quantity demanded equals 8000. The product currently sells for \$3.50 per unit. Suppose manufacturing costs are increasing over time at a rate of 15% and the company plans to increase the price p at this rate as well. Find the rate of change of demand over time.

14. Revenue A company is increasing production at the rate of 25 units per day. The daily demand function is determined by the fact that the price (in dollars) is a linear function of q. At a price of \$70, the demand is 0, and 100 items will be demanded at a price of \$60. Find the rate of change of revenue with respect to time (in days) when the daily production (and sales) is 20 items.

Life Sciences

15. Blood Velocity A cross-country skier has a history of heart problems. She takes nitroglycerin to dilate blood vessels, thus avoiding angina (chest pain) due to blood vessel contraction. Use Poiseuille's law with $k = 555.6$ to find the rate of change of the blood velocity when $R = 0.02$ mm and R is changing at 0.003 mm per minute. Assume r is constant. (See Example 6.)

16. Allometric Growth Suppose x and y are two quantities that vary with time according to the allometric formula $y = nx^m$. (See Exercise 84 in the section on Logarithmic Functions.) Show that the derivatives of x and y are related by the formula

$$\frac{1}{y}\frac{dy}{dt} = m\frac{1}{x}\frac{dx}{dt}.$$

(*Hint:* Take natural logarithms of both sides before taking the derivatives.)

17. Brain Mass The brain mass of a fetus can be estimated using the total mass of the fetus by the function

$$b = 0.22m^{0.87},$$

where m is the mass of the fetus (in grams) and b is the brain mass (in grams). Suppose the brain mass of a 25-g fetus is changing at a rate of 0.25 g per day. Use this to estimate the rate of change of the total mass of the fetus, dm/dt. ***Source:* *Archives d'Anatomie, d'Histologie et d'Embryologie***.

18. Birds The energy cost of bird flight as a function of body mass is given by

$$E = 429m^{-0.35},$$

where m is the mass of the bird (in grams) and E is the energy expenditure (in calories per gram per hour). Suppose that the

mass of a 10-g bird is increasing at a rate of 0.001 g per hour. Find the rate at which the energy expenditure is changing with respect to time. ***Source: Wildlife Feeding and Nutrition***.

19. Metabolic Rate The average daily metabolic rate for captive animals from weasels to elk can be expressed as a function of mass by

$$r = 140.2m^{0.75},$$

where m is the mass of the animal (in kilograms) and r is the metabolic rate (in kcal per day). ***Source: Wildlife Feeding and Nutrition***.

a. Suppose that the mass of a weasel is changing with respect to time at a rate dm/dt. Find dr/dt.

b. Determine dr/dt for a 250-kg elk that is gaining mass at a rate of 2 kg per day.

20. Lizards The energy cost of horizontal locomotion as a function of the body mass of a lizard is given by

$$E = 26.5m^{-0.34},$$

where m is the mass of the lizard (in kilograms) and E is the energy expenditure (in kcal/kg/km). Suppose that the mass of a 5-kg lizard is increasing at a rate of 0.05 kg per day. Find the rate at which the energy expenditure is changing with respect to time. ***Source: Wildlife Feeding and Nutrition***.

Social Sciences

21. Crime Rate Sociologists have found that crime rates are influenced by temperature. In a midwestern town of 100,000 people, the crime rate has been approximated as

$$C = \frac{1}{10}(T - 60)^2 + 100,$$

where C is the number of crimes per month and T is the average monthly temperature in degrees Fahrenheit. The average temperature for May was 76°, and by the end of May the temperature was rising at the rate of 8° per month. How fast is the crime rate rising at the end of May?

22. Memorization Skills Under certain conditions, a person can memorize W words in t minutes, where

$$W(t) = \frac{-0.02t^2 + t}{t + 1}.$$

Find dW/dt when $t = 5$.

Physical Sciences

23. Sliding Ladder A 17-ft ladder is placed against a building. The base of the ladder is slipping away from the building at a rate of 9 ft per minute. Find the rate at which the top of the ladder is sliding down the building at the instant when the bottom of the ladder is 8 ft from the base of the building.

24. Distance

a. One car leaves a given point and travels north at 30 mph. Another car leaves the same point at the same time and travels west at 40 mph. At what rate is the distance between the two cars changing at the instant when the cars have traveled 2 hours?

b. Suppose that, in part a, the second car left 1 hour later than the first car. At what rate is the distance between the two cars changing at the instant when the second car has traveled 1 hour?

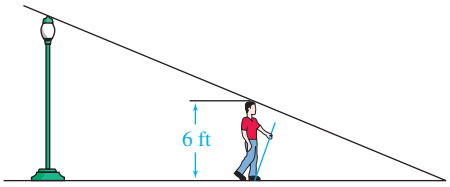
25. Area A rock is thrown into a still pond. The circular ripples move outward from the point of impact of the rock so that the radius of the circle formed by a ripple increases at the rate of 2 ft per minute. Find the rate at which the area is changing at the instant the radius is 4 ft.

26. Volume A spherical snowball is placed in the sun. The sun melts the snowball so that its radius decreases 1/4 in. per hour. Find the rate of change of the volume with respect to time at the instant the radius is 4 in.

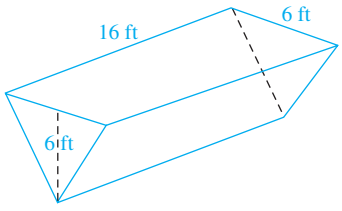
27. Ice Cube An ice cube that is 3 cm on each side is melting at a rate of 2 cm^3 per min. How fast is the length of the side decreasing?

28. Volume A sand storage tank used by the highway department for winter storms is leaking. As the sand leaks out, it forms a conical pile. The radius of the base of the pile increases at the rate of 0.75 in. per minute. The height of the pile is always twice the radius of the base. Find the rate at which the volume of the pile is increasing at the instant the radius of the base is 6 in.

29. Shadow Length A man 6 ft tall is walking away from a lamp post at the rate of 50 ft per minute. When the man is 8 ft from the lamp post, his shadow is 10 ft long. Find the rate at which the length of the shadow is increasing when he is 25 ft from the lamp post. (See the figure.)

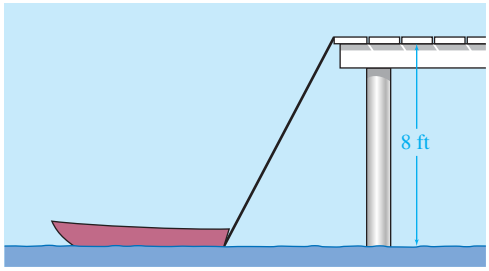


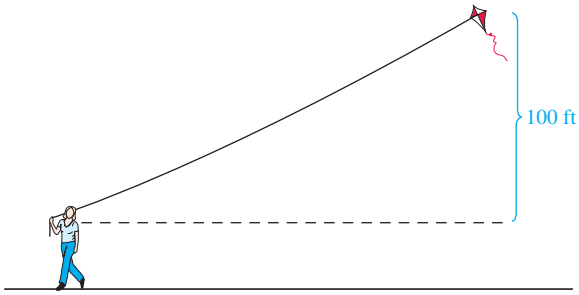
30. Water Level A trough has a triangular cross section. The trough is 6 ft across the top, 6 ft deep, and 16 ft long. Water is being pumped into the trough at the rate of 4 ft^3 per minute.



Find the rate at which the height of the water is increasing at the instant that the height is 4 ft.

31. Velocity A pulley is on the edge of a dock, 8 ft above the water level. (See the figure on the next page.) A rope is being used to pull in a boat. The rope is attached to the boat at water level. The rope is being pulled in at the rate of 1 ft per second. Find the rate at which the boat is approaching the dock at the instant the boat is 8 ft from the dock.





32. Kite Flying Christine O'Brien is flying her kite in a wind that is blowing it east at a rate of 50 ft per minute. She has already let out 200 ft of string, and the kite is flying 100 ft above her hand. How fast must she let out string at this moment to keep the kite flying with the same speed and altitude?

YOUR TURN ANSWERS

1. $dy/dt = -3$.
2. $-7/8$ ft/min
3. Increases 3.4 cm per hour
4. Revenue is increasing at the rate of $40,000 per day.

14.6 Differentials: Linear Approximation

APPLY IT **If the estimated sales of cellular telephones turn out to be inaccurate, approximately how much are profits affected?**

Using differentials, we will answer this question in Example 4.

As mentioned earlier, the symbol Δx represents a change in the variable x. Similarly, Δy represents a change in y. An important problem that arises in many applications is to determine Δy given specific values of x and Δx. This quantity is often difficult to evaluate. In this section we show a method of approximating Δy that uses the derivative dy/dx. In essence, we use the tangent line at a particular value of x to approximate $f(x)$ for values close to x.

For values x_1 and x_2,

$$\Delta x = x_2 - x_1.$$

Solving for x_2 gives

$$x_2 = x_1 + \Delta x.$$

For a function $y = f(x)$, the symbol Δy represents a change in y:

$$\Delta y = f(x_2) - f(x_1).$$

Replacing x_2 with $x_1 + \Delta x$ gives

$$\Delta y = f(x_1 + \Delta x) - f(x_1).$$

If Δx is used instead of h, the derivative of a function f at x_1 could be defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

If the derivative exists, then

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

as long as Δx is close to 0. Multiplying both sides by Δx (assume $\Delta x \neq 0$) gives

$$\Delta y \approx \frac{dy}{dx} \cdot \Delta x.$$

Until now, dy/dx has been used as a single symbol representing the derivative of y with respect to x. In this section, separate meanings for dy and dx are introduced in such a way that their quotient, when $dx \neq 0$, is the derivative of y with respect to x. These meanings of dy and dx are then used to find an approximate value of Δy.

To define dy and dx, look at Figure 22 below, which shows the graph of a function $y = f(x)$. The tangent line to the graph has been drawn at the point P. Let Δx be any nonzero real number (in practical problems, Δx is a small number) and locate the point $x + \Delta x$ on the x-axis. Draw a vertical line through $x + \Delta x$. Let this vertical line cut the tangent line at M and the graph of the function at Q.

Define the new symbol dx to be the same as Δx. Define the new symbol dy to equal the length MR. The slope of PM is $f'(x)$. By the definition of slope, the slope of PM is also dy/dx, so that

$$f'(x) = \frac{dy}{dx},$$

or

$$dy = f'(x)dx.$$

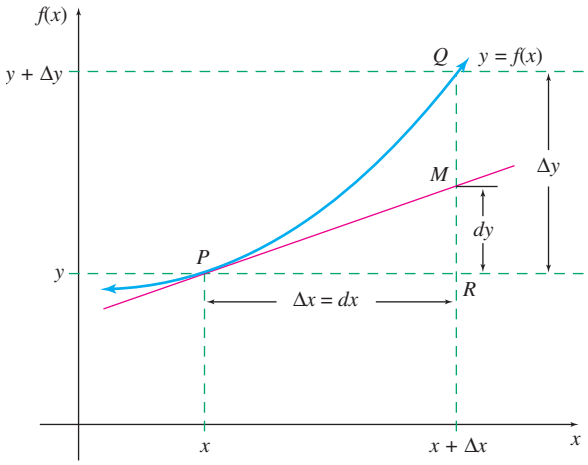


FIGURE 22

In summary, the definitions of the symbols dy and dx are as follows.

Differentials

For a function $y = f(x)$ whose derivative exists, the **differential** of x, written dx, is an arbitrary real number (usually small compared with x); the **differential** of y, written dy, is the product of $f'(x)$ and dx, or

$$\boldsymbol{dy = f'(x)dx}.$$

The usefulness of the differential is suggested by Figure 22. As dx approaches 0, the value of dy gets closer and closer to that of Δy, so that for small nonzero values of dx

$$dy \approx \Delta y,$$

or

$$\Delta y \approx f'(x)dx.$$

EXAMPLE 1 Differential

Find dy for the following functions.

(a) $y = 6x^2$

SOLUTION The derivative is $dy/dx = 12x$ so

$$dy = 12x\,dx.$$

(b) $y = 800x^{-3/4}$, $x = 16$, $dx = 0.01$

SOLUTION

$$\begin{aligned} dy &= -600x^{-7/4}dx \\ &= -600(16)^{-7/4}(0.01) \\ &= -600\left(\frac{1}{2^7}\right)(0.01) = -0.046875 \end{aligned}$$

YOUR TURN 1 Find dy if $y = 300x^{-2/3}$, $x = 8$, and $dx = 0.05$.

TRY YOUR TURN 1

Differentials can be used to approximate function values for a given x-value (in the absence of a calculator or computer). As discussed above,

$$\Delta y = f(x + \Delta x) - f(x).$$

For small nonzero values of Δx, $\Delta y \approx dy$, so that

$$dy \approx f(x + \Delta x) - f(x),$$

or

$$f(x) + dy \approx f(x + \Delta x).$$

Replacing dy with $f'(x)dx$ gives the following result.

Linear Approximation

Let f be a function whose derivative exists. For small nonzero values of Δx,

$$\boldsymbol{dy \approx \Delta y,}$$

and

$$\boldsymbol{f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)dx.}$$

EXAMPLE 2 Approximation

Approximate $\sqrt{50}$.

SOLUTION We will use the linear approximation

$$f(x + \Delta x) \approx f(x) + f'(x)dx$$

for a small value of Δx to form this estimate. We first choose a number x that is close to 50 for which we know its square root. Since $\sqrt{49} = 7$, we let $f(x) = \sqrt{x}$, $x = 49$, and $\Delta x = dx = 1$. Using this information, with the fact that when $f(x) = \sqrt{x} = x^{1/2}$,

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}},$$

we have

$$f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx.$$

Substituting $x = 49$ and $dx = 1$ into the preceding formula gives

$$\begin{aligned} f(50) = f(49 + 1) &\approx \sqrt{49} + \frac{1}{2\sqrt{49}}(1) \\ &= 7 + \frac{1}{14} \\ &= 7\frac{1}{14}. \end{aligned}$$

YOUR TURN 2
Approximate $\sqrt{99}$.

A calculator gives $7\frac{1}{14} \approx 7.07143$ and $\sqrt{50} \approx 7.07107$. Our approximation of $7\frac{1}{14}$ is close to the true answer and does not require a calculator. TRY YOUR TURN 2

While calculators have made differentials less important, the approximation of functions, including linear approximation, is still important in the branch of mathematics known as numerical analysis.

Marginal Analysis

Differentials are used to find an approximate value of the change in the dependent variable corresponding to a given change in the independent variable. When the concept of marginal cost (or profit or revenue) was used to approximate the change in cost for nonlinear functions, the same idea was developed. Thus the differential dy approximates Δy in much the same way as the marginal quantities approximate changes in functions.

For example, for a cost function $C(x)$,

$$dC = C'(x)dx = C'(x)\Delta x.$$

Since $\Delta C \approx dC$,

$$\Delta C \approx C'(x)\Delta x.$$

If the change in production, Δx, is equal to 1, then

$$\begin{aligned} C(x + 1) - C(x) &= \Delta C \\ &\approx C'(x)\Delta x \\ &= C'(x), \end{aligned}$$

which shows that marginal cost $C'(x)$ approximates the cost of the next unit produced, as mentioned earlier.

EXAMPLE 3 Cost

Let $C(x) = 2x^3 + 300$.

(a) Use $C'(x)$ to approximate ΔC when $\Delta x = 1$ and $x = 3$. Then compare ΔC with $C'(3)$.

SOLUTION Since $C(x) = 2x^3 + 300$, the derivative is

$$C'(x) = 6x^2$$

and the marginal cost approximation at $x = 3$ is

$$C'(3) = 6(3^2) = 54.$$

Now, the actual cost of the next unit produced is

$$\Delta C = C(4) - C(3) = 428 - 354 = 74.$$

Here, the approximation of $C'(3)$ for ΔC is poor, since $\Delta x = 1$ is large relative to $x = 3$.

(b) Use $C'(x)$ to approximate ΔC when $\Delta x = 1$ and x $= 50$. Then compare ΔC with $C'(50)$.

SOLUTION The marginal cost approximation at x $= 50$ is

$$C'(50) = 6(50^2) = 15{,}000$$

and the actual cost of the next unit produced is

$$\Delta C = C(51) - C(50) = 265{,}602 - 250{,}300 = 15{,}302.$$

This approximation is quite good since $\Delta x = 1$ is small compared to $x = 50$.

EXAMPLE 4 Profit

An analyst for a manufacturer of electronic devices estimates that the profit (in dollars) from the sale of x cellular telephones is given by

$$P(x) = 4000 \ln x.$$

In a report to management, the analyst projected sales for the coming year to be 10,000 phones, for a total profit of about $36,840. He now realizes that his sales estimate may have been as much as 1000 phones too high. Approximately how far off is his profit estimate?

APPLY IT

SOLUTION Differentials can be used to find the approximate change in P resulting from decreasing x by 1000. This change can be approximated by $dP = P'(x)dx$ where $x = 10{,}000$ and $dx = -1000$. Since $P'(x) = 4000/x$,

$$\begin{aligned}\Delta P \approx dP &= \frac{4000}{x}dx \\ &= \frac{4000}{10{,}000}(-1000) \\ &= -400.\end{aligned}$$

Thus the profit estimate may have been as much as $400 too high. Computing the actual difference with a calculator gives $4000 \ln 9000 - 4000 \ln 10{,}000 \approx -421$, which is close to our approximation.

Error Estimation

The final example in this section shows how differentials are used to estimate errors that might enter into measurements of a physical quantity.

EXAMPLE 5 Error Estimation

In a precision manufacturing process, ball bearings must be made with a radius of 0.6 mm, with a maximum error in the radius of ± 0.015 mm. Estimate the maximum error in the volume of the ball bearing.

SOLUTION The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

If an error of Δr is made in measuring the radius of the sphere, the maximum error in the volume is

$$\Delta V = \frac{4}{3}\pi(r + \Delta r)^3 - \frac{4}{3}\pi r^3.$$

Rather than calculating ΔV, approximate ΔV with dV, where

$$dV = 4\pi r^2 dr.$$

Replacing r with 0.6 and $dr = \Delta r$ with ± 0.015 gives

$$\begin{aligned}dV &= 4\pi(0.6)^2(\pm 0.015) \\ &\approx \pm 0.0679.\end{aligned}$$

The maximum error in the volume is about 0.07 mm^3.

YOUR TURN 3 Repeat Example 5 for $r = 1.25$ mm with a maximum error in the radius of $\pm$ 0.025 mm.

TRY YOUR TURN 3

14.6 EXERCISES

For Exercises 1–8, find *dy* for the given values of x and Δx.

1. $y = 2x^3 - 5x; \quad x = -2, \Delta x = 0.1$
2. $y = 4x^3 - 3x; \quad x = 3, \Delta x = 0.2$
3. $y = x^3 - 2x^2 + 3; \quad x = 1, \Delta x = -0.1$
4. $y = 2x^3 + x^2 - 4x; \quad x = 2, \Delta x = -0.2$
5. $y = \sqrt{3x + 2} \quad x = 4, \Delta x = 0.15$
6. $y = \sqrt{4x - 1}; \quad x = 5, \Delta x = 0.08$
7. $y = \dfrac{2x - 5}{x + 1}; \quad x = 2, \Delta x = -0.03$
8. $y = \dfrac{6x - 3}{2x + 1}; \quad x = 3, \Delta x = -0.04$

Use the differential to approximate each quantity. Then use a calculator to approximate the quantity, and give the absolute value of the difference in the two results to 4 decimal places.

9. $\sqrt{145}$
10. $\sqrt{23}$
11. $\sqrt{0.99}$
12. $\sqrt{17.02}$
13. $e^{0.01}$
14. $e^{-0.002}$
15. $\ln 1.05$
16. $\ln 0.98$

APPLICATIONS

Business and Economics

17. **Demand** The demand for grass seed (in thousands of pounds) at a price of p dollars is

$$D(p) = -3p^3 - 2p^2 + 1500.$$

Use the differential to approximate the changes in demand for the following changes in p.

a. \$2 to \$2.10 **b.** \$6 to \$6.15

18. **Average Cost** The average cost (in dollars) to manufacture x dozen marking pencils is

$$A(x) = 0.04x^3 + 0.1x^2 + 0.5x + 6.$$

Use the differential to approximate the changes in the average cost for the following changes in x.

a. 3 to 4 **b.** 5 to 6

19. **Revenue** A company estimates that the revenue (in dollars) from the sale of x doghouses is given by

$$R(x) = 12{,}000 \ln(0.01x + 1).$$

Use the differential to approximate the change in revenue from the sale of one more doghouse when 100 doghouses are sold.

20. **Profit** The cost function for the company in Exercise 19 is

$$C(x) = 150 + 75x,$$

where x represents the demand for the product. Find the approximate change in profit for a 1-unit change in demand when demand is at a level of 100 doghouses. Use the differential.

21. **Material Requirement** A cube 4 in. on an edge is given a protective coating 0.1 in. thick. About how much coating should a production manager order for 1000 such cubes?

22. **Material Requirement** Beach balls 1 ft in diameter have a thickness of 0.03 in. How much material would be needed to make 5000 beach balls?

Life Sciences

23. **Alcohol Concentration** In Exercise 55 in the section on Polynomial and Rational Functions, we gave the function defined by

$$A(x) = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$$

as the approximate blood alcohol concentration in a 170-lb woman x hours after drinking 2 oz of alcohol on an empty stomach, for x in the interval [0, 5]. ***Source: Medicolegal Aspects of Alcohol Determination in Biological Specimens.***

a. Approximate the change in alcohol level from 1 to 1.2 hours.

b. Approximate the change in alcohol level from 3 to 3.2 hours.

24. **Drug Concentration** The concentration of a certain drug in the bloodstream x hours after being administered is approximately

$$C(x) = \frac{5x}{9 + x^2}.$$

Use the differential to approximate the changes in concentration for the following changes in x.

a. 1 to 1.5 **b.** 2 to 2.25

25. **Bacteria Population** The population of bacteria (in millions) in a certain culture x hours after an experimental nutrient is introduced into the culture is

$$P(x) = \frac{25x}{8 + x^2}.$$

Use the differential to approximate the changes in population for the following changes in x.

a. 2 to 2.5 **b.** 3 to 3.25

26. **Area of a Blood Vessel** The radius of a blood vessel is 1.7 mm. A drug causes the radius to change to 1.6 mm. Find the approximate change in the area of a cross section of the vessel.

27. **Volume of a Tumor** A tumor is approximately spherical in shape. If the radius of the tumor changes from 14 mm to 16 mm, find the approximate change in volume.

28. **Area of an Oil Slick** An oil slick is in the shape of a circle. Find the approximate increase in the area of the slick if its radius increases from 1.2 miles to 1.4 miles.

29. **Area of a Bacteria Colony** The shape of a colony of bacteria on a Petri dish is circular. Find the approximate increase in its area if the radius increases from 20 mm to 22 mm.

30. Gray Wolves Accurate methods of estimating the age of gray wolves are important to scientists who study wolf population dynamics. One method of estimating the age of a gray wolf is to measure the percent closure of the pulp cavity of a canine tooth and then estimate age by

$$A(p) = \frac{1.181p}{94.359 - p},$$

where p is the percent closure and $A(p)$ is the age of the wolf (in years). ***Source: Journal of Wildlife Management***.

a. What is a sensible domain for this function?

b. Use differentials to estimate how long it will take for a gray wolf that first measures a 60% closure to obtain a 65% closure. Compare this with the actual value of about 0.55 years.

31. Pigs Researchers have observed that the mass of a female (gilt) pig can be estimated by the function

$$M(t) = -3.5 + 197.5e^{-e^{-0.01394(t-108.4)}},$$

where t is the age of the pig (in days) and $M(t)$ is the mass of the pig (in kilograms). ***Source: Animal Science***.

a. If a particular gilt is 80 days old, use differentials to estimate how much it will gain before it is 90 days old.

b. What is the actual gain in mass?

Physical Sciences

32. Volume A spherical balloon is being inflated. Find the approximate change in volume if the radius increases from 4 cm to 4.2 cm.

33. Volume A spherical snowball is melting. Find the approximate change in volume if the radius decreases from 3 cm to 2.8 cm.

34. Volume A cubical crystal is growing in size. Find the approximate change in the length of a side when the volume increases from 27 cubic mm to 27.1 cubic mm.

35. Volume An icicle is gradually increasing in length, while maintaining a cone shape with a length 15 times the radius. Find the approximate amount that the volume of the icicle increases when the length increases from 13 cm to 13.2 cm.

General Interest

36. Measurement Error The edge of a square is measured as 3.45 in., with a possible error of ± 0.002 in. Estimate the maximum error in the area of the square.

37. Tolerance A worker is cutting a square from a piece of sheet metal. The specifications call for an area that is 16 cm^2 with an error of no more than 0.01 cm^2. How much error could be tolerated in the length of each side to ensure that the area is within the tolerance?

38. Measurement Error The radius of a circle is measured as 4.87 in., with a possible error of ± 0.040 in. Estimate the maximum error in the area of the circle.

39. Measurement Error A sphere has a radius of 5.81 in., with a possible error of ± 0.003 in. Estimate the maximum error in the volume of the sphere.

40. Tolerance A worker is constructing a cubical box that must contain 125 ft^3, with an error of no more than 0.3 ft^3. How much error could be tolerated in the length of each side to ensure that the volume is within the tolerance?

41. Measurement Error A cone has a known height of 7.284 in. The radius of the base is measured as 1.09 in., with a possible error of ± 0.007 in. Estimate the maximum error in the volume of the cone.

YOUR TURN ANSWERS

1. $-5/16$
2. 9.95
3. About 0.5 mm^3

14 CHAPTER REVIEW

SUMMARY

In this chapter, we began by discussing how to find an absolute maximum or minimum. In contrast to a relative extremum, which is the largest or smallest value of a function on some open interval about the point, an absolute extremum is the largest or smallest value of the function on the entire interval under consideration. We then studied various applications with maximizing or minimizing as the goal. Two more applications, economic lot size and economic order quantity, were covered in a separate section, which also applied the derivative to the economic concept of elasticity of demand. Implicit differentiation is more of a technique than an application, but it underlies related rate problems, in which one or more rates are given and another is to be found. Finally, we studied the differential as a way to find linear approximations of functions.

Finding Absolute Extrema To find absolute extrema for a function f continuous on a closed interval $[a, b]$:

1. Find all critical numbers for f in (a, b).
2. Evaluate f for all critical numbers in (a, b).
3. Evaluate f for the endpoints a and b of the interval.
4. The largest value found in Step 2 or 3 is the maximum, and the smallest value is the minimum.

Solving an Applied Extrema Problem

1. Read the problem carefully. Make sure you understand what is given and what is unknown.
2. If possible, sketch a diagram. Label the various parts.
3. Decide on the variable that must be maximized or minimized. Express that variable as a function of *one* other variable.
4. Find the domain of the function.
5. Find the critical points for the function from Step 3.
6. If the domain is a closed interval, evaluate the function at the endpoints and at each critical number to see which yields the absolute maximum or minimum. If the domain is an open interval, apply the critical point theorem when there is only one critical number. If there is more than one critical number, evaluate the function at the critical numbers and find the limit as the endpoints of the interval are approached to determine if an absolute maximum or minimum exists at one of the critical points.

Elasticity of Demand

Let $q = f(p)$, where q is demand at a price p.

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}.$$

Demand is inelastic if $E < 1$.
Demand is elastic if $E > 1$.
Demand has unit elasticity if $E = 1$.
Total revenue is maximized at the price where demand has unit elasticity.

Implicit Differentiation

To find dy/dx for an equation containing x and y:

1. Differentiate on both sides of the equation with respect to x, keeping in mind that y is assumed to be a function of x.
2. Place all terms with dy/dx on one side of the equals sign and all terms without dy/dx on the other side.
3. Factor out dy/dx, and then solve for dy/dx.

Solving a Related Rate Problem

1. Identify all given quantities, as well as the quantities to be found. Draw a sketch when possible.
2. Write an equation relating the variables of the problem.
3. Use implicit differentiation to find the derivative of both sides of the equation in Step 2 with respect to time.
4. Solve for the derivative, giving the unknown rate of change, and substitute the given values.

Differentials

$dy = f'(x)dx$

Linear Approximation

$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)dx$

KEY TERMS

14.1
absolute maximum
absolute minimum
absolute extremum (or extrema)
extreme value theorem
critical point theorem
graphical optimization

14.2
spawner-recruit function
parent-progeny function
maximum sustainable harvest

14.3
economic lot size
economic order quantity
elasticity of demand
unit elasticity

14.4
explicit function
implicit differentiation

14.5
related rates

14.6
differential

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The absolute maximum of a function always occurs where the derivative has a critical number.

2. A continuous function on a closed interval has an absolute maximum and minimum.

3. A continuous function on an open interval does not have an absolute maximum or minimum.

4. Demand for a product is elastic if the elasticity is greater than 1.

5. Total revenue is maximized at the price where demand has unit elasticity.

6. Implicit differentiation can be used to find dy/dx when x is defined in terms of y.

7. In a related rates problem, all derivatives are with respect to time.

8. In a related rates problem, there can be more than two quantities that vary with time.

9. A differential is a real number.

10. When the change in x is small, the differential of y is approximately the change in y.

PRACTICE AND EXPLORATIONS

Find the absolute extrema if they exist, and all values of x where they occur on the given intervals.

11. $f(x) = -x^3 + 6x^2 + 1; \quad [-1, 6]$

12. $f(x) = 4x^3 - 9x^2 - 3; \quad [-1, 2]$

13. $f(x) = x^3 + 2x^2 - 15x + 3; \quad [-4, 2]$

14. $f(x) = -2x^3 - 2x^2 + 2x - 1; \quad [-3, 1]$

15. When solving applied extrema problems, why is it necessary to check the endpoints of the domain?

16. What is elasticity of demand (in words; no mathematical symbols allowed)? Why is the derivative used to describe elasticity?

17. Find the absolute maximum and minimum of $f(x) = \dfrac{2 \ln x}{x^2}$ on each interval.

a. $[1, 4]$ **b.** $[2, 5]$

18. Find the absolute maximum and minimum of $f(x) = \dfrac{e^{2x}}{x^2}$ on each interval.

a. $[\,1/2\,, 2]$ **b.** $[1, 3]$

19. When is it necessary to use implicit differentiation?

20. When a term involving y is differentiated in implicit differentiation, it is multiplied by dy/dx. Why? Why aren't terms involving x multiplied by dx/dx?

Find dy/dx.

21. $x^2 - 4y^2 = 3x^3y^4$ **22.** $x^2y^3 + 4xy = 2$

23. $2\sqrt{y - 1} = 9x^{2/3} + y$ **24.** $9\sqrt{x} + 4y^3 = 2\sqrt{y}$

25. $\dfrac{6 + 5x}{2 - 3y} = \dfrac{1}{5x}$ **26.** $\dfrac{x + 2y}{x - 3y} = y^{1/2}$

27. $\ln(xy + 1) = 2xy^3 + 4$ **28.** $\ln(x + y) = 1 + x^2 + y^3$

29. Find the equation of the line tangent to the graph of $\sqrt{2y} - 4xy = -22$ at the point $(3, 2)$.

30. Find an equation of the line tangent to the graph of $8y^3 - 4xy^2 = 20$ at the point $(-3, 1)$.

31. What is the difference between a related rate problem and an applied extremum problem?

32. Why is implicit differentiation used in related rate problems?

Find dy/dt.

33. $y = 8x^3 - 7x^2; \quad \dfrac{dx}{dt} = 4, x = 2$

34. $y = \dfrac{9 - 4x}{3 + 2x}; \quad \dfrac{dx}{dt} = -1, x = -3$

35. $y = \dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}; \quad \dfrac{dx}{dt} = -4, x = 4$

36. $\dfrac{x^2 + 5y}{x - 2y} = 2; \quad \dfrac{dx}{dt} = 1, x = 2, y = 0$

37. $y = xe^{3x}; \quad \dfrac{dx}{dt} = -2, x = 1$

38. $y = \dfrac{1}{e^{x^2} + 1}; \quad \dfrac{dx}{dt} = 3, x = 1$

39. What is a differential? What is it used for?

40. Describe when linear approximations are most accurate.

Evaluate dy.

41. $y = \dfrac{3x - 7}{2x + 1}; \quad x = 2, \Delta x = 0.003$

42. $y = 8 - x^2 + x^3; \quad x = -1, \Delta x = 0.02$

43. Suppose x and y are related by the equation

$$-12x + x^3 + y + y^2 = 4.$$

a. Find all critical points on the curve.

b. Determine whether the critical points found in part a are relative maxima or relative minima by taking values of x nearby and solving for the corresponding values of y.

c. Is there an absolute maximum or minimum for x and y in the relationship given in part a? Why or why not?

44. In Exercise 43, implicit differentiation was used to find the relative extrema. The exercise was contrived to avoid various difficulties that could have arisen. Discuss some of the difficulties that might be encountered in such problems, and how these difficulties might be resolved.

APPLICATIONS

Business and Economics

45. **Profit** The total profit (in tens of dollars) from the sale of x hundred boxes of candy is given by

$$P(x) = -x^3 + 10x^2 - 12x.$$

a. Find the number of boxes of candy that should be sold in order to produce maximum profit.

b. Find the maximum profit.

46. **Packaging Design** The packaging department of a corporation is designing a box with a square base and no top. The volume is to be 32 m^3. To reduce cost, the box is to have minimum surface area. What dimensions (height, length, and width) should the box have?

47. **Packaging Design** A company plans to package its product in a cylinder that is open at one end. The cylinder is to have a volume of 27π in^3. What radius should the circular bottom of the cylinder have to minimize the cost of the material?

48. **Packaging Design** Fruit juice will be packaged in cylindrical cans with a volume of 40 in^3 each. The top and bottom of the

can cost $4¢$ per in^2, while the sides cost $3¢$ per in^2. Find the radius and height of the can of minimum cost.

49. Order Quantity A large camera store sells 20,000 batteries annually. It costs $15¢$ to store 1 battery for 1 year and \$12 to place a reorder. Find the number of batteries that should be ordered each time.

50. Order Quantity A store sells 180,000 cases of a product annually. It costs \$12 to store 1 case for 1 year and \$20 to place a reorder. Find the number of cases that should be ordered each time.

51. Lot Size A company produces 128,000 cases of soft drink annually. It costs \$1 to store 1 case for 1 year and \$10 to produce 1 lot. Find the number of lots that should be produced annually.

52. Lot Size In 1 year, a health food manufacturer produces and sells 240,000 cases of vitamins. It costs \$2 to store 1 case for 1 year and \$15 to produce each batch. Find the number of batches that should be produced annually.

53. Elasticity of Demand The demand for butter in Mexico for a particular year can be estimated by the general function

$$\ln q = D - 0.47 \ln p,$$

where p represents the price of a unit of butter and D represents a constant that can be calculated uniquely for a particular year. Here q represents the annual per capita butter demand. Calculate and interpret the elasticity of demand. ***Source: Agricultural Economics***.

54. Elasticity Suppose the demand function for a product is given by $q = A/p^k$, where A and k are positive constants. For what values of k is the demand elastic? Inelastic?

Life Sciences

55. Pollution A circle of pollution is spreading from a broken underwater waste disposal pipe, with the radius increasing at the rate of 4 ft per minute. Find the rate of change of the area of the circle when the radius is 7 ft.

56. Logistic Growth Many populations grow according to the logistic equation

$$\frac{dx}{dt} = rx(N - x),$$

where r is a constant involving the rate of growth and N is the carrying capacity of the environment, beyond which the population decreases. Show that the graph of x has an inflection point where $x = N/2$. (*Hint:* Use implicit differentiation. Then set $d^2x/dt^2 = 0$, and factor.)

57. Dentin Growth The dentinal formation of molars in mice has been studied by researchers in Copenhagen. They determined that the growth curve that best fits dentinal formation for the first molar is

$$M(t) = 1.3386309 - 0.4321173t + 0.0564512t^2 - 0.0020506t^3 + 0.0000315t^4 - 0.0000001785t^5, \quad 5 \le t \le 51,$$

where t is the age of the mouse (in days), and $M(t)$ is the cumulative dentin volume (in 10^{-1} mm^3). ***Source: Journal of Craniofacial Genetics and Developmental Biology***.

a. Use a graphing calculator to sketch the graph of this function on $[5, 51]$ by $[0, 7.5]$.

b. Find the time in which the dentin formation is growing most rapidly. (*Hint:* Find the maximum value of the derivative of this function.)

58. Human Skin Surface The surface of the skin is made up of a network of intersecting lines that form polygons. Researchers have discovered a functional relationship between the age of a female and the number of polygons per area of skin according to

$$P(t) = 237.09 - 8.0398t + 0.20813t^2 - 0.0027563t^3 + 0.000013016t^4, \quad 0 \le t \le 95,$$

where t is the age of the person (in years), and $P(t)$ is the number of polygons for a particular surface area of skin. ***Source: Gerontology.***

a. Use a graphing calculator to sketch a graph of $P(t)$ on $[0, 95]$ by $[0, 300]$.

b. Find the maximum and minimum number of polygons per area predicted by the model.

c. Discuss the accuracy of this model for older people.

Physical Sciences

59. Sliding Ladder A 50-ft ladder is placed against a building. The top of the ladder is sliding down the building at the rate of 2 ft per minute. Find the rate at which the base of the ladder is slipping away from the building at the instant that the base is 30 ft from the building.

60. Spherical Radius A large weather balloon is being inflated with air at the rate of 0.9 ft^3 per minute. Find the rate of change of the radius when the radius is 1.7 ft.

61. Water Level A water trough 2 ft across, 4 ft long, and 1 ft deep has ends in the shape of isosceles triangles. (See the figure.) It is being filled with 3.5 ft^3 of water per minute. Find the rate at which the depth of water in the tank is changing when the water is $1/3$ ft deep.

General Interest

62. Volume Approximate the volume of coating on a sphere of radius 4 in. if the coating is 0.02 in. thick.

63. Area A square has an edge of 9.2 in., with a possible error in the measurement of ± 0.04 in. Estimate the possible error in the area of the square.

64. Package Dimensions UPS has the following rule regarding package dimensions. The length can be no more than 108 in., and the length plus the girth (twice the sum of the width and the height) can be no more than 130 in. If the width of a package is 4 in. more than its height and it has the maximum length plus girth allowed, find the length that produces maximum volume.

65. Pursuit A boat moves north at a constant speed. A second boat, moving at the same speed, pursues the first boat in such a way that it always points directly at the first boat. When the

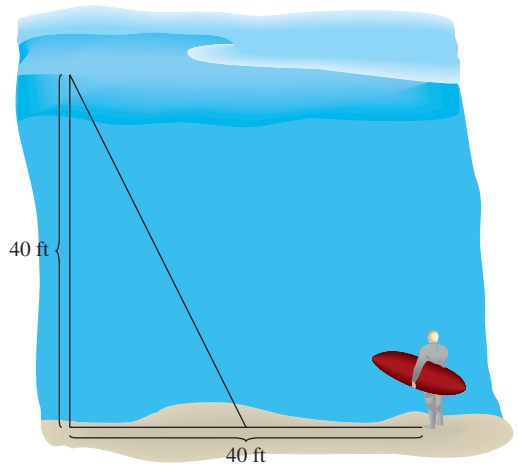
first boat is at the point $(0, 1)$, the second boat is at the point $(6, 2.5)$, with the positive y-axis pointing north. It can then be shown that the curve traced by the second boat, known as a pursuit curve, is given by

$$y = \frac{x^2}{16} - 2 \ln x + \frac{1}{4} + 2 \ln 6.$$

Find the y-coordinate of the southernmost point of the second boat's path. ***Source: Differential Equations: Theory and Applications***.

66. Playground Area The city park department is planning an enclosed play area in a new park. One side of the area will be against an existing building, with no fence needed there. Find the dimensions of the maximum rectangular area that can be made with 900 m of fence.

67. Surfing A mathematician is surfing in Long Beach, New York. He is standing on the shore and wants to paddle out to a spot 40 ft from shore; the closest point on the shore to that spot is 40 ft from where he is now standing. (See the figure.) If he can walk 5 ft per second along the shore and paddle 3 ft per second once he's in the water, how far along the shore should he walk before paddling toward the desired destination if he wants to complete the trip in the shortest possible time? What is the shortest possible time?



68. Repeat Exercise 67, but the closest point on the shore to the desired destination is now 25 ft from where he is standing.

EXTENDED APPLICATION

A TOTAL MODEL FOR A TRAINING PROGRAM

In this application, we set up a mathematical model for determining the total costs in setting up a training program. Then we use calculus to find the time interval between training programs that produces the minimum total cost. The model assumes that the demand for trainees is constant and that the fixed cost of training a batch of trainees is known. Also, it is assumed that people who are trained, but for whom no job is readily available, will be paid a fixed amount per month while waiting for a job to open up.

The model uses the following variables.

D = demand for trainees per month

N = number of trainees per batch

C_1 = fixed cost of training a batch of trainees

C_2 = marginal cost of training per trainee per month

C_3 = salary paid monthly to a trainee who has not yet been given a job after training

m = time interval in months between successive batches of trainees

t = length of training program in months

$Z(m)$ = total monthly cost of program

The total cost of training a batch of trainees is given by $C_1 + NtC_2$. However, $N = mD$, so that the total cost per batch is $C_1 + mDtC_2$.

After training, personnel are given jobs at the rate of D per month. Thus, $N - D$ of the trainees will not get a job the first month, $N - 2D$ will not get a job the second month, and so on. The $N - D$ trainees who do not get a job the first month produce total costs of $(N - D)C_3$, those not getting jobs during the second month produce costs of $(N - 2D)C_3$, and so on. Since $N = mD$, the costs during the first month can be written as

$$(N - D)C_3 = (mD - D)C_3 = (m - 1)DC_3,$$

while the costs during the second month are $(m - 2)DC_3$, and so on. The total cost for keeping the trainees without a job is thus

$$(m - 1)DC_3 + (m - 2)DC_3 + (m - 3)DC_3 + \cdots + 2DC_3 + DC_3,$$

which can be factored to give

$$DC_3[(m - 1) + (m - 2) + (m - 3) + \cdots + 2 + 1].$$

The expression in brackets is the sum of the terms of an arithmetic sequence, discussed in most algebra texts. Using formulas for arithmetic sequences, the expression in brackets can be shown to equal $m(m - 1)/2$, so that we have

$$DC_3\left[\frac{m(m - 1)}{2}\right] \quad \textbf{(1)}$$

as the total cost for keeping jobless trainees.

The total cost per batch is the sum of the training cost per batch, $C_1 + mDtC_2$, and the cost of keeping trainees without a proper job, given by Equation (1). Since we assume that a batch of trainees is trained every m months, the total cost per month, $Z(m)$, is given by

$$Z(m) = \frac{C_1 + mDtC_2}{m} + \frac{DC_3\left[\frac{m(m - 1)}{2}\right]}{m}$$

$$= \frac{C_1}{m} + DtC_2 + DC_3\left(\frac{m - 1}{2}\right).$$

Source: P. L. Goyal and S. K. Goyal.

EXERCISES

1. Find $Z'(m)$.
2. Solve the equation $Z'(m) = 0$.

   As a practical matter, it is usually required that m be a whole number. If m does not come out to be a whole number, then m^+ and m^-, the two whole numbers closest to m, must be chosen. Calculate both $Z(m^+)$ and $Z(m^-)$; the smaller of the two provides the optimum value of Z.

3. Suppose a company finds that its demand for trainees is 3 per month, that a training program requires 12 months, that the fixed cost of training a batch of trainees is $15,000, that the marginal cost per trainee per month is $100, and that trainees are paid $900 per month after training but before going to work. Use your result from Exercise 2 and find m.
4. Since m is not a whole number, find m^+ and m^-.
5. Calculate $Z(m^+)$ and $Z(m^-)$.
6. What is the optimum time interval between successive batches of trainees? How many trainees should be in a batch?
7. The parameters of this model are likely to change over time; it is essential that such changes be incorporated into the model as they change. One way to anticipate this is to create a spreadsheet that gives the manager the total cost of training a batch of trainees for various scenarios. Using the data from Exercise 3 as a starting point, create a spreadsheet that varies these numbers and calculates the total cost of training a group of employees for each scenario. Graph the total cost of training with respect to changes in the various costs associated with training.

DIRECTIONS FOR GROUP PROJECT

Suppose you have read an article in the paper announcing that a new high-tech company is locating in your town. Given that the company is manufacturing very specialized equipment, you realize that it must develop a program to train all new employees. Because you would like to get an internship at this new company, use the information above to develop a hypothetical training program that optimizes the time interval between successive batches of trainees and the number of trainees that should be in each session. Assume that you know the new CEO because you and three of your friends have served her pizza at various times at the local pizza shop (your current jobs) and that she is willing to listen to a proposal that describes your training program. Prepare a presentation for your interview that will describe your training program. Use presentation software such as Microsoft PowerPoint.

15 Integration

15.1 Antiderivatives

15.2 Substitution

15.3 Area and the Definite Integral

15.4 The Fundamental Theorem of Calculus

15.5 The Area Between Two Curves

15.6 Numerical Integration

Chapter 15 Review

Extended Application: Estimating Depletion Dates for Minerals

If we know the rate at which a quantity is changing, we can find the total change over a period of time by integrating. An exercise in Section 3 illustrates how this process can be used to estimate the number of cars that cross the Tappan Zee Bridge in New York state each day, given information about how the rate of cars per hour varies with time. This same concept allows us to determine how far a car has gone, given its speed as a function of time; how much a culture of bacteria will grow; or how much consumers benefit by buying a product at the price determined by supply and demand.

Up to this point in calculus you have solved problems such as

$$f(x) = x^5; \text{ find } f'(x).$$

In this chapter you will be asked to solve problems that are the reverse of these, that is, problems of the form

$$f'(x) = 5x^4; \text{ find } f(x).$$

The derivative and its applications, which you studied in previous chapters, are part of what is called *differential calculus*. The next two chapters are devoted to the other main branch of calculus, *integral calculus*. Integrals have many applications: finding areas; determining the lengths of curved paths; solving complicated probability problems; and calculating the location of an object (such as the distance of a space shuttle from Earth) when its velocity and initial position are known. The Fundamental Theorem of Calculus, presented later in this chapter, will reveal a surprisingly close connection between differential and integral calculus.

15.1 Antiderivatives

APPLY IT **If an object is thrown from the top of the Willis Tower in Chicago, how fast is it going when it hits the ground?**
Using antiderivatives, we will answer this question in Example 11.

Functions used in applications in previous chapters have provided information about a *total amount* of a quantity, such as cost, revenue, profit, temperature, gallons of oil, or distance. Derivatives of these functions provided information about the rate of change of these quantities and allowed us to answer important questions about the extrema of the functions. It is not always possible to find ready-made functions that provide information about the total amount of a quantity, but it is often possible to collect enough data to come up with a function that gives the *rate of change* of a quantity. We know that derivatives give the rate of change when the total amount is known. The reverse of finding a derivative is known as **antidifferentiation**. The goal is to find an *antiderivative,* defined as follows.

Antiderivative

If $F'(x) = f(x)$, then $F(x)$ is an **antiderivative** of $f(x)$.

EXAMPLE 1 **Antiderivative**

(a) If $F(x) = 10x$, then $F'(x) = 10$, so $F(x) = 10x$ is an antiderivative of $f(x) = 10$.
(b) For $F(x) = x^2$, $F'(x) = 2x$, making $F(x) = x^2$ an antiderivative of $f(x) = 2x$.

EXAMPLE 2 Antiderivative

Find an antiderivative of $f(x) = 5x^4$.

SOLUTION To find a function $F(x)$ whose derivative is $5x^4$, work backwards. Recall that the derivative of x^n is nx^{n-1}. If

$$\overbrace{nx^{n-1}}^{\text{is } 5x^4},$$

then $n - 1 = 4$ and $n = 5$, so x^5 is an antiderivative of $5x^4$. TRY YOUR TURN 1

YOUR TURN 1 Find an antiderivative $f(x) = 8x^7$.

EXAMPLE 3 Population

Suppose a population is growing at a rate given by $f(x) = e^x$, where x is time in years from some initial date. Find a function giving the population at time x.

SOLUTION Let the population function be $F(x)$. Then

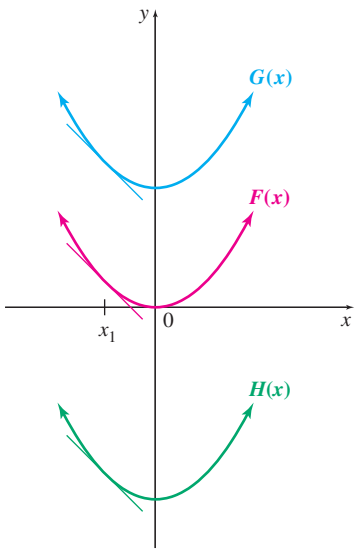
$$f(x) = F'(x) = e^x.$$

The derivative of the function defined by $F(x) = e^x$ is $F'(x) = e^x$, so one possible population function with the given growth rate is $F(x) = e^x$.

The function from Example 1(b), defined by $F(x) = x^2$, is not the only function whose derivative is $f(x) = 2x$. For example,

$$F(x) = x^2, \qquad G(x) = x^2 + 2, \qquad \text{and} \qquad H(x) = x^2 - 4$$

are all antiderivatives of $f(x) = 2x$, and any two of them differ only by a constant. These three functions, shown in Figure 1, have the same derivative, $f(x) = 2x$, and the slopes of their tangent lines at any particular value of x are the same. In fact, for any real number C, the function $F(x) = x^2 + C$ has $f(x) = 2x$ as its derivative. This means that there is a *family* or *class* of functions having $2x$ as a derivative. As the next theorem states, if two functions $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x)$ and $G(x)$ can differ only by a constant.



Slopes of the tangent lines at $x = x_1$ are the same.

FIGURE 1

If $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$ on an interval, then there is a constant C such that

$$\boldsymbol{F(x) - G(x) = C.}$$

(Two antiderivatives of a function can differ only by a constant.) The arbitrary real number C is called an integration constant.

The family of all antiderivatives of the function f is indicated by

$$\int f(x)\,dx = F(x) + C.$$

The symbol $\int$ is the **integral sign**, $f(x)$ is the **integrand**, and $\int f(x)\,dx$ is called an **indefinite integral**, the most general antiderivative of f.

Indefinite Integral

If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

for any real number C.

For example, using this notation,

$$\int 2x\,dx = x^2 + C.$$

The dx in the indefinite integral indicates that $\int f(x)\,dx$ is the "integral of $f(x)$ *with respect* to x" just as the symbol dy/dx denotes the "derivative of y with respect to x." For example, in the indefinite integral $\int 2ax\,dx$, dx indicates that a is to be treated as a constant and x as the variable, so that

$$\int 2ax\,dx = \int a(2x)dx = ax^2 + C.$$

On the other hand,

$$\int 2ax\,da = a^2x + C = xa^2 + C.$$

A more complete interpretation of dx will be discussed later.

The symbol $\int f(x)\,dx$ was created by G. W. Leibniz (1646–1716) in the latter part of the seventeenth century. The $\int$ is an elongated S from *summa,* the Latin word for *sum.* The word *integral* as a term in the calculus was coined by Jakob Bernoulli (1654–1705), a Swiss mathematician who corresponded frequently with Leibniz. The relationship between sums and integrals will be clarified in a later section.

Finding an antiderivative is the reverse of finding a derivative. Therefore, each rule for derivatives leads to a rule for antiderivatives. For example, the power rule for derivatives tells us that

$$\frac{d}{dx}x^5 = 5x^4.$$

Consequently,

$$\int 5x^4\,dx = x^5 + C,$$

the result found in Example 2. Note that the derivative of x^n is found by multiplying x by n and reducing the exponent on x by 1. To find an indefinite integral—that is, to undo what was done—*increase* the exponent by 1 and *divide* by the new exponent, $n + 1$.

FOR REVIEW

Recall that $\dfrac{d}{dx}x^n = nx^{n-1}$.

Power Rule

For any real number $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

(The antiderivative of $f(x) = x^n$ for $n \neq -1$ is found by increasing the exponent n by 1 and dividing x raised to the new power by the new value of the exponent.)

This rule can be verified by differentiating the expression on the right above:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{n+1}{n+1}x^{(n+1)-1} + 0 = x^n.$$

(If $n = -1$, the expression in the denominator is 0, and the above rule cannot be used. Finding an antiderivative for this case is discussed later.)

EXAMPLE 4 Power Rule

Use the power rule to find each indefinite integral.

(a) $\int t^3\,dt$

SOLUTION Use the power rule with $n = 3$.

$$\int t^3\,dt = \frac{t^{3+1}}{3+1} + C = \frac{t^4}{4} + C$$

To check the solution, find the derivative of $t^4/4 + C$. The derivative is t^3, the original function.

(b) $\int \frac{1}{t^2}\,dt$

SOLUTION First, write $1/t^2$ as t^{-2}. Then

$$\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = \frac{t^{-2+1}}{-2+1} + C = \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C.$$

Verify the solution by differentiating $-(1/t) + C$ to get $1/t^2$.

(c) $\int \sqrt{u}\,du$

SOLUTION Since $\sqrt{u} = u^{1/2}$,

$$\int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C.$$

To check this, differentiate $(2/3)u^{3/2} + C$; the derivative is $u^{1/2}$, the original function.

(d) $\int dx$

SOLUTION Write dx as $1 \cdot dx$, and use the fact that $x^0 = 1$ for any nonzero number x to get

$$\int dx = \int 1\,dx = \int x^0\,dx = \frac{x^1}{1} + C = x + C.$$

TRY YOUR TURN 2

YOUR TURN 2

Find $\int \frac{1}{t^4}\,dt$.

FOR REVIEW

Recall that $\frac{d}{dx}[f(x) \pm g(x)] = [f'(x) \pm g'(x)]$ and $\frac{d}{dx}[kf(x)] = kf'(x)$.

As shown earlier, the derivative of the product of a constant and a function is the product of the constant and the derivative of the function. A similar rule applies to indefinite integrals. Also, since derivatives of sums or differences are found term by term, indefinite integrals also can be found term by term.

Constant Multiple Rule and Sum or Difference Rule

If all indicated integrals exist,

$$\int k \cdot f(x)\,dx = k\int f(x)\,dx, \qquad \text{for any real number } k,$$

and

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

(The antiderivative of a constant times a function is the constant times the antiderivative of the function. The antiderivative of a sum or difference of functions is the sum or difference of the antiderivatives.)

CAUTION The constant multiple rule requires k to be a *number*. The rule does not apply to a *variable*. For example,

$$\int x\sqrt{x-1}\,dx \neq x\int\sqrt{x-1}\,dx.$$

EXAMPLE 5 Rules of Integration

Use the rules to find each integral.

(a) $\int 2v^3\,dv$

SOLUTION By the constant multiple rule and the power rule,

$$\int 2v^3\,dv = 2\int v^3\,dv = 2\left(\frac{v^4}{4}\right) + C = \frac{v^4}{2} + C.$$

Because C represents any real number, it is not necessary to multiply it by 2 in the next-to-last step.

(b) $\int \frac{12}{z^5}\,dz$

SOLUTION Rewrite $12/z^5$ as $12z^{-5}$, then find the integral.

$$\int \frac{12}{z^5}\,dz = \int 12z^{-5}\,dz = 12\int z^{-5}\,dz = 12\left(\frac{z^{-4}}{-4}\right) + C$$

$$= -3z^{-4} + C = \frac{-3}{z^4} + C$$

(c) $\int (3z^2 - 4z + 5)\,dz$

SOLUTION By extending the sum and difference rules to more than two terms, we get

$$\int (3z^2 - 4z + 5)\,dz = 3\int z^2\,dz - 4\int z\,dz + 5\int dz$$

$$= 3\left(\frac{z^3}{3}\right) - 4\left(\frac{z^2}{2}\right) + 5z + C$$

$$= z^3 - 2z^2 + 5z + C.$$

Only one constant C is needed in the answer; the three constants from integrating term by term are combined.

YOUR TURN 3

Find $\int (6x^2 + 8x - 9)\,dx$.

TRY YOUR TURN 3

Remember to check your work by taking the derivative of the result. For instance, in Example 5(c) check that $z^3 - 2z^2 + 5z + C$ is the required indefinite integral by taking the derivative

$$\frac{d}{dz}(z^3 - 2z^2 + 5z + C) = 3z^2 - 4z + 5,$$

which agrees with the original information.

EXAMPLE 6 Rules of Integration

Use the rules to find each integral.

(a) $\int \frac{x^2 + 1}{\sqrt{x}}\,dx$

SOLUTION First rewrite the integrand as follows.

$$\int \frac{x^2 + 1}{\sqrt{x}}\,dx = \int \left(\frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx \qquad \text{Rewrite as a sum of fractions.}$$

$$= \int \left(\frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}\right) dx \qquad \sqrt{a} = a^{1/2}$$

$$= \int (x^{3/2} + x^{-1/2})\,dx \qquad \text{Use } \frac{a^m}{a^n} = a^{m-n}.$$

Now find the antiderivative.

$$\int (x^{3/2} + x^{-1/2})\,dx = \frac{x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{5}x^{5/2} + 2x^{1/2} + C$$

(b) $\int (x^2 - 1)^2\,dx$

SOLUTION Square the binomial first, and then find the antiderivative.

$$\int (x^2 - 1)^2\,dx = \int (x^4 - 2x^2 + 1)\,dx$$

$$= \frac{x^5}{5} - \frac{2x^3}{3} + x + C$$

TRY YOUR TURN 4

YOUR TURN 4

Find $\int \frac{x^3 - 2}{\sqrt{x}}\,dx$.

It was shown earlier that the derivative of $f(x) = e^x$ is $f'(x) = e^x$, and the derivative of $f(x) = a^x$ is $f'(x) = (\ln a)a^x$. Also, the derivative of $f(x) = e^{kx}$ is $f'(x) = k \cdot e^{kx}$, and the derivative of $f(x) = a^{kx}$ is $f'(x) = k(\ln a)a^{kx}$. These results lead to the following formulas for indefinite integrals of exponential functions.

Indefinite Integrals of Exponential Functions

$$\int e^x\,dx = e^x + C$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C, \quad k \neq 0$$

For $a > 0$, $a \neq 1$:

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int a^{kx}\,dx = \frac{a^{kx}}{k(\ln a)} + C, \quad k \neq 0$$

(The antiderivative of the exponential function e^x is itself. If x has a coefficient of k, we must divide by k in the antiderivative. If the base is not e, we must divide by the natural logarithm of the base.)

EXAMPLE 7 Exponential Functions

(a) $\displaystyle\int 9e^t\,dt = 9\int e^t\,dt = 9e^t + C$

(b) $\displaystyle\int e^{9t}\,dt = \frac{e^{9t}}{9} + C$

(c) $\displaystyle\int 3e^{(5/4)u}\,du = 3\left(\frac{e^{(5/4)u}}{5/4}\right) + C$

$$= 3\left(\frac{4}{5}\right)e^{(5/4)u} + C$$

$$= \frac{12}{5}e^{(5/4)u} + C$$

(d) $\displaystyle\int 2^{-5x}\,dx = \frac{2^{-5x}}{-5(\ln 2)} + C = -\frac{2^{-5x}}{5(\ln 2)} + C$

The restriction $n \neq -1$ was necessary in the formula for $\int x^n\,dx$ since $n = -1$ made the denominator of $1/(n+1)$ equal to 0. To find $\int x^n\,dx$ when $n = -1$, that is, to find $\int x^{-1}\,dx$, recall the differentiation formula for the logarithmic function: The derivative of $f(x) = \ln|x|$, where $x \neq 0$, is $f'(x) = 1/x = x^{-1}$. This formula for the derivative of $f(x) = \ln|x|$ gives a formula for $\int x^{-1}\,dx$.

Indefinite Integral of x^{-1}

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

(The antiderivative of $f(x) = x^n$ for $n = -1$ is the natural logarithm of the absolute value of x.)

CAUTION Don't neglect the absolute value sign in the natural logarithm when integrating x^{-1}. If x can take on a negative value, $\ln x$ will be undefined there. Note, however, that the absolute value is redundant (but harmless) in an expression such as $\ln|x^2 + 1|$, since $x^2 + 1$ can never be negative.

EXAMPLE 8 Integrals

(a) $\displaystyle\int \frac{4}{x}\,dx = 4\int \frac{1}{x}\,dx = 4\ln|x| + C$

(b) $\displaystyle\int\left(-\frac{5}{x} + e^{-2x}\right)dx = -5\ln|x| - \frac{1}{2}e^{-2x} + C$

TRY YOUR TURN 5

YOUR TURN 5

Find $\displaystyle\int\left(\frac{3}{x} + e^{-3x}\right)dx$.

In all these examples, the antiderivative family of functions was found. In many applications, however, the given information allows us to determine the value of the integration constant C. The next examples illustrate this idea.

EXAMPLE 9 Cost

Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$C'(x) = \frac{50}{\sqrt{x}}$$

and that the fixed cost (the cost before the first book can be produced) is \$25,000. Find the cost function $C(x)$.

SOLUTION Write $50/\sqrt{x}$ as $50/x^{1/2}$ or $50x^{-1/2}$, and then use the indefinite integral rules to integrate the function.

$$C(x) = \int \frac{50}{\sqrt{x}}\,dx = \int 50x^{-1/2}\,dx = 50(2x^{1/2}) + k = 100x^{1/2} + k$$

(Here k is used instead of C to avoid confusion with the cost function $C(x)$.) To find the value of k, use the fact that $C(0)$ is 25,000.

$$\begin{aligned} C(x) &= 100x^{1/2} + k \\ 25{,}000 &= 100 \cdot 0 + k \\ k &= 25{,}000 \end{aligned}$$

With this result, the cost function is $C(x) = 100x^{1/2} + 25{,}000$.

EXAMPLE 10 Demand

Suppose the marginal revenue from a product is given by $400e^{-0.1q} + 8$, where q is the number of products produced.

(a) Find the revenue function for the product.

SOLUTION The marginal revenue is the derivative of the revenue function, so

$$\begin{aligned} R'(q) &= 400e^{-0.1q} + 8 \\ R(q) &= \int (400e^{-0.1q} + 8)\,dq \\ &= 400\,\frac{e^{-0.1q}}{-0.1} + 8q + C \\ &= -4000e^{-0.1q} + 8q + C. \end{aligned}$$

If $q = 0$, then $R = 0$ (no items sold means no revenue), so that

$$\begin{aligned} 0 &= -4000e^{-0.1(0)} + 8(0) + C \\ 0 &= -4000 + 0 + C \\ 4000 &= C. \end{aligned}$$

Thus, the revenue function is

$$R(q) = -4000e^{-0.1q} + 8q + 4000.$$

(b) Find the demand function for this product.

SOLUTION Recall that $R = qp$, where p is the demand function giving the price p as a function of q. Then

$$-4000e^{-0.1q} + 8q + 4000 = qp$$

$$\frac{-4000e^{-0.1q} + 8q + 4000}{q} = p. \qquad \text{Divide by } q.$$

The demand function is $p = \dfrac{-4000e^{-0.1q} + 8q + 4000}{q}$.

In the next example, integrals are used to find the position of a particle when the acceleration of the particle is given.

EXAMPLE 11 Velocity and Acceleration

Recall that if the function $s(t)$ gives the position of a particle at time t, then its velocity $v(t)$ and its acceleration $a(t)$ are given by

$$v(t) = s'(t) \qquad \text{and} \qquad a(t) = v'(t) = s''(t).$$

(a) Suppose the velocity of an object is $v(t) = 6t^2 - 8t$ and that the object is at 5 when time is 0. Find $s(t)$.

SOLUTION Since $v(t) = s'(t)$, the function $s(t)$ is an antiderivative of $v(t)$:

$$s(t) = \int v(t)\,dt = \int (6t^2 - 8t)\,dt$$
$$= 2t^3 - 4t^2 + C$$

for some constant C. Find C from the given information that $s = 5$ when $t = 0$.

$$s(t) = 2t^3 - 4t^2 + C$$
$$5 = 2(0)^3 - 4(0)^2 + C$$
$$5 = C$$
$$s(t) = 2t^3 - 4t^2 + 5$$

(b) Many experiments have shown that when an object is dropped, its acceleration (ignoring air resistance) is constant. This constant has been found to be approximately 32 ft per second every second; that is,

$$a(t) = -32.$$

The negative sign is used because the object is falling. Suppose an object is thrown down from the top of the 1100-ft-tall Willis Tower (formerly known as the Sears Tower) in Chicago. If the initial velocity of the object is -20 ft per second, find $s(t)$, the distance of the object from the ground at time t.

SOLUTION First find $v(t)$ by integrating $a(t)$:

$$v(t) = \int (-32)\,dt = -32t + k.$$

When $t = 0$, $v(t) = -20$:

$$-20 = -32(0) + k$$
$$-20 = k$$

and

$$v(t) = -32t - 20.$$

Be sure to evaluate the constant of integration k before integrating again to get $s(t)$. Now integrate $v(t)$ to find $s(t)$.

$$s(t) = \int (-32t - 20)\,dt = -16t^2 - 20t + C$$

Since $s(t) = 1100$ when $t = 0$, we can substitute these values into the equation for $s(t)$ to get $C = 1100$ and

$$s(t) = -16t^2 - 20t + 1100$$

as the distance of the object from the ground after t seconds.

(c) Use the equations derived in (b) to find the velocity of the object when it hit the ground and how long it took to strike the ground.

APPLY IT

SOLUTION When the object strikes the ground, $s = 0$, so

$$0 = -16t^2 - 20t + 1100.$$

To solve this equation for t, factor out the common factor of -4 and then use the quadratic formula.

$$0 = -4(4t^2 + 5t - 275)$$
$$t = \frac{-5 \pm \sqrt{25 + 4400}}{8} \approx \frac{-5 \pm 66.5}{8}$$

Only the positive value of t is meaningful here: $t \approx 7.69$. It took the object about 7.69 seconds to strike the ground. From the velocity equation, with $t = 7.69$, we find

$$v(t) = -32t - 20$$
$$v(7.69) = -32(7.69) - 20 \approx -266,$$

so the object was falling (as indicated by the negative sign) at about 266 ft per second when it hit the ground. TRY YOUR TURN 6

YOUR TURN 6 Repeat Example 11(b) and 11(c) for the Burj Khalifa in Dubai, which is the tallest building in the world, standing 2717 ft. The initial velocity is -20 ft per second.

EXAMPLE 12 Slope

Find a function f whose graph has slope $f'(x) = 6x^2 + 4$ and goes through the point $(1, 1)$.

SOLUTION Since $f'(x) = 6x^2 + 4$,

$$f(x) = \int (6x^2 + 4)\,dx = 2x^3 + 4x + C.$$

The graph of f goes through $(1, 1)$, so C can be found by substituting 1 for x and 1 for $f(x)$.

$$1 = 2(1)^3 + 4(1) + C$$
$$1 = 6 + C$$
$$C = -5$$

Finally, $f(x) = 2x^3 + 4x - 5$. TRY YOUR TURN 7

YOUR TURN 7 Find an equation of the curve whose tangent line has slope $f'(x) = 3x^{1/2} + 4$ and goes through the point $(1, -2)$.

15.1 EXERCISES

1. What must be true of $F(x)$ and $G(x)$ if both are antiderivatives of $f(x)$?
2. How is the antiderivative of a function related to the function?
3. In your own words, describe what is meant by an integrand.
4. Explain why the restriction $n \neq -1$ is necessary in the rule $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$.

Find the following.

5. $\int 6\,dk$
6. $\int 9\,dy$
7. $\int (2z + 3)\,dz$
8. $\int (3x - 5)\,dx$
9. $\int (6t^2 - 8t + 7)\,dt$
10. $\int (5x^2 - 6x + 3)\,dx$
11. $\int (4z^3 + 3z^2 + 2z - 6)\,dz$
12. $\int (16y^3 + 9y^2 - 6y + 3)\,dy$
13. $\int (5\sqrt{z} + \sqrt{2})\,dz$
14. $\int (t^{1/4} + \pi^{1/4})\,dt$
15. $\int 5x(x^2 - 8)\,dx$
16. $\int x^2(x^4 + 4x + 3)\,dx$
17. $\int (4\sqrt{v} - 3v^{3/2})\,dv$
18. $\int (15x\sqrt{x} + 2\sqrt{x})\,dx$
19. $\int (10u^{3/2} - 14u^{5/2})\,du$
20. $\int (56t^{5/2} + 18t^{7/2})\,dt$
21. $\int \left(\dfrac{7}{z^2}\right) dz$
22. $\int \left(\dfrac{4}{x^3}\right) dx$
23. $\int \left(\dfrac{\pi^3}{y^3} - \dfrac{\sqrt{\pi}}{\sqrt{y}}\right) dy$
24. $\int \left(\sqrt{u} + \dfrac{1}{u^2}\right) du$
25. $\int (-9t^{-2.5} - 2t^{-1})\,dt$
26. $\int (10x^{-3.5} + 4x^{-1})\,dx$
27. $\int \dfrac{1}{3x^2}\,dx$
28. $\int \dfrac{2}{3x^4}\,dx$
29. $\int 3e^{-0.2x}\,dx$
30. $\int -4e^{0.2v}\,dv$
31. $\int \left(\dfrac{-3}{x} + 4e^{-0.4x} + e^{0.1}\right) dx$
32. $\int \left(\dfrac{9}{x} - 3e^{-0.4x}\right) dx$
33. $\int \dfrac{1 + 2t^3}{4t}\,dt$
34. $\int \dfrac{2y^{1/2} - 3y^2}{6y}\,dy$
35. $\int (e^{2u} + 4u)\,du$
36. $\int (v^2 - e^{3v})\,dv$
37. $\int (x + 1)^2\,dx$
38. $\int (2y - 1)^2\,dy$
39. $\int \dfrac{\sqrt{x} + 1}{\sqrt[3]{x}}\,dx$
40. $\int \dfrac{1 - 2\sqrt[3]{z}}{\sqrt[3]{z}}\,dz$
41. $\int 10^x\,dx$
42. $\int 3^{2x}\,dx$
43. Find an equation of the curve whose tangent line has a slope of
$$f'(x) = x^{2/3},$$
given that the point $(1, 3/5)$ is on the curve.
44. The slope of the tangent line to a curve is given by
$$f'(x) = 6x^2 - 4x + 3.$$
If the point $(0, 1)$ is on the curve, find an equation of the curve.

APPLICATIONS

Business and Economics

Cost **Find the cost function for each marginal cost function.**

45. $C'(x) = 4x - 5$; fixed cost is \$8
46. $C'(x) = 0.2x^2 + 5x$; fixed cost is \$10
47. $C'(x) = 0.03e^{0.01x}$; fixed cost is \$8
48. $C'(x) = x^{1/2}$; 16 units cost \$45
49. $C'(x) = x^{2/3} + 2$; 8 units cost \$58
50. $C'(x) = x + 1/x^2$; 2 units cost \$5.50
51. $C'(x) = 5x - 1/x$; 10 units cost \$94.20
52. $C'(x) = 1.2^x(\ln 1.2)$; 2 units cost \$9.44

Demand **Find the demand function for each marginal revenue function. Recall that if no items are sold, the revenue is 0.**

53. $R'(x) = 175 - 0.02x - 0.03x^2$
54. $R'(x) = 50 - 5x^{2/3}$
55. $R'(x) = 500 - 0.15\sqrt{x}$
56. $R'(x) = 600 - 5e^{0.0002x}$
57. **Text Messaging** The approximate rate of change in the number (in billions) of monthly text messages is given by
$$f'(t) = 7.50t - 16.8,$$
where t represents the number of years since 2000. In 2005 ($t = 5$) there were approximately 9.8 billion monthly text messages. ***Source: Cellular Telecommunication & Internet Association.***
   a. Find the function that gives the total number (in billions) of monthly text messages in year t.
   b. According to this function, how many monthly text messages were there in 2009? Compare this with the actual number of 152.7 billion.
58. **Profit** The marginal profit of a small fast-food stand is given, in thousands of dollars, by
$$P'(x) = \sqrt{x} + \frac{1}{2},$$
where x is the sales volume in thousands of hamburgers. The "profit" is $-\$1000$ when no hamburgers are sold. Find the profit function.

59. Profit The marginal profit in dollars on Brie cheese sold at a cheese store is given by

$$P'(x) = x(50x^2 + 30x),$$

where x is the amount of cheese sold, in hundreds of pounds. The "profit" is $-\$40$ when no cheese is sold.

a. Find the profit function.

b. Find the profit from selling 200 lb of Brie cheese.

Life Sciences

60. Biochemical Excretion If the rate of excretion of a biochemical compound is given by

$$f'(t) = 0.01e^{-0.01t},$$

the total amount excreted by time t (in minutes) is $f(t)$.

a. Find an expression for $f(t)$.

b. If 0 units are excreted at time $t = 0$, how many units are excreted in 10 minutes?

61. Flour Beetles A model for describing the population of adult flour beetles involves evaluating the integral

$$\int \frac{g(x)}{x}\,dx,$$

where $g(x)$ is the per-unit-abundance growth rate for a population of size x. The researchers consider the simple case in which $g(x) = a - bx$ for positive constants a and b. Find the integral in this case. ***Source: Ecology.***

62. Concentration of a Solute According to Fick's law, the diffusion of a solute across a cell membrane is given by

$$c'(t) = \frac{kA}{V}[C - c(t)], \tag{1}$$

where A is the area of the cell membrane, V is the volume of the cell, $c(t)$ is the concentration inside the cell at time t, C is the concentration outside the cell, and k is a constant. If c_0 represents the concentration of the solute inside the cell when $t = 0$, then it can be shown that

$$c(t) = (c_0 - C)e^{-kAt/V} + C. \tag{2}$$

a. Use the last result to find $c'(t)$.

b. Substitute back into Equation (1) to show that (2) is indeed the correct antiderivative of (1).

63. Cell Growth Under certain conditions, the number of cancer cells $N(t)$ at time t increases at a rate

$$N'(t) = Ae^{kt},$$

where A is the rate of increase at time 0 (in cells per day) and k is a constant.

a. Suppose $A = 50$, and at 5 days, the cells are growing at a rate of 250 per day. Find a formula for the number of cells after t days, given that 300 cells are present at $t = 0$.

b. Use your answer from part a to find the number of cells present after 12 days.

64. Blood Pressure The rate of change of the volume $V(t)$ of blood in the aorta at time t is given by

$$V'(t) = -kP(t),$$

where $P(t)$ is the pressure in the aorta at time t and k is a constant that depends upon properties of the aorta. The pressure in the aorta is given by

$$P(t) = P_0e^{-mt},$$

where P_0 is the pressure at time $t = 0$ and m is another constant. Letting V_0 be the volume at time $t = 0$, find a formula for $V(t)$.

Social Sciences

65. Bachelor's Degrees The number of bachelor's degrees conferred in the United States has been increasing steadily in recent decades. Based on data from the National Center for Education Statistics, the rate of change of the number of bachelor's degrees (in thousands) can be approximated by the function

$$B'(t) = 0.06048t^2 - 1.292t + 15.86,$$

where t is the number of years since 1970. ***Source: National Center for Education Statistics.***

a. Find $B(t)$, given that about 839,700 degrees were conferred in 1970 ($t = 0$).

b. Use the formula from part a to project the number of bachelor's degrees that will be conferred in 2015 ($t = 45$).

66. Degrees in Dentistry The number of degrees in dentistry (D.D.S. or D.M.D.) conferred to females in the United States has been increasing steadily in recent decades. Based on data from the National Center for Education Statistics, the rate of change of the number of bachelor's degrees can be approximated by the function

$$D'(t) = 29.25e^{0.03572t},$$

where t is the number of years since 1980. ***Source: National Center for Education Statistics.***

a. Find $D(t)$, given that about 700 degrees in dentistry were conferred to females in 1980 ($t = 0$).

b. Use the formula from part a to project the number of degrees in dentistry that will be conferred to females in 2015 ($t = 35$).

Physical Sciences

Exercises 67–71 refer to Example 11 in this section.

67. Velocity For a particular object, $a(t) = 5t^2 + 4$ and $v(0) = 6$. Find $v(t)$.

68. Distance Suppose $v(t) = 9t^2 - 3\sqrt{t}$ and $s(1) = 8$. Find $s(t)$.

69. Time An object is dropped from a small plane flying at 6400 ft. Assume that $a(t) = -32$ ft per second and $v(0) = 0$. Find $s(t)$. How long will it take the object to hit the ground?

70. Distance Suppose $a(t) = 18t + 8$, $v(1) = 15$, and $s(1) = 19$. Find $s(t)$.

71. Distance Suppose $a(t) = (15/2)\sqrt{t} + 3e^{-t}$, $v(0) = -3$, and $s(0) = 4$. Find $s(t)$.

72. Motion Under Gravity Show that an object thrown from an initial height h_0 with an initial velocity v_0 has a height at time t given by the function

$$h(t) = \tfrac{1}{2}gt^2 + v_0t + h_0,$$

where g is the acceleration due to gravity, a constant with value -32 ft/sec^2.

73. Rocket A small rocket was launched straight up from a platform. After 5 seconds, the rocket reached a maximum height of 412 ft. Find the initial velocity and height of the rocket. (*Hint:* See the previous exercise.)

74. Rocket Science In the 1999 movie *October Sky*, Homer Hickum was accused of launching a rocket that started a forest fire. Homer proved his innocence by showing that his rocket could not have flown far enough to reach where the fire started. He used the following reasoning.

a. Using the fact that $a(t) = -32$ (see Example 11(b)), find $v(t)$ and $s(t)$, given $v(0) = v_0$ and $s(0) = 0$. (The initial velocity was unknown, and the initial height was 0 ft.)

b. Homer estimated that the rocket was in the air for 14 seconds. Use $s(14) = 0$ to find v_0.

c. If the rocket left the ground at a 45° angle, the velocity in the horizontal direction would be equal to v_0, the velocity in the vertical direction, so the distance traveled horizontally would be v_0t. (The rocket left the ground at a steeper angle, so this would overestimate the distance from starting to landing point.) Find the distance the rocket would travel horizontally during its 14-second flight.

YOUR TURN ANSWERS

1. x^8 or $x^8 + C$ **2.** $-\dfrac{1}{3t^3} + C$

3. $2x^3 + 4x^2 - 9x + C$ **4.** $\dfrac{2}{7}x^{7/2} - 4x^{1/2} + C$

5. $3\ln|x| - \dfrac{1}{3}e^{-3x} + C$

6. $s(t) = -16t^2 - 20t + 2717$; 12.42 sec; 417 ft/sec

7. $f(x) = 2x^{3/2} + 4x - 8$

15.2 Substitution

APPLY IT **If a formula for the marginal revenue is known, how can a formula for the total revenue be found?**

Using the method of substitution, this question will be answered in Exercise 39.

In earlier chapters you learned all the rules for finding derivatives of elementary functions. By correctly applying those rules, you can take the derivative of any function involving powers of x, exponential functions, and logarithmic functions, combined in any way using the operations of arithmetic (addition, subtraction, multiplication, division, and exponentiation). By contrast, finding the antiderivative is much more complicated. There are a large number of techniques—more than we can cover in this book. Furthermore, for some functions all possible techniques fail. In the last section we saw how to integrate a few simple functions. In this section we introduce a technique known as *substitution* that will greatly expand the set of functions you can integrate.

The substitution technique depends on the idea of a differential, discussed in Chapter 14 on Applications of the Derivative. If $u = f(x)$, the *differential* of u, written du, is defined as

$$du = f'(x)\,dx.$$

For example, if $u = 2x^3 + 1$, then $du = 6x^2\,dx$. In this chapter we will only use differentials as a convenient notational device when finding an antiderivative such as

$$\int (2x^3 + 1)^4 6x^2\,dx.$$

FOR REVIEW

The chain rule, discussed in detail in Chapter 12 on Calculating the Derivative, states that $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

The function $(2x^3 + 1)^4 6x^2$ might remind you of the result when using the chain rule to take the derivative. We will now use differentials and the chain rule in reverse to find the antiderivative. Let $u = 2x^3 + 1$; then $du = 6x^2\,dx$. Now substitute u for $2x^3 + 1$ and du for $6x^2\,dx$ in the indefinite integral.

$$\int (2x^3 + 1)^4 6x^2\,dx = \int \overbrace{(2x^3 + 1)}^{u}{}^4 \overbrace{(6x^2\,dx)}^{du}$$

$$= \int u^4\,du$$

With substitution we have changed a complicated integral into a simple one. This last integral can now be found by the power rule.

$$\int u^4\,du = \frac{u^5}{5} + C$$

Finally, substitute $2x^3 + 1$ for u in the antiderivative to get

$$\int (2x^3 + 1)^4 6x^2\,dx = \frac{(2x^3 + 1)^5}{5} + C.$$

We can check the accuracy of this result by using the chain rule to take the derivative. We get

$$\frac{d}{dx}\left[\frac{(2x^3 + 1)^5}{5} + C\right] = \frac{1}{5} \cdot 5(2x^3 + 1)^4(6x^2) + 0$$

$$= (2x^3 + 1)^4 6x^2.$$

This method of integration is called **integration by substitution**. As shown above, it is simply the chain rule for derivatives in reverse. The results can always be verified by differentiation.

EXAMPLE 1 Substitution

Find $\int 6x(3x^2 + 4)^7\,dx$.

SOLUTION If we choose $u = 3x^2 + 4$, then $du = 6x\,dx$ and the integrand can be written as the product of $(3x^2 + 4)^7$ and $6x\,dx$. Now substitute.

$$\int 6x(3x^2 + 4)^7\,dx = \int (3x^2 + 4)^7 (6x\,dx) = \int u^7\,du$$

Find this last indefinite integral.

$$\int u^7\,du = \frac{u^8}{8} + C$$

Now replace u with $3x^2 + 4$.

$$\int 6x(3x^2 + 4)^7\,dx = \frac{u^8}{8} + C = \frac{(3x^2 + 4)^8}{8} + C$$

To verify this result, find the derivative.

$$\frac{d}{dx}\left[\frac{(3x^2 + 4)^8}{8} + C\right] = \frac{8}{8}(3x^2 + 4)^7(6x) + 0 = (3x^2 + 4)^7(6x)$$

The derivative is the original function, as required.

YOUR TURN 1

Find $\int 8x(4x^2 + 8)^6 dx$.

TRY YOUR TURN 1

EXAMPLE 2 Substitution

Find $\int x^2\sqrt{x^3+1}\,dx$.

SOLUTION

Method 1 Modifying the Integral

An expression raised to a power is usually a good choice for u, so because of the square root or $1/2$ power, let $u = x^3 + 1$; then $du = 3x^2\,dx$. The integrand does not contain the constant 3, which is needed for du. To take care of this, multiply by $3/3$, placing 3 inside the integral sign and $1/3$ outside.

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int 3x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int \sqrt{x^3+1}\,(3x^2\,dx)$$

Now substitute u for $x^3 + 1$ and du for $3x^2\,dx$, and then integrate.

$$\begin{aligned}\frac{1}{3}\int \sqrt{x^3+1}\,(3x^2\,dx) &= \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\int u^{1/2}\,du \\ &= \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} + C = \frac{2}{9}u^{3/2} + C\end{aligned}$$

Since $u = x^3 + 1$,

$$\int x^2\sqrt{x^3+1}\,dx = \frac{2}{9}(x^3+1)^{3/2} + C.$$

Method 2 Eliminating the Constant

As in Method 1, we let $u = x^3 + 1$, so that $du = 3x^2\,dx$. Since there is no 3 in the integral, we divide the equation for du by 3 to get

$$\frac{1}{3}\,du = x^2\,dx.$$

We then substitute u for $x^3 + 1$ and $du/3$ for $x^2\,dx$ to get

$$\int \sqrt{u}\,\frac{1}{3}\,du = \frac{1}{3}\int u^{1/2}\,du$$

and proceed as we did in Method 1. The two methods are just slightly different ways of doing the same thing, but some people prefer one method over the other.

YOUR TURN 2

Find $\int x^3\sqrt{3x^4+10}\,dx$.

TRY YOUR TURN 2

The substitution method given in the examples above *will not always work*. For example, you might try to find

$$\int x^3\sqrt{x^3+1}\,dx$$

by substituting $u = x^3 + 1$, so that $du = 3x^2\,dx$. However, there is no *constant* that can be inserted inside the integral sign to give $3x^2$ alone. This integral, and a great many others, cannot be evaluated by substitution.

With practice, choosing u will become easy if you keep two principles in mind.

1. u should equal some expression in the integral that, when replaced with u, tends to make the integral simpler.
2. u must be an expression whose derivative—disregarding any constant multiplier, such as the 3 in $3x^2$—is also present in the integral.

The substitution should include as much of the integral as possible, as long as its derivative is still present. In Example 1, we could have chosen $u = 3x^2$, but $u = 3x^2 + 4$

is better, because it has the same derivative as $3x^2$ and captures more of the original integral. If we carry this reasoning further, we might try $u = (3x^2 + 4)^4$, but this is a poor choice, for $du = 4(3x^2 + 4)^3(6x)\,dx$, an expression not present in the original integral.

EXAMPLE 3 Substitution

Find $\int \frac{x + 3}{(x^2 + 6x)^2}\,dx$.

SOLUTION Let $u = x^2 + 6x$, so that $du = (2x + 6)\,dx = 2(x + 3)\,dx$. The integral is missing the 2, so multiply by $2 \cdot (1/2)$, putting 2 inside the integral sign and $1/2$ outside.

$$\int \frac{x + 3}{(x^2 + 6x)^2}\,dx = \frac{1}{2}\int \frac{2(x + 3)}{(x^2 + 6x)^2}\,dx$$
$$= \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du = \frac{1}{2}\cdot\frac{u^{-1}}{-1} + C = \frac{-1}{2u} + C$$

Substituting $x^2 + 6x$ for u gives

$$\int \frac{x + 3}{(x^2 + 6x)^2}\,dx = \frac{-1}{2(x^2 + 6x)} + C.$$

TRY YOUR TURN 3

YOUR TURN 3
Find $\int \frac{x + 1}{(4x^2 + 8x)^3}\,dx$.

In Example 3, the quantity $x^2 + 6x$ was raised to a power in the denominator. When such an expression is not raised to a power, the function can often be integrated using the fact that

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}\cdot f'(x).$$

This suggests that such integrals can be solved by letting u equal the expression in the denominator, as long as the derivative of the denominator is present in the numerator (disregarding any constant multiplier as usual). The next example illustrates this idea.

EXAMPLE 4 Substitution

Find $\int \frac{(2x - 3)\,dx}{x^2 - 3x}$.

SOLUTION Let $u = x^2 - 3x$, so that $du = (2x - 3)\,dx$. Then

$$\int \frac{(2x - 3)\,dx}{x^2 - 3x} = \int \frac{du}{u} = \ln|u| + C = \ln|x^2 - 3x| + C.$$

TRY YOUR TURN 4

YOUR TURN 4
Find $\int \frac{x + 3}{x^2 + 6x}\,dx$.

Recall that if $f(x)$ is a function, then by the chain rule, the derivative of the exponential function $y = e^{f(x)}$ is

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}\cdot f'(x).$$

This suggests that the antiderivative of a function of the form $e^{f(x)}$ can be found by letting u be the exponent, as long as $f'(x)$ is also present in the integral (disregarding any constant multiplier as usual).

EXAMPLE 5 Substitution

Find $\int x^2 e^{x^3}\,dx$.

SOLUTION Let $u = x^3$, the exponent on e. Then $du = 3x^2\,dx$. Multiplying by $3/3$ gives

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^{x^3}(3x^2\,dx)$$

$$= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

TRY YOUR TURN 5

YOUR TURN 5
Find $\int x^3 e^{x^4}\,dx$.

The techniques in the preceding examples can be summarized as follows.

Substitution

Each of the following forms can be integrated using the substitution $u = f(x)$.

Form of the Integral	Result
1. $\int [f(x)]^n f'(x)\,dx, \quad n \neq -1$	$\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{[f(x)]^{n+1}}{n+1} + C$
2. $\int \frac{f'(x)}{f(x)}\,dx$	$\int \frac{1}{u}\,du = \ln\lvert u\rvert + C = \ln\lvert f(x)\rvert + C$
3. $\int e^{f(x)} f'(x)\,dx$	$\int e^u\,du = e^u + C = e^{f(x)} + C$

The next example shows a more complicated integral in which none of the previous forms apply, but for which substitution still works.

EXAMPLE 6 Substitution

Find $\int x\sqrt{1-x}\,dx$.

SOLUTION Let $u = 1 - x$. To get the x outside the radical in terms of u, solve $u = 1 - x$ for x to get $x = 1 - u$. Then $dx = -du$ and we can substitute as follows.

$$\int x\sqrt{1-x}\,dx = \int (1-u)\sqrt{u}(-du) = \int (u-1)u^{1/2}\,du$$

$$= \int (u^{3/2} - u^{1/2})\,du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C$$

TRY YOUR TURN 6

YOUR TURN 6
Find $\int x\sqrt{3+x}\,dx$

The substitution method is useful if the integral can be written in one of the following forms, where u is some function of x.

Substitution Method

In general, for the types of problems we are concerned with, there are three cases. We choose u to be one of the following:

1. the quantity under a root or raised to a power;
2. the quantity in the denominator;
3. the exponent on e.

Remember that some integrands may need to be rearranged to fit one of these cases.

TECHNOLOGY NOTE

Some calculators, such as the TI-89 and TI-Nspire CAS, can find indefinite integrals automatically. Many computer algebra systems, such as Maple, Matlab, and Mathematica, also do this. The website www.wolframalpha.com can also be used to symbolically determine indefinite integrals and derivatives of functions. Figure 2 shows the integral in Example 6 performed on a TI-89. The answer looks different but is algebraically equivalent to the answer found in Example 6.

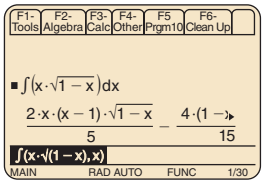


FIGURE 2

EXAMPLE 7 Demand

The research department for a hardware chain has determined that at one store the marginal price of x boxes per week of a particular type of nails is

$$p'(x) = \frac{-4000}{(2x + 15)^3}.$$

Find the demand equation if the weekly demand for this type of nails is 10 boxes when the price of a box of nails is $4.

SOLUTION To find the demand function, first integrate $p'(x)$ as follows.

$$p(x) = \int p'(x)\,dx = \int \frac{-4000}{(2x + 15)^3}\,dx$$

Let $u = 2x + 15$. Then $du = 2\,dx$, and

$$p(x) = \frac{-4000}{2}\int (2x + 15)^{-3}\,2\,dx \qquad \text{Multiply by } 2/2.$$

$$= -2000\int u^{-3}\,du \qquad \text{Substitute.}$$

$$= (-2000)\frac{u^{-2}}{-2} + C$$

$$= \frac{1000}{u^2} + C$$

$$p(x) = \frac{1000}{(2x + 15)^2} + C. \tag{1}$$

Find the value of C by using the given information that $p = 4$ when $x = 10$.

$$4 = \frac{1000}{(2 \cdot 10 + 15)^2} + C$$

$$4 = \frac{1000}{35^2} + C$$

$$4 \approx 0.82 + C$$

$$3.18 \approx C$$

Replacing C with 3.18 in Equation (1) gives the demand function,

$$p(x) = \frac{1000}{(2x + 15)^2} + 3.18.$$

With a little practice, you will find you can skip the substitution step for integrals such as that shown in Example 7, in which the derivative of u is a constant. Recall from the chain rule that when you differentiate a function, such as $p(x) = 1000/(2x + 15)^2 + 3.18$

in the previous example, you multiply by 2, the derivative of $(2x + 15)$. So when taking the antiderivative, simply divide by 2:

$$\int -4000(2x + 15)^{-3}\,dx = \frac{-4000}{2} \cdot \frac{(2x + 15)^{-2}}{-2} + C$$
$$= \frac{1000}{(2x + 15)^2} + C.$$

CAUTION This procedure is valid because of the constant multiple rule presented in the previous section, which says that constant multiples can be brought into or out of integrals, just as they can with derivatives. This procedure is *not* valid with any expression other than a constant.

EXAMPLE 8 Popularity Index

To determine the top 100 popular songs of each year since 1956, Jim Quirin and Barry Cohen developed a function that represents the rate of change on the charts of *Billboard* magazine required for a song to earn a "star" on the *Billboard* "Hot 100" survey. They developed the function

$$f(x) = \frac{A}{B + x},$$

where $f(x)$ represents the rate of change in position on the charts, x is the position on the "Hot 100" survey, and A and B are positive constants. The function

$$F(x) = \int f(x)\,dx$$

is defined as the "Popularity Index." Find $F(x)$. ***Source: Chartmasters' Rock 100.***

SOLUTION Integrating $f(x)$ gives

$$F(x) = \int f(x)\,dx$$
$$= \int \frac{A}{B + x}\,dx$$
$$= A\int \frac{1}{B + x}\,dx.$$

Let $u = B + x$, so that $du = dx$. Then

$$F(x) = A\int \frac{1}{u}\,du = A \ln u + C$$
$$= A \ln(B + x) + C.$$

(The absolute value bars are not necessary, since $B + x$ is always positive here.)

15.2 EXERCISES

1. Integration by substitution is related to what differentiation method? What type of integrand suggests using integration by substitution?

2. The following integrals may be solved using substitution. Choose a function u that may be used to solve each problem. Then find du.

a. $\int (3x^2 - 5)^4\,2x\,dx$ **b.** $\int \sqrt{1 - x}\,dx$

c. $\int \frac{x^2}{2x^3 + 1}\,dx$ **d.** $\int 4x^3 e^{x^4}\,dx$

Use substitution to find each indefinite integral.

3. $\int 4(2x+3)^4\,dx$

4. $\int (-4t+1)^3\,dt$

5. $\int \frac{2\,dm}{(2m+1)^3}$

6. $\int \frac{3\,du}{\sqrt{3u-5}}$

7. $\int \frac{2x+2}{(x^2+2x-4)^4}\,dx$

8. $\int \frac{6x^2\,dx}{(2x^3+7)^{3/2}}$

9. $\int z\sqrt{4z^2-5}\,dz$

10. $\int r\sqrt{5r^2+2}\,dr$

11. $\int 3x^2e^{2x^3}\,dx$

12. $\int re^{-r^2}\,dr$

13. $\int (1-t)e^{2t-t^2}\,dt$

14. $\int (x^2-1)e^{x^3-3x}\,dx$

15. $\int \frac{e^{1/z}}{z^2}\,dz$

16. $\int \frac{e^{\sqrt{y}}}{2\sqrt{y}}\,dy$

17. $\int \frac{t}{t^2+2}\,dt$

18. $\int \frac{-4x}{x^2+3}\,dx$

19. $\int \frac{x^3+2x}{x^4+4x^2+7}\,dx$

20. $\int \frac{t^2+2}{t^3+6t+3}\,dt$

21. $\int \frac{2x+1}{(x^2+x)^3}\,dx$

22. $\int \frac{y^2+y}{(2y^3+3y^2+1)^{2/3}}\,dy$

23. $\int p(p+1)^5\,dp$

24. $\int 4r\sqrt{8-r}\,dr$

25. $\int \frac{u}{\sqrt{u-1}}\,du$

26. $\int \frac{2x}{(x+5)^6}\,dx$

27. $\int (\sqrt{x^2+12x})(x+6)\,dx$

28. $\int (\sqrt{x^2-6x})(x-3)\,dx$

29. $\int \frac{(1+3\ln x)^2}{x}\,dx$

30. $\int \frac{\sqrt{2+\ln x}}{x}\,dx$

31. $\int \frac{e^{2x}}{e^{2x}+5}\,dx$

32. $\int \frac{1}{x(\ln x)}\,dx$

33. $\int \frac{\log x}{x}\,dx$

34. $\int \frac{(\log_2(5x+1))^2}{5x+1}\,dx$

35. $\int x8^{3x^2+1}\,dx$

36. $\int \frac{10^{5\sqrt{x}+2}}{\sqrt{x}}\,dx$

37. Stan and Ollie work on the integral

$$\int 3x^2e^{x^3}\,dx.$$

Stan lets $u = x^3$ and proceeds to get

$$\int e^u\,du = e^u + C = e^{x^3} + C.$$

Ollie tries $u = e^{x^3}$ and proceeds to get

$$\int du = u + C = e^{x^3} + C.$$

Discuss which procedure you prefer, and why.

38. Stan and Ollie work on the integral

$$\int 2x(x^2+2)\,dx.$$

Stan lets $u = x^2 + 2$ and proceeds to get

$$\int u\,du = \frac{u^2}{2} + C = \frac{(x^2+2)^2}{2} + C.$$

Ollie multiplies out the function under the integral and gets

$$\int (2x^3+4x)\,dx = \frac{x^4}{2} + 2x^2 + C.$$

How can they both be right?

APPLICATIONS

Business and Economics

39. APPLY IT Revenue The marginal revenue (in thousands of dollars) from the sale of x MP3 players is given by

$$R'(x) = 4x(x^2+27{,}000)^{-2/3}.$$

a. Find the total revenue function if the revenue from 125 players is \$29,591.

b. How many players must be sold for a revenue of at least \$40,000?

40. Debt A company incurs debt at a rate of

$$D'(t) = 90(t+6)\sqrt{t^2+12t}$$

dollars per year, where t is the amount of time (in years) since the company began. By the fourth year the company had accumulated \$16,260 in debt.

a. Find the total debt function.

b. How many years must pass before the total debt exceeds \$40,000?

41. Cost A company has found that the marginal cost (in thousands of dollars) to produce x central air conditioning units is

$$C'(x) = \frac{60x}{5x^2+e},$$

where x is the number of units produced.

a. Find the cost function, given that the company incurs a fixed cost of \$10,000 even if no units are built.

b. The company will seek a new source of investment income if the cost is more than \$20,000 to produce 5 units. Should they seek this new source?

42. Profit The rate of growth of the profit (in millions of dollars) from a new technology is approximated by

$$P'(x) = xe^{-x^2},$$

where x represents time measured in years. The total profit in the third year that the new technology is in operation is \$10,000.

a. Find the total profit function.

b. What happens to the total amount of profit in the long run?

43. Transportation According to data from the Bureau of Transportation Statistics, the rate of change in the number of local transit vehicles (buses, light rail, etc.), in thousands, in the United States from 1970 to the present can be approximated by

$$f'(t) = 4.0674 \times 10^{-4} t(t - 1970)^{0.4},$$

where t is the year. ***Source: National Transportation Statistics 2006.***

a. Using the fact that in 1970 there were 61,298 such vehicles, find a formula giving the approximate number of local transit vehicles as a function of time.

b. Use the answer to part a to forecast the number of local transit vehicles in the year 2015.

Life Sciences

44. Outpatient Visits According to data from the American Hospital Association, the rate of change in the number of hospital outpatient visits, in millions, in the United States each year from 1980 to the present can be approximated by

$$f'(t) = 0.001483t(t - 1980)^{0.75},$$

where t is the year. ***Source: Hospital Statistics.***

a. Using the fact that in 1980 there were 262,951,000 outpatient visits, find a formula giving the approximate number of outpatient visits as a function of time.

b. Use the answer to part a to forecast the number of outpatient visits in the year 2015.

YOUR TURN ANSWERS

1. $\dfrac{(4x^2 + 8)^7}{7} + C$ **2.** $\dfrac{(3x^4 + 10)^{3/2}}{18} + C$

3. $-\dfrac{1}{16(4x^2 + 8x)^2} + C$ **4.** $\frac{1}{2}\ln|x^2 + 6x| + C$

5. $\frac{1}{4}e^{x^4} + C$

6. $\frac{2}{5}(3 + x)^{5/2} - 2(3 + x)^{3/2} + C$

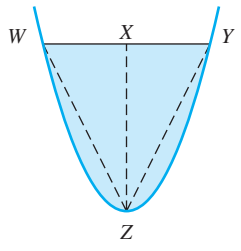
15.3 Area and the Definite Integral

APPLY IT **If we know how the rate that oil is leaking from a machine varies with time, how can we estimate the total amount of leakage over a certain period of time?**

We will answer this question in Example 3 using a method introduced in this section.

To calculate the areas of geometric figures such as rectangles, squares, triangles, and circles, we use specific formulas. In this section we consider the problem of finding the area of a figure or region that is bounded by curves, such as the shaded region in Figure 3.

The brilliant Greek mathematician Archimedes (about 287 B.C.–212 B.C.) is considered one of the greatest mathematicians of all time. His development of a rigorous method known as *exhaustion* to derive results was a forerunner of the ideas of integral calculus. Archimedes used a method that would later be verified by the theory of integration. His method involved viewing a geometric figure as a sum of other figures. For example, he thought of a plane surface area as a figure consisting of infinitely many parallel line segments. Among the results established by Archimedes' method was the fact that the area of a segment of a parabola (shown in color in Figure 3) is equal to $4/3$ the area of a triangle with the same base and the same height.



Area of parabolic segment

$= \frac{4}{3}$ (area of triangle WYZ)

FIGURE 3

EXAMPLE 1 Approximation of Area

Consider the region bounded by the y-axis, the x-axis, and the graph of $f(x) = \sqrt{4 - x^2}$, shown in Figure 4.

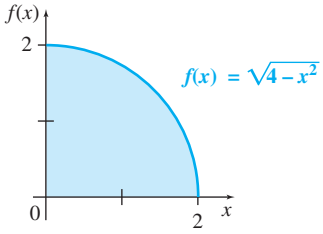


FIGURE 4

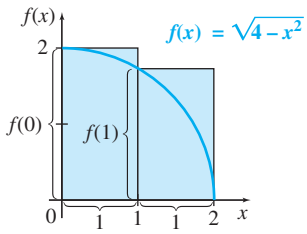


FIGURE 5

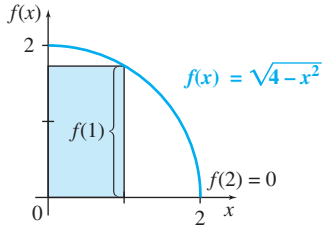


FIGURE 6

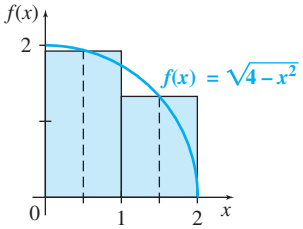


FIGURE 7

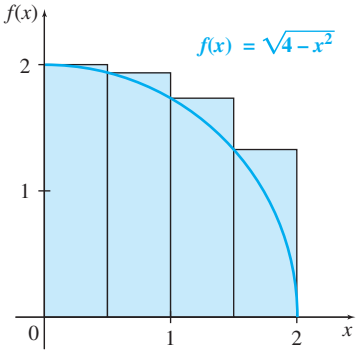


FIGURE 8

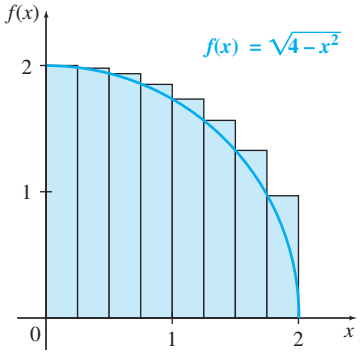


FIGURE 9

(a) Approximate the area of the region using two rectangles. Determine the height of the rectangle by the value of the function at the *left* endpoint.

SOLUTION A very rough approximation of the area of this region can be found by using two rectangles whose heights are determined by the value of the function at the left endpoints, as in Figure 5. The height of the rectangle on the left is $f(0) = 2$ and the height of the rectangle on the right is $f(1) = \sqrt{3}$. The width of each rectangle is 1, making the total area of the two rectangles

$$1 \cdot f(0) + 1 \cdot f(1) = 2 + \sqrt{3} \approx 3.7321 \text{ square units.}$$

Note that $f(x)$ is a decreasing function, and that we will overestimate the area when we evaluate the function at the left endpoint to determine the height of the rectangle in that interval.

(b) Repeat part (a) using the value of the function at the *right* endpoint to determine the height of the rectangle.

SOLUTION Using the right endpoints, as in Figure 6, the area of the two rectangles is

$$1 \cdot f(1) + 1 \cdot f(2) = \sqrt{3} + 0 \approx 1.7321 \text{ square units.}$$

Note that we underestimate the area of this particular region when we use the right endpoints.

If the left endpoint gives an answer too big and the right endpoint an answer too small, it seems reasonable to average the two answers. This produces the method called the *trapezoidal rule*, discussed in more detail later in this chapter. In this example, we get

$$\frac{3.7321 + 1.7321}{2} = 2.7321 \text{ square units.}$$

(c) Repeat part (a) using the value of the function at the *midpoint* of each interval to determine the height of the rectangle.

SOLUTION In Figure 7, the rectangles are drawn with height determined by the midpoint of each interval. This method is called the **midpoint rule**, and gives

$$1 \cdot f(0.5) + 1 \cdot f(1.5) = \sqrt{3.75} + \sqrt{1.75} \approx 3.2594 \text{ square units.}$$

(d) We can improve the accuracy of the previous approximations by increasing the number of rectangles. Repeat part (a) using four rectangles.

SOLUTION Divide the interval from $x = 0$ to $x = 2$ into four equal parts, each of width $1/2$. The height of each rectangle is given by the value of f at the left side of the rectangle, as shown in Figure 8. The area of each rectangle is the width, $1/2$, multiplied by the height. The total area of the four rectangles is

$$\begin{aligned}
&\frac{1}{2} \cdot f(0) + \frac{1}{2} \cdot f\left(\frac{1}{2}\right) + \frac{1}{2} \cdot f(1) + \frac{1}{2} \cdot f\left(1\frac{1}{2}\right) \\
&\quad = \frac{1}{2}(2) + \frac{1}{2}\left(\frac{\sqrt{15}}{2}\right) + \frac{1}{2}(\sqrt{3}) + \frac{1}{2}\left(\frac{\sqrt{7}}{2}\right) \\
&\quad = 1 + \frac{\sqrt{15}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{4} \approx 3.4957 \text{ square units.}
\end{aligned}$$

This approximation looks better, but it is still greater than the actual area.

(e) Repeat part (a) using eight rectangles.

SOLUTION Divide the interval from $x = 0$ to $x = 2$ into 8 equal parts, each of width $1/4$ (see Figure 9). The total area of all of these rectangles is

$$\begin{aligned}
&\frac{1}{4} \cdot f(0) + \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} \cdot f\left(\frac{1}{2}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right) + \frac{1}{4} \cdot f(1) \\
&\qquad + \frac{1}{4} \cdot f\left(\frac{5}{4}\right) + \frac{1}{4} \cdot f\left(\frac{3}{2}\right) + \frac{1}{4} \cdot f\left(\frac{7}{4}\right) \\
&\quad \approx 3.3398 \text{ square units.}
\end{aligned}$$

The process used in Example 1 of approximating the area under a curve by using more and more rectangles to get a better and better approximation can be generalized. To do this, divide the interval from $x = 0$ to $x = 2$ into n equal parts. Each of these n intervals has width

$$\frac{2-0}{n} = \frac{2}{n},$$

so each rectangle has width $2/n$ and height determined by the function value at the left side of the rectangle, or the right side, or the midpoint. We could also average the left and right side values as before. Using a computer or graphing calculator to find approximations to the area for several values of n gives the results in the following table.

	Approximations to the Area			
n	Left Sum	Right Sum	Trapezoidal	Midpoint
2	3.7321	1.7321	2.7321	3.2594
4	3.4957	2.4957	2.9957	3.1839
8	3.3398	2.8398	3.0898	3.1567
10	3.3045	2.9045	3.1045	3.1524
20	3.2285	3.0285	3.1285	3.1454
50	3.1783	3.0983	3.1383	3.1426
100	3.1604	3.1204	3.1404	3.1419
500	3.1455	3.1375	3.1415	3.1416

The numbers in the last four columns of this table represent approximations to the area under the curve, above the x-axis, and between the lines $x = 0$ and $x = 2$. As n becomes larger and larger, all four approximations become better and better, getting closer to the actual area. In this example, the exact area can be found by a formula from plane geometry. Write the given function as

$$y = \sqrt{4 - x^2},$$

then square both sides to get

$$y^2 = 4 - x^2$$
$$x^2 + y^2 = 4,$$

the equation of a circle centered at the origin with radius 2. The region in Figure 4 is the quarter of this circle that lies in the first quadrant. The actual area of this region is one-quarter of the area of the entire circle, or

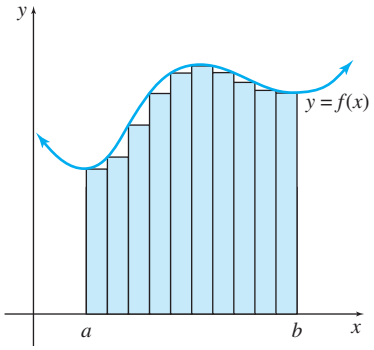
$$\frac{1}{4}\pi(2)^2 = \pi \approx 3.1416.$$

As the number of rectangles increases without bound, the sum of the areas of these rectangles gets closer and closer to the actual area of the region, π. This can be written as

$$\lim_{n\to\infty} (\text{sum of areas of } n \text{ rectangles}) = \pi.$$

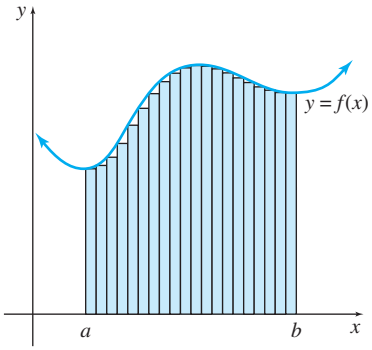
(The value of π was originally found by a process similar to this.)*

*The number π is the ratio of the circumference of a circle to its diameter. It is an example of an *irrational number*, and as such it cannot be expressed as a terminating or repeating decimal. Many approximations have been used for π over the years. A passage in the Bible (1 Kings 7:23) indicates a value of 3. The Egyptians used the value $3.16\underline{0}$, and Archimedes showed that its value must be between $22/7$ and $223/71$. A Hindu writer, Brahmagupta, used $\sqrt{10}$ as its value in the seventh century. The search for the digits of π has continued into modern times. Fabrice Bellard, using a desktop computer, recently computed the value to nearly 2.7 trillion digits.



Ten rectangles of equal width

(a)



Twenty rectangles of equal width

(b)

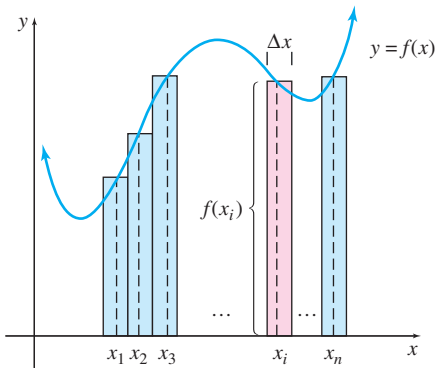
FIGURE 10

Notice in the previous table that for a particular value of n, the midpoint rule gave the best answer (the one closest to the true value of $\pi \approx 3.1416$), followed by the trapezoidal rule, followed by the left and right sums. In fact, the midpoint rule with $n = 20$ gives a value (3.1454) that is slightly more accurate than the left sum with $n = 500$ (3.1455). It is usually the case that the midpoint rule gives a more accurate answer than either the left or the right sum.

Now we can generalize to get a method of finding the area bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 10. To approximate this area, we could divide the region under the curve first into 10 rectangles (Figure 10(a)) and then into 20 rectangles (Figure 10(b)). The sum of the areas of the rectangles gives an approximation to the area under the curve when $f(x) \geq 0$. In the next section we will consider the case in which $f(x)$ might be negative.

To develop a process that would yield the *exact* area, begin by dividing the interval from a to b into n pieces of equal width, using each of these n pieces as the base of a rectangle (see Figure 11). Let x_1 be an arbitrary point in the first interval, x_2 be an arbitrary point in the second interval, and so on, up to the nth interval. In the graph of Figure 11, the symbol Δx is used to represent the width of each of the intervals. Since the length of the entire interval is $b - a$, each of the n pieces has length

$$\Delta x = \frac{b - a}{n}.$$



n rectangles of equal width

FIGURE 11

The pink rectangle is an arbitrary rectangle called the ith rectangle. Its area is the product of its length and width. Since the width of the ith rectangle is Δx and the length of the ith rectangle is given by the height $f(x_i)$,

$$\text{Area of the } i\text{th rectangle} = f(x_i) \cdot \Delta x.$$

The total area under the curve is approximated by the sum of the areas of all n of the rectangles. With sigma notation, the approximation to the total area becomes

$$\text{Area of all } n \text{ rectangles} = \sum_{i=1}^{n} f(x_i) \cdot \Delta x.$$

The exact area is defined to be the limit of this sum (if the limit exists) as the number of rectangles increases without bound:

$$\text{Exact area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.$$

Whenever this limit exists, regardless of whether $f(x)$ is positive or negative, we will call it the *definite integral* of $f(x)$ from a to b. It is written as follows.

FOR REVIEW

Recall from Chapter 1 that the symbol Σ (sigma) indicates "the sum of." Here, we use $\sum_{i=1}^{n} f(x_i)\Delta x$ to indicate the sum $f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$, where we replace i with 1 in the first term, 2 in the second term, and so on, ending with n replacing i in the last term.

The Definite Integral

If f is defined on the interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

provided the limit exists, where $\Delta x = (b - a)/n$ and x_i is *any* value of x in the ith interval.*

The definite integral can be approximated by

$$\sum_{i=1}^{n} f(x_i)\Delta x.$$

If $f(x) \geq 0$ on the interval $[a, b]$, the definite integral gives the area under the curve between $x = a$ and $x = b$. In the midpoint rule, x_i is the midpoint of the ith interval. We may also let x_i be the left endpoint, the right endpoint, or any other point in the ith interval.

In Example 1, the area bounded by the x-axis, the curve $y = \sqrt{4 - x^2}$, and the lines $x = 0$ and $x = 2$ could be written as the definite integral

$$\int_0^2 \sqrt{4 - x^2}\,dx = \pi.$$

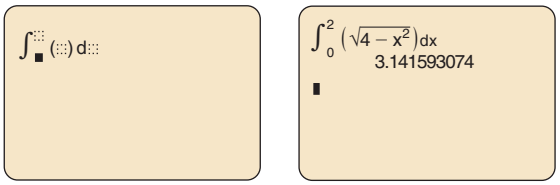
NOTE Notice that unlike the indefinite integral, which is a set of *functions*, the definite integral represents a *number*. The next section will show how antiderivatives are used in finding the definite integral and, thus, the area under a curve.

Keep in mind that finding the definite integral of a function can be thought of as a mathematical process that gives the sum of an infinite number of individual parts (within certain limits). The definite integral represents area only if the function involved is *nonnegative* $(f(x) \geq 0)$ at every x-value in the interval $[a, b]$. There are many other interpretations of the definite integral, and all of them involve this idea of approximation by appropriate sums. In the next section we will consider the definite integral when $f(x)$ might be negative.

As indicated in this definition, although the left endpoint of the ith interval has been used to find the height of the ith rectangle, any number in the ith interval can be used. (A more general definition is possible in which the rectangles do not necessarily all have the same width.) The b above the integral sign is called the **upper limit** of integration, and the a is the **lower limit** of integration. This use of the word *limit* has nothing to do with the limit of the sum; it refers to the limits, or boundaries, on x.

TECHNOLOGY NOTE

Some calculators have a built-in function for evaluating the definite integral. For example, the TI-84 Plus uses the `fnInt` command, found in the `MATH` menu, as shown in Figure 12(a). Figure 12(b) shows the command used for Example 1 and gives the answer 3.141593074, with an error of approximately 0.0000004.



(a) **(b)**

FIGURE 12

*The sum in the definition of the definite integral is an example of a Riemann sum, named for the German mathematician Georg Riemann (1826–1866), who at the age of 20 changed his field of study from theology and the classics to mathematics. Twenty years later he died of tuberculosis while traveling in Italy in search of a cure. The concepts of *Riemann sum* and *Riemann integral* are still studied in rigorous calculus textbooks.

EXAMPLE 2 Approximation of Area

Approximate $\int_0^4 2x\,dx$, the area of the region under the graph of $f(x) = 2x$, above the x-axis, and between $x = 0$ and $x = 4$, by using four rectangles of equal width whose heights are the values of the function at the midpoint of each subinterval.

Method 1
Calculating by Hand

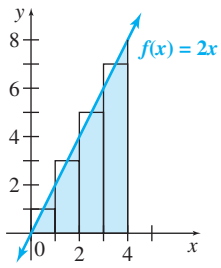


FIGURE 13

SOLUTION

We want to find the area of the shaded region in Figure 13. The heights of the four rectangles given by $f(x_i)$ for $i = 1, 2, 3$, and 4 are as follows.

	Rectangle Heights	
i	x_i	$f(x_i)$
1	$x_1 = 0.5$	$f(0.5) = 1.0$
2	$x_2 = 1.5$	$f(1.5) = 3.0$
3	$x_3 = 2.5$	$f(2.5) = 5.0$
4	$x_4 = 3.5$	$f(3.5) = 7.0$

The width of each rectangle is $\Delta x = (4 - 0)/4 = 1$. The sum of the areas of the four rectangles is

$$\begin{aligned}\sum_{i=1}^{4} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= f(0.5)\Delta x + f(1.5)\Delta x + f(2.5)\Delta x + f(3.5)\Delta x \\ &= 1(1) + 3(1) + 5(1) + 7(1) \\ &= 16.\end{aligned}$$

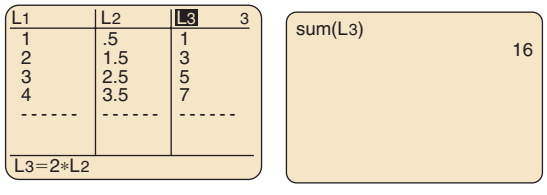
Using the formula for the area of a triangle, $A = (1/2)bh$, with b, the length of the base, equal to 4 and h, the height, equal to 8, gives

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(8) = 16,$$

the exact value of the area. The approximation equals the exact area in this case because our use of the midpoints of each subinterval distributed the error evenly above and below the graph.

Method 2
Graphing Calculator

A graphing calculator can be used to organize the information in this example. For example, the `seq` feature in the LIST OPS menu of the TI-84 Plus calculator can be used to store the values of i in the list L_1. Using the STAT EDIT menu, the entries for x_i can be generated by entering the formula $-.5 + L_1$ as the heading of L_2. Similarly, entering the formula for $f(x_i)$, $2*L_2$, at the top of list L_3 will generate the values of $f(x_i)$ in L_3. (The entries are listed automatically when the formula is entered.) Then the `sum` feature in the LIST MATH menu can be used to add the values in L_3. The resulting screens are shown in Figure 14.



(a) (b)

FIGURE 14

Method 3 Spreadsheet

The calculations in this example can also be done on a spreadsheet. In Microsoft Excel, for example, store the values of i in column A. Put the command "=A1 − .5" into B1; copying this formula into the rest of column B gives the values of x_i. Similarly, use the formula for $f(x_i)$ to fill column C. Column D is the product of Column C and Δx. Sum column D to get the answer. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

YOUR TURN 1 Repeat Example 1 to approximate $\int_1^5 4x\,dx$.

TRY YOUR TURN 1

Total Change

Suppose the function $f(x) = x^2 + 20$ gives the marginal cost of some item at a particular x-value. Then $f(2) = 24$ gives the rate of change of cost at $x = 2$. That is, a unit change in x (at this point) will produce a change of 24 units in the cost function. Also, $f(3) = 29$ means that each unit of change in x (when $x = 3$) will produce a change of 29 units in the cost function.

To find the *total* change in the cost function as x changes from 2 to 3, we could divide the interval from 2 to 3 into n equal parts, using each part as the base of a rectangle as we did above. The area of each rectangle would approximate the change in cost at the x-value that is the left endpoint of the base of the rectangle. Then the sum of the areas of these rectangles would approximate the net total change in cost from $x = 2$ to $x = 3$. The limit of this sum as $n \to \infty$ would give the exact total change.

This result produces another application of the definite integral: the area of the region under the graph of the marginal cost function $f(x)$ that is above the x-axis and between $x = a$ and $x = b$ gives the *net total change in the cost* as x goes from a to b.

Total Change in $F(x)$

If $f(x)$ gives the rate of change of $F(x)$ for x in $[a, b]$, then the **total change** in $F(x)$ as x goes from a to b is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx.$$

In other words, the total change in a quantity can be found from the function that gives the rate of change of the quantity, using the same methods used to approximate the area under a curve.

EXAMPLE 3 Oil Leakage

APPLY IT

Figure 15 shows the rate that oil is leaking from a machine in a large factory (in cubic centimeters per hour) with specific rates over a 12-hour period given in the table. Approximate the total amount of leakage over a 12-hour shift.

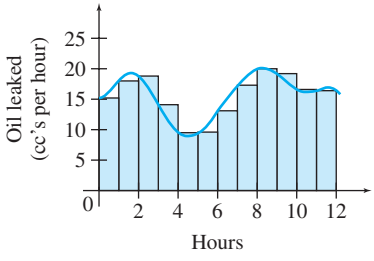


FIGURE 15

Oil Leakage (cc's per hour)			
x	y	x	y
0	15.2	6	13.1
1	18.0	7	17.3
2	18.8	8	20.0
3	14.1	9	19.2
4	9.5	10	16.6
5	9.6	11	16.4

SOLUTION Use approximating rectangles, dividing the interval from 0 to 12 into 12 equal subdivisions. Each subinterval has width 1. Using the left endpoint of each subinterval and the table to determine the height of the rectangle, as shown, the approximation becomes

$$1 \cdot 15.2 + 1 \cdot 18.0 + 1 \cdot 18.8 + 1 \cdot 14.1 + 1 \cdot 9.5 + 1 \cdot 9.6 + 1 \cdot 13.1 + 1 \cdot 17.3$$
$$+ 1 \cdot 20.0 + 1 \cdot 19.2 + 1 \cdot 16.6 + 1 \cdot 16.4 = 187.8.$$

About 187.8 cubic centimeters of oil leak during this time. Mathematically, we could write

$$\int_0^{12} f(x)\,dx \approx 187.8,$$

where $f(x)$ is the function shown in Figure 15.

Recall, velocity is the rate of change in distance from time a to time b. Thus the area under the velocity function defined by $v(t)$ from $t = a$ to $t = b$ gives the distance traveled in that time period.

EXAMPLE 4 Total Distance

A driver traveling on a business trip checks the speedometer each hour. The table shows the driver's velocity at several times.

Approximate the total distance traveled during the 3-hour period using the left endpoint of each interval, then the right endpoint.

Velocity				
Time (hr)	0	1	2	3
Velocity (mph)	0	52	58	60

SOLUTION Using left endpoints, the total distance is

$$0 \cdot 1 + 52 \cdot 1 + 58 \cdot 1 = 110.$$

With right endpoints, we get

$$52 \cdot 1 + 58 \cdot 1 + 60 \cdot 1 = 170.$$

Again, left endpoints give a total that is too small, while right endpoints give a total that is too large. The average, 140 miles, is a better estimate of the total distance traveled.

TRY YOUR TURN 2

YOUR TURN 2 Repeat Example 4 for a driver traveling at the following velocities at various times.

Time (hr)	0	0.5	1	1.5	2
Velocity (mph)	0	50	56	40	48

Before discussing further applications of the definite integral, we need a more efficient method for evaluating it. This method will be developed in the next section.

15.3 EXERCISES

1. Explain the difference between an indefinite integral and a definite integral.

2. Complete the following statement.

$\int_0^4 (x^2 + 3)\,dx = \lim_{n\to\infty}$ _____, where $\Delta x =$ _____, and x_i is _____.

3. Let $f(x) = 2x + 5$, $x_1 = 0$, $x_2 = 2$, $x_3 = 4$, $x_4 = 6$, and $\Delta x = 2$.

a. Find $\sum_{i=1}^{4} f(x_i)\Delta x$.

b. The sum in part a approximates a definite integral using rectangles. The height of each rectangle is given by the value of the function at the left endpoint. Write the definite integral that the sum approximates.

4. Let $f(x) = 1/x$, $x_1 = 1/2$, $x_2 = 1$, $x_3 = 3/2$, $x_4 = 2$, and $\Delta x = 1/2$.

a. Find $\sum_{i=1}^{4} f(x_i)\Delta x$.

b. The sum in part a approximates a definite integral using rectangles. The height of each rectangle is given by the value of the function at the left endpoint. Write the definite integral that the sum approximates.

In Exercises 5–12, approximate the area under the graph of $f(x)$ and above the x-axis using the following methods with $n = 4$. (a) Use left endpoints. (b) Use right endpoints. (c) Average the answers in parts a and b. (d) Use midpoints.

5. $f(x) = 2x + 5$ from $x = 2$ to $x = 4$

6. $f(x) = 3x + 2$ from $x = 1$ to $x = 3$

7. $f(x) = -x^2 + 4$ from $x = -2$ to $x = 2$

8. $f(x) = x^2$ from $x = 1$ to $x = 5$

9. $f(x) = e^x + 1$ from $x = -2$ to $x = 2$

10. $f(x) = e^x - 1$ from $x = 0$ to $x = 4$

11. $f(x) = \dfrac{2}{x}$ from $x = 1$ to $x = 9$

12. $f(x) = \dfrac{1}{x}$ from $x = 1$ to $x = 3$

13. Consider the region below $f(x) = x/2$, above the x-axis, and between $x = 0$ and $x = 4$. Let x_i be the midpoint of the ith subinterval.

a. Approximate the area of the region using four rectangles.

b. Find $\int_0^4 f(x)\,dx$ by using the formula for the area of a triangle.

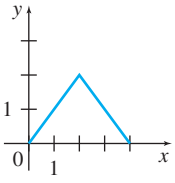
14. Consider the region below $f(x) = 5 - x$, above the x-axis, and between $x = 0$ and $x = 5$. Let x_i be the midpoint of the ith subinterval.

a. Approximate the area of the region using five rectangles.

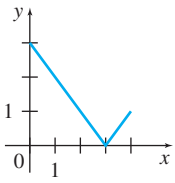
b. Find $\int_0^5 (5 - x)dx$ by using the formula for the area of a triangle.

15. Find $\int_0^4 f(x)\,dx$ for each graph of $y = f(x)$.

a.

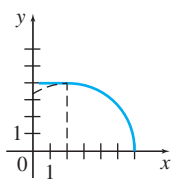


b.

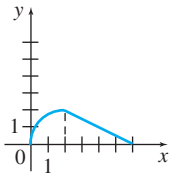


16. Find $\int_0^6 f(x)\,dx$ for each graph of $y = f(x)$, where $f(x)$ consists of line segments and circular arcs.

a.



b.



Find the exact value of each integral using formulas from geometry.

17. $\int_{-4}^{0} \sqrt{16 - x^2}\,dx$

18. $\int_{-3}^{3} \sqrt{9 - x^2}\,dx$

19. $\int_{2}^{5} (1 + 2x)\,dx$

20. $\int_{1}^{3} (5 + x)\,dx$

21. In this exercise, we investigate the value of $\int_0^1 x^2\,dx$ using larger and larger values of n in the definition of the definite integral.

a. First let $n = 10$, so $\Delta x = 0.1$. Fill a list on your calculator with values of x^2 as x goes from 0.1 to 1. (On a TI-84 Plus, use the command `seq(X^2,X,.1,1,.1)→L1`.)

b. Sum the values in the list formed in part a, and multiply by 0.1, to estimate $\int_0^1 x^2\,dx$ with $n = 10$. (On a TI-84 Plus, use the command `.1*sum(L1)`.)

c. Repeat parts a and b with $n = 100$.

d. Repeat parts a and b with $n = 500$.

e. Based on your answers to parts b through d, what do you estimate the value of $\int_0^1 x^2\,dx$ to be?

22. Repeat Exercise 21 for $\int_0^1 x^3\,dx$.

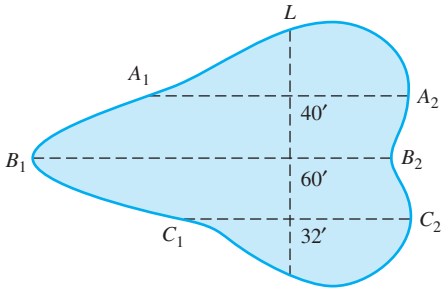
23. The booklet *All About Lawns* published by Ortho Books gives the following instructions for measuring the area of an irregularly shaped region. (See figure in the next column.) ***Source: All About Lawns.***

Irregular Shapes
(within 5% accuracy)
Measure a long (L) axis of the area. Every 10 feet along the length line, measure the width at right angles to the length line. Total widths and multiply by 10.

Area $= (\overline{A_1A_2} + \overline{B_1B_2} + \overline{C_1C_2}$ **etc.**$) \times 10$

$\mathbf{A} = (40' + 60' + 32') \times 10$
$\mathbf{A} = 132' \times 10'$
$\mathbf{A} = 1320$ square feet

How does this method relate to the discussion in this section?

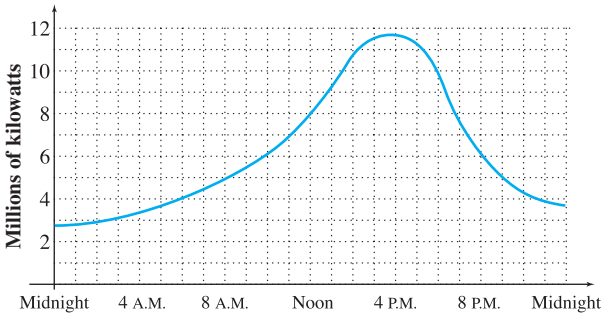


APPLICATIONS

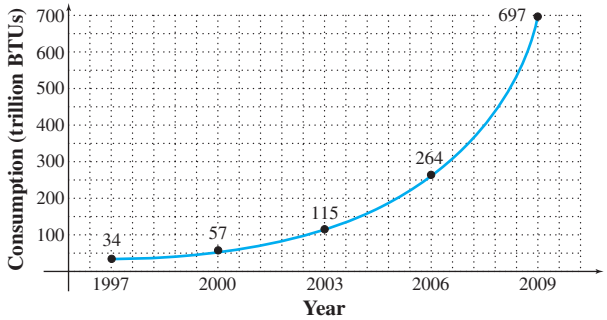
In Exercises 24–28, estimate the area under each curve by summing the area of rectangles. Use the left endpoints, then the right endpoints, then give the average of those answers.

Business and Economics

24. Electricity Consumption The following graph shows the rate of use of electrical energy (in millions of kilowatts) in a certain city on a very hot day. Estimate the total usage of electricity on that day. Let the width of each rectangle be 2 hours.

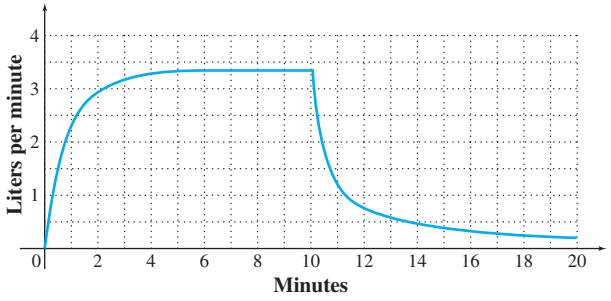


25. Wind Energy Consumption The following graph shows the U.S. wind energy consumption (trillion BTUs) for various years. Estimate the total consumption for the 12-year period from 1997 to 2009 using rectangles of width 3 years. ***Source: Annual Energy Review.***

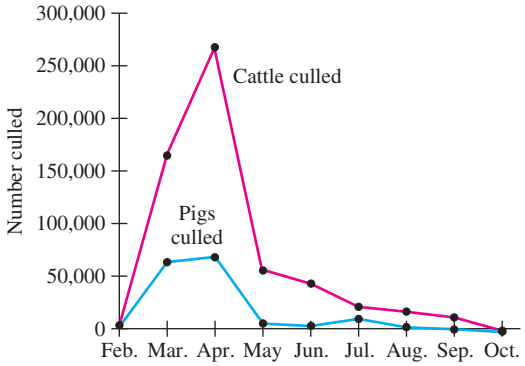


Life Sciences

26. Oxygen Inhalation The graph on the next page shows the rate of inhalation of oxygen (in liters per minute) by a person riding a bicycle very rapidly for 10 minutes. Estimate the total volume of oxygen inhaled in the first 20 minutes after the beginning of the ride. Use rectangles with widths of 1 minute.



27. Foot-and-Mouth Epidemic In 2001, the United Kingdom suffered an epidemic of foot-and-mouth disease. The graph below shows the reported number of cattle (red) and pigs (blue) that were culled each month from mid-February through mid-October in an effort to stop the spread of the disease. ***Source: Department of Environment, Food and Rural Affairs, United Kingdom.***

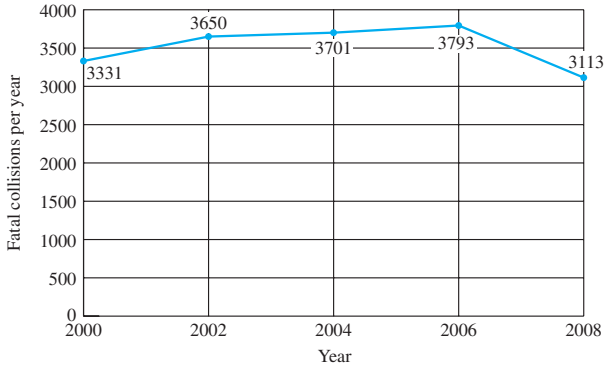


a. Estimate the total number of cattle that were culled from mid-February through mid-October and compare this with 581,801, the actual number of cattle that were culled. Use rectangles that are one month in width, starting with mid-February.

b. Estimate the total number of pigs that were culled from mid-February through mid-October and compare this with 146,145, the actual number of pigs that were culled. Use rectangles that are one month in width starting with mid-February.

Social Sciences

28. Automobile Accidents The graph shows the number of fatal automobile accidents in California for various years. Estimate the total number of accidents in the 8-year period from 2000 to 2008 using rectangles of width 2 years. ***Source: California Highway Patrol.***



Physical Sciences

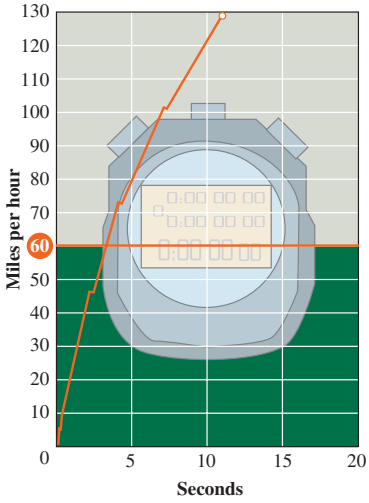
Distance The next two graphs are from the Road & Track website. The curves show the velocity at t seconds after the car accelerates from a dead stop. To find the total distance traveled by the car in reaching 130 mph, we must estimate the definite integral

$$\int_0^T v(t)\,dt,$$

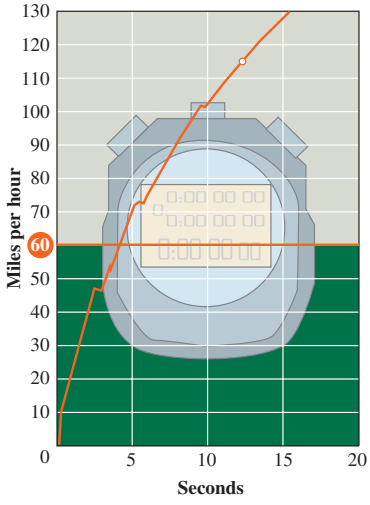
where T represents the number of seconds it takes for the car to reach 130 mph.

Use the graphs to estimate this distance by adding the areas of rectangles and using the midpoint rule. To adjust your answer to miles per hour, divide by 3600 (the number of seconds in an hour). You then have the number of miles that the car traveled in reaching 130 mph. Finally, multiply by 5280 ft per mile to convert the answer to feet. *Source: Road & Track.*

29. Estimate the distance traveled by the Lamborghini Gallardo LP560-4 using the graph below. Use rectangles with widths of 3 seconds, except for the last rectangle, which should have a width of 2 seconds. The circle marks the point where the car has gone a quarter mile. Does this seem correct?



30. Estimate the distance traveled by the Alfa Romeo 8C Competizione using the graph below. Use rectangles with widths of 4 seconds, except for the last rectangle, which should have a width of 3.5 seconds. The circle marks the point where the car has gone a quarter mile. Does this seem correct?



Distance When data are given in tabular form, you may need to vary the size of the interval to calculate the area under the curve. The next two exercises include data from *Car and Driver* magazine. To estimate the total distance traveled by the car (in feet) during the time it took to reach its maximum velocity, estimate the area under the velocity versus time graph, as in the previous two exercises. Use the left endpoint for each time interval (the velocity at the beginning of that interval) and then the right endpoint (the velocity at the end of the interval). Finally, average the two answers together. Calculating and adding up the areas of the rectangles is most easily done on a spreadsheet or graphing calculator. As in the previous two exercises, you will need to multiply by a conversion factor of 5280/3600 = 22/15, since the velocities are given in miles per hour, but the time is in seconds, and we want the answer in feet. *Source: Car and Driver.*

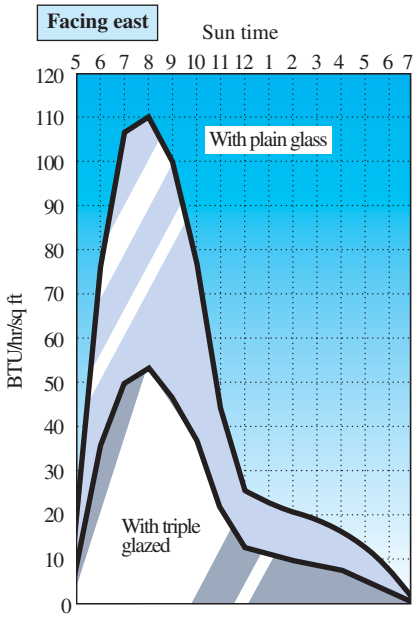
31. Estimate the distance traveled by the Mercedes-Benz S550, using the table below.

Acceleration	Seconds
Zero to 30 mph	2.0
40 mph	2.9
50 mph	4.1
60 mph	5.3
70 mph	6.9
80 mph	8.7
90 mph	10.7
100 mph	13.2
110 mph	16.1
120 mph	19.3
130 mph	23.4

32. Estimate the distance traveled by the Chevrolet Malibu Maxx SS, using the table below.

Acceleration	Seconds
Zero to 30 mph	2.4
40 mph	3.5
50 mph	5.1
60 mph	6.9
70 mph	8.9
80 mph	11.2
90 mph	14.9
100 mph	19.2
110 mph	24.4

Heat Gain The following graphs show the typical heat gain, in BTUs per hour per square foot, for windows (one with plain glass and one that is triple glazed) in Pittsburgh in June, one facing east and one facing south. The horizontal axis gives the time of the day. Estimate the total heat gain per square foot by summing the areas of rectangles. Use rectangles with widths of 2 hours, and let the function value at the midpoint of the subinterval give the height of the rectangle. *Source: Sustainable by Design.*

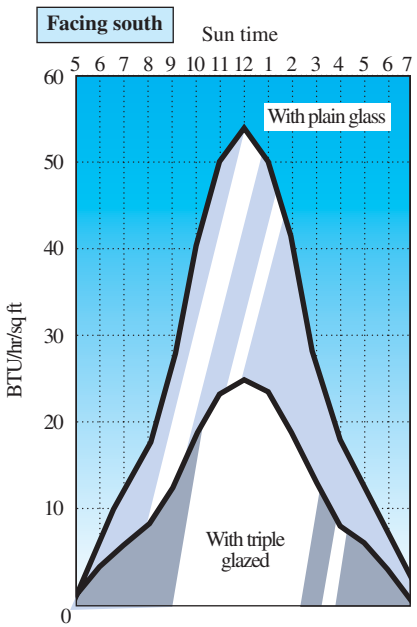


33. a. Estimate the total heat gain per square foot for a plain glass window facing east.

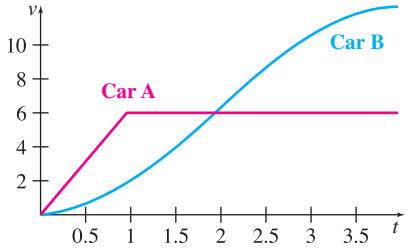
b. Estimate the total heat gain per square foot for a triple glazed window facing east.

34. a. Estimate the total heat gain per square foot for a plain glass window facing south.

b. Estimate the total heat gain per square foot for a triple glazed window facing south.



35. Automobile Velocity Two cars start from rest at a traffic light and accelerate for several minutes. The graph shows their velocities (in feet per second) as a function of time (in seconds). Car A is the one that initially has greater velocity. ***Source: Stephen Monk.***



a. How far has car A traveled after 2 seconds? (*Hint*: Use formulas from geometry.)

b. When is car A farthest ahead of car B?

c. Estimate the farthest that car A gets ahead of car B. For car A, use formulas from geometry. For car B, use $n = 4$ and the value of the function at the midpoint of each interval.

d. Give a rough estimate of when car B catches up with car A.

36. Distance Musk the friendly pit bull has escaped again! Here is her velocity during the first 4 seconds of her romp.

t (sec)	0	1	2	3	4
v (ft/sec)	0	8	13	17	18

Give two estimates for the total distance Musk traveled during her 4-second trip, one using the left endpoint of each interval and one using the right endpoint.

37. Distance The speed of a particle in a test laboratory was noted every second for 3 seconds. The results are shown in the following table. Use the left endpoints and then the right endpoints to estimate the total distance the particle moved in the first three seconds.

t (sec)	0	1	2	3
v (ft/sec)	10	6.5	6	5.5

38. Running In 1987, Canadian Ben Johnson set a world record in the 100-m sprint. (The record was later taken away when he was found to have used an anabolic steroid to enhance his performance.) His speed at various times in the race is given in the following table*. ***Source: Information Graphics***.

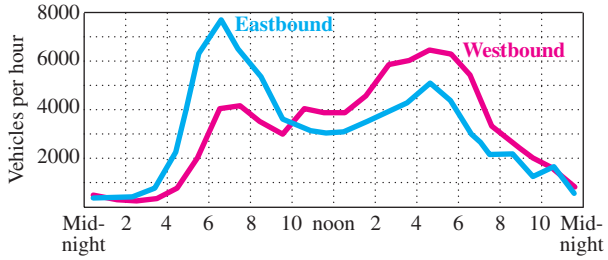
Time (sec)	Speed (mph)
0	0
1.84	12.9
3.80	23.8
6.38	26.3
7.23	26.3
8.96	26.0
9.83	25.7

a. Use the information in the table and left endpoints to estimate the distance that Johnson ran in miles. You will first need to calculate Δt for each interval. At the end, you will need to divide by 3600 (the number of seconds in an hour), since the speed is in miles per hour.

b. Repeat part a, using right endpoints.

c. Wait a minute; we know that the distance Johnson ran is 100 m. Divide this by 1609, the number of meters in a mile, to find how far Johnson ran in miles. Is your answer from part a or part b closer to the true answer? Briefly explain why you think this answer should be more accurate.

39. Traffic The following graph shows the number of vehicles per hour crossing the Tappan Zee Bridge, which spans the Hudson River north of New York City. The graph shows the number of vehicles traveling eastbound (into the city) and westbound (out of the city) as a function of time. ***Source: The New York Times.***



Source: New York Metropolitan Transportation Council

*The world record of 9.58 seconds is currently held by Usain Bolt of Jamaica.

a. Using midpoints on intervals of one hour, estimate the total number of vehicles that cross the bridge going eastbound each day.

b. Repeat the instructions for part a for vehicles going westbound.

c. Discuss whether the answers to parts a and b should be equal, and try to explain any discrepancies.

YOUR TURN ANSWERS

1. 48

2. Left endpoint estimate is 73 miles, right endpoint estimate is 97 miles, and average is 85 miles.

15.4 The Fundamental Theorem of Calculus

APPLY IT **If we know how the rate of consumption of natural gas varies over time, how can we compute the total amount of natural gas used?**

We will answer this question in Example 7.

In the first section of this chapter, you learned about antiderivatives. In the previous section, you learned about the definite integral. In this section, we connect these two separate topics and present one of the most powerful theorems of calculus.

We have seen that, if $f(x) \geq 0$,

$$\int_a^b f(x)\,dx$$

gives the area between the graph of $f(x)$ and the x-axis, from $x = a$ to $x = b$. The definite integral was defined and evaluated in the previous section using the limit of a sum. In that section, we also saw that if $f(x)$ gives the rate of change of $F(x)$, the definite integral $\int_a^b f(x)\,dx$ gives the total change of $F(x)$ as x changes from a to b. If $f(x)$ gives the rate of change of $F(x)$, then $F(x)$ is an antiderivative of $f(x)$. Writing the total change in $F(x)$ from $x = a$ to $x = b$ as $F(b) - F(a)$ shows the connection between antiderivatives and definite integrals. This relationship is called the **Fundamental Theorem of Calculus**.

Fundamental Theorem of Calculus

Let f be continuous on the interval $[a, b]$, and let F be *any* antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

The symbol $F(x)|_a^b$ is used to represent $F(b) - F(a)$. It is important to note that the Fundamental Theorem does not require $f(x) > 0$. The condition $f(x) > 0$ is necessary only when using the Fundamental Theorem to find area. Also, note that the Fundamental Theorem does not *define* the definite integral; it just provides a method for evaluating it.

EXAMPLE 1 Fundamental Theorem of Calculus

First find $\int 4t^3\,dt$ and then find $\int_1^2 4t^3\,dt$.

SOLUTION By the power rule given earlier, the indefinite integral is

$$\int 4t^3\,dt = t^4 + C.$$

YOUR TURN 1 Find $\int_1^3 3x^2 dx$.

By the Fundamental Theorem, the value of the definite integral $\int_1^2 4t^3\,dt$ is found by evaluating $t^4\big|_1^2$, with no constant C required.

$$\int_1^2 4t^3\,dt = t^4\Big|_1^2 = 2^4 - 1^4 = 15$$ TRY YOUR TURN 1

Example 1 illustrates the difference between the definite integral and the indefinite integral. A definite integral is a real number; an indefinite integral is a family of functions in which all the functions are antiderivatives of a function f.

NOTE No constant C is needed, as it is for the indefinite integral, because even if C were added to an antiderivative F, it would be eliminated in the final answer:

$$\begin{aligned}\int_a^b f(x)\,dx &= (F(x) + C)\Big|_a^b \\ &= (F(b) + C) - (F(a) + C) \\ &= F(b) - F(a).\end{aligned}$$

In other words, any antiderivative will give the same answer, so for simplicity, we choose the one with $C = 0$.

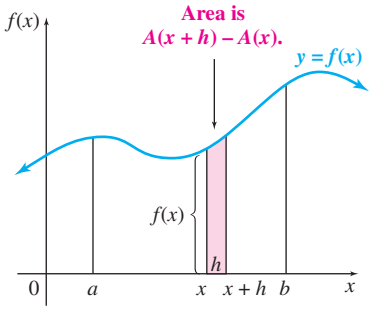


FIGURE 16

To see why the Fundamental Theorem of Calculus is true for $f(x) > 0$ when f is continuous, look at Figure 16. Define the function $A(x)$ as the area between the x-axis and the graph of $y = f(x)$ from a to x. We first show that A is an antiderivative of f; that is $A'(x) = f(x)$.

To do this, let h be a small positive number. Then $A(x + h) - A(x)$ is the shaded area in Figure 16. This area can be approximated with a rectangle having width h and height $f(x)$. The area of the rectangle is $h \cdot f(x)$, and

$$A(x + h) - A(x) \approx h \cdot f(x).$$

Dividing both sides by h gives

$$\frac{A(x + h) - A(x)}{h} \approx f(x).$$

This approximation improves as h gets smaller and smaller. Taking the limit on the left as h approaches 0 gives an exact result.

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x)$$

This limit is simply $A'(x)$, so

$$A'(x) = f(x).$$

This result means that A is an antiderivative of f, as we set out to show.

$A(b)$ is the area under the curve from a to b, and $A(a) = 0$, so the area under the curve can be written as $A(b) - A(a)$. From the previous section, we know that the area under the curve is also given by $\int_a^b f(x)\,dx$. Putting these two results together gives

$$\begin{aligned}\int_a^b f(x)\,dx &= A(b) - A(a) \\ &= A(x)\Big|_a^b\end{aligned}$$

where A is an antiderivative of f. From the note after Example 1, we know that any antiderivative will give the same answer, which proves the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus certainly deserves its name, which sets it apart as the most important theorem of calculus. It is the key connection between differential

calculus and integral calculus, which originally were developed separately without knowledge of this connection between them.

The variable used in the integrand does not matter; each of the following definite integrals represents the number $F(b) - F(a)$.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Key properties of definite integrals are listed below. Some of them are just restatements of properties from Section 1.

Properties of Definite Integrals

If all indicated definite integrals exist,

1. $\displaystyle\int_a^a f(x)\,dx = 0$;
2. $\displaystyle\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$ for any real constant k (constant multiple of a function);
3. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (sum or difference of functions);
4. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any real number c;
5. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

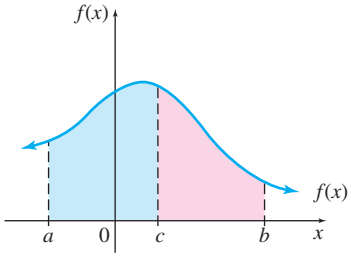


FIGURE 17

For $f(x) \geq 0$, since the distance from a to a is 0, the first property says that the "area" under the graph of f bounded by $x = a$ and $x = a$ is 0. Also, since $\int_a^c f(x)\,dx$ represents the blue region in Figure 17 and $\int_c^b f(x)\,dx$ represents the pink region,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

as stated in the fourth property. While the figure shows $a < c < b$, the property is true for any value of c where both $f(x)$ and $F(x)$ are defined.

An algebraic proof is given here for the third property; proofs of the other properties are left for the exercises. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively,

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= [F(x) + G(x)]\Big|_a^b \\ &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

EXAMPLE 2 Fundamental Theorem of Calculus

Find $\int_2^5 (6x^2 - 3x + 5)\,dx$.

SOLUTION Use the properties above and the Fundamental Theorem, along with properties from Section 1.

$$\begin{aligned}\int_2^5 (6x^2 - 3x + 5)\,dx &= 6\int_2^5 x^2\,dx - 3\int_2^5 x\,dx + 5\int_2^5 dx \\ &= 2x^3\Big|_2^5 - \frac{3}{2}x^2\Big|_2^5 + 5x\Big|_2^5 \\ &= 2(5^3 - 2^3) - \frac{3}{2}(5^2 - 2^2) + 5(5 - 2) \\ &= 2(125 - 8) - \frac{3}{2}(25 - 4) + 5(3) \\ &= 234 - \frac{63}{2} + 15 = \frac{435}{2}\end{aligned}$$

YOUR TURN 2

Find $\int_3^5 (2x^3 - 3x + 4)\,dx$.

TRY YOUR TURN 2

EXAMPLE 3 Fundamental Theorem of Calculus

YOUR TURN 3

Find $\int_1^3 \frac{2}{y}\,dy$.

$$\begin{aligned}\int_1^2 \frac{dy}{y} &= \ln|y|\Big|_1^2 = \ln|2| - \ln|1| \\ &= \ln 2 - \ln 1 \approx 0.6931 - 0 = 0.6931\end{aligned}$$

TRY YOUR TURN 3

EXAMPLE 4 Substitution

Evaluate $\int_0^5 x\sqrt{25 - x^2}\,dx$.

SOLUTION

Method 1 Changing the Limits

Use substitution. Let $u = 25 - x^2$, so that $du = -2x\,dx$. With a definite integral, the limits should be changed, too. The new limits on u are found as follows.

If $x = 5$, then $u = 25 - 5^2 = 0$.
If $x = 0$, then $u = 25 - 0^2 = 25$.

Then

$$\begin{aligned}\int_0^5 x\sqrt{25 - x^2}\,dx &= -\frac{1}{2}\int_0^5 \sqrt{25 - x^2}(-2x\,dx) && \text{Multiply by } -2/-2. \\ &= -\frac{1}{2}\int_{25}^0 \sqrt{u}\,du && \text{Substitute and change limits.} \\ &= -\frac{1}{2}\int_{25}^0 u^{1/2}\,du && \sqrt{u} = u^{1/2} \\ &= -\frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\Big|_{25}^0 && \text{Use the power rule.} \\ &= -\frac{1}{2}\cdot\frac{2}{3}[0^{3/2} - 25^{3/2}] \\ &= -\frac{1}{3}(-125) = \frac{125}{3}.\end{aligned}$$

Method 2
Evaluating the Antiderivative

An alternative method that some people prefer is to evaluate the antiderivative first and then calculate the definite integral. To evaluate the antiderivative in this example, ignore the limits on the original integral and use the substitution $u = 25 - x^2$, so that $du = -2x\,dx$. Then

$$\begin{aligned}\int x\sqrt{25 - x^2}\,dx &= -\frac{1}{2}\int \sqrt{25 - x^2}(-2x\,dx)\\ &= -\frac{1}{2}\int \sqrt{u}\,du\\ &= -\frac{1}{2}\int u^{1/2}\,du\\ &= -\frac{1}{2}\frac{u^{3/2}}{3/2} + C\\ &= -\frac{u^{3/2}}{3} + C\\ &= -\frac{(25 - x^2)^{3/2}}{3} + C.\end{aligned}$$

We will ignore the constant C because it doesn't affect the answer, as we mentioned in the Note following Example 1.

Then, using the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_0^5 x\sqrt{25 - x^2}\,dx &= -\frac{(25 - x^2)^{3/2}}{3}\Bigg|_0^5\\ &= 0 - \left[-\frac{(25)^{3/2}}{3}\right]\\ &= \frac{125}{3}.\end{aligned}$$

YOUR TURN 4 Evaluate $\int_0^4 2x\sqrt{16 - x^2}\,dx$.

TRY YOUR TURN 4

CAUTION Don't confuse these two methods. In Method 1, we never return to the original variable or the original limits of integration. In Method 2, it is essential to return to the original variable and to not change the limits. When using Method 1, we recommend labeling the limits with the appropriate variable to avoid confusion, so the substitution in Example 4 becomes

$$\int_{x=0}^{x=5} x\sqrt{25 - x^2}\,dx = -\frac{1}{2}\int_{u=25}^{u=0} \sqrt{u}\,du.$$

The Fundamental Theorem of Calculus is a powerful tool, but it has a limitation. The problem is that not every function has an antiderivative in terms of the functions and operations you have seen so far. One example of an integral that cannot be evaluated by the Fundamental Theorem of Calculus for this reason is

$$\int_a^b e^{-x^2/2}\,dx,$$

yet this integral is crucial in probability and statistics. Such integrals may be evaluated by numerical integration, which is covered in the last section of this chapter. Fortunately for you, all the integrals in this section can be antidifferentiated using the techniques presented in the first two sections of this chapter.

Area

In the previous section we saw that, if $f(x) \geq 0$ in $[a, b]$, the definite integral $\int_a^b f(x)\,dx$ gives the area below the graph of the function $y = f(x)$, above the x-axis, and between the lines $x = a$ and $x = b$.

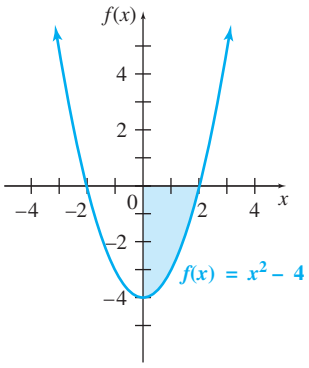


FIGURE 18

To see how to work around the requirement that $f(x) \geq 0$, look at the graph of $f(x) = x^2 - 4$ in Figure 18. The area bounded by the graph of f, the x-axis, and the vertical lines $x = 0$ and $x = 2$ lies below the x-axis. Using the Fundamental Theorem gives

$$\int_0^2 (x^2 - 4)\,dx = \left(\frac{x^3}{3} - 4x\right)\Big|_0^2$$
$$= \left(\frac{8}{3} - 8\right) - (0 - 0) = -\frac{16}{3}.$$

The result is a negative number because $f(x)$ is negative for values of x in the interval $[0, 2]$. Since Δx is always positive, if $f(x) < 0$ the product $f(x) \cdot \Delta x$ is negative, so $\int_0^2 f(x)\,dx$ is negative. Since area is nonnegative, the required area is given by $|-16/3|$ or $16/3$. Using a definite integral, the area could be written as

$$\left|\int_0^2 (x^2 - 4)\,dx\right| = \left|-\frac{16}{3}\right| = \frac{16}{3}.$$

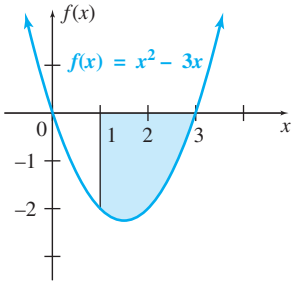


FIGURE 19

EXAMPLE 5 Area

Find the area of the region between the x-axis and the graph of $f(x) = x^2 - 3x$ from $x = 1$ to $x = 3$.

SOLUTION The region is shown in Figure 19. Since the region lies below the x-axis, the area is given by

$$\left|\int_1^3 (x^2 - 3x)\,dx\right|.$$

By the Fundamental Theorem,

$$\int_1^3 (x^2 - 3x)\,dx = \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_1^3 = \left(\frac{27}{3} - \frac{27}{2}\right) - \left(\frac{1}{3} - \frac{3}{2}\right) = -\frac{10}{3}.$$

The required area is $|-10/3| = 10/3$.

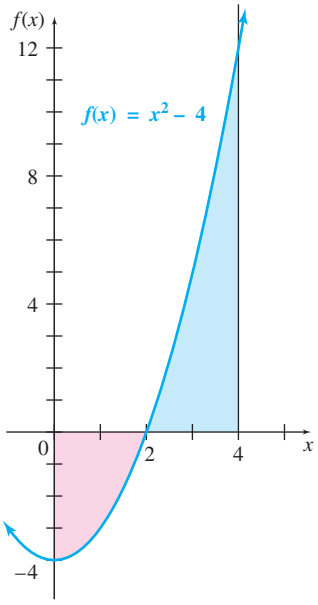


FIGURE 20

EXAMPLE 6 Area

Find the area between the x-axis and the graph of $f(x) = x^2 - 4$ from $x = 0$ to $x = 4$.

SOLUTION Figure 20 shows the required region. Part of the region is below the x-axis. The definite integral over that interval will have a negative value. To find the area, integrate the negative and positive portions separately and take the absolute value of the first result before combining the two results to get the total area. Start by finding the point where the graph crosses the x-axis. This is done by solving the equation

$$x^2 - 4 = 0.$$

The solutions of this equation are 2 and -2. The only solution in the interval $[0, 4]$ is 2. The total area of the region in Figure 20 is

$$\left|\int_0^2 (x^2 - 4)\,dx\right| + \int_2^4 (x^2 - 4)\,dx = \left|\left(\frac{1}{3}x^3 - 4x\right)\Big|_0^2\right| + \left(\frac{1}{3}x^3 - 4x\right)\Big|_2^4$$
$$= \left|\frac{8}{3} - 8\right| + \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)$$
$$= 16.$$

YOUR TURN 5 Repeat Example 6 for the function $f(x) = x^2 - 9$ from $x = 0$ to $x = 6$.

TRY YOUR TURN 5

Incorrectly using one integral over the entire interval to find the area in Example 6 would have given

$$\int_0^4 (x^2 - 4)\,dx = \left(\frac{x^3}{3} - 4x\right)\Big|_0^4 = \left(\frac{64}{3} - 16\right) - 0 = \frac{16}{3},$$

which is not the correct area. This definite integral does not represent any area but is just a real number.

For instance, if $f(x)$ in Example 6 represents the annual rate of profit of a company, then $16/3$ represents the total profit for the company over a 4-year period. The integral between 0 and 2 is $-16/3$; the negative sign indicates a loss for the first two years. The integral between 2 and 4 is $32/3$, indicating a profit. The overall profit is $32/3 - 16/3 = 16/3$, although the total shaded area is $32/3 + |-16/3| = 16$.

Finding Area

In summary, to find the area bounded by $f(x)$, $x = a$, $x = b$, and the x-axis, use the following steps.

1. Sketch a graph.
2. Find any x-intercepts of $f(x)$ in $[a, b]$. These divide the total region into subregions.
3. The definite integral will be *positive* for subregions above the x-axis and *negative* for subregions below the x-axis. Use separate integrals to find the (positive) areas of the subregions.
4. The total area is the sum of the areas of all of the subregions.

In the last section, we saw that the area under a rate of change function $f'(x)$ from $x = a$ to $x = b$ gives the total value of $f(x)$ on $[a, b]$. Now we can use the definite integral to solve these problems.

EXAMPLE 7 Natural Gas Consumption

The yearly rate of consumption of natural gas (in trillions of cubic feet) for a certain city is

$$C'(t) = t + e^{0.01t},$$

where t is time in years and $t = 0$ corresponds to 2000. At this consumption rate, what was the total amount the city used in the 10-year period of 2000 to 2010?

APPLY IT

SOLUTION To find the consumption over the 10-year period from 2000 to 2010, use the definite integral.

$$\begin{aligned}\int_0^{10} (t + e^{0.01t})\,dt &= \left(\frac{t^2}{2} + \frac{e^{0.01t}}{0.01}\right)\Big|_0^{10} \\ &= (50 + 100e^{0.1}) - (0 + 100) \\ &\approx -50 + 100(1.10517) \approx 60.5\end{aligned}$$

Therefore, a total of about 60.5 trillion ft^3 of natural gas was used from 2000 to 2010 at this consumption rate.

15.4 EXERCISES

Evaluate each definite integral.

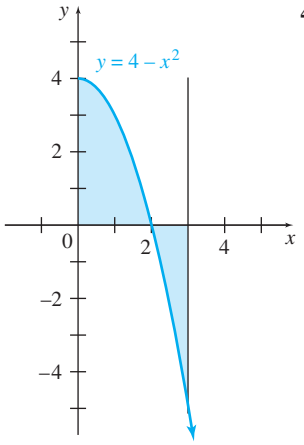
1. $\int_{-2}^{4} (-3)\,dp$
2. $\int_{-4}^{1} \sqrt{2}\,dx$
3. $\int_{-1}^{2} (5t - 3)\,dt$
4. $\int_{-2}^{2} (4z + 3)\,dz$
5. $\int_{0}^{2} (5x^2 - 4x + 2)\,dx$
6. $\int_{-2}^{3} (-x^2 - 3x + 5)\,dx$
7. $\int_{0}^{2} 3\sqrt{4u + 1}\,du$
8. $\int_{3}^{9} \sqrt{2r - 2}\,dr$
9. $\int_{0}^{4} 2(t^{1/2} - t)\,dt$
10. $\int_{0}^{4} -(3x^{3/2} + x^{1/2})\,dx$
11. $\int_{1}^{4} (5y\sqrt{y} + 3\sqrt{y})\,dy$
12. $\int_{4}^{9} (4\sqrt{r} - 3r\sqrt{r})\,dr$
13. $\int_{4}^{6} \frac{2}{(2x - 7)^2}\,dx$
14. $\int_{1}^{4} \frac{-3}{(2p + 1)^2}\,dp$
15. $\int_{1}^{5} (6n^{-2} - n^{-3})\,dn$
16. $\int_{2}^{3} (3x^{-3} - 5x^{-4})\,dx$
17. $\int_{-3}^{-2} \left(2e^{-0.1y} + \frac{3}{y}\right) dy$
18. $\int_{-2}^{-1} \left(\frac{-2}{t} + 3e^{0.3t}\right) dt$
19. $\int_{1}^{2} \left(e^{4u} - \frac{1}{(u + 1)^2}\right) du$
20. $\int_{0.5}^{1} (p^3 - e^{4p})\,dp$
21. $\int_{-1}^{0} y(2y^2 - 3)^5\,dy$
22. $\int_{0}^{3} m^2(4m^3 + 2)^3\,dm$
23. $\int_{1}^{64} \frac{\sqrt{z} - 2}{\sqrt[3]{z}}\,dz$
24. $\int_{1}^{8} \frac{3 - y^{1/3}}{y^{2/3}}\,dy$
25. $\int_{1}^{2} \frac{\ln x}{x}\,dx$
26. $\int_{1}^{3} \frac{\sqrt{\ln x}}{x}\,dx$
27. $\int_{0}^{8} x^{1/3}\sqrt{x^{4/3} + 9}\,dx$
28. $\int_{1}^{2} \frac{3}{x(1 + \ln x)}\,dx$
29. $\int_{0}^{1} \frac{e^{2t}}{(3 + e^{2t})^2}\,dt$
30. $\int_{0}^{1} \frac{e^{2z}}{\sqrt{1 + e^{2z}}}\,dz$

In Exercises 31–40, use the definite integral to find the area between the x-axis and $f(x)$ over the indicated interval. Check first to see if the graph crosses the x-axis in the given interval.

31. $f(x) = 2x - 14;\quad [6, 10]$
32. $f(x) = 4x - 32;\quad [5, 10]$
33. $f(x) = 2 - 2x^2;\quad [0, 5]$
34. $f(x) = 9 - x^2;\quad [0, 6]$
35. $f(x) = x^3;\quad [-1, 3]$
36. $f(x) = x^3 - 2x;\quad [-2, 4]$
37. $f(x) = e^x - 1;\quad [-1, 2]$
38. $f(x) = 1 - e^{-x};\quad [-1, 2]$
39. $f(x) = \frac{1}{x} - \frac{1}{e};\quad [1, e^2]$
40. $f(x) = 1 - \frac{1}{x};\quad [e^{-1}, e]$

Find the area of each shaded region.

41.



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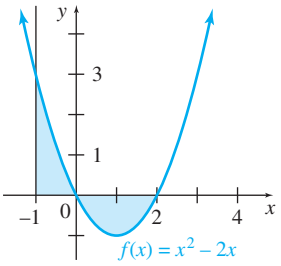

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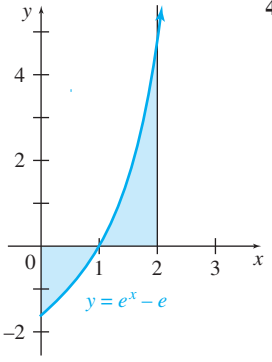
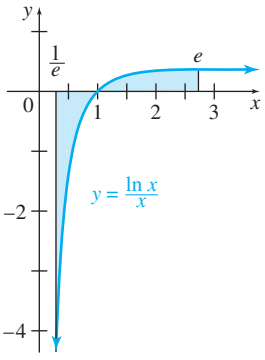
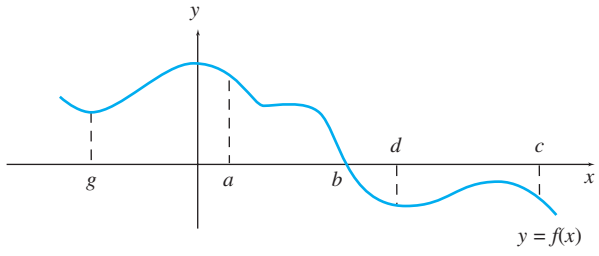


44.



45. Assume $f(x)$ is continuous for $g \le x \le c$ as shown in the figure. Write an equation relating the three quantities

$$\int_a^b f(x)\,dx, \qquad \int_a^c f(x)\,dx, \qquad \int_b^c f(x)\,dx.$$

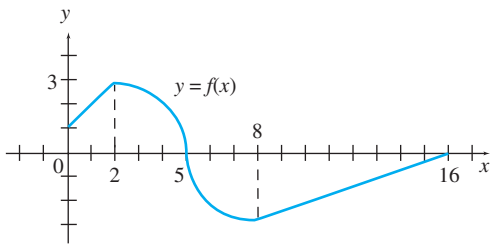


46. Is the equation you wrote for Exercise 45 still true

a. if b is replaced by d?

b. if b is replaced by g?

47. The graph of $f(x)$, shown here, consists of two straight line segments and two quarter circles. Find the value of $\int_0^{16} f(x)\,dx$.



Use the Fundamental Theorem to show that the following are true.

48. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

49. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

50. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

51. Use Exercise 49 to find $\int_{-1}^{4} f(x)\,dx$, given

$$f(x) = \begin{cases} 2x + 3 & \text{if } x \le 0 \\ -\dfrac{x}{4} - 3 & \text{if } x > 0. \end{cases}$$

52. You are given $\int_0^1 e^{x^2}\,dx = 1.46265$ and $\int_0^2 e^{x^2}\,dx = 16.45263$. Use this information to find

a. $\int_{-1}^{1} e^{x^2}\,dx$; **b.** $\int_1^2 e^{x^2}\,dx$.

53. Let $g(t) = t^4$ and define $f(x) = \int_c^x g(t)\,dt$ with $c = 1$.

a. Find a formula for $f(x)$.

b. Verify that $f'(x) = g(x)$. The fact that

$$\frac{d}{dx}\int_c^x g(t)dt = g(x)$$

is true for all continuous functions g is an alternative version of the Fundamental Theorem of Calculus.

c. Let us verify the result in part b for a function whose antiderivative cannot be found. Let $g(t) = e^{t^2}$ and let $c = 0$. Use the integration feature on a graphing calculator to find $f(x)$ for $x = 1$ and $x = 1.01$. Then use the definition of the derivative with $h = 0.01$ to approximate $f'(1)$, and compare it with $g(1)$.

54. Consider the function $f(x) = x(x^2 + 3)^7$.

a. Use the Fundamental Theorem of Calculus to evaluate $\int_{-5}^{5} f(x)\,dx$.

b. Use symmetry to describe how the integral from part a could be evaluated without using substitution or finding an antiderivative.

APPLICATIONS

Business and Economics

55. Profit Karla Harby Communications, a small company of science writers, found that its rate of profit (in thousands of dollars) after t years of operation is given by

$$P'(t) = (3t + 3)(t^2 + 2t + 2)^{1/3}.$$

a. Find the total profit in the first three years.

b. Find the profit in the fourth year of operation.

c. What is happening to the annual profit over the long run?

56. Worker Efficiency A worker new to a job will improve his efficiency with time so that it takes him fewer hours to produce an item with each day on the job, up to a certain point. Suppose the rate of change of the number of hours it takes a worker in a certain factory to produce the xth item is given by

$$H'(x) = 20 - 2x.$$

a. What is the total number of hours required to produce the first 5 items?

b. What is the total number of hours required to produce the first 10 items?

Life Sciences

57. Pollution Pollution from a factory is entering a lake. The rate of concentration of the pollutant at time t is given by

$$P'(t) = 140t^{5/2},$$

where t is the number of years since the factory started introducing pollutants into the lake. Ecologists estimate that the lake can accept a total level of pollution of 4850 units before all the fish life in the lake ends. Can the factory operate for 4 years without killing all the fish in the lake?

58. Spread of an Oil Leak An oil tanker is leaking oil at the rate given (in barrels per hour) by

$$L'(t) = \frac{80\ln(t + 1)}{t + 1},$$

where t is the time (in hours) after the tanker hits a hidden rock (when $t = 0$).

a. Find the total number of barrels that the ship will leak on the first day.

b. Find the total number of barrels that the ship will leak on the second day.

c. What is happening over the long run to the amount of oil leaked per day?

59. Tree Growth After long study, tree scientists conclude that a eucalyptus tree will grow at the rate of $0.6 + 4/(t + 1)^3$ ft per year, where t is time (in years).

a. Find the number of feet that the tree will grow in the second year.

b. Find the number of feet the tree will grow in the third year.

60. Growth of a Substance The rate at which a substance grows is given by

$$R'(x) = 150e^{0.2x},$$

where x is the time (in days). What is the total accumulated growth during the first 3.5 days?

61. Drug Reaction For a certain drug, the rate of reaction in appropriate units is given by

$$R'(t) = \frac{5}{t+1} + \frac{2}{\sqrt{t+1}},$$

where t is time (in hours) after the drug is administered. Find the total reaction to the drug over the following time periods.

a. From $t = 1$ to $t = 12$ **b.** From $t = 12$ to $t = 24$

62. Human Mortality If $f(x)$ is the instantaneous death rate for members of a population at time x, then the number of individuals who survive to age T is given by

$$F(T) = \int_0^T f(x)\,dx.$$

In 1825 the biologist Benjamin Gompertz proposed that $f(x) = kb^x$. Find a formula for $F(T)$. ***Source: Philosophical Transactions of the Royal Society of London.***

63. Cell Division Let the expected number of cells in a culture that have an x percent probability of undergoing cell division during the next hour be denoted by $n(x)$.

a. Explain why $\int_{20}^{30} n(x)\,dx$ approximates the total number of cells with a 20% to 30% chance of dividing during the next hour.

b. Give an integral representing the number of cells that have less than a 60% chance of dividing during the next hour.

c. Let $n(x) = \sqrt{5x+1}$ give the expected number of cells (in millions) with x percent probability of dividing during the next hour. Find the number of cells with a 5 to 10% chance of dividing.

64. Bacterial Growth A population of *E. coli* bacteria will grow at a rate given by

$$w'(t) = (3t+2)^{1/3},$$

where w is the weight (in milligrams) after t hours. Find the change in weight of the population from $t = 0$ to $t = 3$.

65. Blood Flow In an example from an earlier chapter, the velocity v of the blood in a blood vessel was given as

$$v = k(R^2 - r^2),$$

where R is the (constant) radius of the blood vessel, r is the distance of the flowing blood from the center of the blood vessel, and k is a constant. Total blood flow (in millimeters per minute) is given by

$$Q(R) = \int_0^R 2\pi v r\,dr.$$

a. Find the general formula for Q in terms of R by evaluating the definite integral given above.

b. Evaluate $Q(0.4)$.

66. Rams' Horns The average annual increment in the horn length (in centimeters) of bighorn rams born since 1986 can be approximated by

$$y = 0.1762x^2 - 3.986x + 22.68,$$

where x is the ram's age (in years) for x between 3 and 9. Integrate to find the total increase in the length of a ram's horn during this time. ***Source: Journal of Wildlife Management.***

67. Beagles The daily energy requirements of female beagles who are at least 1 year old change with respect to time according to the function

$$E(t) = 753t^{-0.1321},$$

where $E(t)$ is the daily energy requirement (in kJ/$W^{0.67}$), where W is the dog's weight (in kilograms) for a beagle that is t years old. ***Source: Journal of Nutrition.***

a. Assuming 365 days in a year, show that the energy requirement for a female beagle that is t days old is given by

$$E(t) = 1642t^{-0.1321}.$$

b. Using the formula from part a, determine the total energy requirements (in kJ/$W^{0.67}$) for a female beagle between her first and third birthday.

68. Sediment The density of sediment (in grams per cubic centimeter) at the bottom of Lake Coeur d'Alene, Idaho, is given by

$$p(x) = p_0 e^{0.0133x},$$

where x is the depth (in centimeters) and p_0 is the density at the surface. The total mass of a square-centimeter column of sediment above a depth of h cm is given by

$$\int_0^h p(x)\,dx.$$

If $p_0 = 0.85$ g per cm^3, find the total mass above a depth of 100 cm. ***Source: Mathematics Teacher.***

Social Sciences

69. Age Distribution The U.S. Census Bureau gives an age distribution that is approximately modeled (in millions) by the function

$$f(x) = 40.2 + 3.50x - 0.897x^2$$

where x varies from 0 to 9 decades. The population of a given age group can be found by integrating this function over the interval for that age group. ***Source: Ralph DeMarr and the U.S. Census Bureau.***

a. Find the integral of $f(x)$ over the interval [0, 9]. What does this integral represent?

b. Baby boomers are those born between 1945 and 1965, that is, those in the range of 4.5 to 6.5 decades in 2010. Estimate the number of baby boomers.

70. Income Distribution Based on data from the U.S. Census Bureau, an approximate family income distribution for the United States is given by the function

$$f(x) = 0.0353x^3 - 0.541x^2 + 3.78x + 4.29,$$

where x is the annual income in units of \$10,000, with $0 \le x \le 10$. For example, $x = 0.5$ represents an annual family income of \$5000. (*Note*: This function does not give a good representation for family incomes over \$100,000.) The percent of the families with an income in a given range can be found by integrating this function over that range. Find the percentage of families with an income between \$25,000 and \$50,000. ***Source: Ralph DeMarr and the U.S. Census Bureau.***

Physical Sciences

71. Oil Consumption Suppose that the rate of consumption of a natural resource is $c'(t)$, where

$$c'(t) = ke^{rt}.$$

Here t is time in years, r is a constant, and k is the consumption in the year when $t = 0$. In 2010, an oil company sold 1.2 billion barrels of oil. Assume that $r = 0.04$.

a. Write $c'(t)$ for the oil company, letting $t = 0$ represent 2010.

b. Set up a definite integral for the amount of oil that the company will sell in the next 10 years.

c. Evaluate the definite integral of part b.

d. The company has about 20 billion barrels of oil in reserve. To find the number of years that this amount will last, solve the equation

$$\int_0^T 1.2e^{0.04t}\,dt = 20.$$

e. Rework part d, assuming that $r = 0.02$.

72. Oil Consumption The rate of consumption of oil (in billions of barrels) by the company in Exercise 71 was given as

$$1.2e^{0.04t},$$

where $t = 0$ corresponds to 2010. Find the total amount of oil used by the company from 2010 to year T. At this rate, how much will be used in 5 years?

YOUR TURN ANSWERS

1. 26
2. 256
3. 2 ln 3 or ln 9
4. 128/3
5. 54

15.5 The Area Between Two Curves

APPLY IT **If an executive knows how the savings from a new manufacturing process decline over time and how the costs of that process will increase, how can she compute when the net savings will cease and what the total savings will be?**
We will answer this question in Example 4.

Many important applications of integrals require finding the area between two graphs. The method used in previous sections to find the area between the graph of a function and the x-axis from $x = a$ to $x = b$ can be generalized to find such an area. For example, the area between the graphs of $f(x)$ and $g(x)$ from $x = a$ to $x = b$ in Figure 21(a) is the same as the area under the graph of $f(x)$, shown in Figure 21(b), minus the area under the graph of $g(x)$ (see Figure 21(c)). That is, the area between the graphs is given by

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx,$$

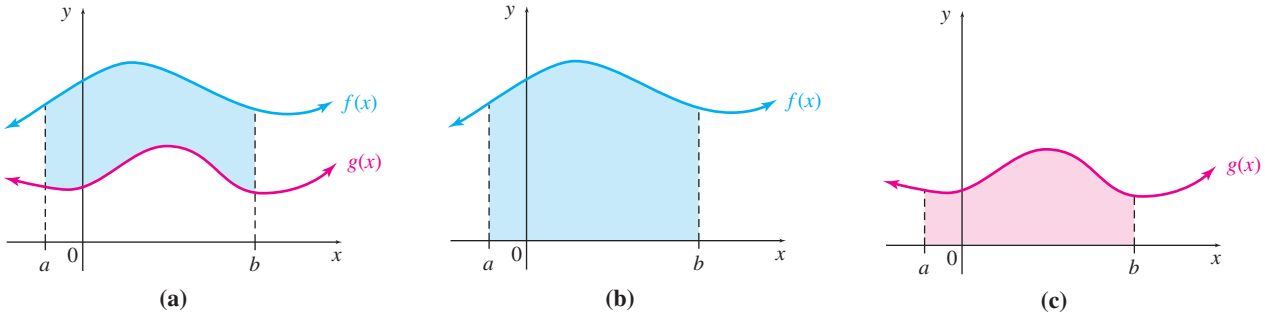


FIGURE 21

which can be written as

$$\int_a^b [f(x) - g(x)]\,dx.$$

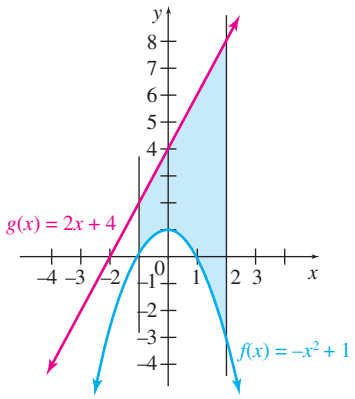


FIGURE 22

Area Between Two Curves

If f and g are continuous functions and $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b [f(x) - g(x)]\,dx.$$

EXAMPLE 1 Area

Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$.

SOLUTION A sketch of the four equations is shown in Figure 22. In general, it is not necessary to spend time drawing a detailed sketch, but only to know whether the two functions intersect, and which function is greater between the intersections. To find out, set the two functions equal.

$$-x^2 + 1 = 2x + 4$$
$$0 = x^2 + 2x + 3$$

Verify by the quadratic formula that this equation has no real roots. Since the graph of f is a parabola opening downward that does not cross the graph of g (a line), the parabola must be entirely under the line, as shown in Figure 22. Therefore $g(x) \geq f(x)$ for x in the interval $[-1, 2]$, and the area is given by

$$\begin{aligned}
\int_{-1}^{2} [g(x) - f(x)]\,dx &= \int_{-1}^{2} [(2x + 4) - (-x^2 + 1)]\,dx \\
&= \int_{-1}^{2} (2x + 4 + x^2 - 1)\,dx \\
&= \int_{-1}^{2} (x^2 + 2x + 3)\,dx \\
&= \frac{x^3}{3} + x^2 + 3x \Big|_{-1}^{2} \\
&= \left(\frac{8}{3} + 4 + 6\right) - \left(\frac{-1}{3} + 1 - 3\right) \\
&= \frac{8}{3} + 10 + \frac{1}{3} + 2 \\
&= 15.
\end{aligned}$$

YOUR TURN 1 Repeat Example 1 for $f(x) = 4 - x^2$, $g(x) = x + 2$, $x = -2$, and $x = 1$.

TRY YOUR TURN 1

NOTE It is not necessary to draw the graphs to determine which function is greater. Since the functions in the previous example do not intersect, we can evaluate them at *any* point to make this determination. For example, $f(0) = 1$ and $g(0) = 4$. Because $g(x) > f(x)$ at $x = 4$, and the two functions are continuous and never intersect, $g(x) > f(x)$ for all x.

EXAMPLE 2 Area

Find the area between the curves $y = x^{1/2}$ and $y = x^3$.

SOLUTION Let $f(x) = x^{1/2}$ and $g(x) = x^3$. As before, set the two equal to find where they intersect.

$$\begin{aligned} x^{1/2} &= x^3 \\ 0 &= x^3 - x^{1/2} \\ 0 &= x^{1/2}(x^{5/2} - 1) \end{aligned}$$

The only solutions are $x = 0$ and $x = 1$. Verify that the graph of f is concave downward, while the graph of g is concave upward, so the graph of f must be greater between 0 and 1. (This may also be verified by taking a point between 0 and 1, such as 0.5, and verifying that $0.5^{1/2} > 0.5^3$.) The graph is shown in Figure 23.

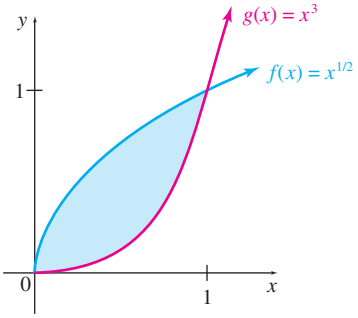


FIGURE 23

The area between the two curves is given by

$$\int_a^b [f(x) - g(x)]\,dx = \int_0^1 (x^{1/2} - x^3)\,dx.$$

Using the Fundamental Theorem,

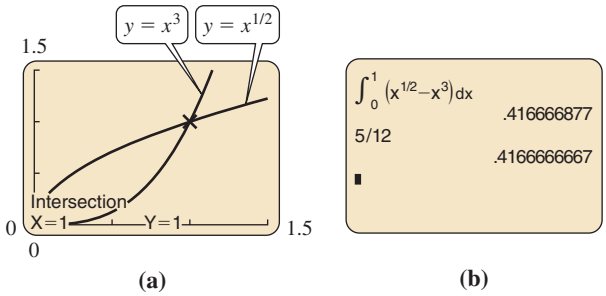
$$\begin{aligned} \int_0^1 (x^{1/2} - x^3)\,dx &= \left(\frac{x^{3/2}}{3/2} - \frac{x^4}{4}\right)\Bigg|_0^1 \\ &= \left(\frac{2}{3}x^{3/2} - \frac{x^4}{4}\right)\Bigg|_0^1 \\ &= \frac{2}{3}(1) - \frac{1}{4} \\ &= \frac{5}{12}. \end{aligned}$$

YOUR TURN 2 Repeat Example 2 for $y = x^{1/4}$ and $y = x^2$.

TRY YOUR TURN 2

TECHNOLOGY NOTE

A graphing calculator is very useful in approximating solutions of problems involving the area between two curves. First, it can be used to graph the functions and identify any intersection points. Then it can be used to approximate the definite integral that represents the area. (A function that gives a numerical approximation to the integral is located in the MATH menu of a TI-84 Plus calculator.) Figure 24 shows the results of using these steps for Example 2. The second window shows that the area closely approximates 5/12.



(a) (b)

FIGURE 24

The difference between two integrals can be used to find the area between the graphs of two functions even if one graph lies below the x-axis. In fact, if $f(x) \geq g(x)$ for all values of x in the interval $[a, b]$, then the area between the two graphs is always given by

$$\int_a^b [f(x) - g(x)]\,dx.$$

To see this, look at the graphs in Figure 25(a), where $f(x) \geq g(x)$ for x in $[a, b]$. Suppose a constant C is added to both functions, with C large enough so that both graphs lie above the x-axis, as in Figure 25(b). The region between the graphs is not changed. By the work above, this area is given by $\int_a^b [f(x) - g(x)]dx$ regardless of where the graphs of $f(x)$ and $g(x)$ are located. As long as $f(x) \geq g(x)$ on $[a, b]$, then the area between the graphs from $x = a$ to $x = b$ will equal $\int_a^b [f(x) - g(x)]dx$.

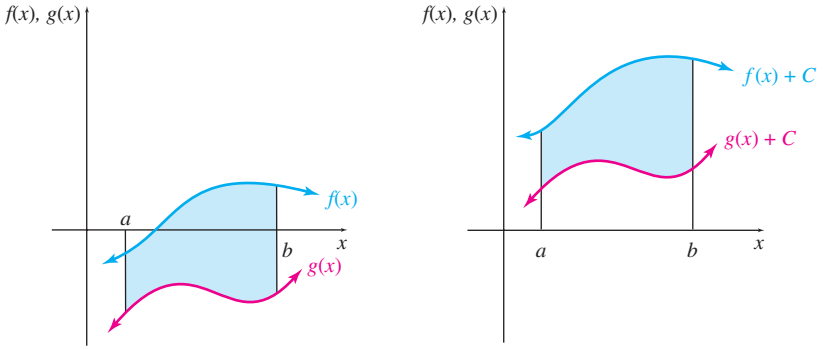


FIGURE 25

EXAMPLE 3 Area

Find the area of the region enclosed by $y = x^2 - 2x$ and $y = x$ on $[0, 4]$.

SOLUTION Verify that the two graphs cross at $x = 0$ and $x = 3$. Because the first graph is a parabola opening upward, the parabola must be below the line between 0 and 3 and above the line between 3 and 4. See Figure 26. (The greater function could also be identified by checking a point between 0 and 3, such as 1, and a point between 3 and 4, such as 3.5. For each of these values of x, we could calculate the corresponding value of y for the two functions and see which is greater.) Because the graphs cross at $x = 3$, the area is found by taking the sum of two integrals as follows.

$$\begin{aligned}
\text{Area} &= \int_0^3 [x - (x^2 - 2x)]\,dx + \int_3^4 [(x^2 - 2x) - x]\,dx \\
&= \int_0^3 (-x^2 + 3x)\,dx + \int_3^4 (x^2 - 3x)\,dx \\
&= \left(\frac{-x^3}{3} + \frac{3x^2}{2}\right)\Big|_0^3 + \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_3^4 \\
&= \left(-9 + \frac{27}{2} - 0\right) + \left(\frac{64}{3} - 24 - 9 + \frac{27}{2}\right) \\
&= \frac{19}{3}
\end{aligned}$$

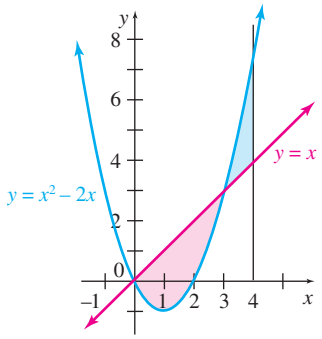


FIGURE 26

YOUR TURN 3 Repeat Example 3 for $y = x^2 - 3x$ and $y = 2x$ on $[0, 6]$.

TRY YOUR TURN 3

In the remainder of this section we will consider some typical applications that require finding the area between two curves.

EXAMPLE 4 Savings

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2,$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t.$$

(a) For how many years will the company realize savings?

APPLY IT

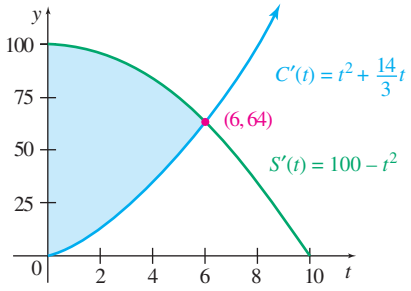


FIGURE 27

SOLUTION Figure 27 shows the graphs of the rate-of-savings and rate-of-cost functions. The rate of cost (marginal cost) is increasing, while the rate of savings (marginal savings) is decreasing. The company should use this new process until the difference between these quantities is zero; that is, until the time at which these graphs intersect. The graphs intersect when

$$C'(t) = S'(t),$$

or

$$t^2 + \frac{14}{3}t = 100 - t^2.$$

Solve this equation as follows.

$$\begin{aligned} 0 &= 2t^2 + \frac{14}{3}t - 100 \\ 0 &= 3t^2 + 7t - 150 && \text{Multiply by } \tfrac{3}{2}. \\ &= (t - 6)(3t + 25) && \text{Factor.} \end{aligned}$$

Set each factor equal to 0 separately to get

$$t = 6 \quad \text{or} \quad t = -25/3.$$

Only 6 is a meaningful solution here. The company should use the new process for 6 years.

(b) What will be the net total savings during this period?

SOLUTION Since the total savings over the 6-year period is given by the area under the rate-of-savings curve and the total additional cost is given by the area under the rate-of-cost curve, the net total savings over the 6-year period is given by the area between the rate-of-cost and the rate-of-savings curves and the lines $t = 0$ and $t = 6$. This area can be evaluated with a definite integral as follows.

$$\begin{aligned} \text{Total savings} &= \int_0^6 \left[(100 - t^2) - \left(t^2 + \frac{14}{3}t \right) \right] dt \\ &= \int_0^6 \left(100 - \frac{14}{3}t - 2t^2 \right) dt \\ &= \left(100t - \frac{7}{3}t^2 - \frac{2}{3}t^3 \right) \Big|_0^6 \\ &= 100(6) - \frac{7}{3}(36) - \frac{2}{3}(216) = 372 \end{aligned}$$

The company will save a total of $372,000 over the 6-year period.

The answer to a problem will not always be an integer. Suppose in solving the quadratic equation in Example 4 we found the solutions to be $t = 6.7$ and $t = -7.3$. It may not be

realistic to use a new process for 6.7 years; it may be necessary to choose between 6 years and 7 years. Since the mathematical model produces a result that is not in the domain of the function in this case, it is necessary to find the total savings after 6 years and after 7 years and then select the best result.

Consumers' Surplus

The market determines the price at which a product is sold. As indicated earlier, the point of intersection of the demand curve and the supply curve for a product gives the equilibrium price. At the equilibrium price, consumers will purchase the same amount of the product that the manufacturers want to sell. Some consumers, however, would be willing to spend more for an item than the equilibrium price. The total of the differences between the equilibrium price of the item and the higher prices that individuals would be willing to pay is thought of as savings realized by those individuals and is called the **consumers' surplus**.

To calculate the total amount that consumers would be willing to pay for q_0 items, first consider the simple case in which everyone is willing to pay exactly p_0, the equilibrium price. Then the total amount everyone would pay would be the price times the quantity, or p_0q_0, which is the green area in Figure 28. In fact, this is the exact total that everyone together pays when the item sells for p_0. Now divide the interval from 0 to q_0 into n intervals, each of width $\Delta q = q_0/n$. Each interval represents Δq people. Specifically, we will assume that the people represented by the ith interval, where i is a number between 1 and n, are those who are willing to pay a price $p_i = D(q_i)$ for the item, where q_i is some quantity on that interval. Then the total amount those people would be willing to pay would be $D(q_i)\Delta q$. The total amount that everyone together would be willing to pay is

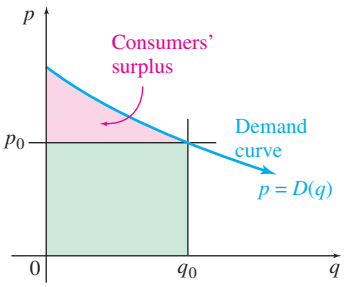


FIGURE 28

$$\sum_{i=1}^{n} D(q_i)\Delta q.$$

In a more realistic situation, the demand curve changes continuously, so we find the total amount that everyone would be willing to pay by taking the limit as n goes to infinity:

$$\lim_{n\to\infty} \sum_{i=1}^{n} D(q_i)\Delta q = \int_0^{q_0} D(q)\, dq.$$

This quantity, which represents the total amount consumers are willing to spend for q_0 items, is the area under the demand curve in Figure 28. The pink shaded area represents the difference between what consumers would be willing to pay and what they actually pay, or the consumers' surplus.

As the figure suggests, the consumers' surplus is given by an area between the two curves $p = D(q)$ and $p = p_0$, so its value can be found with a definite integral as follows.

Consumers' Surplus

If $D(q)$ is a demand function with equilibrium price p_0 and equilibrium demand q_0, then

$$\textbf{Consumers' surplus} = \int_0^{q_0} [D(q) - p_0]\, dq.$$

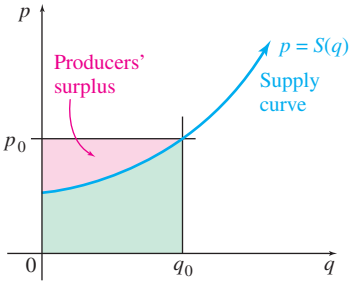


FIGURE 29

Similarly, if some manufacturers would be willing to supply a product at a price *lower* than the equilibrium price p_0, the total of the differences between the equilibrium price and the lower prices at which the manufacturers would sell the product is considered added income for the manufacturers and is called the **producers' surplus**. Figure 29 shows the

(green shaded) total area under the supply curve from $q = 0$ to $q = q_0$, which is the minimum total amount the manufacturers are willing to realize from the sale of q_0 items. The total area under the line $p = p_0$ is the amount actually realized. The difference between these two areas, the producers' surplus, is also given by a definite integral.

Producers' Surplus

If $S(q)$ is a supply function with equilibrium price p_0 and equilibrium supply q_0, then

$$\textbf{Producers' surplus} = \int_0^{q_0} [p_0 - S(q)]\, dq.$$

EXAMPLE 5 Consumers' and Producers' Surplus

Suppose the price (in dollars per ton) for oat bran is

$$D(q) = 400 - e^{q/2},$$

when the demand for the product is q tons. Also, suppose the function

$$S(q) = e^{q/2} - 1$$

gives the price (in dollars per ton) when the supply is q tons. Find the consumers' surplus and the producers' surplus.

SOLUTION Begin by finding the equilibrium quantity. This is done by setting the two equations equal.

$$\begin{aligned} e^{q/2} - 1 &= 400 - e^{q/2} \\ 2e^{q/2} &= 401 \\ e^{q/2} &= \frac{401}{2} \\ q/2 &= \ln\left(\frac{401}{2}\right) \\ q &= 2\ln\left(\frac{401}{2}\right) \approx 10.60163 \end{aligned}$$

The result can be further rounded to 10.60 tons as long as this rounded value is not used in future calculations. At the equilibrium point where the supply and demand are both 10.60 tons, the price is

$$S(10.60163) = e^{10.60163/2} - 1 \approx 199.50,$$

or \$199.50. Verify that this same answer is found by computing $D(10.60163)$. The consumers' surplus, represented by the area shown in Figure 30, is

$$\int_0^{10.60163} [(400 - e^{q/2}) - 199.50]\, dq = \int_0^{10.60163} [200.5 - e^{q/2}]\, dq.$$

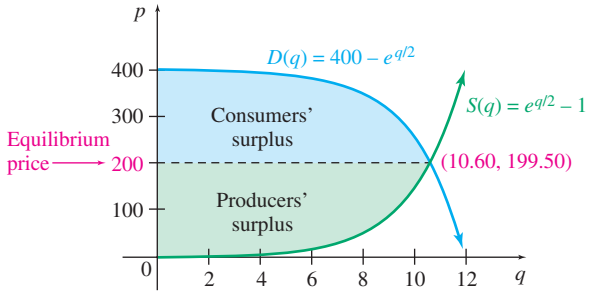


FIGURE 30

Evaluating the definite integral gives

$$\begin{aligned}(200.5q - 2e^{q/2})\Big|_0^{10.60163} &= [200.5(10.60163) - 2e^{10.60163/2}] - (0 - 2)\\ &\approx 1726.63.\end{aligned}$$

Here the consumers' surplus is \$1726.63. The producers' surplus, also shown in Figure 30, is given by

$$\int_0^{10.60163} [199.50 - (e^{q/2} - 1)]\,dq = \int_0^{10.60163} [200.5 - e^{q/2}]\,dq,$$

which is exactly the same as the expression found for the consumers' surplus, so the producers' surplus is also \$1726.63. TRY YOUR TURN 4

YOUR TURN 4 Repeat Example 5 when $D(q) = 600 - e^{q/3}$ and $S(q) = e^{q/3} - 100$.

NOTE In general, the producers' surplus and consumers' surplus are not the same, as they are in Example 5.

15.5 EXERCISES

Find the area between the curves in Exercises 1–24.

1. $x = -2,\quad x = 1,\quad y = 2x^2 + 5,\quad y = 0$
2. $x = 1,\quad x = 2,\quad y = 3x^3 + 2,\quad y = 0$
3. $x = -3,\quad x = 1,\quad y = x^3 + 1,\quad y = 0$
4. $x = -3,\quad x = 0,\quad y = 1 - x^2,\quad y = 0$
5. $x = -2,\quad x = 1,\quad y = 2x,\quad y = x^2 - 3$
6. $x = 0,\quad x = 6,\quad y = 5x,\quad y = 3x + 10$
7. $y = x^2 - 30,\quad y = 10 - 3x$
8. $y = x^2 - 18,\quad y = x - 6$
9. $y = x^2,\quad y = 2x$
10. $y = x^2,\quad y = x^3$
11. $x = 1,\quad x = 6,\quad y = \dfrac{1}{x},\quad y = \dfrac{1}{2}$
12. $x = 0,\quad x = 4,\quad y = \dfrac{1}{x+1},\quad y = \dfrac{x-1}{2}$
13. $x = -1,\quad x = 1,\quad y = e^x,\quad y = 3 - e^x$
14. $x = -1,\quad x = 2,\quad y = e^{-x},\quad y = e^x$
15. $x = -1,\quad x = 2,\quad y = 2e^{2x},\quad y = e^{2x} + 1$
16. $x = 2,\quad x = 4,\quad y = \dfrac{x-1}{4},\quad y = \dfrac{1}{x-1}$
17. $y = x^3 - x^2 + x + 1,\quad y = 2x^2 - x + 1$
18. $y = 2x^3 + x^2 + x + 5,\quad y = x^3 + x^2 + 2x + 5$
19. $y = x^4 + \ln(x + 10),\quad y = x^3 + \ln(x + 10)$
20. $y = x^5 - 2\ln(x + 5),\quad y = x^3 - 2\ln(x + 5)$
21. $y = x^{4/3},\quad y = 2x^{1/3}$
22. $y = \sqrt{x},\quad y = x\sqrt{x}$
23. $x = 0,\quad x = 3,\quad y = 2e^{3x},\quad y = e^{3x} + e^6$
24. $x = 0,\quad x = 3,\quad y = e^x,\quad y = e^{4-x}$

In Exercises 25 and 26, use a graphing calculator to find the values of x where the curves intersect and then to find the area between the two curves.

25. $y = e^x,\quad y = -x^2 - 2x$
26. $y = \ln x,\quad y = x^3 - 5x^2 + 6x - 1$

APPLICATIONS

Business and Economics

27. **Net Savings** Suppose a company wants to introduce a new machine that will produce a rate of annual savings (in dollars) given by

$$S'(x) = 150 - x^2,$$

where x is the number of years of operation of the machine, while producing a rate of annual costs (in dollars) of

$$C'(x) = x^2 + \frac{11}{4}x.$$

a. For how many years will it be profitable to use this new machine?

b. What are the net total savings during the first year of use of the machine?

c. What are the net total savings over the entire period of use of the machine?

28. Net Savings A new smog-control device will reduce the output of sulfur oxides from automobile exhausts. It is estimated that the rate of savings (in millions of dollars per year) to the community from the use of this device will be approximated by

$$S'(x) = -x^2 + 4x + 8,$$

after x years of use of the device. The new device cuts down on the production of sulfur oxides, but it causes an increase in the production of nitrous oxides. The rate of additional costs (in millions of dollars per year) to the community after x years is approximated by

$$C'(x) = \frac{3}{25}x^2.$$

a. For how many years will it pay to use the new device?

b. What will be the net savings over this period of time?

29. Profit Canham Enterprises had an expenditure rate of $E'(x) = e^{0.1x}$ dollars per day and an income rate of $I'(x) = 98.8 - e^{0.1x}$ dollars per day on a particular job, where x was the number of days from the start of the job. The company's profit on that job will equal total income less total expenditures. Profit will be maximized if the job ends at the optimum time, which is the point where the two curves meet. Find the following.

a. The optimum number of days for the job to last

b. The total income for the optimum number of days

c. The total expenditures for the optimum number of days

d. The maximum profit for the job

30. Net Savings A factory of Hollis Sherman Industries has installed a new process that will produce an increased rate of revenue (in thousands of dollars per year) of

$$R'(t) = 104 - 0.4e^{t/2},$$

where t is time measured in years. The new process produces additional costs (in thousands of dollars per year) at the rate of

$$C'(t) = 0.3e^{t/2}.$$

a. When will it no longer be profitable to use this new process?

b. Find the net total savings.

31. Producers' Surplus Find the producers' surplus if the supply function for pork bellies is given by

$$S(q) = q^{5/2} + 2q^{3/2} + 50.$$

Assume supply and demand are in equilibrium at $q = 16$.

32. Producers' Surplus Suppose the supply function for concrete is given by

$$S(q) = 100 + 3q^{3/2} + q^{5/2},$$

and that supply and demand are in equilibrium at $q = 9$. Find the producers' surplus.

33. Consumers' Surplus Find the consumers' surplus if the demand function for grass seed is given by

$$D(q) = \frac{200}{(3q + 1)^2},$$

assuming supply and demand are in equilibrium at $q = 3$.

34. Consumers' Surplus Find the consumers' surplus if the demand function for extra virgin olive oil is given by

$$D(q) = \frac{32{,}000}{(2q + 8)^3},$$

and if supply and demand are in equilibrium at $q = 6$.

35. Consumers' and Producers' Surplus Suppose the supply function for oil is given (in dollars) by

$$S(q) = q^2 + 10q,$$

and the demand function is given (in dollars) by

$$D(q) = 900 - 20q - q^2.$$

a. Graph the supply and demand curves.

b. Find the point at which supply and demand are in equilibrium.

c. Find the consumers' surplus.

d. Find the producers' surplus.

36. Consumers' and Producers' Surplus Suppose the supply function for a certain item is given by

$$S(q) = (q + 1)^2,$$

and the demand function is given by

$$D(q) = \frac{1000}{q + 1}.$$

a. Graph the supply and demand curves.

b. Find the point at which supply and demand are in equilibrium.

c. Find the consumers' surplus.

d. Find the producers' surplus.

37. Consumers' and Producers' Surplus Suppose that with the supply and demand for oil as in Exercise 35, the government sets the price at \$264 per unit.

a. Use the supply function to calculate the quantity that will be produced at the new price.

b. Find the consumers' surplus for the new price, using the quantity found in part a in place of the equilibrium quantity. How much larger is this than the consumers' surplus in Exercise 35?

c. Find the producers' surplus for the new price, using the quantity found in part a in place of the equilibrium quantity. How much smaller is this than the producers' surplus in Exercise 35?

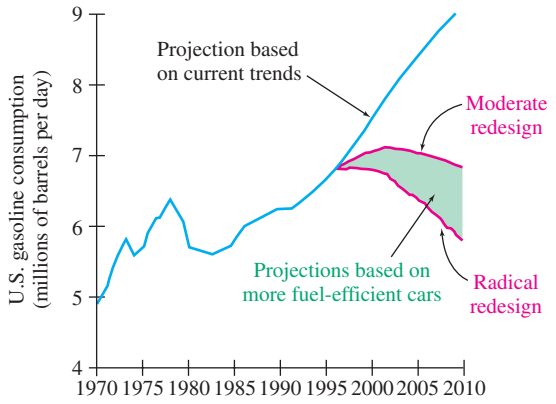
d. Calculate the difference between the total of the consumers' and producers' surplus under the equilibrium price and under the government price. Economists refer to this loss as the *welfare cost* of the government's setting the price.

e. Because of the welfare cost calculated in part d, many economists argue that it is bad economics for the government to set prices. Others point to the increase in the consumers' surplus, calculated in part b, as a justification for such government action. Discuss the pros and cons of this issue.

38. Fuel Economy In an article in the December 1994 *Scientific American* magazine, the authors estimated future gas use. Without a change in U.S. policy, auto fuel use is forecasted to rise along the projection shown at the right in the figure below. The shaded band predicts gas use if the technologies for increased fuel economy are phased in by the year 2010. The moderate estimate (center curve) corresponds to an average of 46 mpg for all cars on the road. ***Source: Scientific American***.

a. Discuss the interpretation of the shaded area and other regions of the graph that pertain to the topic in this section.

b. According to the Energy Information Administration, the U.S. gasoline consumption in 2010 was 9,030,000 barrels per day. Discuss how this affects the areas considered in part a. ***Source: U.S. Energy Information Administration***.



Life Sciences

39. Pollution Pollution begins to enter a lake at time $t = 0$ at a rate (in gallons per hour) given by the formula

$$f(t) = 10(1 - e^{-0.5t}),$$

where t is the time (in hours). At the same time, a pollution filter begins to remove the pollution at a rate

$$g(t) = 0.4t$$

as long as pollution remains in the lake.

a. How much pollution is in the lake after 12 hours?

b. Use a graphing calculator to find the time when the rate that pollution enters the lake equals the rate the pollution is removed.

c. Find the amount of pollution in the lake at the time found in part b.

d. Use a graphing calculator to find the time when all the pollution has been removed from the lake.

40. Pollution Repeat the steps of Exercise 39, using the functions

$$f(t) = 15(1 - e^{-0.05t})$$

and

$$g(t) = 0.3t.$$

Social Sciences

41. Distribution of Income Suppose that all the people in a country are ranked according to their incomes, starting at the bottom. Let x represent the fraction of the community making the lowest income $(0 \le x \le 1)$; $x = 0.4$, therefore, represents the lower 40% of all income producers. Let $I(x)$ represent the proportion of the total income earned by the lowest x of all people. Thus, $I(0.4)$ represents the fraction of total income earned by the lowest 40% of the population. The curve described by this function is known as a *Lorenz curve*. Suppose

$$I(x) = 0.9x^2 + 0.1x.$$

Find and interpret the following.

a. $I(0.1)$ **b.** $I(0.4)$

If income were distributed uniformly, we would have $I(x) = x$. The area under this line of complete equality is $1/2$. As $I(x)$ dips further below $y = x$, there is less equality of income distribution. This inequality can be quantified by the ratio of the area between $I(x)$ and $y = x$ to $1/2$. This ratio is called the *Gini index of income inequality* and equals $2\int_0^1 [x - I(x)]\,dx$.

c. Graph $I(x) = x$ and $I(x) = 0.9x^2 + 0.1x$, for $0 \le x \le 1$, on the same axes.

d. Find the area between the curves.

e. For U.S. families, the Gini index was 0.386 in 1968 and 0.466 in 2008. Describe how the distribution of family incomes has changed over this time. ***Source: U.S. Census***.

Physical Sciences

42. Metal Plate A worker sketches the curves $y = \sqrt{x}$ and $y = x/2$ on a sheet of metal and cuts out the region between the curves to form a metal plate. Find the area of the plate.

YOUR TURN ANSWERS

1. 9/2
2. 7/15
3. 71/3
4. $5103.83; $5103.83

15.6 Numerical Integration

APPLY IT **If the velocity of a vehicle is known only at certain points in time, how can the total distance traveled by the vehicle be estimated?**
Using numerical integration, we will answer this question in Example 3 of this section.

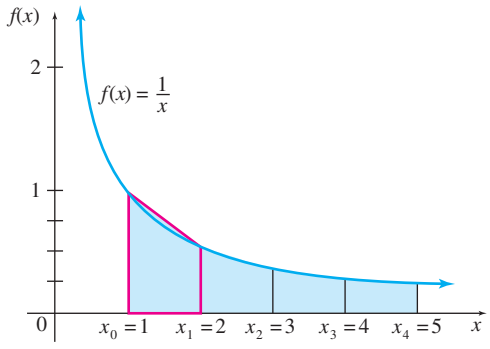
Some integrals cannot be evaluated by any technique. One solution to this problem was presented in Section 3 of this chapter, in which the area under a curve was approximated by summing the areas of rectangles. This method is seldom used in practice because better methods exist that are more accurate for the same amount of work. These methods are referred to as **numerical integration** methods. We discuss two such methods here: the trapezoidal rule and Simpson's rule.

Trapezoidal Rule

Recall, the trapezoidal rule was mentioned briefly in Section 3, where we found approximations with it by averaging the sums of rectangles found by using left endpoints and then using right endpoints. In this section we derive an explicit formula for the trapezoidal rule in terms of function values.* To illustrate the derivation of the trapezoidal rule, consider the integral

$$\int_1^5 \frac{1}{x}\,dx.$$

The shaded region in Figure 31 shows the area representing that integral, the area under the graph $f(x) = 1/x$, above the x-axis, and between the lines $x = 1$ and $x = 5$.



Left-hand trapezoid
$B = f(x_0) = f(1) = 1$
$b = f(x_1) = f(2) = \frac{1}{2}$
$h = 2 - 1 = 1$
Area $= \frac{1}{2}(B + b)h$
$= \frac{1}{2}\left(1 + \frac{1}{2}\right)1$
$= \frac{1}{2} \cdot \frac{3}{2}$
$= \frac{3}{4}$

FIGURE 31

Note that this function can be integrated using the Fundamental Theorem of Calculus. Since $\int (1/x)\,dx = \ln|x| + C$,

$$\int_1^5 \frac{1}{x}\,dx = \ln|x|\Big|_1^5 = \ln 5 - \ln 1 = \ln 5 - 0 = \ln 5 \approx 1.609438.$$

We can also approximate the integral using numerical integration. As shown in the figure, if the area under the curve is approximated with trapezoids rather than rectangles, the approximation should be improved.

As in earlier work, to approximate this area we divide the interval $[1, 5]$ into subintervals of equal widths. To get a first approximation to ln 5 by the trapezoidal rule, find the sum of the

*In American English a trapezoid is a four-sided figure with two parallel sides, contrasted with a trapezium, which has no parallel sides. In British English, however, it is just the opposite. What Americans call a trapezoid is called a trapezium in Great Britain.

Approximations to $\int_1^5 \frac{1}{x}\,dx$	
n	Trapezoidal Approximation
6	1.64360
8	1.62897
10	1.62204
20	1.61262
100	1.60957
1000	1.60944

areas of the four trapezoids shown in Figure 31. From geometry, the area of a trapezoid is half the product of the sum of the bases and the altitude. Each of the trapezoids in Figure 31 has altitude 1. (In this case, the bases of the trapezoid are vertical and the altitudes are horizontal.) Adding the areas gives

$$\ln 5 = \int_1^5 \frac{1}{x}\,dx \approx \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)(1) + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)(1) + \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right)(1) + \frac{1}{2}\left(\frac{1}{4}+\frac{1}{5}\right)(1)$$

$$= \frac{1}{2}\left(\frac{3}{2}+\frac{5}{6}+\frac{7}{12}+\frac{9}{20}\right) \approx 1.68333.$$

To get a better approximation, divide the interval $[1, 5]$ into more subintervals. Generally speaking, the larger the number of subintervals, the better the approximation. The results for selected values of n are shown to 5 decimal places. When $n = 1000$, the approximation agrees with the true value of $\ln 5 \approx 1.609438$ to 5 decimal places.

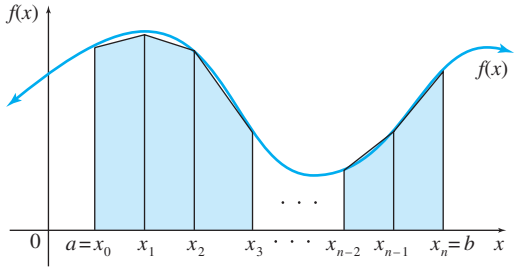


FIGURE 32

Generalizing from this example, let f be a continuous function on an interval $[a, b]$. Divide the interval from a to b into n equal subintervals by the points $a = x_0, x_1, x_2, \ldots, x_n = b$, as shown in Figure 32. Use the subintervals to make trapezoids that approximately fill in the region under the curve. The approximate value of the definite integral $\int_a^b f(x)\,dx$ is given by the sum of the areas of the trapezoids, or

$$\int_a^b f(x)\,dx \approx \frac{1}{2}[f(x_0)+f(x_1)]\left(\frac{b-a}{n}\right) + \frac{1}{2}[f(x_1)+f(x_2)]\left(\frac{b-a}{n}\right)$$

$$+ \cdots + \frac{1}{2}[f(x_{n-1})+f(x_n)]\left(\frac{b-a}{n}\right)$$

$$= \left(\frac{b-a}{n}\right)\left[\frac{1}{2}f(x_0)+\frac{1}{2}f(x_1)+\frac{1}{2}f(x_1)+\frac{1}{2}f(x_2)+\frac{1}{2}f(x_2)+\cdots+\frac{1}{2}f(x_{n-1})+\frac{1}{2}f(x_n)\right]$$

$$= \left(\frac{b-a}{n}\right)\left[\frac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right].$$

This result gives the following rule.

Trapezoidal Rule

Let f be a continuous function on $[a, b]$ and let $[a, b]$ be divided into n equal subintervals by the points $a = x_0, x_1, x_2, \ldots, x_n = b$. Then, by the **trapezoidal rule**,

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)\left[\frac{1}{2}f(x_0)+f(x_1)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right].$$

EXAMPLE 1 Trapezoidal Rule

Use the trapezoidal rule with $n = 4$ to approximate

$$\int_0^2 \sqrt{x^2+1}\,dx.$$

SOLUTION

Method 1 Calculating by Hand

Here $a = 0$, $b = 2$, and $n = 4$, with $(b - a)/n = (2 - 0)/4 = 1/2$ as the altitude of each trapezoid. Then $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, $x_3 = 3/2$, and $x_4 = 2$. Now find the corresponding function values. The work can be organized into a table, as follows.

Calculations for Trapezoidal Rule

i	x_i	$f(x_i)$
0	0	$\sqrt{0^2 + 1} = 1$
1	1/2	$\sqrt{(1/2)^2 + 1} \approx 1.11803$
2	1	$\sqrt{1^2 + 1} \approx 1.41421$
3	3/2	$\sqrt{(3/2)^2 + 1} \approx 1.80278$
4	2	$\sqrt{2^2 + 1} \approx 2.23607$

Substitution into the trapezoidal rule gives

$$\int_0^2 \sqrt{x^2 + 1}\, dx$$
$$\approx \frac{2 - 0}{4}\left[\frac{1}{2}(1) + 1.11803 + 1.41421 + 1.80278 + \frac{1}{2}(2.23607)\right]$$
$$\approx 2.97653.$$

The approximation 2.97653 found above using the trapezoidal rule with $n = 4$ differs from the true value of 2.95789 by 0.01864. As mentioned above, this error would be reduced if larger values were used for n. For example, if $n = 8$, the trapezoidal rule gives an answer of 2.96254, which differs from the true value by 0.00465. Techniques for estimating such errors are considered in more advanced courses.

Method 2 Graphing Calculator

Just as we used a graphing calculator to approximate area using rectangles, we can also use it for the trapezoidal rule. As before, put the values of i in L_1 and the values of x_i in L_2. In the heading for L_3, put $\sqrt{(L_2^2 + 1)}$. Using the fact that $(b - a)/n = (2 - 0)/4 = 0.5$, the command `.5*(.5*L3(1)+sum(L3,2,4)+.5*L3(5))` gives the result 2.976528589. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

Method 3 Spreadsheet

The trapezoidal rule can also be done on a spreadsheet. In Microsoft Excel, for example, store the values of 0 through n in column A. After putting the left endpoint in E1 and Δx in E2, put the command "`=$E$1+A1*$E$2`" into B1; copying this formula into the rest of column B gives the values of x_i. Similarly, use the formula for $f(x_i)$ to fill column C. Using the fact that $n = 5$ in this example, the command "`$E$2*(.5*C1+sum(C2:C4)+.5*C5)`" gives the result 2.976529. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

YOUR TURN 1 Use the trapezoidal rule with $n = 4$ to approximate $\int_1^3 \sqrt{x^2 + 3}\, dx$.

TRY YOUR TURN 1

The trapezoidal rule is not widely used because its results are not very accurate. In fact, the midpoint rule discussed earlier in this chapter is usually more accurate than the trapezoidal rule. We will now consider a method that usually gives more accurate results than either the trapezoidal or midpoint rule.

Simpson's Rule

Another numerical method, *Simpson's rule*, approximates consecutive portions of the curve with portions of parabolas rather than the line segments of the trapezoidal rule. Simpson's rule usually gives a better approximation than the trapezoidal rule for the same number of subintervals. As shown in Figure 33 on the next page, one

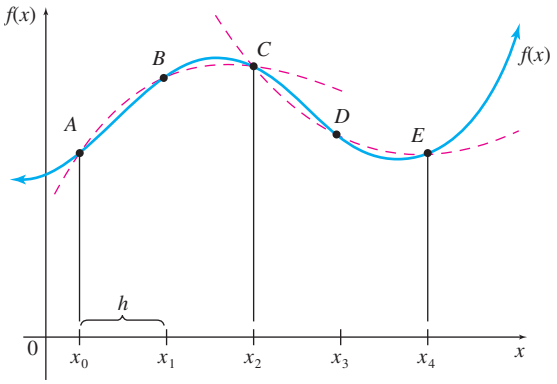


FIGURE 33

parabola is fitted through points A, B, and C, another through C, D, and E, and so on. Then the sum of the areas under these parabolas will approximate the area under the graph of the function. Because of the way the parabolas overlap, it is necessary to have an even number of intervals, and therefore an odd number of points, to apply Simpson's rule.

If h, the length of each subinterval, is $(b - a)/n$, the area under the parabola through points A, B, and C can be found by a definite integral. The details are omitted; the result is

$$\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)].$$

Similarly, the area under the parabola through points C, D, and E is

$$\frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)].$$

When these expressions are added, the last term of one expression equals the first term of the next. For example, the sum of the two areas given above is

$$\frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)].$$

This illustrates the origin of the pattern of the terms in the following rule.

Simpson's Rule

Let f be a continuous function on $[a, b]$ and let $[a, b]$ be divided into an even number n of equal subintervals by the points $a = x_0, x_1, x_2, \ldots, x_n = b$. Then by **Simpson's rule**,

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{3n}\right)[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

Thomas Simpson (1710–1761), a British mathematician, wrote texts on many branches of mathematics. Some of these texts went through as many as ten editions. His name became attached to this numerical method of approximating definite integrals even though the method preceded his work.

CAUTION In Simpson's rule, n (the number of subintervals) must be even.

EXAMPLE 2 Simpson's Rule

Use Simpson's rule with $n = 4$ to approximate

$$\int_0^2 \sqrt{x^2 + 1}\,dx,$$

which was approximated by the trapezoidal rule in Example 1.

SOLUTION As in Example 1, $a = 0$, $b = 2$, and $n = 4$, and the endpoints of the four intervals are $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, $x_3 = 3/2$, and $x_4 = 2$. The table of values is also the same.

Calculations for Simpson's Rule

i	x_i	$f(x_i)$
0	0	1
1	1/2	1.11803
2	1	1.41421
3	3/2	1.80278
4	2	2.23607

Since $(b - a)/(3n) = 2/12 = 1/6$, substituting into Simpson's rule gives

$$\int_0^2 \sqrt{x^2 + 1}\, dx \approx \frac{1}{6}[1 + 4(1.11803) + 2(1.41421) + 4(1.80278) + 2.23607] \approx 2.95796.$$

This differs from the true value by 0.00007, which is less than the trapezoidal rule with $n = 8$. If $n = 8$ for Simpson's rule, the approximation is 2.95788, which differs from the true value by only 0.00001. TRY YOUR TURN 2

YOUR TURN 2 Use Simpson's rule with $n = 4$ to approximate $\int_1^3 \sqrt{x^2 + 3}\, dx$.

NOTE

1. Just as we can use a graphing calculator or a spreadsheet for the trapezoidal rule, we can also use such technology for Simpson's rule. For more details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.
2. Let M represent the midpoint rule approximation and T the trapezoidal rule approximation, using n subintervals in each. Then the formula $S = (2M + T)/3$ gives the Simpson's rule approximation with $2n$ subintervals.

Numerical methods make it possible to approximate

$$\int_a^b f(x)\, dx$$

even when $f(x)$ is not known. The next example shows how this is done.

EXAMPLE 3 Total Distance

As mentioned earlier, the velocity $v(t)$ gives the rate of change of distance $s(t)$ with respect to time t. Suppose a vehicle travels an unknown distance. The passengers keep track of the velocity at 10-minute intervals (every 1/6 of an hour) with the following results.

	Velocity of a Vehicle						
Time in Hours, t	1/6	2/6	3/6	4/6	5/6	1	7/6
Velocity in Miles per Hour, v(t)	45	55	52	60	64	58	47

What is the total distance traveled in the 60-minute period from $t = 1/6$ to $t = 7/6$?

APPLY IT

SOLUTION The distance traveled in t hours is $s(t)$, with $s'(t) = v(t)$. The total distance traveled between $t = 1/6$ and $t = 7/6$ is given by

$$\int_{1/6}^{7/6} v(t)\, dt.$$

Even though this integral cannot be evaluated since we do not have an expression for $v(t)$, either the trapezoidal rule or Simpson's rule can be used to approximate its value and give the total distance traveled. In either case, let $n = 6$, $a = t_0 = 1/6$, and $b = t_6 = 7/6$. By the trapezoidal rule,

$$\int_{1/6}^{7/6} v(t)\,dt \approx \frac{7/6 - 1/6}{6}\left[\frac{1}{2}(45) + 55 + 52 + 60 + 64 + 58 + \frac{1}{2}(47)\right]$$
$$\approx 55.83.$$

By Simpson's rule,

$$\int_{1/6}^{7/6} v(t)\,dt \approx \frac{7/6 - 1/6}{3(6)}[45 + 4(55) + 2(52) + 4(60) + 2(64) + 4(58) + 47]$$
$$= \frac{1}{18}(45 + 220 + 104 + 240 + 128 + 232 + 47) \approx 56.44.$$

The distance traveled in the 1-hour period was about 56 miles.

As already mentioned, Simpson's rule generally gives a better approximation than the trapezoidal rule. As n increases, the two approximations get closer and closer. For the same accuracy, however, a smaller value of n generally can be used with Simpson's rule so that less computation is necessary. Simpson's rule is the method used by many calculators that have a built-in integration feature.

The branch of mathematics that studies methods of approximating definite integrals (as well as many other topics) is called *numerical* analysis. Numerical integration is useful even with functions whose antiderivatives can be determined if the antidifferentiation is complicated and a computer or calculator programmed with Simpson's rule is handy. You may want to program your calculator for both the trapezoidal rule and Simpson's rule. For some calculators, these programs are in the *Graphing Calculator and Excel Spreadsheet Manual* available with this book.

15.6 EXERCISES

In Exercises 1–10, use $n = 4$ to approximate the value of the given integrals by the following methods: (a) the trapezoidal rule, and (b) Simpson's rule. (c) Find the exact value by integration.

1. $\displaystyle\int_0^2 (3x^2 + 2)\,dx$

2. $\displaystyle\int_0^2 (2x^2 + 1)\,dx$

3. $\displaystyle\int_{-1}^3 \frac{3}{5 - x}\,dx$

4. $\displaystyle\int_1^5 \frac{6}{2x + 1}\,dx$

5. $\displaystyle\int_{-1}^2 (2x^3 + 1)\,dx$

6. $\displaystyle\int_0^3 (2x^3 + 1)\,dx$

7. $\displaystyle\int_1^5 \frac{1}{x^2}\,dx$

8. $\displaystyle\int_2^4 \frac{1}{x^3}\,dx$

9. $\displaystyle\int_0^1 4xe^{-x^2}\,dx$

10. $\displaystyle\int_0^4 x\sqrt{2x^2 + 1}\,dx$

11. Find the area under the semicircle $y = \sqrt{4 - x^2}$ and above the x-axis by using $n = 8$ with the following methods.

a. The trapezoidal rule **b.** Simpson's rule

c. Compare the results with the area found by the formula for the area of a circle. Which of the two approximation techniques was more accurate?

12. Find the area between the x-axis and the upper half of the ellipse $4x^2 + 9y^2 = 36$ by using $n = 12$ with the following methods.

a. The trapezoidal rule **b.** Simpson's rule

(*Hint:* Solve the equation for y and find the area of the semiellipse.)

c. Compare the results with the actual area, $3\pi \approx 9.4248$ (which can be found by methods not considered in this text). Which approximation technique was more accurate?

13. Suppose that $f(x) > 0$ and $f''(x) > 0$ for all x between a and b, where $a < b$. Which of the following cases is true of a trapezoidal approximation T for the integral $\int_a^b f(x)\,dx$? Explain.

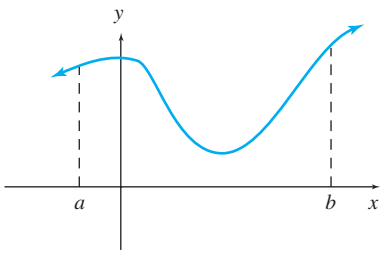
a. $T < \int_a^b f(x)dx$ **b.** $T > \int_a^b f(x)dx$

c. Can't say which is larger

14. Refer to Exercise 13. Which of the three cases applies to these functions?

a. $f(x) = x^2$; $[0, 3]$ **b.** $f(x) = \sqrt{x}$; $[0, 9]$

c.



Exercises 15–18 require both the trapezoidal rule and Simpson's rule. They can be worked without calculator programs if such programs are not available, although they require more calculation than the other problems in this exercise set.

Error Analysis **The difference between the true value of an integral and the value given by the trapezoidal rule or Simpson's rule is known as the error. In numerical analysis, the error is studied to determine how large n must be for the error to be smaller than some specified amount. For both rules, the error is inversely proportional to a power of n, the number of subdivisions. In other words, the error is roughly k/n^p, where k is a constant that depends on the function and the interval, and p is a power that depends only on the method used. With a little experimentation, you can find out what the power p is for the trapezoidal rule and for Simpson's rule.**

15. **a.** Find the exact value of $\int_0^1 x^4\,dx$.

b. Approximate the integral in part a using the trapezoidal rule with $n = 4, 8, 16$, and 32. For each of these answers, find the absolute value of the error by subtracting the trapezoidal rule answer from the exact answer found in part a.

c. If the error is k/n^p, then the error times n^p should be approximately a constant. Multiply the errors in part b times n^p for $p = 1, 2$, etc., until you find a power p yielding the same answer for all four values of n.

16. Based on the results of Exercise 15, what happens to the error in the trapezoidal rule when the number of intervals is doubled?

17. Repeat Exercise 15 using Simpson's rule.

18. Based on the results of Exercise 17, what happens to the error in Simpson's rule when the number of intervals is doubled?

19. For the integral in Exercise 7, apply the midpoint rule with $n = 4$ and Simpson's rule with $n = 8$ to verify the formula $S = (2M + T)/3$.

20. Repeat the instructions of Exercise 19 using the integral in Exercise 8.

APPLICATIONS

Business and Economics

21. **Total Sales** A sales manager presented the following results at a sales meeting.

Year, x	1	2	3	4	5	6	7
Rate of Sales, f(x)	0.4	0.6	0.9	1.1	1.3	1.4	1.6

Find the total sales over the given period as follows.

a. Plot these points. Connect the points with line segments.

b. Use the trapezoidal rule to find the area bounded by the broken line of part a, the x-axis, the line $x = 1$, and the line $x = 7$.

c. Approximate the same area using Simpson's rule.

22. **Total Cost** A company's marginal costs (in hundreds of dollars per year) were as follows over a certain period.

Year, x	1	2	3	4	5	6	7
Marginal Cost, f(x)	9.0	9.2	9.5	9.4	9.8	10.1	10.5

Repeat parts a–c of Exercise 21 for these data to find the total cost over the given period.

Life Sciences

23. **Drug Reaction Rate** The reaction rate to a new drug is given by

$$y = e^{-t^2} + \frac{1}{t+1},$$

where t is time (in hours) after the drug is administered. Find the total reaction to the drug from $t = 1$ to $t = 9$ by letting $n = 8$ and using the following methods.

a. The trapezoidal rule **b.** Simpson's rule

24. **Growth Rate** The growth rate of a certain tree (in feet) is given by

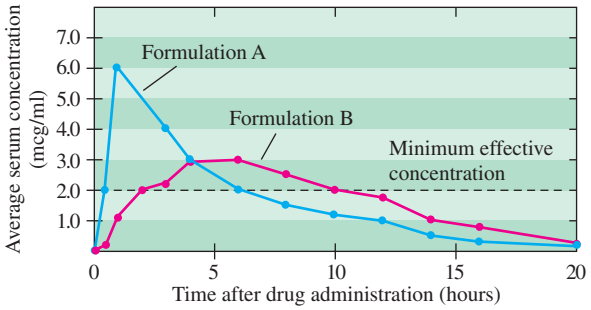
$$y = \frac{2}{t+2} + e^{-t^2/2},$$

where t is time (in years). Find the total growth from $t = 1$ to $t = 7$ by using $n = 12$ with the following methods.

a. The trapezoidal rule **b.** Simpson's rule

Blood Level Curves **In the study of bioavailability in pharmacy, a drug is given to a patient. The level of concentration of the drug is then measured periodically, producing blood level curves such as the ones shown in the figure.**

The areas under the curves give the total amount of the drug available to the patient for each milliliter of blood. Use the trapezoidal rule with $n = 10$ to find the following areas. ***Source: Basics of Bioavailability.***



25. Find the total area under the curve for Formulation A. What does this area represent?

26. Find the total area under the curve for Formulation B. What does this area represent?

27. Find the area between the curve for Formulation A and the minimum effective concentration line. What does your answer represent?

28. Find the area between the curve for Formulation B and the minimum effective concentration line. What does this area represent?

29. Calves The daily milk consumption (in kilograms) for calves can be approximated by the function

$$y = b_0 w^{b_1} e^{-b_2 w},$$

where w is the age of the calf (in weeks) and b_0, b_1, and b_2 are constants. ***Source: Animal Production***.

a. The age in days is given by $t = 7w$. Use this fact to convert the function above to a function in terms of t.

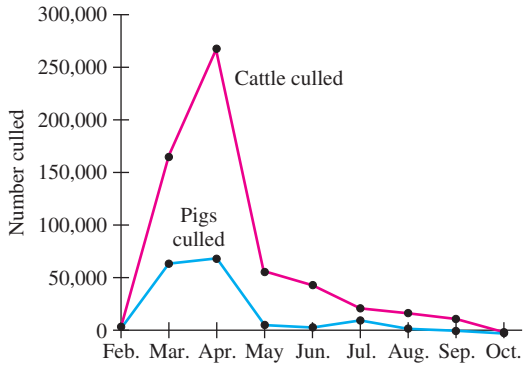
b. For a group of Angus calves, $b_0 = 5.955$, $b_1 = 0.233$, and $b_2 = 0.027$. Use the trapezoidal rule with $n = 10$, and then Simpson's rule with $n = 10$, to find the total amount of milk consumed by one of these calves over the first 25 weeks of life.

c. For a group of Nelore calves, $b_0 = 8.409$, $b_1 = 0.143$, and $b_2 = 0.037$. Use the trapezoidal rule with $n = 10$, and then Simpson's rule with $n = 10$, to find the total amount of milk consumed by one of these calves over the first 25 weeks of life.

30. Foot-and-Mouth Epidemic In 2001, the United Kingdom suffered an epidemic of foot-and-mouth disease. The graph below shows the reported number of cattle (red) and pigs (blue) that were culled each month from mid-February through mid-October in an effort to stop the spread of the disease. In Section 15.3 on Area and the Definite Integral we estimated the number of cattle and pigs that were culled using rectangles. ***Source: Department of Environment, Food and Rural Affairs, United Kingdom.***

a. Estimate the total number of cattle that were culled from mid-February through mid-October and compare this with 581,801, the actual number of cattle that were culled. Use Simpson's rule with interval widths of one month starting with mid-February.

b. Estimate the total number of pigs that were culled from mid-February through mid-October and compare this with 146,145, the actual number of pigs that were culled. Use Simpson's rule with interval widths of one month starting with mid-February.



Social Sciences

31. Educational Psychology The results from a research study in psychology were as follows.

Number of Hours of Study, x	1	2	3	4	5	6	7
Rate of Extra Points Earned on a Test, f(x)	4	7	11	9	15	16	23

Repeat parts a–c of Exercise 21 for these data.

Physical Sciences

32. Chemical Formation The following table shows the results from a chemical experiment.

Concentration of Chemical A, x	1	2	3	4	5	6	7
Rate of Formation of Chemical B, f(x)	12	16	18	21	24	27	32

Repeat parts a–c of Exercise 21 for these data.

If you have a program for Simpson's rule in your graphing calculator, use it with $n = 20$ for Exercises 33–35.

33. Total Revenue An electronics company analyst has determined that the rate per month at which revenue comes in from the calculator division is given by

$$R(x) = 105e^{0.01x} + 32,$$

where x is the number of months the division has been in operation. Find the total revenue between the 12th and 36th months.

34. Milk Consumption As we saw in an earlier chapter, the average individual daily milk consumption for herds of Charolais, Angus, and Hereford calves can be described by a mathematical function. Here we write the consumption in kg/day as a function of the age of the calf in days (t) as

$$M(t) = 3.922t^{0.242}e^{-0.00357t}, \quad 7 \le t \le 182.$$

Find the total amount of milk consumed from 7 to 182 days for a calf. ***Source: Animal Production***.

35. Probability The most important function in probability and statistics is the density function for the standard normal distribution, which is the familiar bell-shaped curve. The function is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

a. The area under this curve between $x = -1$ and $x = 1$ represents the probability that a normal random variable is within 1 standard deviation of the mean. Find this probability.

b. Find the area under this curve between $x = -2$ and $x = 2$, which represents the probability that a normal random variable is within 2 standard deviations of the mean.

c. Find the probability that a normal random variable is within 3 standard deviations of the mean.

YOUR TURN ANSWERS

1. 5.3552 **2.** 5.3477

15 CHAPTER REVIEW

SUMMARY

Earlier chapters dealt with the derivative, one of the two main ideas of calculus. This chapter deals with integration, the second main idea. There are two aspects of integration. The first is indefinite integration, or finding an antiderivative; the second is definite integration, which can be used to find the area under a curve. The Fundamental Theorem of Calculus unites these two ideas by showing that the way to find the area under a curve is to use the antiderivative. Substitution is a technique for finding antiderivatives. Numerical integration can be used to find the definite integral when finding an antiderivative is not feasible. The idea of the definite integral can also be applied to finding the area between two curves.

Antidifferentiation Formulas

Power Rule
$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Constant Multiple Rule
$$\int k \cdot f(x)\,dx = k\int f(x)\,dx, \quad \text{for any real number } k$$

Sum or Difference Rule
$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Integration of x^{-1}
$$\int x^{-1}\,dx = \ln|x| + C$$

Integration of Exponential Functions
$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C, \quad k \neq 0$$

Substitution Method

Choose u to be one of the following:

1. the quantity under a root or raised to a power;
2. the quantity in the denominator;
3. the exponent on e.

Definite Integrals

Definition of the Definite Integral

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = (b-a)/n$ and x_i is any value of x in the ith interval. If $f(x)$ gives the rate of change of $F(x)$ for x in $[a, b]$, then this represents the total change in $F(x)$ as x goes from a to b.

Properties of Definite Integrals

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx, \quad \text{for any real number } k.$
3. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad \text{for any real number } c$
5. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Fundamental Theorem of Calculus

$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$, where f is continuous on $[a, b]$ and F is any antiderivative of f

Area Between Two Curves

$\int_a^b [f(x) - g(x)]\,dx$, where f and g are continuous functions and $f(x) \geq g(x)$ on $[a, b]$

Consumers' Surplus $\int_0^{q_0}[D(q) - p_0]\,dq$, where D is the demand function and p_0 and q_0 are the equilibrium price and demand

Producers' Surplus $\int_0^{q_0}[p_0 - S(q)]\,dq$, where S is the supply function and p_0 and q_0 are the equilibrium price and supply

Trapezoidal Rule
$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{n}\right)\left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]$$

Simpson's Rule
$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{3n}\right)[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)]$$

KEY TERMS

15.1
antidifferentiation
antiderivative
integral sign
integrand
indefinite integral

15.2
integration by substitution

15.3
midpoint rule
definite integral
limits of integration
total change

15.4
Fundamental Theorem of Calculus

15.5
consumers' surplus
producers' surplus

15.6
numerical integration
trapezoidal rule
Simpson's rule

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The indefinite integral is another term for the family of all antiderivatives of a function.

2. The indefinite integral of x^n is $x^{n+1}/(n+1) + C$ for all real numbers n.

3. The indefinite integral $\int xf(x)\,dx$ is equal to $x\int f(x)\,dx$.

4. The velocity function is an antiderivative of the acceleration function.

5. Substitution can often be used to turn a complicated integral into a simpler one.

6. The definite integral gives the instantaneous rate of change of a function.

7. The definite integral gives an approximation to the area under a curve.

8. The definite integral of a positive function is the limit of the sum of the areas of rectangles.

9. The Fundamental Theorem of Calculus gives a relationship between the definite integral and an antiderivative of a function.

10. The definite integral of a function is always a positive quantity.

11. The area between two distinct curves is always a positive quantity.

12. The consumers' surplus and the producers' surplus equal each other.

13. In the trapezoidal rule, the number of subintervals must be even.

14. Simpson's rule usually gives a better approximation than the trapezoidal rule.

PRACTICE AND EXPLORATION

15. Explain the differences between an indefinite integral and a definite integral.

16. Explain under what circumstances substitution is useful in integration.

17. Explain why the limits of integration are changed when u is substituted for an expression in x in a definite integral.

18. Describe the type of integral for which numerical integration is useful.

In Exercises 19–40, find each indefinite integral.

19. $\int (2x + 3)\,dx$

20. $\int (5x - 1)\,dx$

21. $\int (x^2 - 3x + 2)\,dx$

22. $\int (6 - x^2)\,dx$

23. $\int 3\sqrt{x}\,dx$

24. $\int \frac{\sqrt{x}}{2}\,dx$

25. $\int (x^{1/2} + 3x^{-2/3})\,dx$

26. $\int (2x^{4/3} + x^{-1/2})\,dx$

27. $\int \frac{-4}{x^3}\,dx$

28. $\int \frac{5}{x^4}\,dx$

29. $\int -3e^{2x}\,dx$

30. $\int 5e^{-x}\,dx$

31. $\int xe^{3x^2}\,dx$

32. $\int 2xe^{x^2}\,dx$

33. $\int \frac{3x}{x^2 - 1}\,dx$

34. $\int \frac{-x}{2 - x^2}\,dx$

35. $\int \frac{x^2\,dx}{(x^3 + 5)^4}$

36. $\int (x^2 - 5x)^4(2x - 5)\,dx$

37. $\int \frac{x^3}{e^{3x^4}}\,dx$

38. $\int e^{3x^2+4}x\,dx$

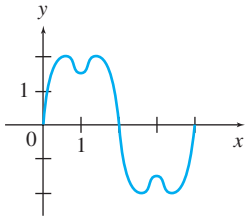
39. $\int \frac{(3\ln x + 2)^4}{x}\,dx$

40. $\int \frac{\sqrt{5\ln x + 3}}{x}\,dx$

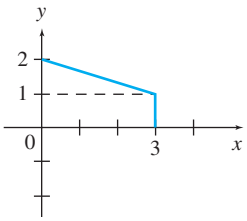
41. Let $f(x) = 3x + 1$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$, and $x_5 = 3$. Find $\sum_{i=1}^{5} f(x_i)$.

42. Find $\int_0^4 f(x)\,dx$ for each graph of $y = f(x)$.

a.



b.



43. Approximate the area under the graph of $f(x) = 2x + 3$ and above the x-axis from $x = 0$ to $x = 4$ using four rectangles. Let the height of each rectangle be the function value on the left side.

44. Find $\int_0^4 (2x + 3)\,dx$ by using the formula for the area of a trapezoid: $A = (1/2)(B + b)h$, where B and b are the lengths of the parallel sides and h is the distance between them. Compare with Exercise 43.

45. In Exercises 29 and 30 of the section on Area and the Definite Integral, you calculated the distance that a car traveled by estimating the integral $\int_0^T v(t)\,dt$.

a. Let $s(t)$ represent the mileage reading on the odometer. Express the distance traveled between $t = 0$ and $t = T$ using the function $s(t)$.

b. Since your answer to part a and the original integral both represent the distance traveled by the car, the two can be set equal. Explain why the resulting equation is a statement of the Fundamental Theorem of Calculus.

46. What does the Fundamental Theorem of Calculus state?

Find each definite integral.

47. $\int_1^2 (3x^2 + 5)\,dx$

48. $\int_1^6 (2x^2 + x)\,dx$

49. $\int_1^5 (3x^{-1} + x^{-3})\,dx$

50. $\int_1^3 (2x^{-1} + x^{-2})\,dx$

51. $\int_0^1 x\sqrt{5x^2 + 4}\,dx$

52. $\int_0^2 x^2(3x^3 + 1)^{1/3}\,dx$

53. $\int_0^2 3e^{-2x}\,dx$

54. $\int_1^5 \frac{5}{2}e^{0.4x}\,dx$

55. Use the substitution $u = 4x^2$ and the equation of a semicircle to evaluate

$$\int_0^{1/2} x\sqrt{1 - 16x^4}\,dx.$$

56. Use the substitution $u = x^2$ and the equation of a semicircle to evaluate

$$\int_0^{\sqrt{2}} 4x\sqrt{4 - x^4}\,dx.$$

In Exercises 57 and 58, use substitution to change the integral into one that can be evaluated by a formula from geometry, and then find the value of the integral.

57. $\int_1^{e^5} \frac{\sqrt{25 - (\ln x)^2}}{x}\,dx$

58. $\int_1^{\sqrt{7}} 2x\sqrt{36 - (x^2 - 1)^2}\,dx$

In Exercises 59–62, find the area between the x-axis and $f(x)$ over each of the given intervals.

59. $f(x) = \sqrt{4x - 3}$; $[1, 3]$

60. $f(x) = (3x + 2)^6$; $[-2, 0]$

61. $f(x) = xe^{x^2}$; $[0, 2]$

62. $f(x) = 1 + e^{-x}$; $[0, 4]$

Find the area of the region enclosed by each group of curves.

63. $f(x) = 5 - x^2$, $g(x) = x^2 - 3$

64. $f(x) = x^2 - 4x$, $g(x) = x - 6$

65. $f(x) = x^2 - 4x$, $g(x) = x + 6$, $x = -2$, $x = 4$

66. $f(x) = 5 - x^2$, $g(x) = x^2 - 3$, $x = 0$, $x = 4$

Use the trapezoidal rule with $n = 4$ to approximate the value of each integral. Then find the exact value and compare the two answers.

67. $\int_1^3 \frac{\ln x}{x}\,dx$

68. $\int_2^{10} \frac{x\,dx}{x - 1}$

69. $\int_0^1 e^x\sqrt{e^x + 4}\,dx$

70. $\int_0^2 xe^{-x^2}dx$

Use Simpson's rule with $n = 4$ to approximate the value of each integral. Compare your answers with the answers to Exercises 67–70.

71. $\int_1^3 \frac{\ln x}{x}\,dx$

72. $\int_2^{10} \frac{x\,dx}{x - 1}$

73. $\int_0^1 e^x\sqrt{e^x + 4}\,dx$

74. $\int_0^2 xe^{-x^2}dx$

75. Find the area of the region between the graphs of $y = \sqrt{x - 1}$ and $2y = x - 1$ from $x = 1$ to $x = 5$ in three ways.

a. Use antidifferentiation.

b. Use the trapezoidal rule with $n = 4$.

c. Use Simpson's rule with $n = 4$.

76. Find the area of the region between the graphs of $y = \frac{1}{x + 1}$ and $y = \frac{x + 2}{2}$ from $x = 0$ to $x = 4$ in three ways.

a. Use antidifferentiation.

b. Use the trapezoidal rule with $n = 4$.

c. Use Simpson's rule with $n = 4$.

77. Let $f(x) = [x(x-1)(x+1)(x-2)(x+2)]^2$.

a. Find $\int_{-2}^{2} f(x)\,dx$ using the trapezoidal rule with $n = 4$.

b. Find $\int_{-2}^{2} f(x)\,dx$ using Simpson's rule with $n = 4$.

c. Without evaluating $\int_{-2}^{2} f(x)\,dx$, explain why your answers to parts a and b cannot possibly be correct.

d. Explain why the trapezoidal rule and Simpson's rule with $n = 4$ give incorrect answers for $\int_{-2}^{2} f(x)\,dx$ with this function.

78. Given $\int_0^2 f(x)\,dx = 3$ and $\int_2^4 f(x)\,dx = 5$, calculate $\int_0^2 f(2x)\,dx$. Choose one of the following. ***Source: Society of Actuaries***.

a. 3/2 **b.** 3 **c.** 4 **d.** 6 **e.** 8

APPLICATIONS

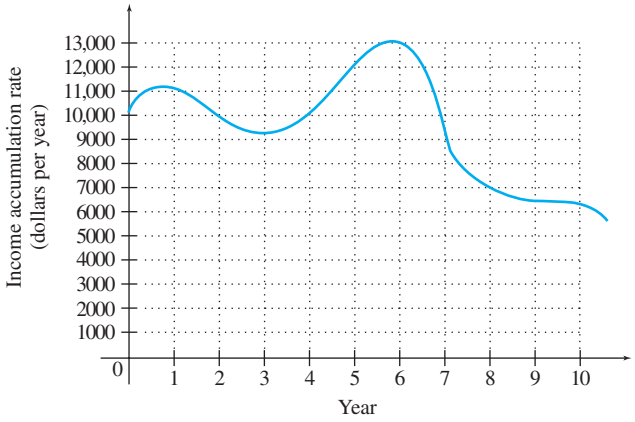
Business and Economics

Cost Find the cost function for each of the marginal cost functions in Exercises 79 and 80.

79. $C'(x) = 3\sqrt{2x-1}$; 13 units cost \$270.

80. $C'(x) = \dfrac{8}{2x+1}$; fixed cost is \$18.

81. Investment The curve shown gives the rate that an investment accumulates income (in dollars per year). Use rectangles of width 2 units and height determined by the function value at the midpoint to find the total income accumulated over 10 years.



82. Utilization of Reserves A manufacturer of electronic equipment requires a certain rare metal. He has a reserve supply of 4,000,000 units that he will not be able to replace. If the rate at which the metal is used is given by

$$f(t) = 100{,}000e^{0.03t},$$

where t is time (in years), how long will it be before he uses up the supply? (*Hint*: Find an expression for the total amount used in t years and set it equal to the known reserve supply.)

83. Sales The rate of change of sales of a new brand of tomato soup (in thousands of dollars per month) is given by

$$S'(x) = 3\sqrt{2x+1} + 3,$$

where x is the time (in months) that the new product has been on the market. Find the total sales after 4 months.

84. Productivity The function defined by

$$f'(x) = -0.1624x + 3.4909$$

approximates marginal U.S. nonfarm productivity from 2000–2009. Productivity is measured as total output per hour compared to a measure of 100 for 2000, and x is the number of years since 2000. ***Source: Bureau of Labor Statistics***.

a. Give the function that describes total productivity in year x.

b. Use your function from part a to find productivity at the end of 2008. In 2009, productivity actually measured 122.3. How does your value using the function compare with this?

85. Producers' and Consumers' Surplus Suppose that the supply function for some commodity is

$$S(q) = q^2 + 5q + 100$$

and the demand function for the commodity is

$$D(q) = 350 - q^2.$$

a. Find the producers' surplus.

b. Find the consumers' surplus.

86. Net Savings A company has installed new machinery that will produce a savings rate (in thousands of dollars per year) of

$$S'(x) = 225 - x^2,$$

where x is the number of years the machinery is to be used. The rate of additional costs (in thousands of dollars per year) to the company due to the new machinery is expected to be

$$C'(x) = x^2 + 25x + 150.$$

For how many years should the company use the new machinery? Find the net savings (in thousands of dollars) over this period.

87. Oil Production The following table shows the amount of crude oil (in billions of barrels) produced in the United States in recent years. ***Source: U.S. Energy Information Administration***.

Year	Crude Oil Produced
2000	2.131
2001	2.118
2002	2.097
2003	2.073
2004	1.983
2005	1.890
2006	1.862
2007	1.848
2008	1.812
2009	1.938

In this exercise we are interested in the total amount of crude oil produced over the 9-year period from mid-2000 to mid-2009, using the data for the 10 years above.

a. One approach is to sum up the numbers in the second column, but only count half of the first and last numbers. Give the answer to this calculation.

b. Approximate the amount of crude oil produced over the 9-year period 2000–2009 by taking the average of the left endpoint sum and the right endpoint sum. Explain why this is equivalent to the calculation done in part a.

c. Explain why the answer from part a is the same as using the trapezoidal rule to approximate the amount of crude oil produced over the 9-year period 2000–2009.

d. Find the equation of the least squares line for this data, letting $x = 0$ correspond to 2000. Then integrate this equation over the interval $[0, 9]$ to estimate the amount of crude oil produced over this time period. Compare with your answer to part a.

88. Inventory At time $t = 0$, a store has 19 units of a product in inventory. The cumulative number of units sold is given by $S(t) = e^{3t} - 1$, where t is measured in weeks. The inventory will be replenished when it drops to 1 unit. The cost of carrying inventory until then is 15 per unit per week (prorated for a portion of a week). Calculate the inventory carrying cost that will be incurred before the inventory is replenished. Choose one of the following. ***Source: Society of Actuaries***.

a. 90 **b.** 199 **c.** 204 **d.** 210 **e.** 294

Life Sciences

89. Population Growth The rate of change of the population of a rare species of Australian spider for one year is given by

$$f'(t) = 100 - t\sqrt{0.4t^2 + 1},$$

where $f(t)$ is the number of spiders present at time t (in months). Find the total number of additional spiders in the first 10 months.

90. Infection Rate The rate of infection of a disease (in people per month) is given by the function

$$I'(t) = \frac{100t}{t^2 + 1},$$

where t is the time (in months) since the disease broke out. Find the total number of infected people over the first four months of the disease.

91. Insect Cannibalism In certain species of flour beetles, the larvae cannibalize the unhatched eggs. In calculating the population cannibalism rate per egg, researchers needed to evaluate the integral

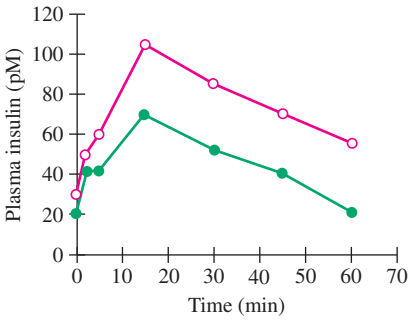
$$\int_0^A c(x)\,dx,$$

where A is the length of the larval stage and $c(x)$ is the cannibalism rate per egg per larva of age x. The minimum value of A for the flour beetle *Tribolium castaneum* is 17.6 days, which is the value we will use. The function $c(x)$ starts at day 0 with a value of 0, increases linearly to the value 0.024 at day 12, and then stays constant. ***Source: Journal of Animal Ecology***. Find the values of the integral using

a. formulas from geometry;

b. the Fundamental Theorem of Calculus.

92. Insulin in Sheep A research group studied the effect of a large injection of glucose in sheep fed a normal diet compared with sheep that were fasting. A graph of the plasma insulin levels (in pM—pico molars, or 10^{-12} of a molar) for both groups is shown below. The red graph designates the fasting sheep and the green graph the sheep fed a normal diet. The researchers compared the area under the curves for the two groups. ***Source: Endocrinology***.



a. For the fasting sheep, estimate the area under the curve using rectangles, first by using the left endpoints, then the right endpoints, and then averaging the two. Note that the width of the rectangles will vary.

b. Repeat part a for the sheep fed a normal diet.

c. How much higher is the area under the curve for the fasting sheep compared with the normal sheep?

93. Milk Production Researchers report that the average amount of milk produced (in kilograms per day) by a 4- to 5-year-old cow weighing 700 kg can be approximated by

$$y = 1.87t^{1.49}e^{-0.189(\ln t)^2},$$

where t is the number of days into lactation. ***Source: Journal of Dairy Science***.

a. Approximate the total amount of milk produced from $t = 1$ to $t = 321$ using the trapezoidal rule with $n = 8$.

b. Repeat part a using Simpson's rule with $n = 8$.

c. Repeat part a using the integration feature of a graphing calculator, and compare your answer with the answers to parts a and b.

Social Sciences

94. Automotive Accidents The table on the next page shows the amount of property damage (in dollars) due to automobile accidents in California in recent years. In this exercise we are interested in the total amount of property damage due to automobile accidents over the 8-year period from mid-2000 to mid-2008, using the data for the 9 years. ***Source: The California Highway Patrol***.

a. One approach is to sum up the numbers in the second column, but only count half of the first and last numbers. Give the answer to this calculation.

b. Approximate the amount of property damage over the 8-year period 2000–2008 by taking the average of the left endpoint sum and the right endpoint sum. Explain why this is equivalent to the calculation done in part a.

Year	Property Damage ($)
2000	309,569
2001	317,567
2002	335,869
2003	331,055
2004	331,208
2005	330,195
2006	325,453
2007	313,357
2008	278,986

c. Explain why the answer from part a is the same as using the trapezoidal rule to approximate the amount of property damage over the 8-year period 2000–2008.

d. Find the equation of the least squares line for this data, letting $x = 0$ correspond to 2000. Then integrate this equation over the interval [0, 8] to estimate the amount of property damage over this time period. Compare with your answer to part a.

Physical Sciences

95. Linear Motion A particle is moving along a straight line with velocity $v(t) = t^2 - 2t$. Its distance from the starting point after 3 seconds is 8 cm. Find $s(t)$, the distance of the particle from the starting point after t seconds.

EXTENDED APPLICATION

ESTIMATING DEPLETION DATES FOR MINERALS

It is becoming more and more obvious that the earth contains only a finite quantity of minerals. The "easy and cheap" sources of minerals are being used up, forcing an ever more expensive search for new sources. For example, oil from the North Slope of Alaska would never have been used in the United States during the 1930s because a great deal of Texas and California oil was readily available.

We said in an earlier chapter that population tends to follow an exponential growth curve. Mineral usage also follows such a curve. Thus, if q represents the rate of consumption of a certain mineral at time t, while q_0 represents consumption when $t = 0$, then

$$q = q_0 e^{kt},$$

where k is the growth constant. For example, the world consumption of petroleum in 1970 was 16,900 million barrels. During this period energy use was growing rapidly, and by 1975 annual world consumption had risen to 21,300 million barrels. We can use these two values to make a rough estimate of the constant k, and we find that over this 5-year span the average value of k was about 0.047, representing 4.7% annual growth. If we let $t = 0$ correspond to the base year 1970, then

$$q = 16{,}900e^{0.047t}$$

is the rate of consumption at time t, assuming that all the trends of the early 1970s have continued. In 1970 a reasonable guess would have put the total amount of oil in provable reserves or likely to be discovered in the future at 1,500,000 million barrels. At the 1970–1975 rate of consumption, in how many years after 1970 would you expect the world's reserves to be depleted? We can use the integral calculus of this chapter to find out. ***Source: Energy Information Administration***.

To begin, we need to know the total quantity of petroleum that would be used between time $t = 0$ and some future time $t = T$. Figure 34 on the following page shows a typical graph of the function $q = q_0 e^{kt}$.

Following the work we did in Section 3, divide the time interval from $t = 0$ to $t = T$ into n subintervals. Let each subinterval have width Δt. Let the rate of consumption for the ith subinterval be approximated by q_i^*. Thus, the approximate total consumption for the subinterval is given by

$$q_i^* \cdot \Delta t,$$

and the total consumption over the interval from time $t = 0$ to $t = T$ is approximated by

$$\sum_{i=1}^{n} q_i^* \cdot \Delta t.$$

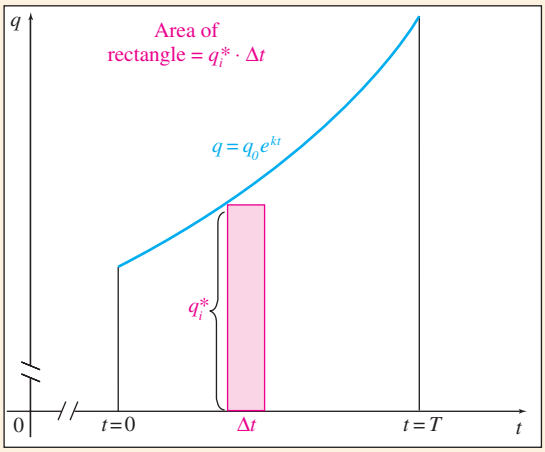


FIGURE 34

The limit of this sum as Δt approaches 0 gives the total consumption from time $t = 0$ to $t = T$. That is,

$$\text{Total consumption} = \lim_{\Delta t \to 0} \sum q_i^* \cdot \Delta t.$$

We have seen, however, that this limit is the definite integral of the function $q = q_0 e^{kt}$ from $t = 0$ to $t = T$, or

$$\text{Total consumption} = \int_0^T q_0 e^{kt}\, dt.$$

We can now evaluate this definite integral.

$$\begin{aligned}\int_0^T q_0 e^{kt}\, dt &= q_0 \int_0^T e^{kt}\, dt = q_0 \left(\frac{e^{kt}}{k} \right) \Bigg|_0^T \\ &= \frac{q_0}{k} e^{kt} \Bigg|_0^T = \frac{q_0}{k} e^{kT} - \frac{q_0}{k} e^0 \\ &= \frac{q_0}{k} e^{kT} - \frac{q_0}{k} (1) \\ &= \frac{q_0}{k} (e^{kT} - 1) \end{aligned} \qquad \textbf{(1)}$$

Now let us return to the numbers we gave for petroleum. We said that $q_0 = 16{,}900$ million barrels, where q_0 represents consumption in the base year of 1970. We have $k = 0.047$ with total petroleum reserves estimated at 1,500,000 million barrels. Thus, using Equation (1) we have

$$1{,}500{,}000 = \frac{16{,}900}{0.047} (e^{0.047T} - 1).$$

Multiply both sides of the equation by 0.047.

$$70{,}500 = 16{,}900 (e^{0.047T} - 1)$$

Divide both sides of the equation by 16,900.

$$4.2 = e^{0.047T} - 1$$

Add 1 to both sides.

$$5.2 = e^{0.047T}$$

Take natural logarithms of both sides.

$$\begin{aligned} \ln 5.2 &= \ln e^{0.047T} \\ &= 0.047T \end{aligned}$$

Finally,

$$T = \frac{\ln 5.2}{0.047} \approx 35.$$

By this result, petroleum reserves would only last 35 years after 1970, that is, until about 2005.

In fact, in the early 1970s some analysts were predicting that reserves would be exhausted before the end of the century, and this was a reasonable guess. But since 1970, more reserves have been discovered. One way to refine our model is to look at the historical data over a longer time span. The following table gives average world annual petroleum consumption in millions of barrels at 5-year intervals from 1970 to 2000. ***Source: Energy Information Administration***.

Year	World Consumption (in millions of barrels)
1970	16,900
1975	21,300
1980	22,900
1985	22,200
1990	24,300
1995	25,700
2000	27,900
2005	30,400

The first step in comparing this data with our exponential model is to estimate a value for the growth constant k. One simple way of doing this is to solve the equation

$$30{,}400 = 16{,}900 \cdot e^{k \cdot 35}.$$

Using natural logarithms just as we did in estimating the time to depletion for $k = 0.036$, we find that

$$k = \frac{\ln \left(\dfrac{30{,}400}{16{,}900} \right)}{35} \approx 0.017.$$

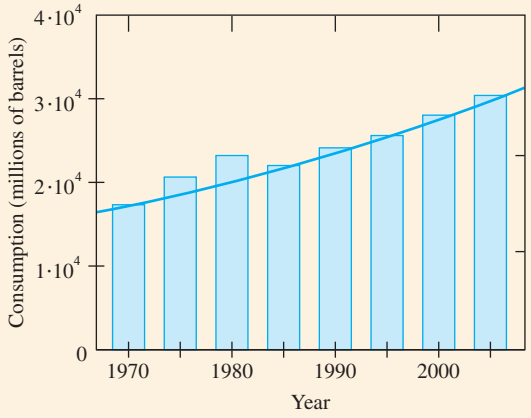


FIGURE 35

So the data from the Bureau of Transportation Statistics suggests a growth constant of about 1.7%. We can check the fit by plotting the function $16{,}900 \cdot e^{0.017t}$ along with a bar graph of the consumption data, shown in Figure 35. The fit looks reasonably good, but over this short range of 35 years, the exponential model is close to a linear model, and the growth in consumption is certainly not smooth.

The exponential model rests on the assumption of a constant growth rate. As already noted, we might expect instead that the growth rate would change as the world comes closer to exhausting its reserves. In particular, scarcity might drive up the price of oil and thus reduce consumption. We can use integration to explore an alternative model in which the factor k changes over time, so that k becomes $k(t)$, a function of time.

As an illustration, we explore a model in which the growth constant k declines toward 0 over time. We'll use 1970 as our base year, so the variable t will count years since 1970. We need a simple positive function $k(t)$ that tends toward 0 as t gets large. To get some numbers to work with, assume that the growth rate was 2% in 1970 and declined to 1% by 1995. There are many possible choices for the function $k(t)$, but a convenient one is

$$k(t) = \frac{0.5}{t + 25}.$$

Using integration to turn the instantaneous rate of consumption into the total consumption up to time T, we can write

$$\begin{aligned}\text{Total consumption} &= 16{,}900\int_0^T e^{k(t)\cdot t}\,dt \\ &= 16{,}900\int_0^T e^{0.5t/(t+25)}\,dt.\end{aligned}$$

We'd like to find out when the world will use up its estimated reserves, but as just noted, the estimates have increased since the 1970s. It is estimated that the current global petroleum reserves are 3,000,000 million barrels. ***Source: Geotimes***. So we need to solve

$$3{,}000{,}000 = 16{,}900\int_0^T e^{0.5t/(t+25)}\,dt \qquad \textbf{(2)}$$

But this problem is much harder to solve than the corresponding problem for constant growth, because *there is no formula for evaluating this definite integral!* The function

$$g(t) = e^{0.5t/(t+25)}$$

doesn't have an antiderivative that we can write down in terms of functions that we know how to compute.

Here the numerical integration techniques discussed in Section 6 come to the rescue. We can use one of the integration rules to *approximate* the integral numerically for various values of T, and with some trial and error we can estimate how long the reserves will last. If you have a calculator or computer algebra system that does numerical integration, you can pick some T values and evaluate the right-hand side of Equation (2). Here are the results produced by one computer algebra system:

For $T = 120$ the integral is about 2,797,000.

For $T = 130$ the integral is about 3,053,000.

For $T = 140$ the integral is about 3,311,000.

So using this model we would estimate that starting in 1970 the petroleum reserves would last for about 130 years, that is, until 2100.

Our integration tools are essential in building and exploring models of resource use, but the difference in our two predictions (35 years vs. 130 years) illustrates the difficulty of making accurate predictions. A model that performs well on historical data may not take the changing dynamics of resource use into account, leading to forecasts that are either unduly gloomy or too optimistic.

EXERCISES

1. Find the number of years that the estimated petroleum reserves would last if used at the same rate as in the base year.

2. How long would the estimated petroleum reserves last if the growth constant was only 2% instead of 4.7%?

Estimate the length of time until depletion for each mineral.

3. Bauxite (the ore from which aluminum is obtained): estimated reserves in base year 15,000,000 thousand tons; rate of consumption in base year 63,000 thousand tons; growth constant 6%

4. Bituminous coal: estimated world reserves 2,000,000 million tons; rate of consumption in base year 2200 million tons; growth constant 4%

5. a. Verify that the function $k(t)$ defined on the previous page has the right values at $k = 0$ and $k = 25$.

b. Find a similar function that has $k(0) = 0.03$ and $k(25) = 0.02$.

6. a. Use the function you defined in Exercise 5 b to write an integral for world petroleum consumption from 1970 until T years after 1970.

b. If you have access to a numerical integrator, compute some values of your integral and estimate the time required to exhaust the reserve of 3,000,000 million barrels.

7. A reasonable assumption is that over time scarcity might drive up the price of oil and thus reduce consumption. Comment on the fact that the rate of oil consumption actually increased in 2002, connecting current events and economic forecasts to the short-term possibility of a reduction in consumption.

8. Develop a spreadsheet that shows the time to exhaustion for various values of k.

9. Go to the website WolframAlpha.com and enter "integrate." Follow the instructions to find the time to exhaustion for various values of k. Discuss how the solution compares with the solutions provided by a graphing calculator and by Microsoft Excel.

DIRECTIONS FOR GROUP PROJECT

Suppose that you and three other students are spending a summer as interns for a local congresswoman. During your internship you realize that the information contained in your calculus class could be used to help with a new bill under consideration. The primary purpose of the bill is to require, by law, that all cars manufactured after a certain date get at least 60 miles per gallon of gasoline. Prepare a report that uses the information above to make a case for or against a bill of this nature.

16 Further Techniques and Applications of Integration

16.1 Integration by Parts

16.2 Volume and Average Value

16.3 Continuous Money Flow

16.4 Improper Integrals

16.5 Solutions of Elementary and Separable Differential Equations

Chapter 16 Review

Extended Application: Estimating Learning Curves in Manufacturing with Integrals

It might seem that definite integrals with infinite limits have only theoretical interest, but in fact these *improper* integrals provide answers to many practical questions. An example in Section 4 models an environmental cleanup process in which the amount of pollution entering a stream decreases by a constant fraction each year. An improper integral gives the total amount of pollutant that will ever enter the river.

In the previous chapter we discussed indefinite and definite integrals and presented rules for finding the antiderivatives of several types of functions. We showed how numerical methods can be used for functions that cannot be integrated by the techniques presented there. In this chapter we develop additional methods of integrating functions. We also show how to evaluate an integral that has one or both limits at infinity. These new techniques allow us to consider additional applications of integration such as volumes of solids of revolution, the average value of a function, and continuous money flow.

16.1 Integration by Parts

APPLY IT **If we know the rate of growth of a patch of moss, how can we calculate the area the moss covers?**

We will use integration by parts to answer this question in Exercise 42.

The technique of *integration by parts* often makes it possible to reduce a complicated integral to a simpler integral. We know that if u and v are both differentiable functions, then uv is also differentiable and, by the product rule for derivatives,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

This expression can be rewritten, using differentials, as

$$d(uv) = u\,dv + v\,du.$$

Integrating both sides of this last equation gives

$$\int d(uv) = \int u\,dv + \int v\,du,$$

or

$$uv = \int u\,dv + \int v\,du.$$

Rearranging terms gives the following formula.

Integration by Parts

If u and v are differentiable functions, then

$$\int u\,dv = uv - \int v\,du.$$

The process of finding integrals by this formula is called **integration by parts**. There are two ways to do integration by parts: the standard method and column integration. Both methods are illustrated in the following example.

EXAMPLE 1 Integration by Parts

Find $\int xe^{5x}\,dx$.

Method 1 Standard Method

SOLUTION

Although this integral cannot be found by using any method studied so far, it can be found with integration by parts. First write the expression $xe^{5x}\,dx$ as a product of two functions u and dv in such a way that $\int dv$ can be found. One way to do this is to choose

the two functions x and e^{5x}. Both x and e^{5x} can be integrated, but $\int x\,dx$, which is $x^2/2$, is more complicated than x itself, while the derivative of x is 1, which is simpler than x. Since e^{5x} remains the same (except for the coefficient) whether it is integrated or differentiated, it is best here to choose

$$dv = e^{5x}\,dx \qquad \text{and} \qquad u = x.$$

FOR REVIEW

In Section 15.2 on Substitution, we pointed out that when the chain rule is used to find the derivative of the function e^{kx}, we multiply by k, so when finding the antiderivative of e^{kx}, we divide by k. Thus $\int e^{5x}\,dx = e^{5x}/5 + C$. Keeping this technique in mind makes integration by parts simpler.

Then

$$du = dx,$$

and v is found by integrating dv:

$$v = \int dv = \int e^{5x}\,dx = \frac{e^{5x}}{5}.$$

We need not introduce the constant of integration until the last step, because only one constant is needed. Now substitute into the formula for integration by parts and complete the integration.

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{xe^{5x}}_{u\ \ dv}\,dx = \underbrace{x}_{u}\underbrace{\left(\frac{e^{5x}}{5}\right)}_{v} - \int \underbrace{\frac{e^{5x}}{5}}_{v}\,\underbrace{dx}_{du}$$

$$= \frac{xe^{5x}}{5} - \frac{e^{5x}}{25} + C$$

$$= \frac{e^{5x}}{25}(5x - 1) + C \qquad \text{Factor out } e^{5x}/25.$$

The constant C was added in the last step. As before, check the answer by taking its derivative.

Method 2 Column Integration

A technique called **column integration**, or *tabular integration*, is equivalent to integration by parts but helps in organizing the details.* We begin by creating two columns. The first column, labeled D, contains u, the part to be differentiated in the original integral. The second column, labeled I, contains the rest of the integral: that is, the part to be integrated, but without the dx. To create the remainder of the first column, write the derivative of the function in the first row underneath it in the second row. Now write the derivative of the function in the second row underneath it in the third row. Proceed in this manner down the first column, taking derivatives until you get a 0. Form the second column in a similar manner, except take an antiderivative at each row, until the second column has the same number of rows as the first.

To illustrate this process, consider our goal of finding $\int xe^{5x}\,dx$. Here $u = x$, so e^{5x} is left for the second column. Taking derivatives down the first column and antiderivatives down the second column results in the following table.

D	I
x	e^{5x}
1	$e^{5x}/5$
0	$e^{5x}/25$

*This technique appeared in the 1988 movie *Stand and Deliver*.

Next, draw a diagonal line from each term (except the last) in the left column to the term in the row below it in the right column. Label the first such line with "+", the next with "−", and continue alternating the signs as shown.

D		I
x	+	e^{5x}
1	−	$e^{5x}/5$
0		$e^{5x}/25$

Then multiply the terms on opposite ends of each diagonal line. Finally, sum up the products just formed, adding the "+" terms and subtracting the "−" terms.

$$\begin{aligned}\int xe^{5x}\,dx &= x(e^{5x}/5) - 1(e^{5x}/25) + C \\ &= \frac{xe^{5x}}{5} - \frac{e^{5x}}{25} + C \\ &= \frac{e^{5x}}{25}(5x - 1) + C \qquad \text{Factor out } e^{5x}/25.\end{aligned}$$

Compare these steps with those of Method 1 and convince yourself that the process is the same.

YOUR TURN 1
Find $\int xe^{-2x}\,dx$.

TRY YOUR TURN 1

Conditions for Integration by Parts

Integration by parts can be used only if the integrand satisfies the following conditions.

1. The integrand can be written as the product of two factors, u and dv.
2. It is possible to integrate dv to get v and to differentiate u to get du.
3. The integral $\int v\,du$ can be found.

EXAMPLE 2 Integration by Parts

Find $\int \ln x\,dx$ for $x > 0$.

Method 1 Standard Method

SOLUTION

No rule has been given for integrating $\ln x$, so choose

$$dv = dx \quad \text{and} \quad u = \ln x.$$

Then

$$v = x \quad \text{and} \quad du = \frac{1}{x}\,dx,$$

and, since $uv = vu$, we have

$$\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = \underbrace{x \ln x}_{v \cdot u} - \int \underbrace{x \cdot \frac{1}{x}\,dx}_{v \cdot du}$$

$$\begin{aligned}&= x \ln x - \int dx \\ &= x \ln x - x + C.\end{aligned}$$

Method 2
Column Integration

Column integration works a little differently here. As in Method 1, choose $\ln x$ as the part to differentiate. The part to be integrated must be 1. (Think of $\ln x$ as $1 \cdot \ln x$.) No matter how many times $\ln x$ is differentiated, the result is never 0. In this case, stop as soon as the natural logarithm is gone.

D	I
$\ln x$	1
$1/x$	x

Draw diagonal lines with alternating + and − as before. On the last line, because the left column does not contain a 0, draw a horizontal line.

D		I
$\ln x$	+	1
$1/x$	−	x

The presence of a horizontal line indicates that the product is to be integrated, just as the original integral was represented by the first row of the two columns.

$$\begin{aligned}\int \ln x\, dx &= (\ln x)x - \int \frac{1}{x} \cdot x\, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C.\end{aligned}$$

Note that when setting up the columns, a horizontal line is drawn only when a 0 does not eventually appear in the left column.

YOUR TURN 2
Find $\int \ln 2x\, dx$.

TRY YOUR TURN 2

Sometimes integration by parts must be applied more than once, as in the next example.

EXAMPLE 3 Integration by Parts

Find $\int (2x^2 + 5)e^{-3x}\, dx$.

Method 1
Standard Method

SOLUTION

Choose

$$dv = e^{-3x}\, dx \quad \text{and} \quad u = 2x^2 + 5.$$

Then

$$v = \frac{-e^{-3x}}{3} \quad \text{and} \quad du = 4x\, dx.$$

Substitute these values into the formula for integration by parts.

$$\int u\, dv = uv - \int v\, du$$

$$\begin{aligned}\int (2x^2 + 5)e^{-3x}\, dx &= (2x^2 + 5)\left(\frac{-e^{-3x}}{3}\right) - \int \left(\frac{-e^{-3x}}{3}\right) 4x\, dx \\ &= -(2x^2 + 5)\left(\frac{e^{-3x}}{3}\right) + \frac{4}{3}\int xe^{-3x}\, dx\end{aligned}$$

Now apply integration by parts to the last integral, letting

$$dv = e^{-3x}\, dx \quad \text{and} \quad u = x,$$

so

$$v = \frac{-e^{-3x}}{3} \quad \text{and} \quad du = dx.$$

$$\begin{aligned}
\int (2x^2 + 5)e^{-3x}\,dx &= -(2x^2 + 5)\left(\frac{e^{-3x}}{3}\right) + \frac{4}{3}\int xe^{-3x}\,dx \\
&= -(2x^2 + 5)\left(\frac{e^{-3x}}{3}\right) + \frac{4}{3}\left[x\left(\frac{-e^{-3x}}{3}\right) - \int\left(\frac{-e^{-3x}}{3}\right)dx\right] \\
&= -(2x^2 + 5)\left(\frac{e^{-3x}}{3}\right) + \frac{4}{3}\left[-\frac{x}{3}e^{-3x} - \left(\frac{e^{-3x}}{9}\right)\right] + C \\
&= -(2x^2 + 5)\left(\frac{e^{-3x}}{3}\right) - \frac{4}{9}xe^{-3x} - \frac{4}{27}e^{-3x} + C \\
&= [-(2x^2 + 5)(9) - 4x(3) - 4]\frac{e^{-3x}}{27} + C \qquad \text{Factor out } e^{-3x}/27. \\
&= (-18x^2 - 12x - 49)\frac{e^{-3x}}{27} + C \qquad \text{Simplify.}
\end{aligned}$$

Method 2
Column Integration

Choose $2x^2 + 5$ as the part to be differentiated, and put e^{-3x} in the integration column.

D		I
$2x^2 + 5$	+	e^{-3x}
$4x$	−	$-e^{-3x}/3$
4	+	$e^{-3x}/9$
0		$-e^{-3x}/27$

Multiplying and adding as before yields

$$\begin{aligned}
\int (2x^2 + 5)e^{-3x}\,dx &= (2x^2 + 5)(-e^{-3x}/3) - 4x(e^{-3x}/9) + 4(-e^{-3x}/27) + C \\
&= (-18x^2 - 12x - 49)\frac{e^{-3x}}{27} + C.
\end{aligned}$$

YOUR TURN 3

Find $\int (3x^2 + 4)e^{2x}\,dx$.

TRY YOUR TURN 3

With the functions discussed so far in this book, choosing u and dv (or the parts to be differentiated and integrated) is relatively simple. In general, the following strategy should be used.

First see if the integration can be performed using substitution. If substitution does not work:

- See if $\ln x$ is in the integral. If it is, set $u = \ln x$ and dv equal to the rest of the integral. (Equivalently, put $\ln x$ in the D column and the rest of the function in the I column.)
- If $\ln x$ is not present, see if the integral contains x^k, where k is any positive integer, or any other polynomial. If it does, set $u = x^k$ (or the polynomial) and dv equal to the rest of the integral. (Equivalently, put x^k in the D column and the rest of the function in the I column.)

EXAMPLE 4 Definite Integral

Find $\int_1^e \frac{\ln x}{x^2}\,dx$.

SOLUTION First find the indefinite integral using integration by parts by the standard method. (You may wish to verify this using column integration.) Whenever $\ln x$ is present, it is selected as u, so let

$$u = \ln x \quad \text{and} \quad dv = \frac{1}{x^2}\,dx.$$

FOR REVIEW

Recall that $\int x^n\,dx = x^{n+1}/(n+1) + C$ for all $n \neq -1$, so $\int 1/x^2\,dx = \int x^{-2}\,dx = x^{-1}/(-1) + C = -1/x + C$.

Then

$$du = \frac{1}{x}\,dx \qquad \text{and} \qquad v = -\frac{1}{x}.$$

Substitute these values into the formula for integration by parts, and integrate the second term on the right.

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{\ln x}{x^2}\,dx = (\ln x)\frac{-1}{x} - \int\left(-\frac{1}{x}\cdot\frac{1}{x}\right)dx$$

$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$= \frac{-\ln x - 1}{x} + C$$

Now find the definite integral.

$$\int_1^e \frac{\ln x}{x^2}\,dx = \left.\frac{-\ln x - 1}{x}\right|_1^e$$

$$= \left(\frac{-1-1}{e}\right) - \left(\frac{0-1}{1}\right)$$

$$= \frac{-2}{e} + 1 \approx 0.2642411177$$

YOUR TURN 4

Find $\int_1^e x^2 \ln x\,dx$.

TRY YOUR TURN 4

TECHNOLOGY NOTE

Definite integrals can be found with a graphing calculator using the function integral feature or by finding the area under the graph of the function between the limits. For example, using the `fnInt` feature of the TI-84 Plus calculator to find the integral in Example 4 gives 0.2642411177. Using the area under the graph approach gives 0.26424112, the same result rounded.

Many integrals cannot be found by the methods presented so far. For example, consider the integral

$$\int \frac{1}{4 - x^2}\,dx.$$

Substitution of $u = 4 - x^2$ will not help, because $du = -2x\,dx$, and there is no x in the numerator of the integral. We could try integration by parts, using $dv = dx$ and $u = (4 - x^2)^{-1}$. Integration gives $v = x$ and differentiation gives $du = 2x\,dx/(4 - x^2)^2$, with

$$\int \frac{1}{4 - x^2}\,dx = \frac{x}{4 - x^2} - \int \frac{2x^2}{(4 - x^2)^2}\,dx.$$

The integral on the right is more complicated than the original integral, however. A second use of integration by parts on the new integral would only make matters worse. Since we cannot choose $dv = (4 - x^2)^{-1}\,dx$ because it cannot be integrated by the methods studied so far, integration by parts is not possible for this problem.

This integration can be performed using one of the many techniques of integration beyond the scope of this text.* Tables of integrals can also be used, but technology is rapidly

*For example, see Thomas, George B., Maurice D. Weir, and Joel Hass, *Thomas' Calculus*, 12th ed., Pearson, 2010.

making such tables obsolete and even reducing the importance of techniques of integration. The following example shows how the table of integrals given in the appendix of this book may be used.

EXAMPLE 5 Tables of Integrals

Find $\int \frac{1}{4 - x^2}\,dx$.

SOLUTION Using formula 7 in the table of integrals in the appendix, with $a = 2$, gives

$$\int \frac{1}{4 - x^2}\,dx = \frac{1}{4} \cdot \ln\left|\frac{2 + x}{2 - x}\right| + C.$$

TRY YOUR TURN 5

YOUR TURN 5

Find $\int \frac{1}{x\sqrt{4 + x^2}}\,dx$.

TECHNOLOGY NOTE

We mentioned in the previous chapter how computer algebra systems and some calculators can perform integration. Using a TI-89, the answer to the above integral is

$$\frac{\ln\left(\frac{|x + 2|}{|x - 2|}\right)}{4}.$$

(The C is not included.) Verify that this is equivalent to the answer given in Example 5.

If you don't have a calculator or computer program that integrates symbolically, there is a Web site (http://integrals.wolfram.com), as of this writing, that finds indefinite integrals using the computer algebra system Mathematica. It includes instructions on how to enter your function. When the previous integral was entered, it returned the answer

$$\frac{1}{4}(\log(-x - 2) - \log(x - 2)).$$

Note that Mathematica does not include the C or the absolute value, and that natural logarithms are written as log. Verify that this answer is equivalent to the answer given by the TI-89 and the answer given in Example 5.

Unfortunately, there are integrals that cannot be antidifferentiated by any technique, in which case numerical integration must be used. (See the last section of the previous chapter.) In this book, for simplicity, all integrals to be antidifferentiated can be done with substitution or by parts, except for Exercises 23–28 in this section.

16.1 EXERCISES

Use integration by parts to find the integrals in Exercises 1–10.

1. $\int xe^x\,dx$

2. $\int (x + 6)e^x\,dx$

3. $\int (4x - 12)e^{-8x}\,dx$

4. $\int (6x + 3)e^{-2x}\,dx$

5. $\int x \ln x\,dx$

6. $\int x^3 \ln x\,dx$

7. $\int_0^1 \frac{2x + 1}{e^x}\,dx$

8. $\int_0^3 \frac{3 - x}{3e^x}\,dx$

9. $\int_1^9 \ln 3x\,dx$

10. $\int_1^2 \ln 5x\,dx$

11. Find the area between $y = (x - 2)e^x$ and the x-axis from $x = 2$ to $x = 4$.

12. Find the area between $y = (x + 1)\ln x$ and the x-axis from $x = 1$ to $x = e$.

Exercises 13–22 are mixed—some require integration by parts, while others can be integrated by using techniques discussed in the chapter on Integration.

13. $\int x^2e^{2x}\,dx$

14. $\int \frac{x^2\,dx}{2x^3 + 1}$

15. $\int x^2\sqrt{x + 4}\,dx$

16. $\int (2x - 1)\ln(3x)\,dx$

17. $\int (8x + 10)\ln(5x)\,dx$

18. $\int x^3e^{x^4}\,dx$

19. $\int_1^2 (1 - x^2)e^{2x}\,dx$

20. $\int_0^1 \frac{x^2\,dx}{2x^3 + 1}$

21. $\int_0^1 \frac{x^3\,dx}{\sqrt{3 + x^2}}$

22. $\int_0^5 x\sqrt[3]{x^2 + 2}\,dx$

Use the table of integrals, or a computer or calculator with symbolic integration capabilities, to find each indefinite integral.

23. $\int \frac{16}{\sqrt{x^2 + 16}}\,dx$

24. $\int \frac{10}{x^2 - 25}\,dx$

25. $\int \frac{3}{x\sqrt{121 - x^2}}\,dx$

26. $\int \frac{2}{3x(3x - 5)}\,dx$

27. $\int \frac{-6}{x(4x + 6)^2}\,dx$

28. $\int \sqrt{x^2 + 15}\,dx$

29. What rule of differentiation is related to integration by parts?

30. Explain why the two methods of solving Example 2 are equivalent.

31. Suppose that u and v are differentiable functions of x with $\int_0^1 v\,du = 4$ and the following functional values.

x	$u(x)$	$v(x)$
0	2	1
1	3	−4

Use this information to determine $\int_0^1 u\,dv$.

32. Suppose that u and v are differentiable functions of x with $\int_1^{20} v\,du = -1$ and the following functional values.

x	$u(x)$	$v(x)$
1	5	−2
20	15	6

Use this information to determine $\int_1^{20} u\,dv$.

33. Suppose we know that the functions r and s are everywhere differentiable and that $r(0) = 0$. Suppose we also know that for $0 \le x \le 2$, the area between the x-axis and the nonnegative function $h(x) = s(x)\frac{dr}{dx}$ is 5, and that on the same interval, the area between the x-axis and the nonnegative function $k(x) = r(x)\frac{ds}{dx}$ is 10. Determine $r(2)s(2)$.

34. Suppose we know that the functions u and v are everywhere differentiable and that $u(3) = 0$. Suppose we also know that for $1 \le x \le 3$, the area between the x-axis and the nonnegative function $h(x) = u(x)\frac{dv}{dx}$ is 15, and that on the same interval, the area between the x-axis and the nonnegative function $k(x) = v(x)\frac{du}{dx}$ is 20. Determine $u(1)v(1)$.

35. Use integration by parts to derive the following formula from the table of integrals.

$$\int x^n \cdot \ln|x|\,dx = x^{n+1}\left[\frac{\ln|x|}{n+1} - \frac{1}{(n+1)^2}\right] + C, \quad n \neq -1$$

36. Use integration by parts to derive the following formula from the table of integrals.

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx + C, \quad a \neq 0$$

37. **a.** One way to integrate $\int x\sqrt{x+1}\,dx$ is to use integration by parts. Do so to find the antiderivative.

b. Another way to evaluate the integral in part a is by using the substitution $u = x + 1$. Do so to find the antiderivative.

c. Compare the results from the two methods. If they do not look the same, explain how this can happen. Discuss the advantages and disadvantages of each method.

38. Using integration by parts,

$$\begin{aligned}\int \frac{1}{x}\,dx &= \int \frac{1}{x}\cdot 1\,dx \\ &= \frac{1}{x}\cdot x - \int\left(-\frac{1}{x^2}\right)x\,dx \\ &= 1 + \int \frac{1}{x}\,dx.\end{aligned}$$

Subtracting $\int \frac{1}{x}\,dx$ from both sides we conclude that $0 = 1$. What is wrong with this logic? ***Source: Sam Northshield.***

APPLICATIONS

Business and Economics

39. **Rate of Change of Revenue** The rate of change of revenue (in dollars per calculator) from the sale of x calculators is

$$R'(x) = (x + 1)\ln(x + 1).$$

Find the total revenue from the sale of the first 12 calculators. (*Hint:* In this exercise, it simplifies matters to write an antiderivative of $x + 1$ as $(x + 1)^2/2$ rather than $x^2/2 + x$.)

Life Sciences

40. **Reaction to a Drug** The rate of reaction to a drug is given by

$$r'(t) = 2t^2e^{-t},$$

where t is the number of hours since the drug was administered. Find the total reaction to the drug from $t = 1$ to $t = 6$.

41. **Growth of a Population** The rate of growth of a microbe population is given by

$$m'(t) = 27te^{3t},$$

where t is time in days. What is the total accumulated growth during the first 2 days?

42. **APPLY IT** **Rate of Growth** The area covered by a patch of moss is growing at a rate of

$$A'(t) = \sqrt{t}\ln t$$

cm^2 per day, for $t \ge 1$. Find the additional amount of area covered by the moss between 4 and 9 days.

43. Thermic Effect of Food As we saw in an earlier chapter, a person's metabolic rate tends to go up after eating a meal and then, after some time has passed, it returns to a resting metabolic rate. This phenomenon is known as the thermic effect of food, and the effect (in kJ per hour) for one individual is

$$F(t) = -10.28 + 175.9te^{-t/1.3},$$

where t is the number of hours that have elapsed since eating a meal. ***Source: American Journal of Clinical Nutrition***. Find the total thermic energy of a meal for the next six hours after a meal by integrating the thermic effect function between $t = 0$ and $t = 6$.

44. Rumen Fermentation The rumen is the first division of the stomach of a ruminant, or cud-chewing animal. An article on the rumen microbial system reports that the fraction of the soluble material passing from the rumen without being fermented during the first hour after its ingestion could be calculated by the integral

$$\int_0^1 ke^{-kt}(1 - t)dt,$$

where k measures the rate that the material is fermented. ***Source: Annual Review of Ecology and Systematics***.

a. Determine the above integral, and evaluate it for the following values of k used in the article: $1/12$, $1/24$, and $1/48$ hour.

b. The fraction of intermediate material left in the rumen at 1 hour that escapes digestion by passage between 1 and 6 hours is given by

$$\int_1^6 ke^{-kt}(6 - t)/5\,dt.$$

Determine this integral, and evaluate it for the values of k given in part a.

YOUR TURN ANSWERS

1. $-e^{-2x}(2x + 1)/4 + C$ **2.** $x \ln 2x - x + C$

3. $(6x^2 - 6x + 11)e^{2x}/4 + C$ **4.** $(2e^3 + 1)/9$

5. $(-1/2) \ln |(2 + \sqrt{4 + x^2})/x| + C$

16.2 Volume and Average Value

APPLY IT **If we have a formula giving the price of a common stock as a function of time, how can we find the average price of the stock over a certain period of time?**

We will answer this question in Example 4 using concepts developed in this section, in which we will discover how to find the average value of a function, as well as how to compute the volume of a solid.

Volume

Figure 1 shows the regions below the graph of some function $y = f(x)$, above the x-axis, and between $x = a$ and $x = b$. We have seen how to use integrals to find the area of such a region. Suppose this region is revolved about the x-axis as shown in Figure 2. The resulting figure is called a **solid of revolution**. In many cases, the volume of a solid of revolution can be found by integration.

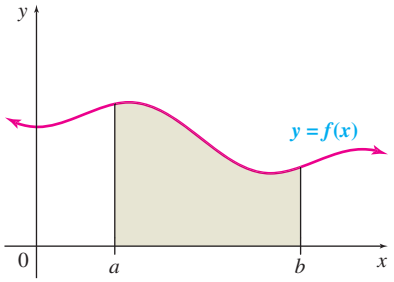


FIGURE 1

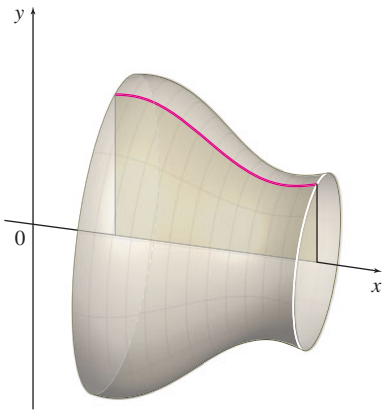


FIGURE 2

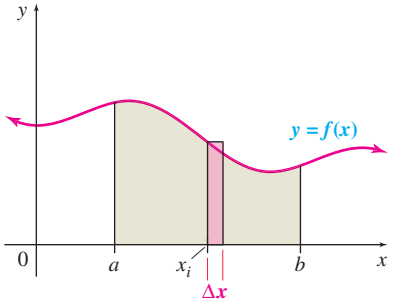
To begin, divide the interval $[a, b]$ into n subintervals of equal width Δx by the points $a = x_0, x_1, x_2, \ldots, x_i, \ldots, x_n = b$. Then think of slicing the solid into n slices of equal thickness Δx, as shown in Figure 3(a). If the slices are thin enough, each slice is very close to being a right circular cylinder, as shown in Figure 3(b). The formula for the volume of a right circular cylinder is $\pi r^2 h$, where r is the radius of the circular base and h is the height of the cylinder. As shown in Figure 4, the height of each slice is Δx. (The height is horizontal here, since the cylinder is on its side.) The radius of the circular base of each slice is $f(x_i)$. Thus, the volume of the slice is closely approximated by $\pi[f(x_i)]^2\Delta x$. The volume of the solid of revolution will be approximated by the sum of the volumes of the slices:

$$V \approx \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x.$$

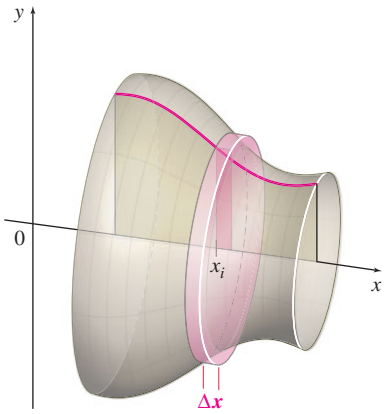
By definition, the volume of the solid of revolution is the limit of this sum as the thickness of the slices approaches 0, or

$$V = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x.$$

This limit, like the one discussed earlier for area, is a definite integral.



(a)



(b)

FIGURE 3

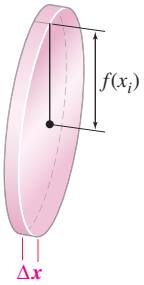


FIGURE 4

Volume of a Solid of Revolution

If $f(x)$ is nonnegative and R is the region between $f(x)$ and the x-axis from $x = a$ to $x = b$, the volume of the solid formed by rotating R about the x-axis is given by

$$V = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \int_a^b \pi[f(x)]^2 \, dx.$$

The technique of summing disks to approximate volumes was originated by Johannes Kepler (1571–1630), a famous German astronomer who discovered three laws of planetary motion. He estimated volumes of wine casks used at his wedding by means of solids of revolution.

EXAMPLE 1 Volume

Find the volume of the solid of revolution formed by rotating about the x-axis the region bounded by $y = x + 1$, $y = 0$, $x = 1$, and $x = 4$.

SOLUTION The region and the solid are shown in Figure 5. Notice that the orientation of the x-axis is slightly different in Figure 5(b) than in Figure 5(a) to emphasize the three-dimensionality of the figure. Use the formula given above for the volume, with $a = 1$, $b = 4$, and $f(x) = x + 1$.

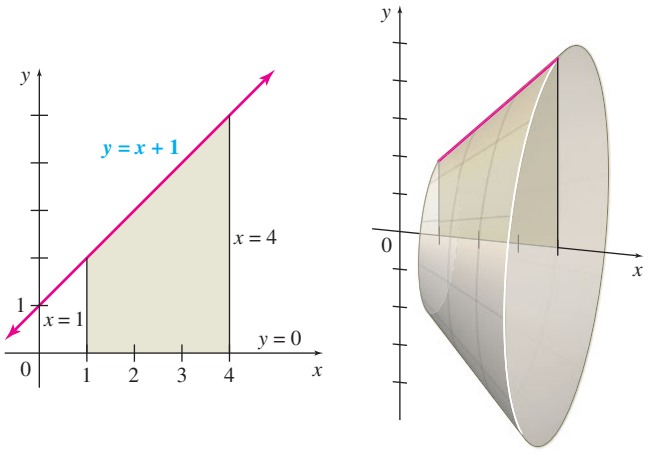
$$V = \int_1^4 \pi(x+1)^2\,dx = \pi\left[\frac{(x+1)^3}{3}\right]\Bigg|_1^4$$

$$= \frac{\pi}{3}(5^3 - 2^3)$$

$$= \frac{117\pi}{3} = 39\pi$$

YOUR TURN 1 Find the volume of the solid of revolution formed by rotating about the x-axis the region bounded by $y = x^2 + 1$, $y = 0$, $x = -1$, and $x = 1$.

TRY YOUR TURN 1



(a) (b)

FIGURE 5

EXAMPLE 2 Volume

Find the volume of the solid of revolution formed by rotating about the x-axis the area bounded by $f(x) = 4 - x^2$ and the x-axis.

SOLUTION The region and the solid are shown in Figure 6 on the next page. Find a and b from the x-intercepts. If $y = 0$, then $x = 2$ or $x = -2$, so that $a = -2$ and $b = 2$. The volume is

$$V = \int_{-2}^{2} \pi(4 - x^2)^2\,dx$$

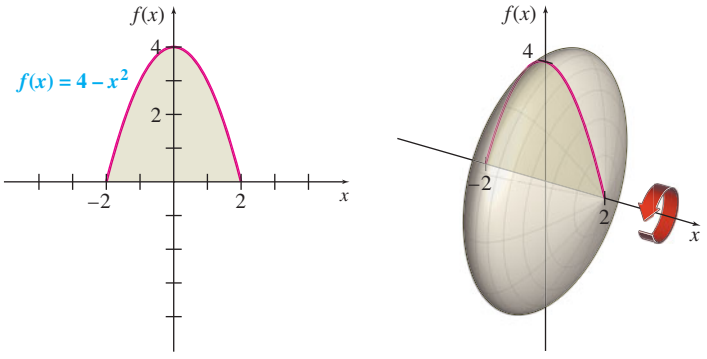
$$= \int_{-2}^{2} \pi(16 - 8x^2 + x^4)\,dx$$

$$= \pi\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Bigg|_{-2}^{2}$$

$$= \frac{512\pi}{15}.$$

TECHNOLOGY NOTE

A graphing calculator with the `fnInt` feature gives the value as 107.2330292, which agrees with the approximation of $512\pi/15$ to the 7 decimal places shown.



(a) (b)

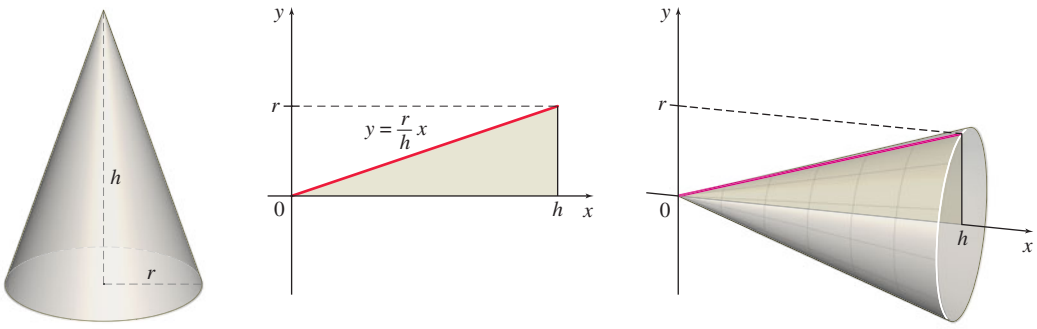
FIGURE 6

EXAMPLE 3 Volume

Find the volume of a right circular cone with height h and base radius r.

SOLUTION Figure 7(a) shows the required cone, while Figure 7(b) shows an area that could be rotated about the x-axis to get such a cone. The cone formed by the rotation is shown in Figure 7(c). Here $y = f(x)$ is the equation of the line through $(0, 0)$ and (h, r). The slope of this line is r/h, and since the y-intercept is 0, the equation of the line is

$$y = \frac{r}{h}x.$$



(a) (b) (c)

FIGURE 7

Then the volume is

$$V = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx = \pi\int_0^h \frac{r^2x^2}{h^2}\,dx$$

$$= \pi\frac{r^2x^3}{3h^2}\Big|_0^h \qquad \text{Since } r \text{ and } h \text{ are constants}$$

$$= \frac{\pi r^2 h}{3}.$$

This is the familiar formula for the volume of a right circular cone.

Average Value of a Function

The average of the n numbers $v_1, v_2, v_3, \ldots, v_i, \ldots, v_n$ is given by

$$\frac{v_1 + v_2 + v_3 + \cdots + v_n}{n} = \frac{\sum_{i=1}^{n} v_i}{n}.$$

For example, to compute an average temperature, we could take readings at equally spaced intervals and average the readings.

The average value of a function f on $[a, b]$ can be defined in a similar manner; divide the interval $[a, b]$ into n subintervals, each of width Δx. Then choose an x-value, x_i, in each subinterval, and find $f(x_i)$. The average function value for the n subintervals and the given choices of x_i is

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{\sum_{i=1}^{n} f(x_i)}{n}.$$

Since $(b - a)/n = \Delta x$, multiply the expression on the right side of the equation by $(b - a)/(b - a)$ and rearrange the expression to get

$$\frac{b - a}{b - a} \cdot \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{b - a} \sum_{i=1}^{n} f(x_i) \left(\frac{b - a}{n} \right) = \frac{1}{b - a} \sum_{i=1}^{n} f(x_i) \Delta x.$$

Now, take the limit as $n \to \infty$. If the limit exists, then

$$\lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} f(x_i) \Delta x = \frac{1}{b - a} \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

The following definition summarizes this discussion.

Average Value of a Function

The **average value of a function** f on the interval $[a, b]$ is

$$\frac{1}{b - a} \int_a^b f(x)\, dx,$$

provided the indicated definite integral exists.

In Figure 8 the quantity $\bar{y}$ represents the average height of the irregular region. The average height can be thought of as the height of a rectangle with base $b - a$. For $f(x) \geq 0$, this rectangle has area $\bar{y}(b - a)$, which equals the area under the graph of $f(x)$ from $x = a$ to $x = b$, so that

$$\bar{y}(b - a) = \int_a^b f(x)\, dx.$$

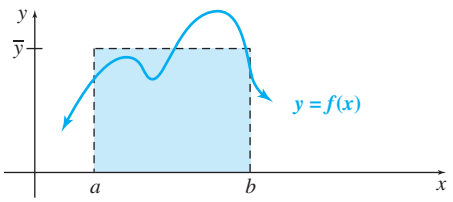


FIGURE 8

EXAMPLE 4 Average Price

A stock analyst plots the price per share of a certain common stock as a function of time and finds that it can be approximated by the function

$$S(t) = 25 - 5e^{-0.01t},$$

where t is the time (in years) since the stock was purchased. Find the average price of the stock over the first six years.

APPLY IT

SOLUTION Use the formula for average value with $a = 0$ and $b = 6$. The average price is

$$\begin{aligned}\frac{1}{6-0}\int_0^6 (25 - 5e^{-0.01t})\,dt &= \frac{1}{6}\left(25t - \frac{5}{-0.01}e^{-0.01t}\right)\Big|_0^6 \\ &= \frac{1}{6}(25t + 500e^{-0.01t})\Big|_0^6 \\ &= \frac{1}{6}(150 + 500e^{-0.06} - 500) \\ &\approx 20.147,\end{aligned}$$

or approximately \$20.15.

YOUR TURN 2 Find the average value of the function $f(x) = x + \sqrt{x}$ on the interval $[1, 4]$.

TRY YOUR TURN 2

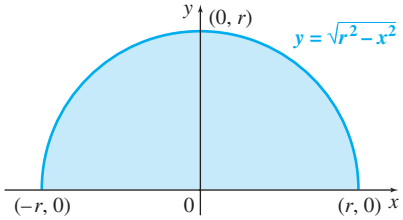
16.2 EXERCISES

Find the volume of the solid of revolution formed by rotating about the x-axis each region bounded by the given curves.

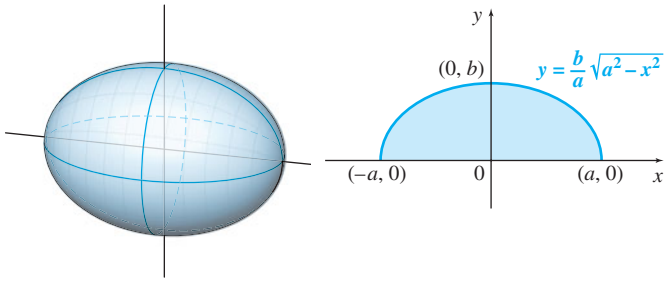
1. $f(x) = x, \quad y = 0, \quad x = 0, \quad x = 3$
2. $f(x) = 3x, \quad y = 0, \quad x = 0, \quad x = 2$
3. $f(x) = 2x + 1, \quad y = 0, \quad x = 0, \quad x = 4$
4. $f(x) = x - 4, \quad y = 0, \quad x = 4, \quad x = 10$
5. $f(x) = \frac{1}{3}x + 2, \quad y = 0, \quad x = 1, \quad x = 3$
6. $f(x) = \frac{1}{2}x + 4, \quad y = 0, \quad x = 0, \quad x = 5$
7. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 1, \quad x = 4$
8. $f(x) = \sqrt{x + 5}, \quad y = 0, \quad x = 1, \quad x = 3$
9. $f(x) = \sqrt{2x + 1}, \quad y = 0, \quad x = 1, \quad x = 4$
10. $f(x) = \sqrt{4x + 2}, \quad y = 0, \quad x = 0, \quad x = 2$
11. $f(x) = e^x, \quad y = 0, \quad x = 0, \quad x = 2$
12. $f(x) = 2e^x, \quad y = 0, \quad x = -2, \quad x = 1$
13. $f(x) = \frac{2}{\sqrt{x}}, \quad y = 0, \quad x = 1, \quad x = 3$
14. $f(x) = \frac{2}{\sqrt{x + 2}}, \quad y = 0, \quad x = -1, \quad x = 2$
15. $f(x) = x^2, \quad y = 0, \quad x = 1, \quad x = 5$
16. $f(x) = \frac{x^2}{2}, \quad y = 0, \quad x = 0, \quad x = 4$
17. $f(x) = 1 - x^2, \quad y = 0$
18. $f(x) = 2 - x^2, \quad y = 0$

The function defined by $y = \sqrt{r^2 - x^2}$ has as its graph a semicircle of radius r with center at $(0, 0)$ (see the figure). In Exercises 19–21, find the volume that results when each semicircle is rotated about the x-axis. (The result of Exercise 21 gives a formula for the volume of a sphere with radius r.)

19. $f(x) = \sqrt{1 - x^2}$
20. $f(x) = \sqrt{36 - x^2}$
21. $f(x) = \sqrt{r^2 - x^2}$



22. Find a formula for the volume of an ellipsoid. See Exercises 19–21 and the following figures.



Ellipsoid

23. Use the methods of this section to find the volume of a cylinder with height h and radius r.

Find the average value of each function on the given interval.

24. $f(x) = 2 - 3x^2$; $[1, 3]$ **25.** $f(x) = x^2 - 4$; $[0, 5]$

26. $f(x) = (2x - 1)^{1/2}$; $[1, 13]$

27. $f(x) = \sqrt{x + 1}$; $[3, 8]$ **28.** $f(x) = e^{0.1x}$; $[0, 10]$

29. $f(x) = e^{x/7}$; $[0, 7]$ **30.** $f(x) = x \ln x$; $[1, e]$

31. $f(x) = x^2e^{2x}$; $[0, 2]$

In Exercises 32 and 33, use the integration feature on a graphing calculator to find the volume of the solid of revolution by rotating about the x-axis each region bounded by the given curves.

32. $f(x) = \dfrac{1}{4 + x^2}$, $y = 0$, $x = -2$, $x = 2$

33. $f(x) = e^{-x^2}$, $y = 0$, $x = -1$, $x = 1$

APPLICATIONS

Business and Economics

34. Average Price Otis Taylor plots the price per share of a stock that he owns as a function of time and finds that it can be approximated by the function

$$S(t) = t(25 - 5t) + 18,$$

where t is the time (in years) since the stock was purchased. Find the average price of the stock over the first five years.

35. Average Price A stock analyst plots the price per share of a certain common stock as a function of time and finds that it can be approximated by the function

$$S(t) = 37 + 6e^{-0.03t},$$

where t is the time (in years) since the stock was purchased. Find the average price of the stock over the first six years.

36. Average Inventory The Yasuko Okada Fragrance Company (YOFC) receives a shipment of 400 cases of specialty perfume early Monday morning of every week. YOFC sells the perfume to retail outlets in California at a rate of about 80 cases per day during each business day (Monday through Friday). What is the average daily inventory for YOFC? (*Hint:* Find a function that represents the inventory for any given business day and then integrate.)

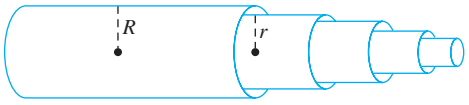
37. Average Inventory The DeMarco Pasta Company receives 600 cases of imported San Marzano tomato sauce every 30 days. The number of cases of sauce on inventory t days after the shipment arrives is

$$N(t) = 600 - 20\sqrt{30t}.$$

Find the average daily inventory.

Life Sciences

38. Blood Flow The figure shows the blood flow in a small artery of the body. The flow of blood is *laminar* (in layers), with the velocity very low near the artery walls and highest in the center of the artery. In this model of blood flow, we calculate the total flow in the artery by thinking of the flow as being made up of many layers of concentric tubes sliding one on the other.



Suppose R is the radius of an artery and r is the distance from a given layer to the center. Then the velocity of blood in a given layer can be shown to equal

$$v(r) = k(R^2 - r^2),$$

where k is a numerical constant.

Since the area of a circle is $A = \pi r^2$, the change in the area of the cross section of one of the layers, corresponding to a small change in the radius, Δr, can be approximated by differentials. For $dr = \Delta r$, the differential of the area A is

$$dA = 2\pi r\, dr = 2\pi r\, \Delta r,$$

where Δr is the thickness of the layer. The total flow in the layer is defined to be the product of velocity and cross-section area, or

$$F(r) = 2\pi rk(R^2 - r^2)\Delta r.$$

a. Set up a definite integral to find the total flow in the artery.

b. Evaluate this definite integral.

39. Drug Reaction The intensity of the reaction to a certain drug, in appropriate units, is given by

$$R(t) = te^{-0.1t},$$

where t is time (in hours) after the drug is administered. Find the average intensity during the following hours.

a. Second hour

b. Twelfth hour

c. Twenty-fourth hour

40. Bird Eggs The average length and width of various bird eggs are given in the following table. ***Source: NCTM.***

Bird Name	Length (cm)	Width (cm)
Canada goose	8.6	5.8
Robin	1.9	1.5
Turtledove	3.1	2.3
Hummingbird	1.0	1.0
Raven	5.0	3.3

a. Assume for simplicity that a bird's egg is roughly the shape of an ellipsoid. Use the result of Exercise 22 to estimate the volume of an egg of each bird.

i. Canada goose

ii. Robin

iii. Turtledove

iv. Hummingbird

v. Raven

b. In Exercise 12 of Section 1.3, we showed that the average length (in centimeters) of an egg of width w cm is given by

$$l = 1.585w - 0.487.$$

Using this result and the ideas in part a, show that the average volume of an egg of width w centimeters is given by

$$V = \pi(1.585w^3 - 0.487w^2)/6.$$

Use this formula to calculate the average volume for the bird eggs in part a, and compare with your results from part a.

Social Sciences

41. Production Rate Suppose the number of items a new worker on an assembly line produces daily after t days on the job is given by

$$I(t) = 45\ln(t + 1).$$

Find the average number of items produced daily by this employee after the following numbers of days.

a. 5 **b.** 9 **c.** 30

42. Typing Speed The function $W(t) = -3.75t^2 + 30t + 40$ describes a typist's speed (in words per minute) over a time interval $[0, 5]$.

a. Find $W(0)$.

b. Find the maximum W value and the time t when it occurs.

c. Find the average speed over $[0, 5]$.

Physical Sciences

43. Earth's Volume Most people assume that the Earth has a spherical shape. It is actually more of an ellipsoid shape, but not an exact ellipsoid, since there are numerous mountains and valleys. Researchers have found that a *datum*, or a reference ellipsoid, that is offset from the center of the Earth can be used to accurately map different regions. According to one datum, called the Geodetic Reference System 1980, this reference ellipsoid assumes an equatorial radius of 6,378,137 m and a polar radius of 6,356,752.3141 m. ***Source: Geodesy Information System***. Use the result of Exercise 22 to estimate the volume of the Earth.

YOUR TURN ANSWERS

1. $56\pi/15$ **2.** $73/18$

16.3 Continuous Money Flow

APPLY IT **Given a changing rate of annual income and a certain rate of interest, how can we find the present value of the income?**

We will answer this question in Example 2 using the concept of continuous money flow.

In an earlier chapter we looked at the concepts of present value and future value when a lump sum of money is deposited in an account and allowed to accumulate interest. In some situations, however, money flows into and out of an account almost continuously over a period of time. Examples include income in a store, bank receipts and payments, and highway tolls. Although the flow of money in such cases is not exactly continuous, it can be treated as though it were continuous, with useful results.

EXAMPLE 1 Total Income

The income from a soda machine (in dollars per year) is growing exponentially. When the machine was first installed, it was producing income at a rate of \$500 per year. By the end of the first year, it was producing income at a rate of \$510.10 per year. Find the total income produced by the machine during its first 3 years of operation.

SOLUTION Let t be the time (in years) since the installation of the machine. The assumption of exponential growth, coupled with the initial value of 500, implies that the rate of change of income is of the form

$$f(t) = 500e^{kt},$$

where k is some constant. To find k, use the value at the end of the first year.

$$\begin{aligned} f(1) &= 500e^{k(1)} = 510.10 \\ e^k &= 1.0202 \qquad \text{Divide by 500.} \\ k &= \ln 1.0202 \qquad \text{Take ln of both sides.} \\ &\approx 0.02 \end{aligned}$$

Therefore, we have

$$f(t) = 500e^{0.02t}.$$

Since the rate of change of incomes is given, the total income can be determined by using the definite integral.

$$\begin{aligned} \text{Total income} &= \int_0^3 500e^{0.02t}\,dt \\ &= \frac{500}{0.02}e^{0.02t}\Big|_0^3 \\ &= 25{,}000e^{0.02t}\Big|_0^3 = 25{,}000(e^{0.06} - 1) \approx 1545.91 \end{aligned}$$

Thus, the soda machine will produce \$1545.91 total income in its first three years of operation. TRY YOUR TURN 1

YOUR TURN 1 Find the total income over the first 2 years in Example 1 with the initial rate changed to \$810 per year and the rate at the end of the first year changed to \$797.94 per year.

The money in Example 1 is not received as a one-time lump sum payment of \$1545.91. Instead, it comes in on a regular basis, perhaps daily, weekly, or monthly. In discussions of such problems it is usually assumed that the income is received continuously over a period of time.

Total Money Flow

Let the continuous function $f(t)$ represent the rate of flow of money per unit of time. If t is in years and $f(t)$ is in dollars per year, the area under $f(t)$ between two points in time gives the total dollar flow over the given time interval.

The function $f(t) = 2000$, shown in Figure 9, represents a uniform rate of money flow of \$2000 per year. The graph of this money flow is a horizontal line; the *total money flow* over a specified time T is given by the rectangular area below the graph of $f(t)$ and above the t-axis between $t = 0$ and $t = T$. For example, the total money flow over $T = 5$ years would be $2000(5) = 10{,}000$, or \$10,000.

The area in the uniform rate example could be found by using an area formula from geometry. For a variable function like the function in Example 1, however, a definite integral is needed to find the total money flow over a specific time interval. For the function

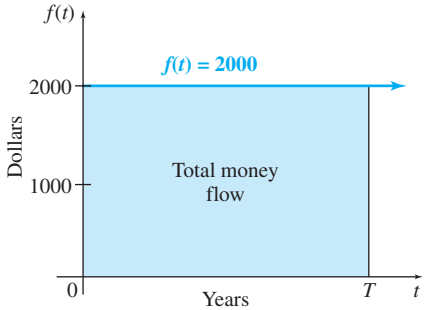


FIGURE 9

$f(t) = 2000e^{0.08t}$, for example, the total money flow over a 5-year period would be given by

$$\int_0^5 2000e^{0.08t}\,dt \approx 12{,}295.62,$$

or \$12,295.62. See Figure 10.

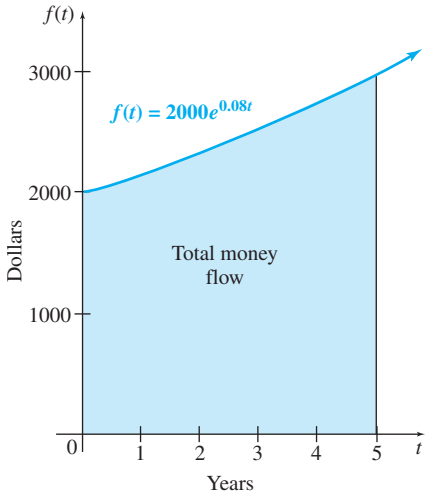


FIGURE 10

Total Money Flow

If $f(t)$ is the rate of money flow, then the **total money flow** over the time interval from $t = 0$ to $t = T$ is given by

$$\int_0^T f(t)\,dt.$$

This "total money flow" does not take into account the interest the money could earn after it is received. It is simply the total income.

Present Value of Money Flow

As mentioned earlier, an amount of money that can be deposited today at a specified interest rate to yield a given sum in the future is called the *present value* of this future sum. The future sum may be called the *future value* or *final amount*. To find the **present value of a continuous money flow** with interest compounded continuously, let $f(t)$ represent the rate of the continuous flow. In Figure 11 on the next page, the time axis from 0 to T is divided into n subintervals, each of width Δt. The amount of money that flows during any interval of time is given by the area between the t-axis and the graph of $f(t)$ over the specified time interval. The area of each subinterval is approximated by the area of a rectangle with height $f(t_i)$, where t_i is the left endpoint of the ith subinterval. The area of each rectangle is $f(t_i)\Delta t$, which (approximately) gives the amount of money flow over that subinterval.

Earlier, we saw that the present value P of an amount A compounded continuously for t years at a rate of interest r is $P = Ae^{-rt}$. Letting t_i represent the time and replacing A with $f(t_i)\Delta t$, the present value of the money flow over the ith subinterval is approximately equal to

$$P_i = [f(t_i)\Delta t]e^{-rt_i}.$$

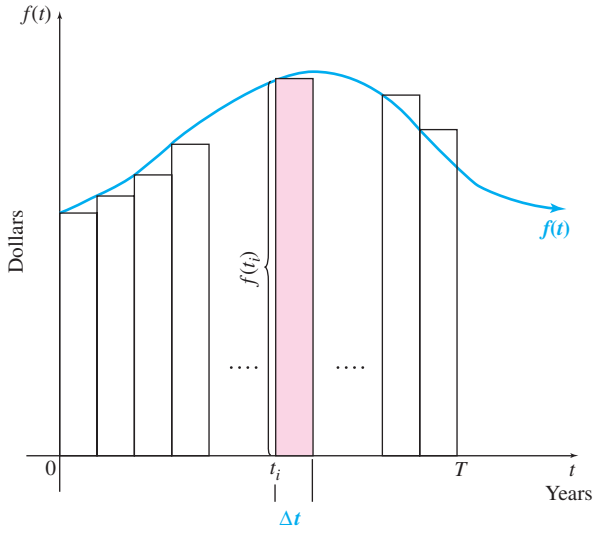


FIGURE 11

The total present value is approximately equal to the sum

$$\sum_{i=1}^{n} [f(t_i)\Delta t]e^{-rt_i}.$$

This approximation is improved as n increases; taking the limit of the sum as n increases without bound gives the present value

$$P = \lim_{n\to\infty} \sum_{i=1}^{n} [f(t_i)\Delta t]e^{-rt_i}.$$

This limit of a summation is given by the following definite integral.

Present Value of Money Flow

If $f(t)$ is the rate of continuous money flow at an interest rate r for T years, then the present value is

$$P = \int_0^T f(t)e^{-rt}\,dt.$$

To understand present value of money flow, consider an account that earns interest and has a continuous money flow. The present value of the money flow is the amount that would have to be deposited into a second account that has the same interest rate but does not have a continuous money flow, so the two accounts have the same amount of money after a specified time.

EXAMPLE 2 Present Value of Income

A company expects its rate of annual income during the next three years to be given by

$$f(t) = 75{,}000t, \quad 0 \le t \le 3.$$

What is the present value of this income over the 3-year period, assuming an annual interest rate of 8% compounded continuously?

APPLY IT

SOLUTION Use the formula for present value, with $f(t) = 75{,}000t$, $T = 3$, and $r = 0.08$.

$$P = \int_0^3 75{,}000te^{-0.08t}\,dt = 75{,}000\int_0^3 te^{-0.08t}\,dt$$

Using integration by parts, verify that

$$\int te^{-0.08t}\,dt = -12.5te^{-0.08t} - 156.25e^{-0.08t} + C.$$

Therefore,

$$\begin{aligned}75{,}000\int_0^3 te^{-0.08t}\,dt &= 75{,}000\left(-12.5te^{-0.08t} - 156.25e^{-0.08t}\right)\Big|_0^3\\ &= 75{,}000[-12.5(3)e^{-0.08(3)} - 156.25e^{-0.08(3)} - (0 - 156.25)]\\ &\approx 75{,}000(-29.498545 - 122.910603 + 156.25)\\ &\approx 288{,}064,\end{aligned}$$

or about $288,000. Notice that the actual income over the 3-year period is given by

$$\text{Total money flow} = \int_0^3 75{,}000t\,dt = \frac{75{,}000t^2}{2}\Big|_0^3 = 337{,}500,$$

or $337,500. This means that it would take a lump-sum deposit of $288,064 today paying a continuously compounded interest rate of 8% over a 3-year period to equal the total cash flow of $337,500 with interest. This approach is used as a basis for determining insurance claims involving income considerations.

YOUR TURN 2 Find the present value of an income given by $f(t) = 50{,}000t$ over the next 5 years if the interest rate is 3.5%.

TRY YOUR TURN 2

Accumulated Amount of Money Flow at Time T

To find the **accumulated amount of money flow** with interest at any time t, start with the formula $A = Pe^{rt}$, let $t = T$, and in place of P substitute the expression for present value of money flow. The result is the following formula.

Accumulated Amount of Money Flow at Time T

If $f(t)$ is the rate of money flow at an interest rate r at time t, the accumulated amount of money flow at time T is

$$A = e^{rT}\int_0^T f(t)e^{-rt}\,dt.$$

Here, the accumulated amount of money A represents the accumulated value or final amount of the money flow *including* interest received on the money after it comes in. (Recall, total money flow *does not* take the interest into account.)

It turns out that most money flows can be expressed as (or at least approximated by) exponential or polynomial functions. When these are multiplied by e^{-rt}, the result is a function that can be integrated. The next example illustrates uniform flow, where $f(t)$ is a constant function. (This is a special case of the polynomial function.)

EXAMPLE 3 Accumulated Amount of Money Flow

If money is flowing continuously at a constant rate of $2000 per year over 5 years at 6% interest compounded continuously, find the following.

(a) The total money flow over the 5-year period

SOLUTION The total money flow is given by $\int_0^T f(t)\,dt$. Here $f(t) = 2000$ and $T = 5$.

$$\int_0^5 2000\,dt = 2000t\Big|_0^5 = 2000(5) = 10{,}000$$

The total money flow over the 5-year period is $10,000.

FOR REVIEW

In this example we use the following two rules for exponents.

1. $a^m \cdot a^n = a^{m+n}$
2. $a^0 = 1$

(b) The accumulated amount of money flow, compounded continuously, at time $T = 5$

SOLUTION At $T = 5$ with $r = 0.06$, the amount is

$$A = e^{rT}\int_0^T f(t)e^{-rt}\,dt = e^{(0.06)5}\int_0^5 (2000)e^{-0.06t}\,dt$$

$$= (e^{0.3})(2000)\int_0^5 e^{-0.06t}\,dt = e^{0.3}(2000)\left(\frac{1}{-0.06}\right)\left(e^{-0.06t}\Big|_0^5\right)$$

$$= \frac{2000e^{0.3}}{-0.06}(e^{-0.3} - 1) = \frac{2000}{-0.06}(1 - e^{0.3}) \qquad (e^{0.03})(e^{-0.03}) = 1$$

$$\approx 11{,}661.96,$$

or \$11,661.96. The answer to part (a), \$10,000, was the amount of money flow over the 5-year period. The \$11,661.96 gives that amount with interest compounded continuously over the 5-year period.

(c) The total interest earned

SOLUTION This is simply the accumulated amount of money flow minus the total amount of flow, or

$$\$11{,}661.96 - \$10{,}000.00 = \$1661.96.$$

(d) The present value of the amount with interest

SOLUTION Use $P = \int_0^T f(t)e^{-rt}\,dt$ with $f(t) = 2000$, $r = 0.06$, and $T = 5$.

$$P = \int_0^5 2000e^{-0.06t}\,dt = 2000\left(\frac{e^{-0.06t}}{-0.06}\right)\Big|_0^5$$

$$= \frac{2000}{-0.06}(e^{-0.3} - 1)$$

$$\approx 8639.39,$$

The present value of the amount with interest in 5 years is \$8639.39, which can be checked by substituting \$11,661.96 for A in $A = Pe^{rt}$. The present value, P, could have been found by dividing the amount found in (b) by $e^{rT} = e^{0.3}$. Check that this would give the same result. TRY YOUR TURN 3

YOUR TURN 3 Find the accumulated amount of money flow for the income and interest rate in Your Turn 2 in this section.

If the rate of money flow is increasing or decreasing exponentially, then $f(t) = Ce^{kt}$, where C is a constant that represents the initial amount and k is the (nominal) continuous rate of change, which may be positive or negative.

EXAMPLE 4 Accumulated Amount of Money Flow

A continuous money flow starts at a rate of \$1000 per year and increases exponentially at 2% per year.

(a) Find the accumulated amount of money flow at the end of 5 years at 10% interest compounded continuously.

SOLUTION Here $C = 1000$ and $k = 0.02$, so that $f(t) = 1000e^{0.02t}$. Using $r = 0.10$ and $T = 5$,

$$A = e^{(0.10)5}\int_0^5 1000e^{0.02t}e^{-0.10t}\,dt$$

$$= (e^{0.5})(1000)\int_0^5 e^{-0.08t}\,dt \qquad e^{0.02t} \cdot e^{-0.10t} = e^{-0.08t}$$

$$= 1000e^{0.5}\left(\frac{e^{-0.08t}}{-0.08}\right)\Big|_0^5$$

$$= \frac{1000e^{0.5}}{-0.08}(e^{-0.4} - 1) = \frac{1000}{-0.08}(e^{0.1} - e^{0.5}) \approx 6794.38,$$

or \$6794.38.

(b) Find the present value at 5% interest compounded continuously.

SOLUTION Using $f(t) = 1000e^{0.02t}$ with $r = 0.05$ and $T = 5$ in the present value expression,

$$\begin{aligned} P &= \int_0^5 1000e^{0.02t}e^{-0.05t}\,dt \\ &= 1000\int_0^5 e^{-0.03t}\,dt = 1000\left(\frac{e^{-0.03t}}{-0.03}\Big|_0^5\right) \\ &= \frac{1000}{-0.03}(e^{-0.15} - 1) \approx 4643.07, \end{aligned}$$

or \$4643.07.

If the rate of change of the continuous money flow is given by the polynomial function $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$, the expressions for present value and accumulated amount can be integrated using integration by parts.

EXAMPLE 5 Present Value of Money Flow

The rate of change of a continuous flow of money is given by

$$f(t) = 1000t^2 + 100t.$$

Find the present value of this money flow at the end of 10 years at 10% compounded continuously.

SOLUTION Evaluate

$$P = \int_0^{10} (1000t^2 + 100t)e^{-0.10t}\,dt.$$

Using integration by parts, verify that

$$\int (1000t^2 + 100t)e^{-0.10t}\,dt =$$

$$(-10{,}000t^2 - 1000t)e^{-0.1t} - (200{,}000t + 10{,}000)e^{-0.1t} - 2{,}000{,}000e^{-0.1t} + C.$$

Thus,

$$\begin{aligned} P &= (-10{,}000t^2 - 1000t)e^{-0.1t} - (200{,}000t + 10{,}000)e^{-0.1t} \\ &\quad - 2{,}000{,}000e^{-0.1t}\Big|_0^{10} \\ &= (-1{,}000{,}000 - 10{,}000)e^{-1} - (2{,}000{,}000 + 10{,}000)e^{-1} \\ &\quad - 2{,}000{,}000e^{-1} - (0 - 10{,}000 - 2{,}000{,}000) \\ &\approx 163{,}245.21. \end{aligned}$$

YOUR TURN 4 Find the present value at the end of 8 years of the continuous flow of money given by $f(t) = 200t^2 + 100t + 50$ at 5% compounded continuously.

TRY YOUR TURN 4

16.3 EXERCISES

Each of the functions in Exercises 1–14 represents the rate of flow of money in dollars per year. Assume a 10-year period at 8% compounded continuously and find the following: (a) the present value; (b) the accumulated amount of money flow at $t = 10$.

1. $f(t) = 1000$

2. $f(t) = 300$

3. $f(t) = 500$

4. $f(t) = 2000$

5. $f(t) = 400e^{0.03t}$

6. $f(t) = 800e^{0.05t}$

7. $f(t) = 5000e^{-0.01t}$

8. $f(t) = 1000e^{-0.02t}$

9. $f(t) = 25t$

10. $f(t) = 50t$

11. $f(t) = 0.01t + 100$

12. $f(t) = 0.05t + 500$

13. $f(t) = 1000t - 100t^2$

14. $f(t) = 2000t - 150t^2$

APPLICATIONS

Business and Economics

15. Accumulated Amount of Money Flow An investment is expected to yield a uniform continuous rate of money flow of $20,000 per year for 3 years. Find the accumulated amount at an interest rate of 4% compounded continuously.

16. Present Value A real estate investment is expected to produce a uniform continuous rate of money flow of $8000 per year for 6 years. Find the present value at the following rates, compounded continuously.

a. 2% **b.** 5% **c.** 8%

17. Money Flow The rate of a continuous flow of money starts at $5000 and decreases exponentially at 1% per year for 8 years. Find the present value and final amount at an interest rate of 8% compounded continuously.

18. Money Flow The rate of a continuous money flow starts at $1000 and increases exponentially at 5% per year for 4 years. Find the present value and accumulated amount if interest earned is 3.5% compounded continuously.

19. Present Value A money market fund has a continuous flow of money at a rate of $f(t) = 1500 - 60t^2$, reaching 0 in 5 years. Find the present value of this flow if interest is 5% compounded continuously.

20. Accumulated Amount of Money Flow Find the amount of a continuous money flow in 3 years if the rate is given by $f(t) = 1000 - t^2$ and if interest is 5% compounded continuously.

YOUR TURN ANSWERS

1. $1595.94 **2.** $556,653

3. $663,111 **4.** $28,156.02

16.4 Improper Integrals

APPLY IT **If we know the rate at which a pollutant is dumped into a stream, how can we compute the total amount released given that the rate of dumping is decreasing over time?**

In this section we will learn how to answer questions such as this one, which is answered in Example 3.

Sometimes it is useful to be able to integrate a function over an infinite period of time. For example, we might want to find the total amount of income generated by an apartment building into the indefinite future or the total amount of pollution into a bay from a source that is continuing indefinitely. In this section we define integrals with one or more infinite limits of integration that can be used to solve such problems.

The graph in Figure 12(a) shows the area bounded by the curve $f(x) = x^{-3/2}$, the x-axis, and the vertical line $x = 1$. Think of the shaded region below the curve as extending indefinitely to the right. Does this shaded region have an area?

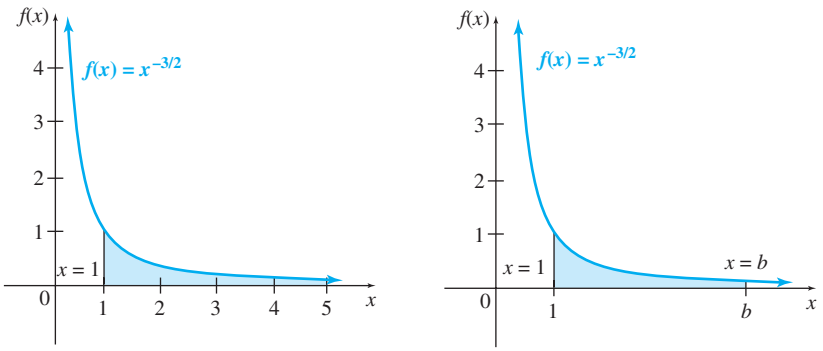


FIGURE 12

To see if the area of this region can be defined, introduce a vertical line at $x = b$, as shown in Figure 12(b). This vertical line gives a region with both upper and lower limits of integration. The area of this new region is given by the definite integral

$$\int_1^b x^{-3/2}\,dx.$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_1^b x^{-3/2}\,dx &= \left(-2x^{-1/2}\right)\Big|_1^b \\ &= -2b^{-1/2} - \left(-2\cdot 1^{-1/2}\right) \\ &= -2b^{-1/2} + 2 = 2 - \frac{2}{b^{1/2}}.\end{aligned}$$

FOR REVIEW

In Section 11.1 on Limits we saw that for any positive real number n,

$$\lim_{b\to\infty}\frac{1}{b^n} = 0.$$

Suppose we now let the vertical line $x = b$ in Figure 12(b) move farther to the right. That is, suppose $b \to \infty$. The expression $-2/b^{1/2}$ would then approach 0, and

$$\lim_{b\to\infty}\left(2 - \frac{2}{b^{1/2}}\right) = 2 - 0 = 2.$$

This limit is defined to be the *area* of the region shown in Figure 12(a), so that

$$\int_1^\infty x^{-3/2}\,dx = 2.$$

An integral of the form

$$\int_a^\infty f(x)\,dx, \qquad \int_{-\infty}^b f(x)\,dx, \qquad \text{or} \qquad \int_{-\infty}^\infty f(x)\,dx$$

is called an *improper integral*. These **improper integrals** are defined as follows.

Improper Integrals

If f is continuous on the indicated interval and if the indicated limits exist, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx,$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

for real numbers a, b, and c, where c is arbitrarily chosen.

If the expressions on the right side exist, the integrals are **convergent**; otherwise, they are **divergent**. A convergent integral has a value that is a real number. A divergent integral does not, often because the area under the curve is infinitely large.

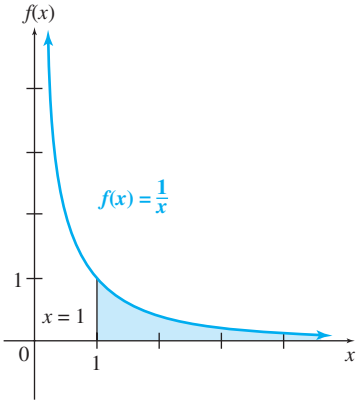


FIGURE 13

EXAMPLE 1 Improper Integrals

Evaluate each integral.

(a) $\displaystyle\int_1^\infty \frac{dx}{x}$

SOLUTION A graph of this region is shown in Figure 13. By the definition of an improper integral,

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x}.$$

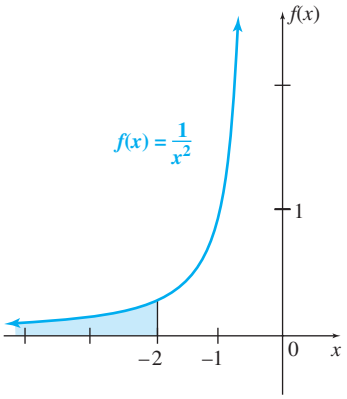


FIGURE 14

YOUR TURN 1 Find each integral.

(a) $\int_8^\infty \frac{1}{x^{1/3}}\,dx$ (b) $\int_8^\infty \frac{1}{x^{4/3}}\,dx$

Find $\int_1^b \frac{dx}{x}$ by the Fundamental Theorem of Calculus.

$$\int_1^b \frac{dx}{x} = \ln|x|\Big|_1^b = \ln|b| - \ln|1| = \ln|b| - 0 = \ln|b|$$

As $b \to \infty$, $\ln|b| \to \infty$, so $\lim_{b\to\infty} \ln|b|$ does not exist. Since the limit does not exist, $\int_1^\infty \frac{dx}{x}$ is divergent.

(b) $$\int_{-\infty}^{-2} \frac{1}{x^2}\,dx = \lim_{a\to-\infty}\int_a^{-2} \frac{1}{x^2}\,dx = \lim_{a\to-\infty}\left(\frac{-1}{x}\right)\Big|_a^{-2}$$
$$= \lim_{a\to-\infty}\left(\frac{1}{2} + \frac{1}{a}\right) = \frac{1}{2}$$

A graph of this region is shown in Figure 14. Since the limit exists, this integral converges. **TRY YOUR TURN 1**

It may seem puzzling that the areas under the curves $f(x) = 1/x^{3/2}$ and $f(x) = 1/x^2$ are finite, while $f(x) = 1/x$ has an infinite amount of area. At first glance the graphs of these functions appear similar. The difference is that although all three functions get small as x becomes infinitely large, $f(x) = 1/x$ does not become small enough fast enough. In the graphing calculator screen in Figure 15, notice how much faster $1/x^2$ becomes small compared with $1/x$.

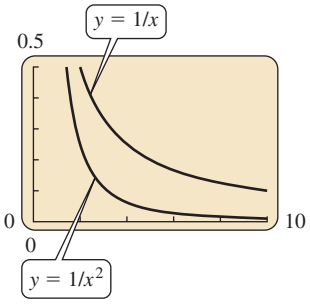


FIGURE 15

CAUTION Since graphing calculators provide only approximations, using them to find improper integrals is tricky and requires skill and care. Although their approximations may be good in some cases, they are wrong in others, and they cannot tell us for certain that an improper integral does not exist. See Exercises 39–41.

EXAMPLE 2 Improper Integral

Find $\int_{-\infty}^{\infty} 4e^{-3x}\,dx$.

SOLUTION In the definition of an improper integral with limits of $-\infty$ and ∞, the value of c is arbitrary, so we'll choose the simple value $c = 0$. We can then write the integral as

$$\int_{-\infty}^{\infty} 4e^{-3x}\,dx = \int_{-\infty}^{0} 4e^{-3x}\,dx + \int_0^{\infty} 4e^{-3x}\,dx$$

and evaluate each of the two improper integrals on the right. If they both converge, the original integral will equal their sum. To show you all the details while maintaining the suspense, we will evaluate the second integral first.

By definition,

$$\int_0^{\infty} 4e^{-3x}\,dx = \lim_{b\to\infty}\int_0^b 4e^{-3x}\,dx = \lim_{b\to\infty}\left(\frac{-4}{3}e^{-3x}\right)\Big|_0^b$$
$$= \lim_{b\to\infty}\left(\frac{-4}{3e^{3b}} + \frac{4}{3}\right) = 0 + \frac{4}{3} = \frac{4}{3}.$$

FOR REVIEW

Recall that

$$\lim_{p\to\infty}\frac{1}{e^p} = 0$$

and

$$\lim_{p\to-\infty}\frac{1}{e^p} = \infty.$$

Similarly, the second integral is evaluated as

$$\int_{-\infty}^{0} 4e^{-3x}\,dx = \lim_{b\to-\infty}\int_{b}^{0} 4e^{-3x}\,dx = \lim_{b\to-\infty}\left(\frac{-4}{3}e^{-3x}\right)\Big|_{b}^{0}$$
$$= \lim_{b\to-\infty}\left(-\frac{4}{3}+\frac{4}{3e^{3b}}\right) = \infty.$$

YOUR TURN 2
Find $\int_0^\infty 5e^{-2x}\,dx$.

Since one of the two improper integrals diverges, the original improper integral diverges. TRY YOUR TURN 2

The following examples describe applications of improper integrals.

EXAMPLE 3 Pollution

The rate at which a pollutant is being dumped into a stream at time t is given by P_0e^{-kt}, where P_0 is the rate that the pollutant is initially released into the stream. Suppose $P_0 = 1000$ and $k = 0.06$. Find the total amount of the pollutant that will be released into the stream into the indefinite future.

APPLY IT

SOLUTION Find

$$\int_0^\infty P_0e^{-kt}\,dt = \int_0^\infty 1000e^{-0.06t}\,dt.$$

This integral is similar to one of the integrals used to solve Example 2 and may be evaluated by the same method.

$$\int_0^\infty 1000e^{-0.06t}\,dt = \lim_{b\to\infty}\int_0^b 1000e^{-0.06t}\,dt$$
$$= \lim_{b\to\infty}\left(\frac{1000}{-0.06}e^{-0.06t}\right)\Big|_0^b$$
$$= \lim_{b\to\infty}\left(\frac{1000}{-0.06e^{0.06b}}-\frac{1000}{-0.06}e^{0}\right) = \frac{-1000}{-0.06} \approx 16{,}667$$

A total of approximately 16,667 units of the pollutant will be released over time.

The *capital value* of an asset is often defined as the present value of all future net earnings of the asset. In other words, suppose an asset provides a continuous money flow that is invested in an account earning a certain rate of interest. A lump sum is invested in a second account earning the same rate of interest, but with no money flow, so that as $t \to \infty$, the amounts in the two accounts approach each other. The lump sum necessary to make this happen is the capital value of the asset. If $R(t)$ gives the annual rate at which earnings are produced by an asset at time t, then the present value formula from Section 3 gives the **capital value** as

$$\int_0^\infty R(t)e^{-rt}\,dt,$$

where r is the annual rate of interest, compounded continuously.

EXAMPLE 4 Capital Value

Suppose income from a rental property is generated at the annual rate of $4000 per year. Find the capital value of this property at an interest rate of 10% compounded continuously.

SOLUTION This is a continuous income stream with a rate of flow of \$4000 per year, so $R(t) = 4000$. Also, $r = 0.10$ or 0.1. The capital value is given by

$$\begin{aligned}\int_0^\infty 4000e^{-0.1t}\,dt &= \lim_{b\to\infty}\int_0^b 4000e^{-0.1t}\,dt \\ &= \lim_{b\to\infty}\left(\frac{4000}{-0.1}e^{-0.1t}\right)\Bigg|_0^b \\ &= \lim_{b\to\infty}\left(-40{,}000e^{-0.1b} + 40{,}000\right) = 40{,}000,\end{aligned}$$

or \$40,000.

16.4 EXERCISES

Determine whether each improper integral converges or diverges, and find the value of each that converges.

1. $\displaystyle\int_3^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_3^\infty \frac{1}{(x+1)^3}\,dx$
3. $\displaystyle\int_4^\infty \frac{2}{\sqrt{x}}\,dx$
4. $\displaystyle\int_{27}^\infty \frac{2}{\sqrt[3]{x}}\,dx$
5. $\displaystyle\int_{-\infty}^{-1} \frac{2}{x^3}\,dx$
6. $\displaystyle\int_{-\infty}^{-4} \frac{3}{x^4}\,dx$
7. $\displaystyle\int_1^\infty \frac{1}{x^{1.0001}}\,dx$
8. $\displaystyle\int_1^\infty \frac{1}{x^{0.999}}\,dx$
9. $\displaystyle\int_{-\infty}^{-10} x^{-2}\,dx$
10. $\displaystyle\int_{-\infty}^{-1} (x-2)^{-3}\,dx$
11. $\displaystyle\int_{-\infty}^{-1} x^{-8/3}\,dx$
12. $\displaystyle\int_{-\infty}^{-27} x^{-5/3}\,dx$
13. $\displaystyle\int_0^\infty 8e^{-8x}\,dx$
14. $\displaystyle\int_0^\infty 50e^{-50x}\,dx$
15. $\displaystyle\int_{-\infty}^0 1000e^{x}\,dx$
16. $\displaystyle\int_{-\infty}^0 5e^{60x}\,dx$
17. $\displaystyle\int_{-\infty}^{-1} \ln|x|\,dx$
18. $\displaystyle\int_1^\infty \ln|x|\,dx$
19. $\displaystyle\int_0^\infty \frac{dx}{(x+1)^2}$
20. $\displaystyle\int_0^\infty \frac{dx}{(4x+1)^3}$
21. $\displaystyle\int_{-\infty}^{-1} \frac{2x-1}{x^2-x}\,dx$
22. $\displaystyle\int_1^\infty \frac{4x+6}{x^2+3x}\,dx$
23. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$
24. $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$
25. $\displaystyle\int_0^\infty xe^{4x}\,dx$
26. $\displaystyle\int_{-\infty}^0 xe^{0.2x}\,dx$ (*Hint:* Recall that $\lim_{x\to-\infty} xe^x = 0$.)
27. $\displaystyle\int_{-\infty}^\infty x^3e^{-x^4}\,dx$ (*Hint:* Recall from Exercise 66 in Section 11.1 on Limits that $\lim_{x\to\infty} x^ne^{-x} = 0$.)
28. $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$ (*Hint:* Recall that when $x < 0$, $|x| = -x$.)
29. $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$
30. $\displaystyle\int_{-\infty}^\infty \frac{2x+4}{x^2+4x+5}\,dx$

Find the area between the graph of the given function and the x-axis over the given interval, if possible.

31. $f(x) = \dfrac{1}{x-1}$, for $(-\infty, 0]$
32. $f(x) = e^{-x}$, for $(-\infty, e]$
33. $f(x) = \dfrac{1}{(x-1)^2}$, for $(-\infty, 0]$
34. $f(x) = \dfrac{3}{(x-1)^3}$, for $(-\infty, 0]$
35. Find $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$.
36. Find $\displaystyle\int_{-\infty}^\infty \frac{x}{(1+x^2)^2}\,dx$.
37. Show that $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \le 1$.
38. Example 1(b) leads to a paradox. On the one hand, the unbounded region in that example has an area of 1/2, so theoretically it could be colored with ink. On the other hand, the boundary of that region is infinite, so it cannot be drawn with a finite amount of ink. This seems impossible, because coloring the region automatically colors the boundary. Explain why it is possible to color the region.
39. Consider the functions $f(x) = 1/\sqrt{1+x^2}$ and $g(x) = 1/\sqrt{1+x^4}$.
    a. Use your calculator to approximate $\int_1^b f(x)\,dx$ for $b = 20, 50, 100, 1000$, and 10,000.
    b. Based on your answers from part a, would you guess that $\int_1^\infty f(x)\,dx$ is convergent or divergent?

c. Use your calculator to approximate $\int_1^b g(x)\,dx$ for $b = 20, 50, 100, 1000$, and 10,000.

d. Based on your answers from part c, would you guess that $\int_1^\infty g(x)\,dx$ is convergent or divergent?

e. Show how the answer to parts b and d might be guessed by comparing the integrals with others whose convergence or divergence is known. (*Hint:* For large x, the difference between $1 + x^2$ and x^2 is relatively small.)

Note: The first integral is indeed divergent, and the second convergent, with an approximate value of 0.9270.

40. a. Use your calculator to approximate $\int_0^b e^{-x^2}\,dx$ for $b = 1$, 5, 10, and 20.

b. Based on your answers to part a, does $\int_0^\infty e^{-x^2}\,dx$ appear to be convergent or divergent? If convergent, what seems to be its approximate value?

c. Explain why this integral should be convergent by comparing e^{-x^2} with e^{-x} for $x > 1$.

Note: The integral is convergent, with a value of $\sqrt{\pi}/2$.

41. a. Use your calculator to approximate $\int_0^b e^{-0.00001x}\,dx$ for $b = 10$, 50, 100, and 1000.

b. Based on your answers to part a, does $\int_0^\infty e^{-0.00001x}\,dx$ appear to be convergent or divergent?

c. To what value does the integral actually converge?

APPLICATIONS

Business and Economics

Capital Value Find the capital values of the properties in Exercises 42–43.

42. A castle for which annual rent of $225,000 will be paid in perpetuity; the interest rate is 6% compounded continuously

43. A fort on a strategic peninsula in the North Sea; the annual rent is $1,000,000, paid in perpetuity; the interest rate is 5% compounded continuously

44. Capital Value Find the capital value of an asset that generates $7200 yearly income if the interest rate is as follows.

a. 5% compounded continuously

b. 10% compounded continuously

45. Capital Value An investment produces a perpetual stream of income with a flow rate of

$$R(t) = 1200e^{0.03t}.$$

Find the capital value at an interest rate of 7% compounded continuously.

46. Capital Value Suppose income from an investment starts (at time 0) at $6000 a year and increases linearly and continuously at a rate of $200 a year. Find the capital value at an interest rate of 5% compounded continuously.

47. Scholarship The Drucker family wants to establish an ongoing scholarship award at a college. Each year in June $3000 will be awarded, starting 1 year from now. What amount must the Druckers provide the college, assuming funds will be invested at 10% compounded continuously?

Social Sciences

48. Drug Reaction The rate of reaction to a drug is given by

$$r'(t) = 2t^2e^{-t},$$

where t is the number of hours since the drug was administered. Find the total reaction to the drug over all the time since it was administered, assuming this is an infinite time interval. (*Hint:* $\lim_{t\to\infty} t^k e^{-t} = 0$ for all real numbers k.)

49. Drug Epidemic In an epidemiological model used to study the spread of drug use, a single drug user is introduced into a population of N non-users. Under certain assumptions, the number of people expected to use drugs as a result of direct influence from each drug user is given by

$$S = N\int_0^\infty \frac{a(1 - e^{-kt})}{k} e^{-bt}\,dt,$$

where a, b, and k are constants. Find the value of S. ***Source: Mathematical Biology***.

50. Present Value When harvesting a population, such as fish, the present value of the resource is given by

$$P = \int_0^\infty e^{-rt}n(t)y(t)dt,$$

where r is a discount factor, $n(t)$ is the net revenue at time t, and $y(t)$ is the harvesting effort. Suppose $y(t) = K$ and $n(t) = at + b$. Find the present value. ***Source: Some Mathematical Questions in Biology***.

Physical Sciences

Radioactive Waste Radioactive waste is entering the atmosphere over an area at a decreasing rate. Use the improper integral

$$\int_0^\infty Pe^{-kt}dt$$

with $P = 50$ to find the total amount of the waste that will enter the atmosphere for each value of k.

51. $k = 0.06$

52. $k = 0.04$

YOUR TURN ANSWERS

1. (a) Divergent (b) 3/2 **2.** 5/2

16.5 Solutions of Elementary and Separable Differential Equations

APPLY IT **How can we predict the future population of a flock of mountain goats?**
Using differential equations, we will answer this question in Example 6.

Suppose that an economist wants to develop an equation that will forecast interest rates. By studying data on previous changes in interest rates, she hopes to find a relationship between the level of interest rates and their rate of change. A function giving the rate of change of interest rates would be the derivative of the function describing the level of interest rates. A **differential equation** is an equation that involves an unknown function $y = f(x)$ and a finite number of its derivatives. Solving the differential equation for y would give the unknown function to be used for forecasting interest rates.

Differential equations have been important in the study of physical science and engineering since the eighteenth century. More recently, differential equations have become useful in the social sciences, life sciences, and economics for solving problems about population growth, ecological balance, and interest rates. In this section, we will introduce a method for solving a particular class of differential equations and give examples of its application.

Usually a solution of an equation is a *number*. A solution of a differential equation, however, is a *function* that satisfies the equation.

EXAMPLE 1 Solving a Differential Equation

Find all solutions of the differential equation

$$\frac{dy}{dx} = 3x^2 - 2x. \tag{1}$$

SOLUTION To say that a function $y(x)$ is a solution of Equation (1) simply means that the derivative of the function y is $3x^2 - 2x$. This is the same as saying that y is an antiderivative of $3x^2 - 2x$, or

$$y = \int (3x^2 - 2x)\,dx = x^3 - x^2 + C. \tag{2}$$

We can verify that the function given by Equation (2) is a solution by taking its derivative. The result is the differential equation (1). TRY YOUR TURN 1

YOUR TURN 1 Find all solutions of the differential equation $\frac{dy}{dx} = 12x^5 + \sqrt{x} + e^{5x}$.

FOR REVIEW
For review on finding antiderivatives, see the first two sections in Chapter 15 on Integration.

Each different value of C in Equation (2) leads to a different solution of Equation (1), showing that a differential equation can have an infinite number of solutions. Equation (2) is the **general solution** of the differential equation (1). Some of the solutions of Equation (1) are graphed in Figure 16.

The simplest kind of differential equation has the form

$$\frac{dy}{dx} = g(x).$$

Since Equation (1) has this form, the solution of Equation (1) suggests the following generalization.

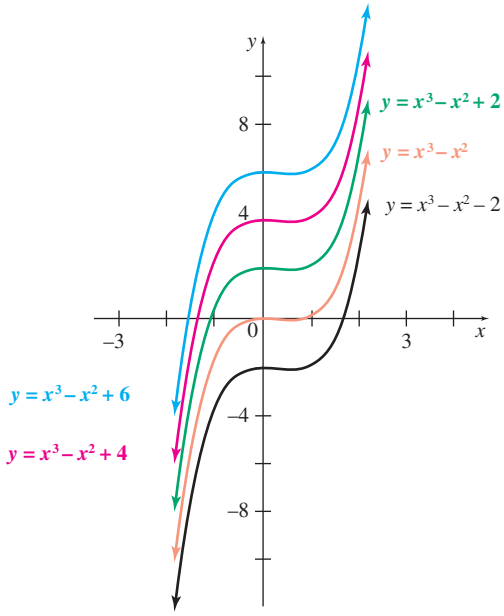


FIGURE 16

General Solution of $\dfrac{dy}{dx} = g(x)$

The general solution of the differential equation $\dfrac{dy}{dx} = g(x)$ is

$$y = \int g(x)\,dx.$$

EXAMPLE 2 Population

The population P of a flock of birds is growing exponentially so that

$$\frac{dP}{dt} = 20e^{0.05t},$$

where t is time in years. Find P in terms of t if there were 20 birds in the flock initially.

SOLUTION To solve the differential equation, first determine the antiderivative of $20e^{0.05t}$, that is,

$$P = \int 20e^{0.05t}\,dt = \frac{20}{0.05}e^{0.05t} + C = 400e^{0.05t} + C.$$

Initially, there were 20 birds, so $P = 20$ at time $t = 0$. We can substitute this information into the equation to determine the value of C that satisfies this condition.

$$20 = 400e^0 + C$$
$$-380 = C$$

Therefore, $P = 400e^{0.05t} - 380$. Verify that P is a solution to the differential equation by taking its derivative. The result should be the original differential equation.

NOTE
We can write the condition $P = 20$ when $t = 0$ as $P(0) = 20$. We will use this notation from now on.

In Example 2, the given information was used to produce a solution with a specific value of C. Such a solution is called a **particular solution** of the differential equation. The given information, $P = 20$ when $t = 0$, is called an **initial condition**. An **initial value problem** is a differential equation with a value of y given at $x = x_0$ (or $t = t_0$ in this case), where x_0 is any real number.

Sometimes a differential equation must be rewritten in the form

$$\frac{dy}{dx} = g(x)$$

before it can be solved.

EXAMPLE 3 Initial Value Problem

Find the particular solution of

$$\frac{dy}{dx} - 2x = \frac{2x}{\sqrt{x^2 + 3}},$$

given that $y(1) = 8$.

SOLUTION This differential equation is not in the proper form, but we can easily fix this by adding $2x$ to both sides of the equation. That is,

$$\frac{dy}{dx} = 2x + \frac{2x}{\sqrt{x^2 + 3}}.$$

To find the general solution, integrate this expression, using the substitution $u = x^2 + 3$ in the second term.

$$\begin{aligned} y &= \int \left(2x + \frac{2x}{\sqrt{x^2 + 3}}\right) dx \\ &= x^2 + \int \frac{1}{\sqrt{u}}\, du && du = 2x\,dx \\ &= x^2 + 2\sqrt{u} + C && \text{Use the power rule with } n = -1/2. \\ &= x^2 + 2\sqrt{x^2 + 3} + C \end{aligned}$$

Now use the initial condition to find the value of C. Substituting 8 for y and 1 for x gives

$$\begin{aligned} 8 &= 1^2 + 2\sqrt{1^2 + 3} + C \\ 8 &= 5 + C \\ C &= 3. \end{aligned}$$

The particular solution is $y = x^2 + 2\sqrt{x^2 + 3} + 3$. Verify that $y(1) = 8$ and that differentiating y leads to the original differential equation. TRY YOUR TURN 2

YOUR TURN 2 Find the particular solution of $\dfrac{dy}{dx} - 12x^3 = 6x^2$, given that $y(2) = 60$.

So far in this section, we have used a method that is essentially the same as that used in the section on antiderivatives, when we first started the topic of integration. But not all differential equations can be solved so easily. For example, if interest on an investment is compounded continuously, then the investment grows at a rate proportional to the amount of money present. If A is the amount in an account at time t, then for some constant k, the differential equation

$$\frac{dA}{dt} = kA \tag{3}$$

gives the rate of growth of A with respect to t. This differential equation is different from those discussed previously, which had the form

$$\frac{dy}{dx} = g(x).$$

CAUTION Since the right-hand side of Equation (3) is a function of A, rather than a function of t, it would be completely invalid to simply integrate both sides as we did before. The previous method only works when the side opposite the derivative is simply a function of the independent variable.

Equation (3) is an example of a more general differential equation we will now learn to solve; namely, those that can be written in the form

$$\frac{dy}{dx} = \frac{p(x)}{q(y)}.$$

Suppose we think of dy/dx as a fraction dy over dx. This is incorrect, of course; the derivative is actually the limit of a small change in y over a small change in x, but the notation is chosen so that this interpretation gives a correct answer, as we shall see. Multiply on both sides by $q(y)\,dx$ to get

$$q(y)\,dy = p(x)\,dx.$$

In this form all terms involving y (including dy) are on one side of the equation and all terms involving x (and dx) are on the other side. A differential equation that can be put into this form is said to be *separable*, since the variables x and y can be separated. After separation, a **separable differential equation** may be solved by integrating each side with respect to the variable given. This method is known as **separation of variables**.

$$\int q(y)\,dy = \int p(x)\,dx$$

$$Q(y) = P(x) + C,$$

where P and Q are antiderivatives of p and q. (We don't need a constant of integration on the left side of the equation; it can be combined with the constant of integration on the right side as C.) To show that this answer is correct, differentiate implicitly with respect to x.

$$Q'(y)\frac{dy}{dx} = P'(x) \qquad \text{Use the chain rule on the left side.}$$

$$q(y)\frac{dy}{dx} = p(x) \qquad q(y) \text{ is the derivative of } Q(y) \text{ and } p(x) \text{ is the derivative of } P(x).$$

$$\frac{dy}{dx} = \frac{p(x)}{q(y)}$$

This last equation is the one we set out to solve.

EXAMPLE 4 Separation of Variables

Find the general solution of

$$y\frac{dy}{dx} = x^2.$$

SOLUTION Begin by separating the variables to get

$$y\,dy = x^2\,dx.$$

The general solution is found by determining the antiderivatives of each side.

$$\int y\,dy = \int x^2\,dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

$$y^2 = \frac{2}{3}x^3 + 2C$$

$$y^2 = \frac{2}{3}x^3 + K$$

YOUR TURN 3 Find the general solution of $\dfrac{dy}{dx} = \dfrac{x^2 + 1}{xy^2}$.

The constant K was substituted for $2C$ in the last step. The solution is left in implicit form, not solved explicitly for y. In general, we will use C for the arbitrary constant, so that our final answer would be written as $y^2 = (2/3)x^3 + C$. It would also be nice to solve for y by taking the square root of both sides, but since we don't know anything about the sign of y, we don't know whether the solution is $y = \sqrt{(2/3)x^3 + C}$ or $y = -\sqrt{(2/3)x^3 + C}$. If y were raised to the third power, we could solve for y by taking the cube root of both sides, since the cube root is unique. TRY YOUR TURN 3

EXAMPLE 5 Separation of Variables

Find the general solution of the differential equation for interest compounded continuously,

$$\frac{dA}{dt} = kA.$$

SOLUTION Separating variables leads to

$$\frac{1}{A}\,dA = k\,dt.$$

To solve this equation, determine the antiderivative of each side.

$$\int \frac{1}{A}\,dA = \int k\,dt$$
$$\ln|A| = kt + C$$

(Here A represents a nonnegative quantity, so the absolute value is unnecessary, but we wish to show how to solve equations for which this may not be true.) Use the definition of logarithm to write the equation in exponential form as

$$|A| = e^{kt+C} = e^{kt}e^{C}. \qquad \text{Use the property } e^{m+n} = e^m e^n.$$

Finally, use the definition of absolute value to get

$$A = e^{kt}e^{C} \quad \text{or} \quad A = -e^{kt}e^{C}.$$

Because e^C and $-e^C$ are constants, replace them with the constant M, which may be any nonzero real number. (We use M rather than K because we already have a constant k in this example. We could also relabel M as C, as we did in Example 4.) The resulting equation,

$$A = Me^{kt},$$

not only describes interest compounded continuously but also defines the exponential growth or decay function that was discussed in Chapter 10 on Nonlinear Functions.

FOR REVIEW

In Chapter 10 on Nonlinear Functions, we saw that the amount of money in an account with interest compounded continuously is given by

$$A = Pe^{rt},$$

where P is the initial amount in the account, r is the annual interest rate, and t is the time in years. Observe that this is the same as the equation for the amount of money in an account derived here, where P and r have been replaced with M and k, respectively. For more applications of exponential growth and decay, see Chapter 10 on Nonlinear Functions.

CAUTION Notice that $y = 0$ is also a solution to the differential equation in Example 5, but after we divide by A (which is not possible if $A = 0$) and integrate, the resulting equation $|A| = e^{kt+C}$ does not allow y to equal 0. In this example, the lost solution can be recovered in the final answer if we allow M to equal 0, a value that was previously excluded. When dividing by an expression in separation of variables, look for solutions that would make this expression 0 and may be lost.

Recall that equations of the form $A = Me^{kt}$ arise in situations where the rate of change of a quantity is proportional to the amount present at time t, which is precisely what the differential equation (3) describes. The constant k is called the **growth constant**, while M represents the amount present at time $t = 0$. A positive value of k indicates growth, while a negative value of k indicates decay. The equation was often written in the form $y = y_0e^{kt}$ in Chapter 10 on Nonlinear Functions, where we discussed other applications of this equation, such as radioactive decay.

As a model of population growth, the equation $y = Me^{kt}$ is not realistic over the long run for most populations. As shown by graphs of functions of the form $y = Me^{kt}$, with both M and k positive, growth would be unbounded. Additional factors, such as space restrictions or a limited amount of food, tend to inhibit growth of populations as time goes on. In an alternative model that assumes a maximum population of size N, the rate of growth of a population is proportional to how close the population is to that maximum, that is, to the difference between N and y. These assumptions lead to the differential equation

$$\frac{dy}{dt} = k(N - y),$$

whose solution is the limited growth function mentioned in the last section of Chapter 10 on Nonlinear Functions.

EXAMPLE 6 Population

A certain nature reserve can support no more than 4000 mountain goats. Assume that the rate of growth is proportional to how close the population is to this maximum, with a growth rate of 20 percent. There are currently 1000 goats in the area.

(a) Solve the general limited growth differential equation, and then write a function describing the goat population at time t.

SOLUTION To solve the equation

$$\frac{dy}{dt} = k(N - y),$$

first separate the variables.

$$\frac{dy}{N - y} = k\,dt$$

$$\int \frac{dy}{N - y} = \int k\,dt \qquad \text{Integrate both sides.}$$

$$-\ln(N - y) = kt + C \qquad \text{We assume the population is less than } N\text{, so } N - y > 0.$$

$$\ln(N - y) = -kt - C$$

$$N - y = e^{-kt - C} \qquad \text{Apply the function } e^x \text{ to both sides.}$$

$$N - y = e^{-kt}e^{-C} \qquad \text{Use the property } e^{m+n} = e^m e^n.$$

$$y = N - e^{-kt}e^{-C} \qquad \text{Solve for } y.$$

$$y = N - Me^{-kt} \qquad \text{Relabel the constant } e^{-C} \text{ as } M.$$

Now apply the initial condition that $y(0) = y_0$.

$$y_0 = N - Me^{-k\cdot 0} = N - M \qquad e^0 = 1$$

$$M = N - y_0 \qquad \text{Solve for } M.$$

Substituting this value of M into the previous solution gives

$$y = N - (N - y_0)e^{-kt}.$$

The graph of this function is shown in Figure 17.

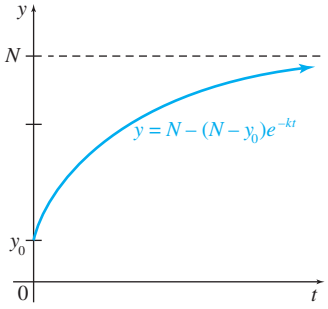


FIGURE 17

For the goat problem, the maximum population is $N = 4000$, the initial population is $y_0 = 1000$, and the growth rate constant is 20%, or $k = 0.2$. Therefore,

$$y = 4000 - (4000 - 1000)e^{-0.2t} = 4000 - 3000e^{-0.2t}.$$

APPLY IT

YOUR TURN 4 In Example 6, find the goat population in 5 years if the reserve can support 6000 goats, the growth rate is 15%, and there are currently 1200 goats in the area.

(b) What will the goat population be in 5 years?

SOLUTION In 5 years, the population will be

$$\begin{aligned} y &= 4000 - 3000e^{-(0.2)(5)} = 4000 - 3000e^{-1} \\ &\approx 4000 - 1103.6 = 2896.4, \end{aligned}$$

or about 2900 goats. TRY YOUR TURN 4

Logistic Growth

Let y be the size of a certain population at time t. In the standard model for unlimited growth given by Equation (3), the rate of growth is proportional to the current population size. The constant k, the growth rate constant, is the difference between the birth and death rates of the population. The unlimited growth model predicts that the population's growth rate is a constant, k.

Growth usually is not unlimited, however, and the population's growth rate is usually not constant because the population is limited by environmental factors to a maximum size N, called the **carrying capacity** of the environment for the species. In the limited growth model already given,

$$\frac{dy}{dt} = k(N - y),$$

the rate of growth is proportional to the remaining room for growth, $N - y$.

In the **logistic growth model**

$$\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right)y \tag{4}$$

the rate of growth is proportional to both the current population size y and a factor $(1 - y/N)$ that is equal to the remaining room for growth, $N - y$, divided by N. Equation (4) is called the **logistic equation**. Notice that $(1 - y/N) \to 1$ as $y \to 0$, and the differential equation can be approximated as

$$\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right)y \approx k(1)y = ky.$$

In other words, when y is small, the growth of the population behaves as if it were unlimited. On the other hand, $(1 - y/N) \to 0$ as $y \to N$, so

$$\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right)y \approx k(0)y = 0.$$

That is, population growth levels off as y nears the maximum population N. Thus, the logistic Equation (4) is the unlimited growth Equation (3) with a damping factor $(1 - y/N)$ to account for limiting environmental factors when y nears N. Let y_0 denote the initial population size. Under the assumption $0 < y < N$, the general solution of Equation (4) is

$$y = \frac{N}{1 + be^{-kt}}, \tag{5}$$

where $b = (N - y_0)/y_0$ (see Exercise 31). This solution, called a **logistic curve**, is shown in Figure 18. This function was introduced in Section 12.4 on Derivatives of Exponential Functions in the form

$$G(t) = \frac{m}{1 + (\frac{m}{G_0})e^{-kmt}},$$

where m is the limiting value of the population, G_0 is the initial number present, and k is a positive constant.

As expected, the logistic curve begins exponentially and subsequently levels off. Another important feature is the point of inflection $((\ln b)/k, N/2)$, where dy/dx is a maximum (see Exercise 33). Notice that the point of inflection is when the population is half of the carrying capacity and that at this point, the population is increasing most rapidly.

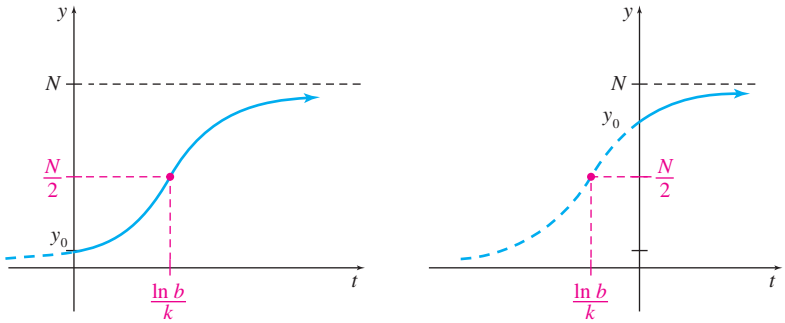


FIGURE 18

Logistic equations arise frequently in the study of populations. In about 1840 the Belgian sociologist P. F. Verhulst fitted a logistic curve to U.S. census figures and made predictions about the population that were subsequently proved to be quite accurate. American biologist Raymond Pearl (circa 1920) found that the growth of a population of fruit flies in a limited space could be modeled by the logistic equation

$$\frac{dy}{dt} = 0.2y - \frac{0.2}{1035}y^2.$$

TECHNOLOGY NOTE

Some calculators can fit a logistic curve to a set of data points. For example, the TI-84 Plus has this capability, listed as `Logistic` in the STAT CALC menu, along with other types of regression. See Exercises 40, 47, 48, and 52.

Logistic growth is an example of how a model is modified over time as new insights occur. The model for population growth changed from the early exponential curve $y = Me^{kt}$ to the logistic curve

$$y = \frac{N}{1 + be^{-kt}}.$$

Many other quantities besides population grow logistically. That is, their initial rate of growth is slow, but as time progresses, their rate of growth increases to a maximum value and subsequently begins to decline and to approach zero.

EXAMPLE 7 Logistic Curve

Rapid technological advancements in the last 20 years have made many products obsolete practically overnight. J. C. Fisher and R. H. Pry successfully described the phenomenon of a technically superior new product replacing another product by the logistic equation

$$\frac{dz}{dt} = k(1 - z)z, \tag{6}$$

where z is the market share of the new product and $1 - z$ is the market share of the other product. ***Source: Technological Forecasting and Social Change***. The new product will initially have little or no market share; that is, $z_0 \approx 0$. Thus, the constant b in Equation (5) will have to be determined in a different way. Let t_0 be the time at which $z = 1/2$. Under the assumption $0 < z < 1$, the general solution of Equation (6) is

$$z = \frac{1}{1 + be^{-kt}}, \tag{7}$$

where $b = e^{kt_0}$ (see Exercise 32).

TECHNOLOGY NOTE

Fisher and Pry applied their model to the fraction of fabric consumed in the United States that was synthetic. Their data is shown in the table below. At the time of the study, natural fabrics in clothing were being replaced with synthetic fabric. Using the logistic regression function on a TI-84 Plus calculator, with t as the number of years since 1930, the best logistic function to fit this data can be shown to be

$$z = \frac{1.293}{1 + 21.80e^{-0.06751t}}.$$

A graph of the data and this function is shown in Figure 19(a). Although the data fits the function well, this function is not of the form studied by Fisher and Pry because it has a numerator of 1.293, rather than 1. This function predicts that the percentage of fabric that is synthetic approaches 129%! A more appropriate function can be found by rewriting Equation (7) as

$$be^{-kt} = \frac{1}{z} - 1,$$

and then finding the best fit exponential function to the points $(t, 1/z - 1)$. This leads to the equation

$$z = \frac{1}{1 + 18.93(0.92703)^t} = \frac{1}{1 + 18.93e^{(\ln 0.92703)t}} = \frac{1}{1 + 18.93e^{-0.07577t}}.$$

As Figure 19(b) shows, this more realistic model also fits the data well. The dangers of extrapolating beyond the data are illustrated by this equation's prediction that 95.8% of fabrics in the United States would be synthetic by 2010. In fact, cotton is still more popular than synthetic fabrics. ***Source: Fabrics Manufacturers***.

	Synthetic Fabric as Percent of U.S. Consumption								
Year	1930	1935	1940	1945	1950	1955	1960	1965	1967
Fraction synthetic	0.044	0.079	0.10	0.14	0.22	0.28	0.29	0.43	0.47

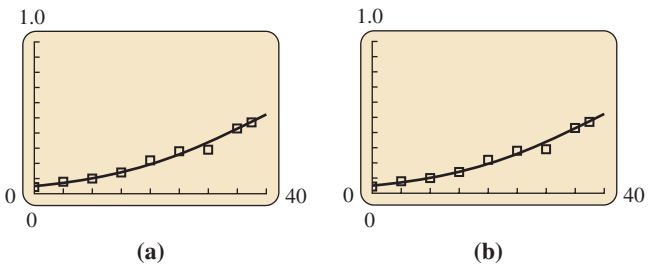


FIGURE 19

16.5 EXERCISES

Find the general solution for each differential equation. Verify that each solution satisfies the original differential equation.

1. $\frac{dy}{dx} = -4x + 6x^2$
2. $\frac{dy}{dx} = 4e^{-3x}$
3. $4x^3 - 2\frac{dy}{dx} = 0$
4. $3x^2 - 3\frac{dy}{dx} = 2$
5. $y\frac{dy}{dx} = x^2$
6. $y\frac{dy}{dx} = x^2 - x$
7. $\frac{dy}{dx} = 2xy$
8. $\frac{dy}{dx} = x^2y$
9. $\frac{dy}{dx} = 3x^2y - 2xy$
10. $(y^2 - y)\frac{dy}{dx} = x$
11. $\frac{dy}{dx} = \frac{y}{x}, x > 0$
12. $\frac{dy}{dx} = \frac{y}{x^2}$
13. $\frac{dy}{dx} = \frac{y^2 + 6}{2y}$
14. $\frac{dy}{dx} = \frac{e^{y^2}}{y}$
15. $\frac{dy}{dx} = y^2e^{2x}$
16. $\frac{dy}{dx} = \frac{e^x}{e^y}$

Find the particular solution for each initial value problem.

17. $\frac{dy}{dx} + 3x^2 = 2x; \quad y(0) = 5$
18. $\frac{dy}{dx} = 4x^3 - 3x^2 + x; \quad y(1) = 0$
19. $2\frac{dy}{dx} = 4xe^{-x}; \quad y(0) = 42$
20. $x\frac{dy}{dx} = x^2e^{3x}; \quad y(0) = \frac{8}{9}$
21. $\frac{dy}{dx} = \frac{x^3}{y}; \quad y(0) = 5$
22. $x^2\frac{dy}{dx} - y\sqrt{x} = 0; \quad y(1) = e^{-2}$
23. $(2x + 3)y = \frac{dy}{dx}; \quad y(0) = 1$
24. $\frac{dy}{dx} = \frac{x^2 + 5}{2y - 1}; \quad y(0) = 11$
25. $\frac{dy}{dx} = \frac{2x + 1}{y - 3}; \quad y(0) = 4$
26. $x^2\frac{dy}{dx} = y; \quad y(1) = -1$
27. $\frac{dy}{dx} = \frac{y^2}{x}; \quad y(e) = 3$
28. $\frac{dy}{dx} = x^{1/2}y^2; \quad y(4) = 9$
29. $\frac{dy}{dx} = (y - 1)^2e^{x-1}; \quad y(1) = 2$
30. $\frac{dy}{dx} = (x + 2)^2e^y; \quad y(1) = 0$

31. **a.** Solve the logistic Equation (4) in this section by observing that

$$\frac{1}{y} + \frac{1}{N - y} = \frac{N}{(N - y)y}.$$

b. Assume $0 < y < N$. Verify that $b = (N - y_0)/y_0$ in Equation (5), where y_0 is the initial population size.

c. Assume $0 < N < y$ for all y. Verify that $b = (y_0 - N)/y_0$.

32. Suppose that $0 < z < 1$ for all z. Solve the logistic Equation (6) as in Exercise 31. Verify that $b = e^{kx_0}$, where x_0 is the time at which $z = 1/2$.

33. Suppose that $0 < y_0 < N$. Let $b = (N - y_0)/y_0$, and let $y(x) = N/(1 + be^{-kx})$ for all x. Show the following.

a. $0 < y(x) < N$ for all x.

b. The lines $y = 0$ and $y = N$ are horizontal asymptotes of the graph.

c. $y(x)$ is an increasing function.

d. $((\ln b)/k, N/2)$ is a point of inflection of the graph.

e. dy/dx is a maximum at $x_0 = (\ln b)/k$.

34. Suppose that $0 < N < y_0$. Let $b = (y_0 - N)/y_0$ and let

$$y(x) = \frac{N}{1 - be^{-kx}} \quad \text{for all } x \neq \frac{\ln b}{k}.$$

See the figure. Show the following.

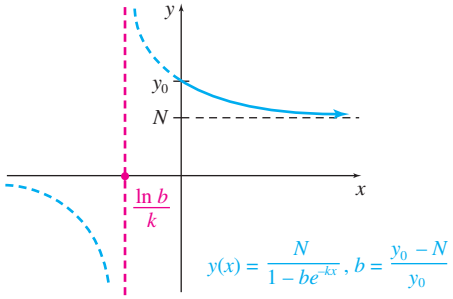
a. $0 < b < 1$

b. The lines $y = 0$ and $y = N$ are horizontal asymptotes of the graph.

c. The line $x = (\ln b)/k$ is a vertical asymptote of the graph.

d. $y(x)$ is decreasing on $((\ln b)/k, \infty)$ and on $(-\infty, (\ln b)/k)$.

e. $y(x)$ is concave upward on $((\ln b)/k, \infty)$ and concave downward on $(-\infty, (\ln b)/k)$.



APPLICATIONS

Business and Economics

35. **Profit** The marginal profit of a certain company is given by

$$\frac{dy}{dx} = \frac{100}{32 - 4x},$$

where x represents the amount of money (in thousands of dollars) that the company spends on advertising. Find the profit for each advertising expenditure if the profit is \$1000 when nothing is spent on advertising.

a. \$3000 **b.** \$5000

c. Can advertising expenditures ever reach \$8000 according to this model? Explain why or why not.

36. Sales Decline Sales (in thousands) of a certain product are declining at a rate proportional to the amount of sales, with a decay constant of 15% per year.

a. Write a differential equation to express the rate of sales decline.

b. Find a general solution to the equation in part a.

c. How much time will pass before sales become 25% of their original value?

37. Inflation If inflation grows continuously at a rate of 5% per year, how long will it take for $1 to lose half its value?

Elasticity of Demand Elasticity of demand was discussed in Chapter 14 on Applications of the Derivative, where it was defined as

$$E = -\frac{p}{q} \cdot \frac{dq}{dp},$$

for demand q and price p. Find the general demand equation $q = f(p)$ for each elasticity function. (*Hint*: Set each elasticity function equal to $-\frac{p}{q} \cdot \frac{dq}{dp}$, then solve for q. Write the constant of integration as $\ln C$ in Exercise 39.)

38. $E = \dfrac{4p^2}{q^2}$

39. $E = 2$

40. Internet Usage During the early days of the Internet, growth in the number of users worldwide could be approximated by an exponential function. The following table gives the number of worldwide users of the Internet. *Source: Internet World Stats.*

Year	Number of Users (in millions)
1995	16
1996	36
1997	70
1998	147
1999	248
2000	361
2001	513
2002	587
2003	719
2004	817
2005	1018
2006	1093
2007	1319
2008	1574
2009	1802

Use a calculator with exponential and logistic regression capabilities to complete the following.

a. Letting t represent the years since 1990, plot the number of worldwide users of the Internet on the y-axis against the year on the t-axis. Discuss the shape of the graph.

b. Use the exponential regression function on your calculator to determine the exponential equation that best fits the data. Plot the exponential equation on the same graph as the data points. Discuss the appropriateness of fitting an exponential function to these data.

c. Use the logistic regression function on your calculator to determine the logistic equation that best fits the data. Plot the logistic equation on the same graph. Discuss the appropriateness of fitting a logistic function to these data. Which graph better fits the data?

d. Assuming that the logistic function found in part c continues to be accurate, what seems to be the limiting size of the number of worldwide Internet users?

41. Life Insurance A life insurance company invests $5000 in a bank account in order to fund a death benefit of $20,000. Growth in the investment over time can be modeled by the differential equation

$$\frac{dA}{dt} = Ai$$

where i is the interest rate and $A(t)$ is the amount invested at time t (in years). Calculate the interest rate that the investment must earn in order for the company to fund the death benefit in 24 years. Choose one of the following. *Source: Society of Actuaries*.

a. $\dfrac{-\ln 2}{12}$ **b.** $\dfrac{-\ln 2}{24}$ **c.** $\dfrac{\ln 2}{24}$ **d.** $\dfrac{\ln 2}{12}$ **e.** $\dfrac{\ln 2}{6}$

Life Sciences

42. Tracer Dye The amount of a tracer dye injected into the bloodstream decreases exponentially, with a decay constant of 3% per minute. If 6 cc are present initially, how many cubic centimeters are present after 10 minutes? (Here k will be negative.)

43. Soil Moisture The evapotranspiration index I is a measure of soil moisture. An article on 10- to 14-year-old heath vegetation described the rate of change of I with respect to W, the amount of water available, by the equation

$$\frac{dI}{dW} = 0.088(2.4 - I).$$

Source: Australian Journal of Botany.

a. According to the article, I has a value of 1 when $W = 0$. Solve the initial value problem.

b. What happens to I as W becomes larger and larger?

44. Fish Population An isolated fish population is limited to 4000 by the amount of food available. If there are now 320 fish and the population is growing with a growth constant of 2% a year, find the expected population at the end of 10 years.

Dieting A person's weight depends both on the daily rate of energy intake, say C calories per day, and on the daily rate of energy consumption, typically between 15 and 20 calories per pound per day. Using an average value of 17.5 calories per pound per day, a person weighing w pounds uses $17.5w$ calories per day. If $C = 17.5w$, then weight remains constant, and weight gain or loss occurs according to whether C is greater or less than $17.5w$. *Source: The College Mathematics Journal.*

45. To determine how fast a change in weight will occur, the most plausible assumption is that dw/dt is proportional to the net excess (or deficit) $C - 17.5w$ in the number of calories per day.

a. Assume C is constant and write a differential equation to express this relationship. Use k to represent the constant of proportionality. What does C being constant imply?

b. The units of dw/dt are pounds per day, and the units of $C - 17.5w$ are calories per day. What units must k have?

c. Use the fact that 3500 calories is equivalent to 1 lb to rewrite the differential equation in part a.

d. Solve the differential equation.

e. Let w_0 represent the initial weight and use it to express the coefficient of $e^{-0.005t}$ in terms of w_0 and C.

46. (Refer to Exercise 45.) Suppose someone initially weighing 180 lb adopts a diet of 2500 calories per day.

a. Write the weight function for this individual.

b. Graph the weight function on the window $[0, 300]$ by $[120, 200]$. What is the asymptote? This value of w is the equilibrium weight w_{eq}. According to the model, can a person ever achieve this weight?

c. How long will it take a dieter to reach a weight just 2 lb more than w_{eq}?

47. H1N1 Virus The cumulative number of deaths worldwide due to the H1N1 virus, or swine flu, at various days into the epidemic are listed below, where April 21, 2009 was day 1. *Source: BBC.*

Day	Deaths	Day	Deaths
14	27	148	3696
28	74	163	4334
43	114	182	4804
57	164	206	6704
71	315	221	8450
84	580	234	9797
99	1049	266	14,024
117	2074	274	14,378
132	2967		

Use a calculator with logistic regression capability to complete the following.

a. Plot the number of deaths y against the number of days t. Discuss the appropriateness of fitting a logistic function to this data.

b. Use the logistic regression function on your calculator to determine the logistic equation that best fits the data.

c. Plot the logistic regression function from part b on the same graph as the data points. Discuss how well the logistic equation fits the data.

d. Assuming the logistic equation found in part b continues to be accurate, what seems to be the limiting size of the number of deaths due to this outbreak of the H1N1 virus?

e. Discuss whether a logistic model is more appropriate than an exponential model for estimating the number of deaths due to the H1N1 virus.

48. Population Growth The following table gives the historic and projected populations (in millions) of China and India. *Source: United Nations.*

Year	China	India
1950	545	372
1960	646	448
1970	816	553
1980	981	693
1990	1142	862
2000	1267	1043
2010	1354	1214
2020	1431	1367
2030	1462	1485
2040	1455	1565
2050	1417	1614

Use a calculator with logistic regression capability to complete the following.

a. Letting t represent the years since 1950, plot the Chinese population on the y-axis against the year on the t-axis. Discuss the appropriateness of fitting a logistic function to these data.

b. Use the logistic regression function on your calculator to determine the logistic equation that best fits the data. Plot the logistic function on the same graph as the data points. Discuss how well the logistic function fits the data.

c. Assuming the logistic equation found in part b continues to be accurate, what seems to be the limiting size of the Chinese population?

d. Repeat parts a–c using the population for India.

49. U.S. Hispanic Population A recent report by the U.S. Census Bureau predicts that the U.S. Hispanic population will increase from 35.6 million in 2000 to 102.6 million in 2050. *Source: U.S. Census Bureau.* Assuming the unlimited growth model $dy/dt = ky$ fits this population growth, express the population y as a function of the year t. Let 2000 correspond to $t = 0$.

50. U.S. Asian Population (Refer to Exercise 49.) The report also predicted that the U.S. Asian population would increase from 10.7 million in 2000 to 33.4 million in 2050. *Source: U.S. Census Bureau.* Repeat Exercise 49 using this data.

51. Spread of a Rumor Suppose the rate at which a rumor spreads—that is, the number of people who have heard the rumor over a period of time—increases with the number of people who have heard it. If y is the number of people who have heard the rumor, then

$$\frac{dy}{dt} = ky,$$

where t is the time in days.

a. If y is 1 when $t = 0$, and y is 5 when $t = 2$, find k.

Using the value of k from part a, find y for each time.

b. $t = 3$ **c.** $t = 5$ **d.** $t = 10$

52. World Population The following table gives the population of the world at various times over the last two centuries, plus projections for this century. ***Source: The New York Times***.

Year	Population (billions)
1804	1
1927	2
1960	3
1974	4
1987	5
1999	6
2011	7
2025	8
2041	9
2071	10

Use a calculator with logistic regression capability to complete the following.

a. Use the logistic regression function on your calculator to determine the logistic equation that best fits the data.

b. Plot the logistic function found in part a and the original data in the same window. Does the logistic function seem to fit the data from 1927 on? Before 1927?

c. To get a better fit, subtract 0.99 from each value of the population in the table. (This makes the population in 1804 small, but not 0 or negative.) Find a logistic function that fits the new data.

d. Plot the logistic function found in part c and the modified data in the same window. Does the logistic function now seem to be a better fit than in part b?

e. Based on the results from parts c and d, predict the limiting value of the world's population as time increases. For comparison, the *New York Times* article predicts a value of 10.73 billion. (*Hint*: After taking the limit, remember to add the 0.99 that was removed earlier.)

f. Based on the results from parts c and d, predict the limiting value of the world population as you go further and further back in time. Does that seem reasonable? Explain.

53. Worker Productivity A company has found that the rate at which a person new to the assembly line produces items is

$$\frac{dy}{dx} = 7.5e^{-0.3y},$$

where x is the number of days the person has worked on the line. How many items can a new worker be expected to produce on the eighth day if he produces none when $x = 0$?

Physical Sciences

54. Radioactive Decay The amount of a radioactive substance decreases exponentially, with a decay constant of 3% per month.

a. Write a differential equation to express the rate of change.

b. Find a general solution to the differential equation from part a.

c. If there are 75 g at the start of the decay process, find a particular solution for the differential equation from part a.

d. Find the amount left after 10 months.

55. Snowplow One morning snow began to fall at a heavy and constant rate. A snowplow started out at 8:00 A.M. At 9:00 A.M. it had traveled 2 miles. By 10:00 A.M. it had traveled 3 miles. Assuming that the snowplow removes a constant volume of snow per hour, determine the time at which it started snowing. (*Hint:* Let t denote the time since the snow started to fall, and let T be the time when the snowplow started out. Let x, the distance the snowplow has traveled, and h, the height of the snow, be functions of t. The assumption that a constant volume of snow per hour is removed implies that the speed of the snowplow times the height of the snow is a constant. Set up and solve differential equations involving dx/dt and dh/dt.) ***Source: The American Mathematical Monthly***.

Newton's Law of Cooling **Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference in temperature between the object and the surrounding medium. Thus, if T is the temperature of the object after t hours and T_M is the (constant) temperature of the surrounding medium, then**

$$\frac{dT}{dt} = -k(T - T_M),$$

where k is a constant. Use this equation in Exercises 56–59.

56. Show that the solution of this differential equation is

$$T = Ce^{-kt} + T_M,$$

where C is a constant.

57. According to the solution of the differential equation for Newton's law of cooling, what happens to the temperature of an object after it has been in a surrounding medium with constant temperature for a long period of time? How well does this agree with reality?

Newton's Law of Cooling **When a dead body is discovered, one of the first steps in the ensuing investigation is for a medical examiner to determine the time of death as closely as possible. Have you ever wondered how this is done? If the temperature of the medium (air, water, or whatever) has been fairly constant and less than 48 hours have passed since the death, Newton's law of cooling can be used. The medical examiner does not actually solve the equation for each case. Instead, a table based on the formula is consulted. Use Newton's law of cooling to work the following exercises. *Source: The College Mathematics Journal.***

58. Assume the temperature of a body at death is 98.6°F, the temperature of the surrounding air is 68°F, and at the end of one hour the body temperature is 90°F.

a. What is the temperature of the body after 2 hours?

b. When will the temperature of the body be 75°F?

c. Approximately when will the temperature of the body be within 0.01° of the surrounding air?

59. Suppose the air temperature surrounding a body remains at a constant 10°F, $C = 88.6$, and $k = 0.24$.

a. Determine a formula for the temperature at any time t.

b. Use a graphing calculator to graph the temperature T as a function of time t on the window $[0, 30]$ by $[0, 100]$.

c. When does the temperature of the body decrease more rapidly: just after death, or much later? How do you know?

d. What will the temperature of the body be after 4 hours?

e. How long will it take for the body temperature to reach 40°F? Use your calculator graph to verify your answer.

YOUR TURN ANSWERS

1. $y = 2x^6 + (2/3)x^{3/2} + e^{5x}/5 + C$ **2.** $y = 3x^4 + 2x^3 - 4$

3. $y = ((3/2)x^2 + 3\ln|x| + C)^{1/3}$ **4.** 3733

16 CHAPTER REVIEW

SUMMARY

In this chapter, we introduced another technique of integration and some applications of integration. The technique is known as integration by parts, which is derived from the product rule for derivatives. We also developed definite integral formulas to calculate the volume of a solid of revolution and the average value of a function on some interval. We then used definite integrals to study continuous money flow and we learned how to evaluate improper integrals that have upper or lower limits of ∞ or $-\infty$. Finally, we solved a particular class of differential equations using the method of separation of variables, and we used the technique to investigate several applications.

Integration by Parts If u and v are differentiable functions, then

$$\int u\,dv = uv - \int v\,du.$$

Volume of a Solid of Revolution If $f(x)$ is nonnegative and R is the region between $f(x)$ and the x-axis from $x = a$ to $x = b$, the volume of the solid formed by rotating R about the x-axis is given by

$$V = \int_a^b \pi[f(x)]^2\,dx.$$

Average Value of a Function The average value of a function f on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx,$$

provided the indicated definite integral exists.

Total Money Flow If $f(t)$ is the rate of money flow, then the total money flow over the time interval from $t = 0$ to $t = T$ is given by

$$\int_0^T f(t)\,dt.$$

Present Value of Money Flow If $f(t)$ is the rate of continuous money flow at an interest rate r for T years, then the present value is

$$P = \int_0^T f(t)e^{-rt}\,dt.$$

Accumulated Amount of Money Flow at Time T If $f(t)$ is the rate of money flow at an interest rate r at time t, the accumulated amount of money flow at time T is

$$A = e^{rT}\int_0^T f(t)e^{-rt}\,dt.$$

Improper Integrals If f is continuous on the indicated interval and if the indicated limits exist, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx,$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

for real numbers a, b, and c, where c is arbitrarily chosen.

Capital Value If $R(t)$ gives the annual rate at which earnings are produced by an asset at time t, the capital value is given by

$$\int_0^\infty R(t)e^{-rt}\,dt,$$

where r is the annual rate of interest, compounded continuously.

General Solution of $\frac{dy}{dx} = g(x)$ The general solution of the differential equation $dy/dx = g(x)$ is

$$y = \int g(x)\,dx.$$

Separable Differential Equation An equation is separable if it can be written in the form

$$\frac{dy}{dx} = \frac{p(x)}{q(y)}.$$

By separating the variables, it is transformed into the equation

$$\int q(y)\,dy = \int p(x)\,dx.$$

KEY TERMS

16.1
integration by parts
column integration

16.2
solid of revolution
average value of a function

16.3
total money flow
present value of continuous money flow
accumulated amount of money flow

16.4
improper integral
convergent integral
divergent integral

16.5
differential equation
general solution
particular solution
initial condition
initial value problem
separable differential equation
capital value
separation of variables
growth constant
carrying capacity
logistic growth model
logistic equation
logistic curve

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. Integration by parts should be used to evaluate $\int_0^1 \frac{x^2}{x^3+1}\,dx$.

2. Integration by parts should be used to evaluate $\int_0^1 xe^{10x}\,dx$.

3. We would need to apply the method of integration by parts twice to determine

$$\int x^3 e^{-x^2} dx.$$

4. Integration by parts should be used to determine $\int \ln(4x)\,dx$.

5. The average value of the function $f(x) = 2x^2 + 3$ on $[1, 4]$ is given by

$$\frac{1}{3}\int_1^4 \pi(2x^2+3)^2\,dx.$$

6. The volume of the solid formed by revolving the function $f(x) = \sqrt{x^2+1}$ about the x-axis on the interval $[1, 2]$ is given by

$$\int_1^2 \pi\sqrt{x^2+1}\,dx.$$

7. The volume of the solid formed by revolving the function $f(x) = x + 4$ about the x-axis on the interval $[-4, 5]$ is given by

$$\int_{-4}^5 \pi(x+4)^2\,dx.$$

8. If $f(t) = 1000e^{0.05t}$ represents the rate of flow of money for a vending machine over the first five years of income, then the total money flow for that time period is given by

$$\int_0^5 1000e^{0.05t}\,dt.$$

9. If a company expects an annual flow of money during the next five years to be $f(t) = 1000e^{0.05t}$, the present value of this income, assuming an annual interest rate of 4.5% compounded continuously is given by

$$\int_0^5 1000e^{0.005t}\,dt.$$

10. $\int_{-\infty}^{\infty} xe^{-2x}\,dx = \lim_{c\to\infty}\int_{-c}^{c} xe^{-2x}\,dx$

11. To determine a particular solution to a differential equation, you first must find a general solution to the differential equation.

12. The function $y = e^{2x} + 5$ satisfies the differential equation $\dfrac{dy}{dx} = 2y$.

13. The function $y = \dfrac{100}{1 + 99e^{-5t}}$ satisfies the differential equation $\dfrac{dy}{dt} = 5\left(1 - \dfrac{y}{100}\right)y$.

14. It is possible to solve the following differential equation using the method of separation of variables.
$x\dfrac{dy}{dx} = (x + 1)(y + 1)$

PRACTICE AND EXPLORATIONS

15. Describe the type of integral for which integration by parts is useful.

16. Compare finding the average value of a function with finding the average of n numbers.

17. What is an improper integral? Explain why improper integrals must be treated in a special way.

Find each integral, using techniques from this or the previous chapter.

18. $\displaystyle\int x(8 - x)^{3/2}\,dx$

19. $\displaystyle\int \frac{3x}{\sqrt{x - 2}}\,dx$

20. $\displaystyle\int xe^{x}\,dx$

21. $\displaystyle\int (3x + 6)e^{-3x}\,dx$

22. $\displaystyle\int \ln|4x + 5|\,dx$

23. $\displaystyle\int (x - 1)\ln|x|\,dx$

24. $\displaystyle\int \frac{x}{25 - 9x^2}\,dx$

25. $\displaystyle\int \frac{x}{\sqrt{16 + 8x^2}}\,dx$

26. $\displaystyle\int_1^e x^3 \ln x\,dx$

27. $\displaystyle\int_0^1 x^2 e^{x/2}\,dx$

28. Find the area between $y = (3 + x^2)e^{2x}$ and the x-axis from $x = 0$ to $x = 1$.

29. Find the area between $y = x^3(x^2 - 1)^{1/3}$ and the x-axis from $x = 1$ to $x = 3$.

Find the volume of the solid of revolution formed by rotating each bounded region about the x-axis.

30. $f(x) = 3x - 1,\quad y = 0,\quad x = 2$

31. $f(x) = \sqrt{x - 4},\quad y = 0,\quad x = 13$

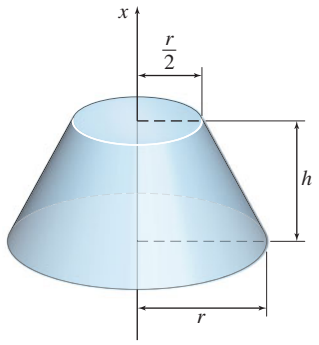
32. $f(x) = e^{-x},\quad y = 0,\quad x = -2,\quad x = 1$

33. $f(x) = \dfrac{1}{\sqrt{x - 1}},\quad y = 0,\quad x = 2,\quad x = 4$

34. $f(x) = 4 - x^2,\quad y = 0,\quad x = -1,\quad x = 1$

35. $f(x) = \dfrac{x^2}{4},\quad y = 0,\quad x = 4$

36. A frustum is what remains of a cone when the top is cut off by a plane parallel to the base. Suppose a right circular frustum (that is, one formed from a right circular cone) has a base with radius r, a top with radius $r/2$, and a height h. (See the figure below.) Find the volume of this frustum by rotating about the x-axis the region below the line segment from $(0, r)$ to $(h, r/2)$.



37. How is the average value of a function found?

38. Find the average value of $f(x) = \sqrt{x + 1}$ over the interval $[0, 8]$.

39. Find the average value of $f(x) = 7x^2(x^3 + 1)^6$ over the interval $[0, 2]$.

Find the value of each integral that converges.

40. $\displaystyle\int_{10}^{\infty} x^{-1}\,dx$

41. $\displaystyle\int_{-\infty}^{-5} x^{-2}\,dx$

42. $\displaystyle\int_0^{\infty} \frac{dx}{(3x + 1)^2}$

43. $\displaystyle\int_1^{\infty} 6e^{-x}\,dx$

44. $\displaystyle\int_{-\infty}^{0} \frac{x}{x^2 + 3}\,dx$

45. $\displaystyle\int_4^{\infty} \ln(5x)\,dx$

Find the area between the graph of each function and the x-axis over the given interval, if possible.

46. $f(x) = \dfrac{5}{(x-2)^2}$, for $(-\infty, 1]$

47. $f(x) = 3e^{-x}$, for $[0, \infty)$

48. How is the present value of money flow found? The accumulated amount of money flow?

49. What is a differential equation? What is it used for?

50. What is the difference between a particular solution and a general solution to a differential equation?

Find the general solution for each differential equation.

51. $\dfrac{dy}{dx} = 3x^2 + 6x$

52. $\dfrac{dy}{dx} = 4x^3 + 6x^5$

53. $\dfrac{dy}{dx} = 4e^{2x}$

54. $\dfrac{dy}{dx} = \dfrac{1}{3x+2}$

55. $\dfrac{dy}{dx} = \dfrac{3x+1}{y}$

56. $\dfrac{dy}{dx} = \dfrac{e^x + x}{y-1}$

57. $\dfrac{dy}{dx} = \dfrac{2y+1}{x}$

58. $\dfrac{dy}{dx} = \dfrac{3-y}{e^x}$

Find the particular solution for each initial value problem. (Some solutions may give y implicitly.)

59. $\dfrac{dy}{dx} = x^2 - 6x; \quad y(0) = 3$

60. $\dfrac{dy}{dx} = 5(e^{-x} - 1); \quad y(0) = 17$

61. $\dfrac{dy}{dx} = (x+2)^3 e^y; \quad y(0) = 0$

62. $\dfrac{dy}{dx} = (3 - 2x)y; \quad y(0) = 5$

63. $\dfrac{dy}{dx} = \dfrac{1-2x}{y+3}; \quad y(0) = 16$

64. $\sqrt{x}\dfrac{dy}{dx} = xy; \quad y(1) = 4$

APPLICATIONS

Business and Economics

65. Total Revenue The rate of change of revenue from the sale of x toaster ovens is

$$R'(x) = x(x-50)^{1/2}.$$

Find the total revenue from the sale of the 50th to the 75th ovens.

Present Value of Money Flow **Each function in Exercises 66–69 represents the rate of flow of money (in dollars per year) over the given time period, compounded continuously at the given annual interest rate. Find the present value in each case.**

66. $f(t) = 5000$, 8 years, 9%

67. $f(t) = 25{,}000$, 12 years, 10%

68. $f(t) = 150e^{0.04t}$, 5 years, 6%

69. $f(t) = 15t$, 18 months, 8%

Accumulated Amount of Money Flow at Time T **Assume that each function gives the rate of flow of money in dollars per year over the given period, with continuous compounding at the given annual interest rate. Find the accumulated amount of money flow at the end of the time period.**

70. $f(t) = 1000$, 5 years, 6%

71. $f(t) = 500e^{-0.04t}$, 8 years, 10%

72. $f(t) = 20t$, 6 years, 4%

73. $f(t) = 1000 + 200t$, 10 years, 9%

74. Money Flow An investment scheme is expected to produce a continuous flow of money, starting at \$1000 and increasing exponentially at 5% a year for 7 years. Find the present value at an interest rate of 11% compounded continuously.

75. Money Flow The proceeds from the sale of a building will yield a uniform continuous flow of \$10,000 a year for 10 years. Find the final amount at an interest rate of 10.5% compounded continuously.

76. Capital Value Find the capital value of an office building for which annual rent of \$50,000 will be paid in perpetuity, if the interest rate is 9%.

77. Marginal Sales The marginal sales (in hundreds of dollars) of a computer software company are given by

$$\frac{dy}{dx} = 6e^{0.3x},$$

where x is the number of months the company has been in business. Assume that sales were 0 initially.

a. Find the sales after 6 months.

b. Find the sales after 12 months.

78. Production Rate The rate at which a new worker in a certain factory produces items is given by

$$\frac{dy}{dx} = 0.1(150 - y),$$

where y is the number of items produced by the worker per day, x is the number of days worked, and the maximum production per day is 150 items. Assume that the worker produces 15 items at the beginning of the first day on the job ($x = 0$).

a. Find the number of items the new worker will produce in 10 days.

b. Determine the number of days for a new worker to produce 100 items per day.

79. Continuous Withdrawals A retirement savings account contains $300,000 and earns 5% interest compounded continuously. The retiree makes *continuous* withdrawals of $20,000 per year.

a. Write a differential equation to describe the situation.

b. How much will be left in the account after 10 years?

80. In Exercise 79, approximately how long will it take to use up the account?

Life Sciences

81. Drug Reaction The reaction rate to a new drug t hours after the drug is administered is

$$r'(t) = 0.5te^{-t}.$$

Find the total reaction over the first 5 hours.

82. Oil Leak Pollution An oil leak from an uncapped well is polluting a bay at a rate of $f(t) = 125e^{-0.025t}$ gallons per year. Use an improper integral to find the total amount of oil that will enter the bay, assuming the well is never capped.

83. Effect of Insecticide After use of an experimental insecticide, the rate of decline of an insect population is

$$\frac{dy}{dt} = \frac{-10}{1 + 5t},$$

where t is the number of hours after the insecticide is applied. Assume that there were 50 insects initially.

a. How many are left after 24 hours?

b. How long will it take for the entire population to die?

84. Growth of a Mite Population A population of mites grows at a rate proportional to the number present, y. If the growth constant is 10% and 120 mites are present at time $t = 0$ (in weeks), find the number present after 6 weeks.

85. Population Growth Let

$$y = \frac{N}{1 + be^{-kt}}.$$

If y is y_1, y_2, and y_3 at times t_1, t_2, and $t_3 = 2t_2 - t_1$ (that is, at three equally spaced times), then prove that

$$N = \frac{1/y_1 + 1/y_3 - 2/y_2}{1/(y_1y_3) - 1/y_2^2}.$$

Population Growth In the following table of U.S. Census figures, y is the population in millions. *Source: U.S. Census Bureau.*

Year	y	Year	y
1790	3.9	1910	92.0
1800	5.3	1920	105.7
1810	7.2	1930	122.8
1820	9.6	1940	131.7
1830	12.9	1950	150.7
1840	17.1	1960	179.3
1850	23.2	1970	203.3
1860	31.4	1980	226.5
1870	39.8	1990	248.7
1880	50.2	2000	281.4
1890	62.9	2010	308.7
1900	76.0		

86. Use Exercise 85 and the table to find the following.

a. Find N using the years 1800, 1850, and 1900.

b. Find N using the years 1850, 1900, and 1950.

c. Find N using the years 1870, 1920, and 1970.

d. Explain why different values of N were obtained in parts a–c. What does this suggest about the validity of this model and others?

87. Let $t = 0$ correspond to 1870, and let every decade correspond to an increase in t of 1. Use the table from Exercise 85.

a. Use 1870, 1920, and 1970 to find N, 1870 to find b, and 1920 to find k in the equation

$$y = \frac{N}{1 + be^{-kt}}.$$

b. Estimate the population of the United States in 2010 and compare your estimate to the actual population in 2010.

c. Predict the populations of the United States in 2030 and 2050.

88. Let $t = 0$ correspond to 1790, and let every decade correspond to an increase in t of 1. Use a calculator with logistic regression capability to complete the following. Use the table from Exercise 85.

a. Plot the data points. Do the points suggest that a logistic function is appropriate here?

b. Use the logistic regression function on your calculator to determine the logistic equation that best fits the data.

c. Plot the logistic equation from part a on the same graph as the data points. How well does the logistic equation seem to fit the data?

d. What seems to be the limiting size of the U.S. population?

Social Sciences

89. Education Researchers have proposed that the amount a full-time student is educated (x) changes with respect to the student's age t according to the differential equation

$$\frac{dx}{dt} = 1 - kx,$$

where k is a constant measuring the rate that education depreciates due to forgetting or technological obsolescence. *Source: Operations Research*.

a. Solve the equation using the method of separation of variables.

b. What does x approach over time?

90. Spread of a Rumor A rumor spreads through the offices of a company with 200 employees, starting in a meeting with 10 people. After 3 days, 35 people have heard the rumor.

a. Write an equation for the number of people who have heard the rumor in t days. (*Hint:* Assume a logistic model.)

b. How many people have heard the rumor in 5 days?

Physical Sciences

91. Average Temperatures Suppose the temperature (degrees F) in a river at a point x meters downstream from a factory that is discharging hot water into the river is given by

$$T(x) = 160 - 0.05x^2.$$

Find the average temperature over each interval.

a. $[0, 10]$ **b.** $[10, 40]$ **c.** $[0, 40]$

92. Newton's Law of Cooling A roast at a temperature of 40°F is put in a 300°F oven. After 1 hour the roast has reached a temperature of 150°F. Newton's law of cooling states that

$$\frac{dT}{dt} = k(T - T_M),$$

where T is the temperature of an object, the surrounding medium has temperature T_M at time t, and k is a constant. Use Newton's law to find the temperature of the roast after 2 hours.

93. In Exercise 92, how long does it take for the roast to reach a temperature of 250°F?

EXTENDED APPLICATION

ESTIMATING LEARNING CURVES IN MANUFACTURING WITH INTEGRALS

In the previous chapter you have seen how the trapezoidal rule uses sums of areas of polygons to approximate the area under a smooth curve, that is, a definite integral. In this Extended Application we look at the reverse process, using an integral to estimate a sum, in the context of estimating production costs.

As a manufacturer produces more units of a new product, the individual units generally become cheaper to produce, because with experience, production workers gain skill and speed, and managers spot opportunities for improved efficiency. This decline in unit costs is often called an *experience curve* or *learning curve*. This curve is important when a manufacturer negotiates a contract with a buyer.

Here's an example, based on an actual contract that came before the Armed Services Board of Contract Appeals. ***Source: Armed Services Board of Contract Appeals***. The Navy asked the ITT Defense Communications Division to bid on the manufacture of several different kinds of mobile telephone switchboards, including 280 of the model called the SB 3865. ITT figured that the cost of making a single SB 3865 was around $300,000. But they couldn't submit a bid of $300,000 × 280 or $84 million, because multiple units should have a lower unit price. So ITT used a learning curve to estimate an average unit cost of $135,300 for all 280 switchboards and submitted a bid of $135,300 × 280 or $37.9 million.

The contract gave the Navy an option to purchase 280 SB units over three years, but in fact it bought fewer. Suppose the Navy bought 140 SBs. Should it pay half of the original price of $37.9 million? No: ITT's bid was based on the efficiencies of a 280-unit run, so 140 units should be repriced to yield *more* than half the full price. A repricing clause in the contract specified that a learning curve would be used to reprice partial orders, and when the Navy ordered less than the full amount, ITT invoked this clause to reprice the switchboards. The question in dispute at the hearing was which learning curve to use.

There are two common learning curve models. The *unit learning curve* model assumes that each time the number of units doubles, say from n to $2n$, the cost of producing the last unit is some constant fraction r of the cost for the nth unit. Usually the fraction r is given as a percent. If $r = 90\%$ (typical for big pieces of hardware), then the contract would refer to a "90% learning curve." The *cumulative learning curve* model assumes that when the number of units doubles, the *average cost* of producing all $2n$ units is some constant fraction of the *average cost* of the first n units. The Navy's contract with ITT didn't specify which model was to be used—it just referred to "a 90% learning curve." The government used the unit model and ITT used the cumulative average model, and ITT calculated a fair price millions of dollars higher than the government's price!

In practice, ITT used a calculator program to make its estimate, and the government used printed tables, but both the program and the tables were derived using calculus. To see how the computation works, we'll derive the government's unit learning curve.

Each unit has a different cost, with the first unit being the most expensive and the 280th unit the least expensive. So the cost of the nth unit, call it $C(n)$, is a function of n. To find a fair price for n units we'll add up all the unit prices. That is, we will compute $C(1) + C(2) + \cdots + C(n-1) + C(n)$. Before we can do that, we need a formula for $C(n)$ in terms of n, but all we know about $C(n)$ is that

$$C(2n) = r \cdot C(n)$$

for *every* n, with $r = 0.90$. This sort of equation is called a *functional equation:* It relates two different values of the function without giving an explicit formula for the function. In Exercise 3 you'll see how you might discover a solution to this functional equation, but here we'll just give the result:

$$C(n) = C(1) \cdot n^b, \text{ where } b = \frac{\ln r}{\ln 2} \approx -0.152.$$

Thus, to find the price for making 280 units, we need to add up all the values of $C(1) \cdot n^b$ as n ranges from 1 to 280. There's just one problem: We don't know $C(1)$! The only numbers that ITT gave were the *average* cost per unit for 280 units, namely \$135,300, and the total price of \$37.9 million. But if we write out the formula for the 280-unit price in terms of $C(1)$, we can figure out $C(1)$ by dividing. Here's how it works.

The cost C is a function of an integer variable n, since the contractor can't deliver fractional units of the hardware. But the function $C(x) = C(1) \cdot x^{-0.152}$ is a perfectly good function of the real variable x, and the sum of the first 280 values of $C(n)$ should be close to $\int_1^{280} C(x)\,dx = C(1) \cdot \int_1^{280} x^{-0.152}\,dx$. In Exercise 4 you'll see how to derive the following improved estimate:

$$C(1)\left(\int_1^{281} x^{-0.152}\,dx + \frac{1 - 281^{-0.152}}{2}\right).$$

The integrand is a power function, so you know how to evaluate the integral exactly.

$$\int_1^{281} x^{-0.152}\,dx = \frac{1}{0.848} x^{0.848}\Big|_1^{281} = \frac{1}{0.848}\left(281^{0.848} - 1^{0.848}\right)$$

Thus, the sum is approximately

$$C(1)\left[\frac{1}{0.848}\left(281^{0.848} - 1^{0.848}\right) + \frac{1 - 281^{-0.152}}{2}\right] \approx C(1) \cdot 139.75.$$

Since ITT's price for 280 units was \$37.9 million,

$$C(1) = \frac{\$37{,}900{,}000}{139.75} \approx \$271{,}000.$$

Now that we know $C(1)$, we can reprice an order of 140 units by adding up the first 140 values of $C(n)$. An estimate exactly like the one above tells us that according to the government's model, a fair price for 140 units is about \$21 million. As we expected, this is more than half the 280-unit price, in fact about \$2 million more.

EXERCISES

1. According to the formula for $C(n)$, what is the unit price of the 280th unit, to the nearest thousand dollars?

2. Suppose that instead of using natural logarithms to compute b, we use logarithms with a base of 10 and define $b = (\log r)/(\log 2)$. Does this change the value of b?

3. All power functions satisfy an equation similar to our functional equation: If $f(x) = ax^b$, then $f(2x) = a(2x)^b = a2^b \cdot x^b = 2^b \cdot f(x)$. How can you choose a and b to make $C(x) = ax^b$ a solution to the functional equation $C(2n) = r \cdot C(n)$?

4. Figure 20 indicates how you could use the integral

$$\int_1^5 \frac{1}{x}\,dx + \frac{1 - \frac{1}{5}}{2}$$

as an estimate for the sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.

The graph shows the function $y = \frac{1}{x}$.

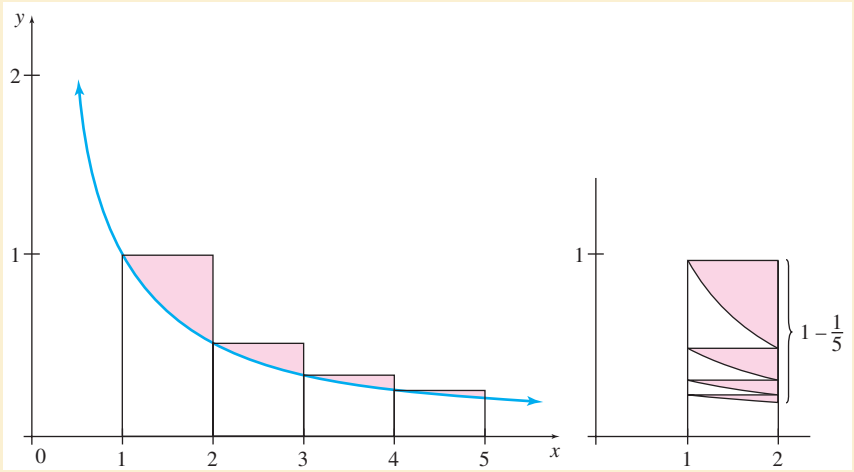


FIGURE 20

a. Write a justification for the integral estimate. (Your argument will also justify the integral estimate for the $C(n)$ sum.) Based on your explanation, does the integral expression overestimate the sum?

b. You know how to integrate the function $1/x$. Compute the integral estimate and the actual value. What is the percentage error in the estimate?

5. Go to the website WolframAlpha.com and enter: "integrate." Follow the instructions and use them along with the improved estimate to verify the fair price for 140 units of about $21 million given in the text. Then enter "sum" into Wolfram| Alpha and follow the instructions to verify the fair price by summing the first 140 values of $C(n)$. Discuss how the solution compares with the solutions provided by the integration and summation features on a graphing calculator, as well as the solution provided by adding up the values using a spreadsheet.

DIRECTIONS FOR GROUP PROJECT

Suppose that you and three other students have an internship with a manufacturing company that is submitting a bid to make several thousand units of some highly technical equipment. The one problem with the bid is that the number of units that will be purchased is only an estimate and that the actual number needed may greatly vary from the estimate. Using the information given above, prepare a presentation for an internal sales meeting that will describe the case listed above and its applications to the bid at hand. Make your presentation realistic in the sense that the product you are manufacturing should have a name, average price, and so on. Then show how integrals can be used to estimate learning curves in this situation and produce a pricing structure for the bid. Presentation software, such as Microsoft PowerPoint, should be used.

17 Multivariable Calculus

17.1 Functions of Several Variables

17.2 Partial Derivatives

17.3 Maxima and Minima

17.4 Lagrange Multipliers

17.5 Total Differentials and Approximations

17.6 Double Integrals

Chapter 17 Review

Extended Application: Using Multivariable Fitting to Create a Response Surface Design

Safe diving requires an understanding of how the increased pressure below the surface affects the body's intake of nitrogen. An exercise in Section 2 of this chapter investigates a formula for nitrogen pressure as a function of two variables, depth and dive time. Partial derivatives tell us how this function behaves when one variable is held constant as the other changes. Dive tables based on the formula help divers to choose a safe time for a given depth, or a safe depth for a given time.

We have thus far limited our study of calculus to functions of one variable. There are other phenomena that require more than one variable to adequately model the situation. For example, the price of an electronics device depends on how long it has been on the market, the number of competing devices, labor costs, demand, and many other factors. In this case, the price is a function of more than one variable. To analyze and better understand situations like this, we will extend the ideas of calculus, including differentiation and integration, to functions of more than one variable.

17.1 Functions of Several Variables

APPLY IT

How are the amounts of labor and capital needed to produce a certain number of items related?

We will study this question in Example 8 using a production function that depends on the two independent variables of labor and capital.

The major difficulty students have with this section, and indeed with this entire chapter, is that they cannot visualize surfaces in 3-dimensional space, even though they live there. Fortunately, such visualization is not necessary for doing the exercises in this chapter. A student who wants to explore what various surfaces look like can use any of the commercial or public domain computer programs available, such as WolframAlpha.com.

If a company produces x items at a cost of \$10 per item, then the total cost $C(x)$ of producing the items is given by

$$C(x) = 10x.$$

The cost is a function of one independent variable, the number of items produced. If the company produces two products, with x of one product at a cost of \$10 each, and y of another product at a cost of \$15 each, then the total cost to the firm is a function of *two* independent variables, x and y. By generalizing $f(x)$ notation, the total cost can be written as $C(x, y)$, where

$$C(x, y) = 10x + 15y.$$

When $x = 5$ and $y = 12$ the total cost is written $C(5, 12)$, with

$$C(5, 12) = 10 \cdot 5 + 15 \cdot 12 = 230.$$

A general definition follows.

Function of Two or More Variables

The expression $z = f(x, y)$ is a **function of two variables** if a unique value of z is obtained from each ordered pair of real numbers (x, y). The variables x and y are **independent variables**, and z is the **dependent variable**. The set of all ordered pairs of real numbers (x, y) such that $f(x, y)$ exists is the **domain** of f; the set of all values of $f(x, y)$ is the **range**. Similar definitions could be given for functions of three, four, or more independent variables.

EXAMPLE 1 Evaluating Functions

Let $f(x, y) = 4x^2 + 2xy + 3/y$ and find the following.

(a) $f(-1, 3)$

SOLUTION Replace x with -1 and y with 3.

$$f(-1, 3) = 4(-1)^2 + 2(-1)(3) + \frac{3}{3} = 4 - 6 + 1 = -1$$

(b) $f(2, 0)$

SOLUTION Because of the quotient $3/y$, it is not possible to replace y with 0, so $f(2, 0)$ is undefined. By inspection, we see that the domain of the function is the set of all (x, y) such that $y \neq 0$.

(c) $\dfrac{f(x + h, y) - f(x, y)}{h}$

SOLUTION Calculate as follows:

$$\frac{f(x + h, y) - f(x, y)}{h} = \frac{4(x + h)^2 + 2(x + h)y + 3/y - [4x^2 + 2xy + 3/y]}{h}$$

$$= \frac{4x^2 + 8xh + 4h^2 + 2xy + 2hy + 3/y - 4x^2 - 2xy - 3/y}{h}$$

$$= \frac{8xh + 4h^2 + 2hy}{h} \quad \text{Simplify the numerator.}$$

$$= \frac{h(8x + 4h + 2y)}{h} \quad \text{Factor } h \text{ from the numerator.}$$

$$= 8x + 4h + 2y.$$

TRY YOUR TURN 1

YOUR TURN 1 For the function in Example 1, find $f(2, 3)$.

EXAMPLE 2 Volume of a Can

Let r and h represent the radius and height of a can in cm. The volume of the can is then a function of the two variables r and h given by

$$V(r, h) = \pi r^2 h.$$

Find $V(3, 11)$.

SOLUTION Replace r with 3 and h with 11 to get

$$V(3, 11) = \pi \cdot 3^2 \cdot 11 = 99\pi \approx 311 \text{ cm}^3.$$

This says that a can with radius 3 cm and height 11 cm has a volume of approximately 311 cm^3.

EXAMPLE 3 Evaluating a Function

Let $f(x, y, z) = 4xz - 3x^2y + 2z^2$. Find $f(2, -3, 1)$.

SOLUTION Replace x with 2, y with -3, and z with 1.

$$f(2, -3, 1) = 4(2)(1) - 3(2)^2(-3) + 2(1)^2 = 8 + 36 + 2 = 46$$

TRY YOUR TURN 2

YOUR TURN 2 For the function in Example 3, find $f(1, 2, 3)$.

Graphing Functions of Two Independent Variables

Functions of one independent variable are graphed by using an x-axis and a y-axis to locate points in a plane. The plane determined by the x- and y-axes is called the *xy-plane*. A third axis is needed to graph functions of two independent variables—the z-axis, which goes through the origin in the xy-plane and is perpendicular to both the x-axis and the y-axis.

Figure 1 shows one possible way to draw the three axes. In Figure 1, the yz-plane is in the plane of the page, with the x-axis perpendicular to the plane of the page.

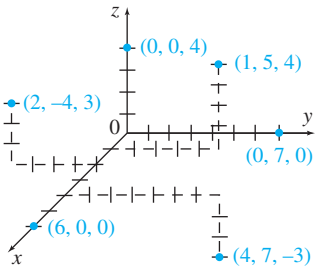


FIGURE 1

Just as we graphed ordered pairs earlier we can now graph **ordered triples** of the form (x, y, z). For example, to locate the point corresponding to the ordered triple $(2, -4, 3)$, start at the origin and go 2 units along the positive x-axis. Then go 4 units in a negative direction (to the left) parallel to the y-axis. Finally, go up 3 units parallel to the z-axis. The point representing $(2, -4, 3)$ is shown in Figure 1, together with several other points. The region of three-dimensional space where all coordinates are positive is called the **first octant**.

In Chapter 1 we saw that the graph of $ax + by = c$ (where a and b are not both 0) is a straight line. This result generalizes to three dimensions.

FOR REVIEW

Graph the following lines. Refer to Section 1.1 if you need to review.

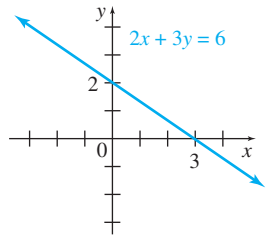
1. $2x + 3y = 6$

2. $x = 4$

3. $y = 2$

Answers

1.



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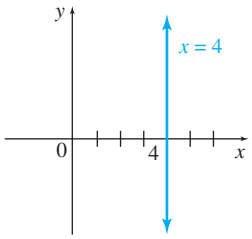
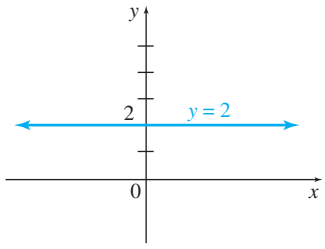


3.



YOUR TURN 3 Graph $x + 2y + 3z = 12$ in the first octant.

Plane

The graph of

$$ax + by + cz = d$$

is a **plane** if a, b, and c are not all 0.

EXAMPLE 4 Graphing a Plane

Graph $2x + y + z = 6$.

SOLUTION The graph of this equation is a plane. Earlier, we graphed straight lines by finding x- and y-intercepts. A similar idea helps in graphing a plane. To find the x-intercept, which is the point where the graph crosses the x-axis, let $y = 0$ and $z = 0$.

$$2x + 0 + 0 = 6$$
$$x = 3$$

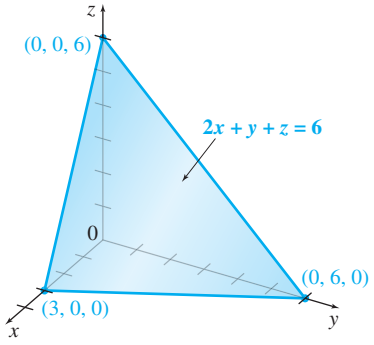


FIGURE 2

The point $(3, 0, 0)$ is on the graph. Letting $x = 0$ and $z = 0$ gives the point $(0, 6, 0)$, while $x = 0$ and $y = 0$ lead to $(0, 0, 6)$. The plane through these three points includes the triangular surface shown in Figure 2. This surface is the first-octant part of the plane that is the graph of $2x + y + z = 6$. The plane does not stop at the axes but extends without bound.

TRY YOUR TURN 3

EXAMPLE 5 Graphing a Plane

Graph $x + z = 6$.

SOLUTION To find the x-intercept, let $y = 0$ and $z = 0$, giving $(6, 0, 0)$. If $x = 0$ and $y = 0$, we get the point $(0, 0, 6)$. Because there is no y in the equation $x + z = 6$, there can be no y-intercept. A plane that has no y-intercept is parallel to the y-axis. The first-octant portion of the graph of $x + z = 6$ is shown in Figure 3.

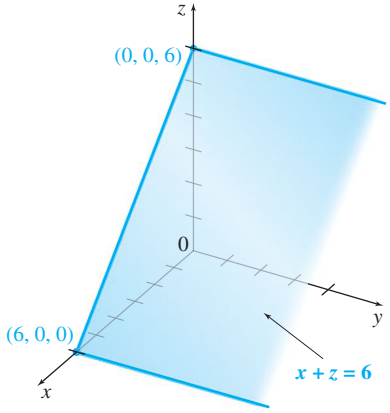


FIGURE 3

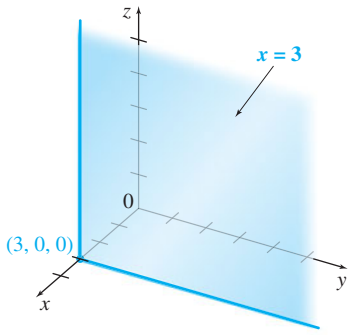


FIGURE 4

EXAMPLE 6 Graphing Planes

Graph each equation in the first octant.

(a) $x = 3$

SOLUTION This graph, which goes through $(3, 0, 0)$, can have no y-intercept and no z-intercept. It is, therefore, a plane parallel to the y-axis and the z-axis and, therefore, to the yz-plane. The first-octant portion of the graph is shown in Figure 4.

(b) $y = 4$

SOLUTION This graph goes through $(0, 4, 0)$ and is parallel to the xz-plane. The first-octant portion of the graph is shown in Figure 5.

(c) $z = 1$

SOLUTION The graph is a plane parallel to the xy-plane, passing through $(0, 0, 1)$. Its first-octant portion is shown in Figure 6.

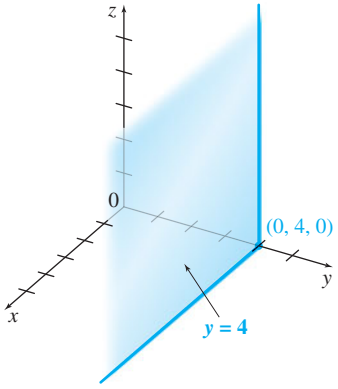


FIGURE 5

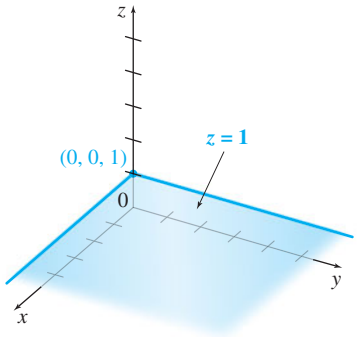


FIGURE 6

The graph of a function of one variable, $y = f(x)$, is a curve in the plane. If x_0 is in the domain of f, the point $(x_0, f(x_0))$ on the graph lies directly above or below the number x_0 on the x-axis, as shown in Figure 7.

The graph of a function of two variables, $z = f(x, y)$, is a **surface** in three-dimensional space. If (x_0, y_0) is in the domain of f, the point $(x_0, y_0, f(x_0, y_0))$ lies directly above or below the point (x_0, y_0) in the xy-plane, as shown in Figure 8.

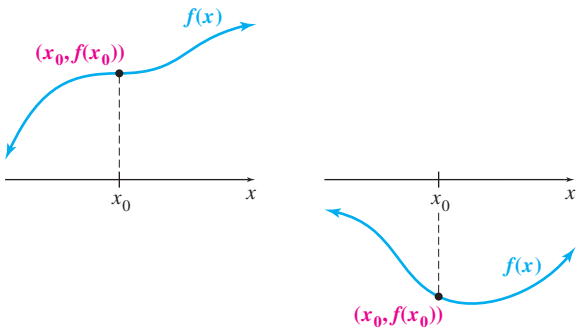


FIGURE 7

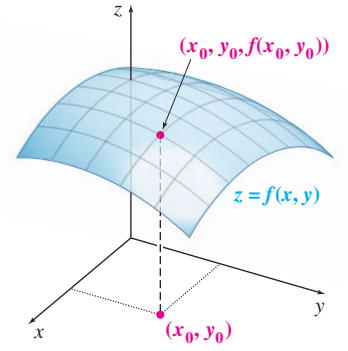


FIGURE 8

Although computer software is available for drawing the graphs of functions of two independent variables, you can often get a good picture of the graph without it by finding various **traces**—the curves that result when a surface is cut by a plane. The ***xy*-trace** is the intersection of the surface with the xy-plane. The ***yz*-trace** and ***xz*-trace** are defined similarly. You can also determine the intersection of the surface with planes parallel to the xy-plane. Such planes are of the form $z = k$, where k is a constant, and the curves that result when they cut the surface are called **level curves**.

EXAMPLE 7 Graphing a Function

Graph $z = x^2 + y^2$.

SOLUTION The yz-plane is the plane in which every point has a first coordinate of 0, so its equation is $x = 0$. When $x = 0$, the equation becomes $z = y^2$, which is the equation of a parabola in the yz-plane, as shown in Figure 9(a). Similarly, to find the intersection of the surface with the xz-plane (whose equation is $y = 0$), let $y = 0$ in the equation. It then becomes $z = x^2$, which is the equation of a parabola in the xz-plane, as shown in Figure 9(a). The xy-trace (the intersection of the surface with the plane $z = 0$) is the single point $(0, 0, 0)$ because $x^2 + y^2$ is equal to 0 only when $x = 0$ and $y = 0$.

Next, we find the level curves by intersecting the surface with the planes $z = 1$, $z = 2$, $z = 3$, etc. (all of which are parallel to the xy-plane). In each case, the result is a circle:

$$x^2 + y^2 = 1, \qquad x^2 + y^2 = 2, \qquad x^2 + y^2 = 3,$$

and so on, as shown in Figure 9(b). Drawing the traces and level curves on the same set of axes suggests that the graph of $z = x^2 + y^2$ is the bowl-shaped figure, called a **paraboloid**, that is shown in Figure 9(c).

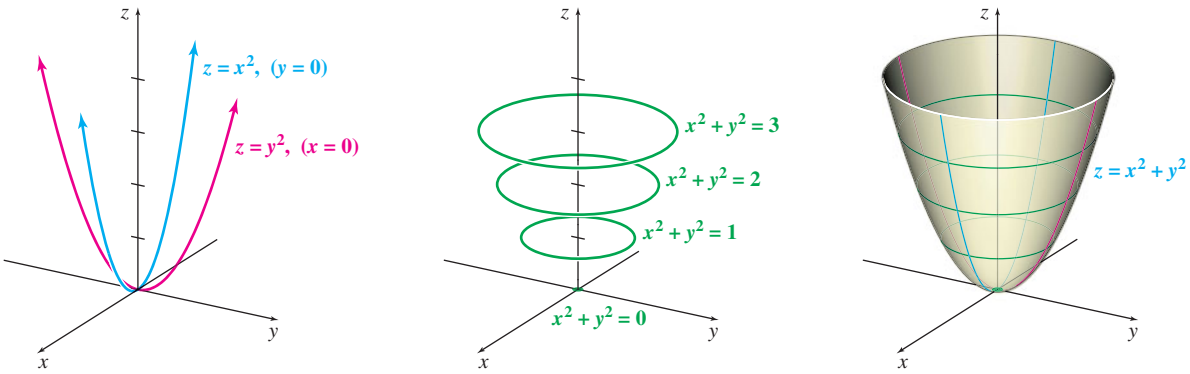


FIGURE 9

Figure 10 on the next page shows the level curves from Example 7 plotted in the xy-plane. The picture can be thought of as a topographical map that describes the surface generated by $z = x^2 + y^2$, just as the topographical map in Figure 11 describes the surface of the land in a part of New York state.

One application of level curves occurs in economics with production functions. A **production function** $z = f(x, y)$ is a function that gives the quantity z of an item produced as a function of x and y, where x is the amount of labor and y is the amount of capital (in appropriate units) needed to produce z units. If the production function has the special form $z = P(x, y) = Ax^a y^{1-a}$, where A is a constant and $0 < a < 1$, the function is called

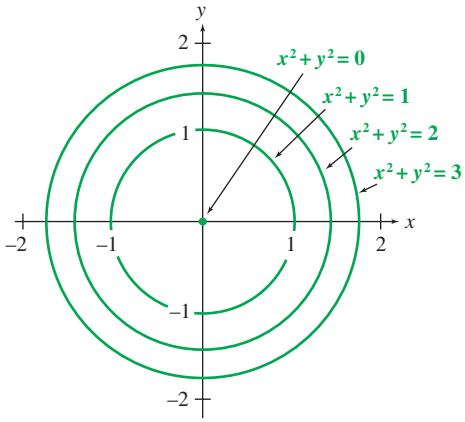


FIGURE 10

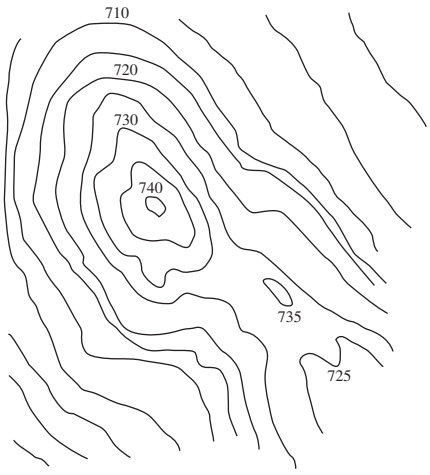


FIGURE 11

a **Cobb-Douglas production function**. This function was developed in 1928 by economist Paul H. Douglas (1892–1976), who later became a senator for the state of Illinois, and mathematician Charles W. Cobb. For production functions, level curves are used to indicate combinations of the values of x and y that produce the same value of production z.

EXAMPLE 8 Cobb-Douglas Production Function

Find the level curve at a production of 100 items for the Cobb-Douglas production function $z = x^{2/3}y^{1/3}$.

APPLY IT

SOLUTION Let $z = 100$ to get

$$100 = x^{2/3}y^{1/3}$$

$$\frac{100}{x^{2/3}} = y^{1/3}.$$

Now cube both sides to express y as a function of x.

$$y = \frac{100^3}{x^2} = \frac{1{,}000{,}000}{x^2}$$

YOUR TURN 4 Find the level curve at a production of 27 items in the form $y = f(x)$ for the Cobb-Douglas production function $z = x^{1/4}y^{3/4}$.

TRY YOUR TURN 4

The level curve of height 100 found in Example 8 is shown graphed in three dimensions in Figure 12(a) and on the familiar xy-plane in Figure 12(b). The points of the graph correspond to those values of x and y that lead to production of 100 items.

The curve in Figure 12 is called an *isoquant,* for *iso* (equal) and *quant* (amount). In Example 8, the "amounts" all "equal" 100.

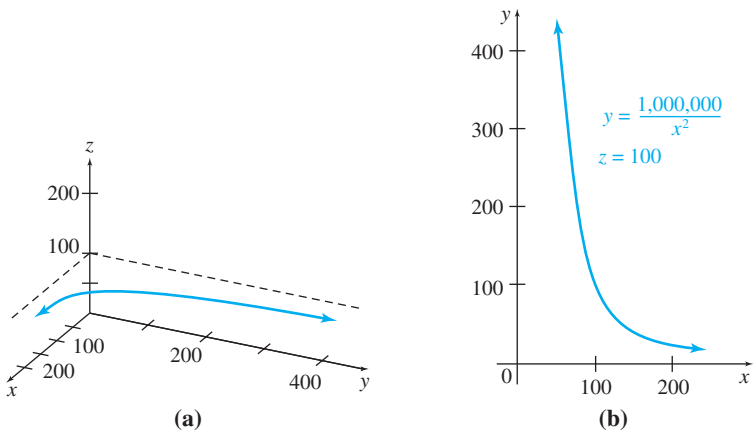
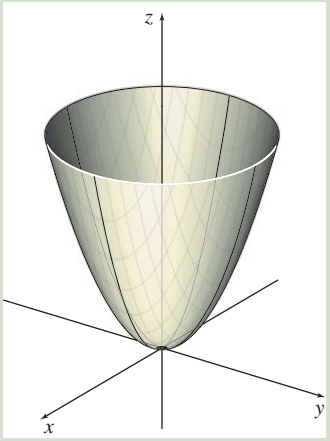


FIGURE 12

Because of the difficulty of drawing the graphs of more complicated functions, we merely list some common equations and their graphs. We encourage you to explore why these graphs look the way they do by studying their traces, level curves, and axis intercepts. These graphs were drawn by computer, a very useful method of depicting three-dimensional surfaces.

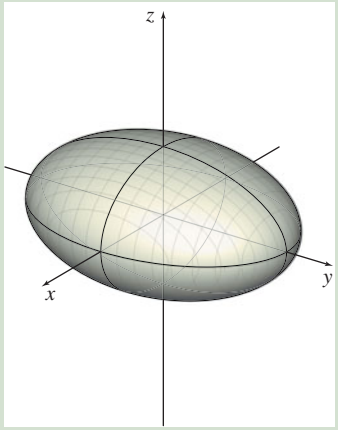
Paraboloid, $z = x^2 + y^2$

xy-trace: point
yz-trace: parabola
xz-trace: parabola



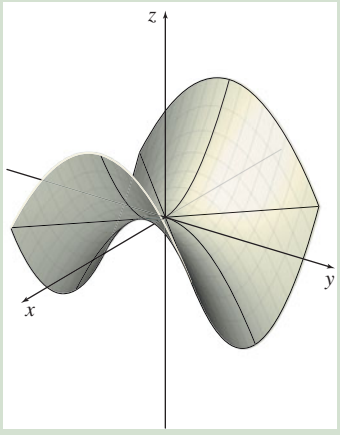
Ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

xy-trace: ellipse
yz-trace: ellipse
xz-trace: ellipse



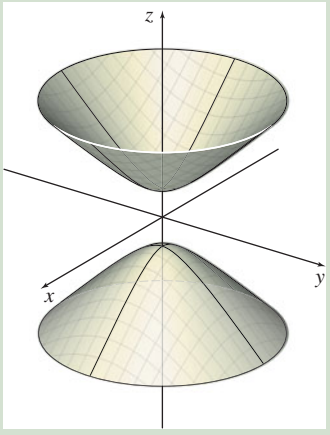
Hyperbolic Paraboloid, $z = x^2 - y^2$
(sometimes called a *saddle*)

xy-trace: two intersecting lines
yz-trace: parabola
xz-trace: parabola



Hyperboloid of Two Sheets, $-x^2 - y^2 + z^2 = 1$

xy-trace: none
yz-trace: hyperbola
xz-trace: hyperbola



Notice that not all the graphs correspond to functions of two variables. In the ellipsoid, for example, if x and y are both 0, then z can equal c or $-c$, whereas a function can take on only one value. We can, however, interpret the graph as a **level surface** for a function of three variables. Let

$$w(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

Then $w = 1$ produces the level surface of the ellipsoid shown, just as $z = c$ gives level curves for the function $z = f(x, y)$.

Another way to draw the graph of a function of two variables is with a graphing calculator. Figure 13 shows the graph of $z = x^2 + y^2$ generated by a TI-89. Figure 14 shows the same graph drawn by the computer program Maple™.

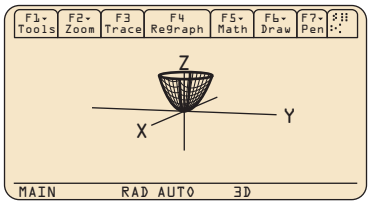


FIGURE 13

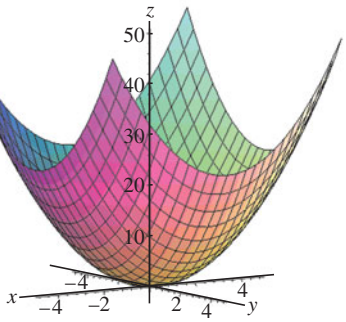


FIGURE 14

17.1 EXERCISES

1. Let $f(x, y) = 2x - 3y + 5$. Find the following.

a. $f(2, -1)$ **b.** $f(-4, 1)$ **c.** $f(-2, -3)$ **d.** $f(0, 8)$

2. Let $g(x, y) = x^2 - 2xy + y^3$. Find the following.

a. $g(-2, 4)$ **b.** $g(-1, -2)$ **c.** $g(-2, 3)$ **d.** $g(5, 1)$

3. Let $h(x, y) = \sqrt{x^2 + 2y^2}$. Find the following.

a. $h(5, 3)$ **b.** $h(2, 4)$ **c.** $h(-1, -3)$ **d.** $h(-3, -1)$

4. Let $f(x, y) = \dfrac{\sqrt{9x + 5y}}{\log x}$. Find the following.

a. $f(10, 2)$ **b.** $f(100, 1)$

c. $f(1000, 0)$ **d.** $f\left(\dfrac{1}{10}, 5\right)$

Graph the first-octant portion of each plane.

5. $x + y + z = 9$ **6.** $x + y + z = 15$

7. $2x + 3y + 4z = 12$ **8.** $4x + 2y + 3z = 24$

9. $x + y = 4$ **10.** $y + z = 5$

11. $x = 5$ **12.** $z = 4$

Graph the level curves in the first quadrant of the *xy*-plane for the following functions at heights of $z = 0$, $z = 2$, and $z = 4$.

13. $3x + 2y + z = 24$ **14.** $3x + y + 2z = 8$

15. $y^2 - x = -z$ **16.** $2y - \dfrac{x^2}{3} = z$

17. Discuss how a function of three variables in the form $w = f(x, y, z)$ might be graphed.

18. Suppose the graph of a plane $ax + by + cz = d$ has a portion in the first octant. What can be said about a, b, c, and d?

19. In the chapter on Nonlinear Functions, the vertical line test was presented, which tells whether a graph is the graph of a function. Does this test apply to functions of two variables? Explain.

20. A graph that was not shown in this section is the *hyperboloid of one sheet,* described by the equation $x^2 + y^2 - z^2 = 1$. Describe it as completely as you can.

Match each equation in Exercises 21–26 with its graph in a–f on the next page

21. $z = x^2 + y^2$ **22.** $z^2 - y^2 - x^2 = 1$

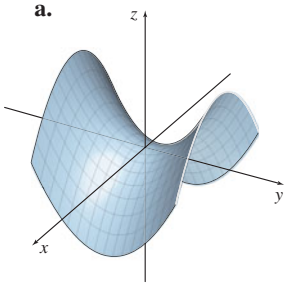
23. $x^2 - y^2 = z$

24. $z = y^2 - x^2$

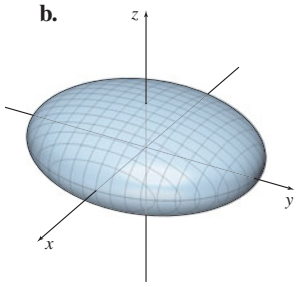
25. $\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{4} = 1$

26. $z = 5(x^2 + y^2)^{-1/2}$

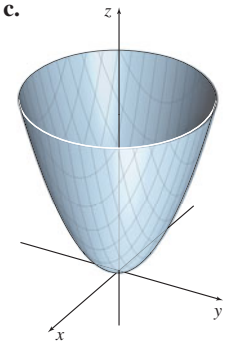
a.



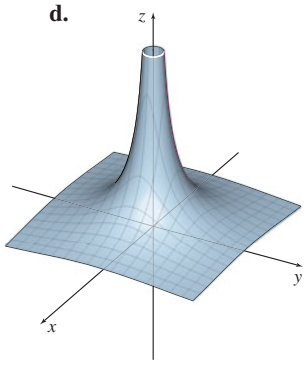
b.



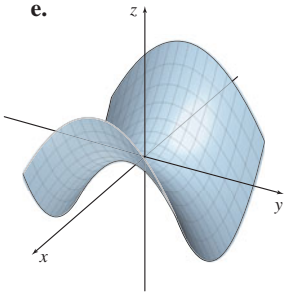
c.



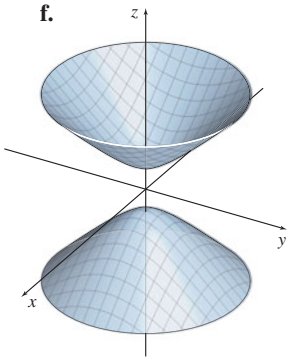
d.



e.



f.



27. Let $f(x, y) = 4x^2 - 2y^2$, and find the following.

a. $\dfrac{f(x + h, y) - f(x, y)}{h}$

b. $\dfrac{f(x, y + h) - f(x, y)}{h}$

c. $\lim\limits_{h \to 0} \dfrac{f(x + h, y) - f(x, y)}{h}$

d. $\lim\limits_{h \to 0} \dfrac{f(x, y + h) - f(x, y)}{h}$

28. Let $f(x, y) = 5x^3 + 3y^2$, and find the following.

a. $\dfrac{f(x + h, y) - f(x, y)}{h}$

b. $\dfrac{f(x, y + h) - f(x, y)}{h}$

c. $\lim\limits_{h \to 0} \dfrac{f(x + h, y) - f(x, y)}{h}$

d. $\lim\limits_{h \to 0} \dfrac{f(x, y + h) - f(x, y)}{h}$

29. Let $f(x, y) = xye^{x^2 + y^2}$. Use a graphing calculator or spreadsheet to find each of the following and give a geometric interpretation of the results. (*Hint:* First factor e^2 from the limit and then evaluate the quotient at smaller and smaller values of h.)

a. $\lim\limits_{h \to 0} \dfrac{f(1 + h, 1) - f(1, 1)}{h}$ **b.** $\lim\limits_{h \to 0} \dfrac{f(1, 1 + h) - f(1, 1)}{h}$

30. The following table provides values of the function $f(x, y)$. However, because of potential errors in measurement, the functional values may be slightly inaccurate. Using the statistical package included with a graphing calculator or spreadsheet and critical thinking skills, find the function $f(x, y) = a + bx + cy$ that best estimates the table where a, b, and c are integers. (*Hint:* Do a linear regression on each column with the value of y fixed and then use these four regression equations to determine the coefficient c.)

x \ y	0	1	2	3
0	4.02	7.04	9.98	13.00
1	6.01	9.06	11.98	14.96
2	7.99	10.95	14.02	17.09
3	9.99	13.01	16.01	19.02

APPLICATIONS

Business and Economics

31. Production Production of a digital camera is given by

$$P(x, y) = 100\left(\frac{3}{5}x^{-2/5} + \frac{2}{5}y^{-2/5}\right)^{-5},$$

where x is the amount of labor in work-hours and y is the amount of capital. Find the following.

a. What is the production when 32 work-hours and 1 unit of capital are provided?

b. Find the production when 1 work-hour and 32 units of capital are provided.

c. If 32 work-hours and 243 units of capital are used, what is the production output?

Individual Retirement Accounts **The multiplier function**

$$M = \frac{(1 + i)^n(1 - t) + t}{[1 + (1 - t)i]^n}$$

compares the growth of an Individual Retirement Account (IRA) with the growth of the same deposit in a regular savings account. The function M depends on the three variables n, i, and t, where n represents the number of years an amount is left at interest, i represents the interest rate in both types of accounts, and t represents the income tax rate. Values of $M > 1$ indicate that the IRA grows faster than the savings account. Let $M = f(n, i, t)$ and find the following.

32. Find the multiplier when funds are left for 25 years at 5% interest and the income tax rate is 33%. Which account grows faster?

33. What is the multiplier when money is invested for 40 years at 6% interest and the income tax rate is 28%? Which account grows faster?

Production **Find the level curve at a production of 500 for the production functions in Exercises 34 and 35. Graph each level curve in the *xy*-plane.**

34. In their original paper, Cobb and Douglas estimated the production function for the United States to be $z = 1.01x^{3/4}y^{1/4}$, where x represents the amount of labor and y the amount of capital. ***Source: American Economic Review***.

35. A study of the connection between immigration and the fiscal problems associated with the aging of the baby boom generation considered a production function of the form $z = x^{0.6}y^{0.4}$, where x represents the amount of labor and y the amount of capital. ***Source: Journal of Political Economy***.

36. Production For the function in Exercise 34, what is the effect on z of doubling x? Of doubling y? Of doubling both?

37. Cost If labor (x) costs \$250 per unit, materials (y) cost \$150 per unit, and capital (z) costs \$75 per unit, write a function for total cost.

Life Sciences

38. Heat Loss The rate of heat loss (in watts) in harbor seal pups has been approximated by

$$H(m,T,A) = \frac{15.2m^{0.67}(T - A)}{10.23 \ln m - 10.74},$$

where m is the body mass of the pup (in kg), and T and A are the body core temperature and ambient water temperature, respectively (in °C). Find the heat loss for the following data. ***Source: Functional Ecology***.

a. Body mass = 21 kg; body core temperature = 36°C; ambient water temperature = 4°C

b. Body mass = 29 kg; body core temperature = 38°C; ambient water temperature = l6°C

39. Body Surface Area The surface area of a human (in square meters) has been approximated by

$$A = 0.024265h^{0.3964}m^{0.5378},$$

where h is the height (in cm) and m is the mass (in kg). Find A for the following data. ***Source: The Journal of Pediatrics***.

a. Height, 178 cm; mass, 72 kg

b. Height, 140 cm; mass, 65 kg

c. Height, 160 cm; mass, 70 kg

d. Using your mass and height, find your own surface area.

40. Dinosaur Running An article entitled "How Dinosaurs Ran" explains that the locomotion of different sized animals can be compared when they have the same Froude number, defined as

$$F = \frac{v^2}{gl},$$

where v is the velocity, g is the acceleration of gravity $(9.81 \text{ m per sec}^2)$, and l is the leg length (in meters). ***Source: Scientific American***.

a. One result described in the article is that different animals change from a trot to a gallop at the same Froude number, roughly 2.56. Find the velocity at which this change occurs for a ferret, with a leg length of 0.09 m, and a rhinoceros, with a leg length of 1.2 m.

b. Ancient footprints in Texas of a sauropod, a large herbivorous dinosaur, are roughly 1 m in diameter, corresponding to a leg length of roughly 4 m. By comparing the stride divided by the leg length with that of various modern creatures, it can be determined that the Froude number for these dinosaurs is roughly 0.025. How fast were the sauropods traveling?

41. Pollution Intolerance According to research at the Great Swamp in New York, the percentage of fish that are intolerant to pollution can be estimated by the function

$$P(W, R, A) = 48 - 2.43W - 1.81R - 1.22A,$$

where W is the percentage of wetland, R is the percentage of residential area, and A is the percentage of agricultural area surrounding the swamp. ***Source: Northeastern Naturalist***.

a. Use this function to estimate the percentage of fish that will be intolerant to pollution if 5 percent of the land is classified as wetland, 15 percent is classified as residential, and 0 percent is classified as agricultural. (*Note*: The land can also be classified as forest land.)

b. What is the maximum percentage of fish that will be intolerant to pollution?

c. Develop two scenarios that will drive the percentage of fish that are intolerant to pollution to zero.

d. Which variable has the greatest influence on P?

42. Dengue Fever In tropical regions, dengue fever is a significant health problem that affects nearly 100 million people each year. Using data collected from the 2002 dengue epidemic in Colima, Mexico, researchers have estimated that the incidence I (number of new cases in a given year) of dengue can be predicted by the following function.

$$I(p, a, m, n, e) = (25.54 + 0.04p - 7.92a + 2.62m + 4.46n + 0.15e)^2,$$

where p is the precipitation (mm), a is the mean temperature (°C), m is the maximum temperature (°C), n is the minimum temperature (°C), and e is the evaporation (mm). ***Source: Journal of Environmental Health***.

a. Estimate the incidence of a dengue fever outbreak for a region with 80 mm of rainfall, average temperature of 23°C, maximum temperature of 34°C, minimum temperature of 16°C, and evaporation of 50 mm.

b. Which variable has a negative influence on the incidence of dengue? Describe this influence and what can be inferred mathematically about the biology of the fever.

43. Deer-Vehicle Accidents Using data collected by the U.S. Forest Service, the annual number of deer-vehicle accidents for any given county in Ohio can be estimated by the function

$$A(L, T, U, C) = 53.02 + 0.383L + 0.0015T + 0.0028U - 0.0003C,$$

where A is the estimated number of accidents, L is the road length (in kilometers), T is the total county land area (in hundred-acres (Ha)), U is the urban land area (in hundred-acres), and C is the number of hundred-acres of crop land. ***Source: Ohio Journal of Science.***

a. Use this formula to estimate the number of deer-vehicle accidents for Mahoning County, where $L = 266$ km, $T = 107{,}484$ Ha, $U = 31{,}697$ Ha, and $C = 24{,}870$ Ha. The actual value was 396.

b. Given the magnitude and nature of the input numbers, which of the variables have the greatest potential to influence the number of deer-vehicle accidents? Explain your answer.

44. Deer Harvest Using data collected by the U.S. Forest Service, the annual number of deer that are harvested for any given county in Ohio can be estimated by the function

$$N(R, C) = 329.32 + 0.0377R - 0.0171C,$$

where N is the estimated number of harvested deer, R is the rural land area (in hundred-acres), and C is the number of hundred-acres of crop land. ***Source: Ohio Journal of Science.***

a. Use this formula to estimate the number of harvested deer for Tuscarawas County, where $R = 141{,}319$ Ha and $C = 37{,}960$ Ha. The actual value was 4925 deer harvested.

b. Sketch the graph of this function in the first octant.

45. Agriculture Pregnant sows tethered in stalls often show high levels of repetitive behavior, such as bar biting and chain chewing, indicating chronic stress. Researchers from Great Britain have developed a function that estimates the relationship between repetitive behavior, the behavior of sows in adjacent stalls, and food allowances such that

$$\ln(T) = 5.49 - 3.00 \ln(F) + 0.18 \ln(C),$$

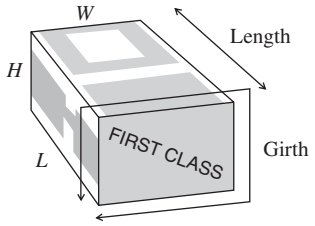
where T is the percent of time spent in repetitive behavior, F is the amount of food given to the sow (in kilograms per day), and C is the percent of time that neighboring sows spent bar biting and chain chewing. ***Source: Applied Animal Behaviour Science.***

a. Solve the above expression for T.

b. Find and interpret T when $F = 2$ and $C = 40$.

General Interest

46. Postage Rates Extra postage is charged for parcels sent by U.S. mail that are more than 84 in. in length and girth combined. (Girth is the distance around the parcel perpendicular to its length. See the figure.) Express the combined length and girth as a function of L, W, and H.



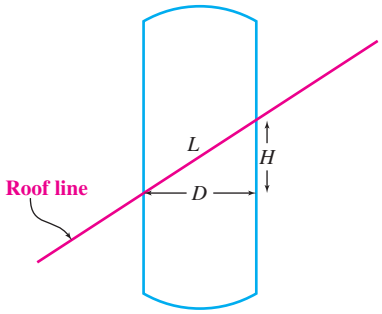
47. Required Material Refer to the figure for Exercise 46. Assume L, W, and H are in feet. Write a function in terms of L, W, and H that gives the total area of the material required to build the box.

48. Elliptical Templates The holes cut in a roof for vent pipes require elliptical templates. A formula for determining the length of the major axis of the ellipse is given by

$$L = f(H, D) = \sqrt{H^2 + D^2},$$

where D is the (outside) diameter of the pipe and H is the "rise" of the roof per D units of "run"; that is, the slope of the roof is H/D. (See the figure below.) The width of the ellipse (minor axis) equals D. Find the length and width of the ellipse required to produce a hole for a vent pipe with a diameter of 3.75 in. in roofs with the following slopes.

a. 3/4 **b.** 2/5

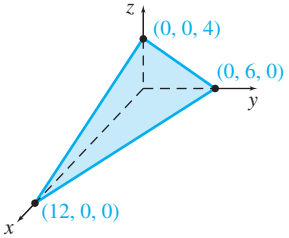


YOUR TURN ANSWERS

1. 29

2. 24

3.



4. $y = 81/x^{1/3}$.

17.2 Partial Derivatives

APPLY IT

What is the change in productivity if labor is increased by one work-hour? What if capital is increased by one unit?

We will answer this question in Example 5 using the concept of partial derivatives.

FOR REVIEW

You may want to review Chapter 12 on Calculating the Derivative for methods used to find some of the derivatives in this section.

Earlier, we found that the derivative dy/dx gives the rate of change of y with respect to x. In this section, we show how derivatives are found and interpreted for multivariable functions.

A small firm makes only two products, radios and CD players. The profits of the firm are given by

$$P(x, y) = 40x^2 - 10xy + 5y^2 - 80,$$

where x is the number of radios sold and y is the number of CD players sold. How will a change in x or y affect P?

Suppose that sales of radios have been steady at 10 units; only the sales of CD players vary. The management would like to find the marginal profit with respect to y, the number of CD players sold. Recall that marginal profit is given by the derivative of the profit function. Here, x is fixed at 10. Using this information, we begin by finding a new function, $f(y) = P(10, y)$. Let $x = 10$ to get

$$\begin{aligned} f(y) = P(10, y) &= 40(10)^2 - 10(10)y + 5y^2 - 80 \\ &= 3920 - 100y + 5y^2. \end{aligned}$$

The function $f(y)$ shows the profit from the sale of y CD players, assuming that x is fixed at 10 units. Find the derivative df/dy to get the marginal profit with respect to y.

$$\frac{df}{dy} = -100 + 10y$$

In this example, the derivative of the function $f(y)$ was taken with respect to y only; we assumed that x was fixed. To generalize, let $z = f(x, y)$. An intuitive definition of the *partial derivatives* of f with respect to x and y follows.

Partial Derivatives (Informal Definition)

The **partial derivative of f with respect to x** is the derivative of f obtained by treating x as a variable and y as a constant.

The **partial derivative of f with respect to y** is the derivative of f obtained by treating y as a variable and x as a constant.

The symbols $f_x(x, y)$ (no prime is used), $\partial z/\partial x$, z_x, and $\partial f/\partial x$ are used to represent the partial derivative of $z = f(x, y)$ with respect to x, with similar symbols used for the partial derivative with respect to y.

Generalizing from the definition of the derivative given earlier, partial derivatives of a function $z = f(x, y)$ are formally defined as follows.

Partial Derivatives (Formal Definition)

Let $z = f(x, y)$ be a function of two independent variables. Let all indicated limits exist. Then the partial derivative of f with respect to x is

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

and the partial derivative of f with respect to y is

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

If the indicated limits do not exist, then the partial derivatives do not exist.

Similar definitions could be given for functions of more than two independent variables.

EXAMPLE 1 Partial Derivatives

Let $f(x, y) = 4x^2 - 9xy + 6y^3$. Find $f_x(x, y)$ and $f_y(x, y)$.

SOLUTION To find $f_x(x, y)$, treat y as a constant and x as a variable. The derivative of the first term, $4x^2$, is $8x$. In the second term, $-9xy$, the constant coefficient of x is $-9y$, so the derivative with x as the variable is $-9y$. The derivative of $6y^3$ is zero, since we are treating y as a constant. Thus,

$$f_x(x, y) = 8x - 9y.$$

Now, to find $f_y(x, y)$, treat y as a variable and x as a constant. Since x is a constant, the derivative of $4x^2$ is zero. In the second term, the coefficient of y is $-9x$ and the derivative of $-9xy$ is $-9x$. The derivative of the third term is $18y^2$. Thus,

$$f_y(x, y) = -9x + 18y^2.$$

TRY YOUR TURN 1

YOUR TURN 1 Let $f(x,y) = 2x^2y^3 + 6x^5y^4$. Find $f_x(x,y)$ and $f_y(x,y)$.

The next example shows how the chain rule can be used to find partial derivatives.

EXAMPLE 2 Partial Derivatives

Let $f(x, y) = \ln |x^2 + 3y|$. Find $f_x(x, y)$ and $f_y(x, y)$.

SOLUTION Recall the formula for the derivative of a natural logarithm function. If $g(x) = \ln |x|$, then $g'(x) = 1/x$. Using this formula and the chain rule,

$$f_x(x, y) = \frac{1}{x^2 + 3y} \cdot \frac{\partial}{\partial x}(x^2 + 3y) = \frac{1}{x^2 + 3y} \cdot 2x = \frac{2x}{x^2 + 3y},$$

and

$$f_y(x, y) = \frac{1}{x^2 + 3y} \cdot \frac{\partial}{\partial y}(x^2 + 3y) = \frac{1}{x^2 + 3y} \cdot 3 = \frac{3}{x^2 + 3y}.$$

TRY YOUR TURN 2

YOUR TURN 2 Let $f(x,y) = e^{3x^2y}$. Find $f_x(x,y)$ and $f_y(x,y)$.

The notation

$$f_x(a, b) \quad \text{or} \quad \frac{\partial f}{\partial y}(a, b)$$

represents the value of the partial derivative when $x = a$ and $y = b$, as shown in the next example.

EXAMPLE 3 Evaluating Partial Derivatives

Let $f(x, y) = 2x^2 + 9xy^3 + 8y + 5$. Find the following.

(a) $f_x(-1, 2)$

SOLUTION First, find $f_x(x, y)$ by holding y constant.

$$f_x(x, y) = 4x + 9y^3$$

Now let $x = -1$ and $y = 2$.

$$f_x(-1, 2) = 4(-1) + 9(2)^3 = -4 + 72 = 68$$

(b) $\dfrac{\partial f}{\partial y}(-4, -3)$

SOLUTION Since $\partial f/\partial y = 27xy^2 + 8$,

$$\frac{\partial f}{\partial y}(-4, -3) = 27(-4)(-3)^2 + 8 = 27(-36) + 8 = -964.$$

(c) All values of x and y such that both $f_x(x, y) = 0$ and $f_y(x, y) = 0$. (The importance of such points will be shown in the next section.)

SOLUTION From parts (a) and (b),

$$f_x(x, y) = 4x + 9y^3 = 0 \quad \text{and} \quad f_y(x, y) = 27xy^2 + 8 = 0.$$

Solving the first equation for x yields $x = -9y^3/4$. Substituting this into the second equation yields

$$\begin{aligned}
27\left(\frac{-9y^3}{4}\right)y^2 + 8 &= 0 \\
\frac{-243y^5}{4} + 8 &= 0 \\
\frac{-243y^5}{4} &= -8 \\
y^5 &= \frac{32}{243} \\
y &= \frac{2}{3}. \qquad \text{Take the fifth root of both sides.}
\end{aligned}$$

Substituting $y = 2/3$ yields $x = -9y^3/4 = -9(2/3)^3/4 = -2/3$. Thus, $f_x(x, y) = 0$ and $f_y(x, y) = 0$ when $x = -2/3$ and $y = 2/3$.

(d) $f_x(x, y)$ using the formal definition of the partial derivative.

SOLUTION Calculate as follows:

$$\begin{aligned}
\frac{f(x+h, y) - f(x, y)}{h} &= \frac{2(x+h)^2 + 9(x+h)y^3 + 8y + 5 - (2x^2 + 9xy^3 + 8y + 5)}{h} \\
&= \frac{2x^2 + 4xh + 2h^2 + 9xy^3 + 9hy^3 + 8y + 5 - 2x^2 - 9xy^3 - 8y - 5}{h} \\
&= \frac{4xh + 2h^2 + 9hy^3}{h} \qquad \text{Simplify the numerator.} \\
&= \frac{h(4x + 2h + 9y^3)}{h} \qquad \text{Factor } h \text{ from the numerator.} \\
&= 4x + 2h + 9y^3.
\end{aligned}$$

Now take the limit as h goes to 0. Thus,

$$f_x(x, y) = \lim_{h\to 0}\frac{f(x + h, y) - f(x, y)}{h} = \lim_{h\to 0}(4x + 2h + 9y^3)$$
$$= 4x + 9y^3,$$

the same answer we found in part (a).

YOUR TURN 3 Let $f(x,y) = xye^{x^2+y^3}$. Find $f_x(2,1)$ and $f_y(2,1)$.

TRY YOUR TURN 3

In some cases, the difference quotient may not simplify as easily as it did in Example 3(d). In such cases, the derivative may be approximated by putting a small value for h into $[f(x + h) - f(x)]/h$. In Example 3(d), with $x = -1$ and $y = 2$, the values $h = 10^{-4}$ and 10^{-5} give approximations for $f_x(-1, 2)$ as 68.0002 and 68.00002, respectively, compared with the exact value of 68 found in Example 3(a).

The derivative of a function of one variable can be interpreted as the slope of the tangent line to the graph at that point. With some modification, the same is true of partial derivatives of functions of two variables. At a point on the graph of a function of two variables, $z = f(x, y)$, there may be many tangent lines, all of which lie in the same tangent plane, as shown in Figure 15.

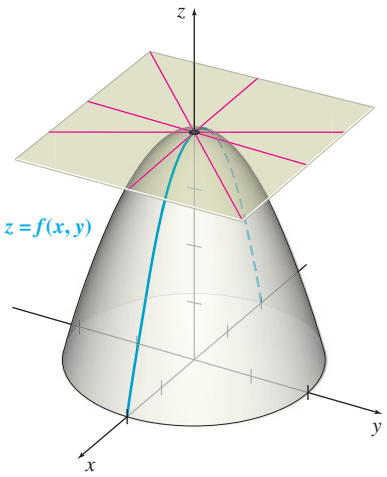


FIGURE 15

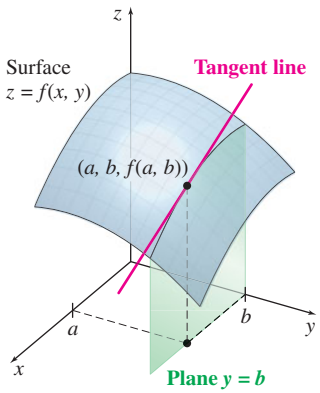


FIGURE 16

In any particular direction, however, there will be only one tangent line. We use partial derivatives to find the slope of the tangent lines in the x- and y-directions as follows.

Figure 16 shows a surface $z = f(x, y)$ and a plane that is parallel to the xz-plane. The equation of the plane is $y = b$. (This corresponds to holding y fixed.) Since $y = b$ for points on the plane, any point on the curve that represents the intersection of the plane and the surface must have the form $(x, y, z) = (x, b, f(x, b))$. Thus, this curve can be described as $z = f(x, b)$. Since b is constant, $z = f(x, b)$ is a function of one variable. When the derivative of $z = f(x, b)$ is evaluated at $x = a$, it gives the slope of the line tangent to this curve at the point $(a, b, f(a, b))$, as shown in Figure 16. Thus, the partial derivative of f with respect to x, $f_x(a, b)$, gives the rate of change of the surface $z = f(x, y)$ in the x-direction at the point $(a, b, f(a, b))$. In the same way, the partial derivative with respect to y will give the slope of the line tangent to the surface in the y-direction at the point $(a, b, f(a, b))$.

Rate of Change

The derivative of $y = f(x)$ gives the rate of change of y with respect to x. In the same way, if $z = f(x, y)$, then $f_x(x, y)$ gives the rate of change of z with respect to x, if y is held constant.

EXAMPLE 4 Water Temperature

Suppose that the temperature of the water at the point on a river where a nuclear power plant discharges its hot waste water is approximated by

$$T(x, y) = 2x + 5y + xy - 40,$$

where x represents the temperature of the river water (in degrees Celsius) before it reaches the power plant and y is the number of megawatts (in hundreds) of electricity being produced by the plant.

(a) Find and interpret $T_x(9, 5)$.

SOLUTION First, find the partial derivative $T_x(x, y)$.

$$T_x(x, y) = 2 + y$$

This partial derivative gives the rate of change of T with respect to x. Replacing x with 9 and y with 5 gives

$$T_x(9, 5) = 2 + 5 = 7.$$

Just as marginal cost is the approximate cost of one more item, this result, 7, is the approximate change in temperature of the output water if input water temperature changes by 1 degree, from $x = 9$ to $x = 9 + 1 = 10$, while y remains constant at 5 (500 megawatts of electricity produced).

(b) Find and interpret $T_y(9, 5)$.

SOLUTION The partial derivative $T_y(x, y)$ is

$$T_y(x, y) = 5 + x.$$

This partial derivative gives the rate of change of T with respect to y as

$$T_y(9, 5) = 5 + 9 = 14.$$

This result, 14, is the approximate change in temperature resulting from a 1-unit increase in production of electricity from $y = 5$ to $y = 5 + 1 = 6$ (from 500 to 600 megawatts), while the input water temperature x remains constant at 9°C.

As mentioned in the previous section, if $P(x, y)$ gives the output P produced by x units of labor and y units of capital, $P(x, y)$ is a production function. The partial derivatives of this production function have practical implications. For example, $\partial P/\partial x$ gives the marginal productivity of labor. This represents the rate at which the output is changing with respect to labor for a fixed capital investment. That is, if the capital investment is held constant and labor is increased by 1 work-hour, $\partial P/\partial x$ will yield the approximate change in the production level. Likewise, $\partial P/\partial y$ gives the marginal productivity of capital, which represents the rate at which the output is changing with respect to a one-unit change in capital for a fixed labor value. So if the labor force is held constant and the capital investment is increased by 1 unit, $\partial P/\partial y$ will approximate the corresponding change in the production level.

EXAMPLE 5 Production Function

A company that manufactures computers has determined that its production function is given by

$$P(x, y) = 0.1xy^2 \ln(2x + 3y + 2),$$

where x is the size of the labor force (measured in work-hours per week) and y is the amount of capital (measured in units of \$1000) invested. Find the marginal productivity of labor and capital when $x = 50$ and $y = 20$, and interpret the results.

APPLY IT

SOLUTION The marginal productivity of labor is found by taking the derivative of P with respect to x.

$$\frac{\partial P}{\partial x} = 0.1\left[\frac{xy^2}{2x+3y+2}\cdot 2 + y^2\ln(2x+3y+2)\right] \quad \text{Use the product and chain rules.}$$

$$\frac{\partial P}{\partial x}(50,20) = 0.1\left[\frac{50(20)^2}{2(50)+3(20)+2}\cdot 2 + 20^2\ln(2(50)+3(20)+2)\right] \approx 228$$

Thus, if the capital investment is held constant at \$20,000 and labor is increased from 50 to 51 work-hours per week, production will increase by about 228 units. In the same way, the marginal productivity of capital is $\partial P/\partial y$.

$$\frac{\partial P}{\partial y} = 0.1\left[\frac{xy^2}{2x+3y+2}\cdot 3 + 2xy\ln(2x+3y+2)\right] \quad \text{Use the product and chain rules.}$$

$$\frac{\partial P}{\partial y}(50,20) = 0.1\left[\frac{50(20)^2}{2(50)+3(20)+2}\cdot 3 + 2(50)(20)\ln(2(50)+3(20)+2)\right] \approx 1055$$

If work-hours are held constant at 50 hours per week and the capital investment is increased from \$20,000 to \$21,000, production will increase by about 1055 units.

Second-Order Partial Derivatives

The second derivative of a function of one variable is very useful in determining relative maxima and minima. **Second-order partial derivatives** (partial derivatives of a partial derivative) are used in a similar way for functions of two or more variables. The situation is somewhat more complicated, however, with more independent variables. For example, $f(x, y) = 4x + x^2y + 2y$ has two first-order partial derivatives,

$$f_x(x, y) = 4 + 2xy \qquad \text{and} \qquad f_y(x, y) = x^2 + 2.$$

Since each of these has two partial derivatives, one with respect to y and one with respect to x, there are *four* second-order partial derivatives of function f. The notations for these four second-order partial derivatives are given below.

Second-Order Partial Derivatives

For a function $z = f(x, y)$, if the indicated partial derivative exists, then

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = z_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = z_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} = f_{xy}(x, y) = z_{xy} \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y} = f_{yx}(x, y) = z_{yx}$$

NOTE For most functions found in applications and for all of the functions in this book, the second-order partial derivatives $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are equal. This is always true when $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous. Therefore, it is not necessary to be particular about the order in which these derivatives are found.

EXAMPLE 6 Second-Order Partial Derivatives

Find all second-order partial derivatives for

$$f(x, y) = -4x^3 - 3x^2y^3 + 2y^2.$$

SOLUTION First find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = -12x^2 - 6xy^3 \quad \text{and} \quad f_y(x, y) = -9x^2y^2 + 4y$$

To find $f_{xx}(x, y)$, take the partial derivative of $f_x(x, y)$ with respect to x.

$$f_{xx}(x, y) = -24x - 6y^3$$

Take the partial derivative of $f_y(x, y)$ with respect to y; this gives f_{yy}.

$$f_{yy}(x, y) = -18x^2y + 4$$

Find $f_{xy}(x, y)$ by starting with $f_x(x, y)$, then taking the partial derivative of $f_x(x, y)$ with respect to y.

$$f_{xy}(x, y) = -18xy^2$$

Finally, find $f_{yx}(x, y)$ by starting with $f_y(x, y)$; take its partial derivative with respect to x.

$$f_{yx}(x, y) = -18xy^2$$

TRY YOUR TURN 4

YOUR TURN 4 Let $f(x, y) = x^2e^{7y} + x^4y^5$. Find all second partial derivatives.

EXAMPLE 7 Second-Order Partial Derivatives

Let $f(x, y) = 2e^x - 8x^3y^2$. Find all second-order partial derivatives.

SOLUTION Here $f_x(x, y) = 2e^x - 24x^2y^2$ and $f_y(x, y) = -16x^3y$. (Recall: If $g(x) = e^x$, then $g'(x) = e^x$.) Now find the second-order partial derivatives.

$$f_{xx}(x, y) = 2e^x - 48xy^2 \qquad f_{xy}(x, y) = -48x^2y$$
$$f_{yy}(x, y) = -16x^3 \qquad f_{yx}(x, y) = -48x^2y$$

Partial derivatives of functions with more than two independent variables are found in a similar manner. For instance, to find $f_z(x, y, z)$ for $f(x, y, z)$, hold x and y constant and differentiate with respect to z.

EXAMPLE 8 Second-Order Partial Derivatives

Let $f(x, y, z) = 2x^2yz^2 + 3xy^2 - 4yz$. Find $f_x(x, y, z)$, $f_y(x, y, z)$, $f_{xz}(x, y, z)$, and $f_{yz}(x, y, z)$.

SOLUTION

$$f_x(x, y, z) = 4xyz^2 + 3y^2$$
$$f_y(x, y, z) = 2x^2z^2 + 6xy - 4z$$

To find $f_{xz}(x, y, z)$, differentiate $f_x(x, y, z)$ with respect to z.

$$f_{xz}(x, y, z) = 8xyz$$

Differentiate $f_y(x, y, z)$ with respect to z to get $f_{yz}(x, y, z)$.

$$f_{yz}(x, y, z) = 4x^2z - 4$$

17.2 EXERCISES

1. Let $z = f(x, y) = 6x^2 - 4xy + 9y^2$. Find the following using the formal definition of the partial derivative.

a. $\dfrac{\partial z}{\partial x}$ **b.** $\dfrac{\partial z}{\partial y}$ **c.** $\dfrac{\partial f}{\partial x}(2, 3)$ **d.** $f_y(1, -2)$

2. Let $z = g(x, y) = 8x + 6x^2y + 2y^2$. Find the following using the formal definition of the partial derivative.

a. $\dfrac{\partial g}{\partial x}$ **b.** $\dfrac{\partial g}{\partial y}$ **c.** $\dfrac{\partial z}{\partial y}(-3, 0)$ **d.** $g_x(2, 1)$

In Exercises 3–20, find $f_x(x, y)$ and $f_y(x, y)$. Then find $f_x(2, -1)$ and $f_y(-4, 3)$. Leave the answers in terms of e in Exercises 7–10, 15–16, and 19–20.

3. $f(x, y) = -4xy + 6y^3 + 5$

4. $f(x, y) = 9x^2y^2 - 4y^2$

5. $f(x, y) = 5x^2y^3$

6. $f(x, y) = -3x^4y^3 + 10$

7. $f(x, y) = e^{x+y}$

8. $f(x, y) = 4e^{3x+2y}$

9. $f(x, y) = -6e^{4x-3y}$

10. $f(x, y) = 8e^{7x-y}$

11. $f(x, y) = \dfrac{x^2 + y^3}{x^3 - y^2}$

12. $f(x, y) = \dfrac{3x^2y^3}{x^2 + y^2}$

13. $f(x, y) = \ln|1 + 5x^3y^2|$

14. $f(x, y) = \ln|4x^4 - 2x^2y^2|$

15. $f(x, y) = xe^{x^2y}$

16. $f(x, y) = y^2e^{x+3y}$

17. $f(x, y) = \sqrt{x^4 + 3xy + y^4 + 10}$

18. $f(x, y) = (7x^2 + 18xy^2 + y^3)^{1/3}$

19. $f(x, y) = \dfrac{3x^2y}{e^{xy} + 2}$

20. $f(x, y) = (7e^{x+2y} + 4)(e^{x^2} + y^2 + 2)$

Find all second-order partial derivatives for the following.

21. $f(x, y) = 4x^2y^2 - 16x^2 + 4y$

22. $g(x, y) = 5x^4y^2 + 12y^3 - 9x$

23. $R(x, y) = 4x^2 - 5xy^3 + 12y^2x^2$

24. $h(x, y) = 30y + 5x^2y + 12xy^2$

25. $r(x, y) = \dfrac{6y}{x + y}$

26. $k(x, y) = \dfrac{-7x}{2x + 3y}$

27. $z = 9ye^x$

28. $z = -6xe^y$

29. $r = \ln|x + y|$

30. $k = \ln|5x - 7y|$

31. $z = x \ln|xy|$

32. $z = (y + 1) \ln|x^3y|$

For the functions defined as follows, find all values of x and y such that both $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

33. $f(x, y) = 6x^2 + 6y^2 + 6xy + 36x - 5$

34. $f(x, y) = 50 + 4x - 5y + x^2 + y^2 + xy$

35. $f(x, y) = 9xy - x^3 - y^3 - 6$

36. $f(x, y) = 2200 + 27x^3 + 72xy + 8y^2$

Find $f_x(x, y, z)$, $f_y(x, y, z)$, $f_z(x, y, z)$, and $f_{yz}(x, y, z)$ for the following.

37. $f(x, y, z) = x^4 + 2yz^2 + z^4$

38. $f(x, y, z) = 6x^3 - x^2y^2 + y^5$

39. $f(x, y, z) = \dfrac{6x - 5y}{4z + 5}$

40. $f(x, y, z) = \dfrac{2x^2 + xy}{yz - 2}$

41. $f(x, y, z) = \ln|x^2 - 5xz^2 + y^4|$

42. $f(x, y, z) = \ln|8xy + 5yz - x^3|$

In Exercises 43 and 44, approximate the indicated derivative for each function by using the definition of the derivative with small values of h.

43. $f(x, y) = (x + y/2)^{x+y/2}$

a. $f_x(1, 2)$ **b.** $f_y(1, 2)$

44. $f(x, y) = (x + y^2)^{2x+y}$

a. $f_x(2, 1)$ **b.** $f_y(2, 1)$

APPLICATIONS

Business and Economics

45. Manufacturing Cost Suppose that the manufacturing cost of a personal digital assistant (PDA) is approximated by

$$M(x, y) = 45x^2 + 40y^2 - 20xy + 50,$$

where x is the cost of electronic chips and y is the cost of labor. Find the following.

a. $M_y(4, 2)$ **b.** $M_x(3, 6)$ **c.** $(\partial M/\partial x)(2, 5)$ **d.** $(\partial M/\partial y)(6, 7)$

46. Revenue The revenue from the sale of x units of a sedative and y units of an antibiotic is given by

$$R(x, y) = 5x^2 + 9y^2 - 4xy.$$

Suppose 9 units of sedative and 5 units of antibiotic are sold.

a. What is the approximate effect on revenue if 10 units of sedative and 5 units of antibiotic are sold?

b. What is the approximate effect on revenue if the amount of antibiotic sold is increased to 6 units, while sedative sales remain constant?

47. Sales A car dealership estimates that the total weekly sales of its most popular model is a function of the car's list price, p, and the interest rate in percent, i, offered by the manufacturer. The approximate weekly sales are given by

$$f(p, i) = 99p - 0.5pi - 0.0025p^2.$$

a. Find the weekly sales if the average list price is \$19,400 and the manufacturer is offering an 8% interest rate.

b. Find and interpret $f_p(p, i)$ and $f_i(p, i)$.

c. What would be the effect on weekly sales if the price is \$19,400 and interest rates rise from 8% to 9%?

48. Marginal Productivity Suppose the production function of a company is given by

$$P(x, y) = 250\sqrt{x^2 + y^2},$$

where x represents units of labor and y represents units of capital. Find the following when 6 units of labor and 8 units of capital are used.

a. The marginal productivity of labor

b. The marginal productivity of capital

49. Marginal Productivity A manufacturer estimates that production (in hundreds of units) is a function of the amounts x and y of labor and capital used, as follows.

$$f(x, y) = \left(\frac{1}{4}x^{-1/4} + \frac{3}{4}y^{-1/4}\right)^{-4}$$

a. Find the number of units produced when 16 units of labor and 81 units of capital are utilized.

b. Find and interpret $f_x(16, 81)$ and $f_y(16, 81)$.

c. What would be the approximate effect on production of increasing labor by 1 unit from 16 units of labor with 81 units of capital?

50. Marginal Productivity The production function z for the United States was once estimated as

$$z = x^{0.7}y^{0.3},$$

where x stands for the amount of labor and y the amount of capital. Find the marginal productivity of labor and of capital.

51. Marginal Productivity A similar production function for Canada is

$$z = x^{0.4}y^{0.6},$$

with x, y, and z as in Exercise 50. Find the marginal productivity of labor and of capital.

52. Marginal Productivity A manufacturer of automobile batteries estimates that his total production (in thousands of units) is given by

$$f(x, y) = 3x^{1/3}y^{2/3},$$

where x is the number of units of labor and y is the number of units of capital utilized.

a. Find and interpret $f_x(64, 125)$ and $f_y(64, 125)$ if the current level of production uses 64 units of labor and 125 units of capital.

b. Use your answer from part a to calculate the approximate effect on production of increasing labor to 65 units while holding capital at the current level.

c. Suppose that sales have been good and management wants to increase either capital or labor by 1 unit. Which option would result in a larger increase in production?

Life Sciences

53. Calorie Expenditure The average energy expended for an animal to walk or run 1 km can be estimated by the function

$$f(m, v) = 25.92m^{0.68} + \frac{3.62m^{0.75}}{v},$$

where $f(m, v)$ is the energy used (in kcal per hour), m is the mass (in g), and v is the speed of movement (in km per hour) of the animal. ***Source: Wildlife Feeding and Nutrition***.

a. Find $f(300, 10)$.

b. Find $f_m(300, 10)$ and interpret.

c. If a mouse could run at the same speed that an elephant walks, which animal would expend more energy? How can partial derivatives be used to explore this question?

54. Heat Loss The rate of heat loss (in watts) in harbor seal pups has been approximated by

$$H(m,T,A) = \frac{15.2m^{0.67}(T - A)}{10.23 \ln m - 10.74},$$

where m is the body mass of the pup (in kg), and T and A are the body core temperature and ambient water temperature, respectively (in °C). Find the approximate change in heat loss under the following conditions. ***Source: Functional Ecology***.

a. The body core temperature increases from 37°C to 38°, while the ambient water temperature remains at 8°C and the body mass remains at 24 kg.

b. The ambient water temperature increases from 10°C to 11°, while the body core temperature remains at 37°C and the body mass remains at 26 kg.

55. Body Surface Area The surface area of a human (in square meters) has been approximated by

$$A = 0.024265h^{0.3964}m^{0.5378},$$

where h is the height (in cm) and m is the mass (in kg). ***Source: The Journal of Pediatrics***.

a. Find the approximate change in surface area when the mass changes from 72 kg to 73 kg, while the height remains at 180 cm.

b. Find the approximate change in surface area when the height changes from 160 cm to 161 cm, while the mass remains at 70 kg.

56. Blood Flow According to the Fick Principle, the quantity of blood pumped through the lungs depends on the following variables (in milliliters):

b = quantity of oxygen used by the body in one minute

a = quantity of oxygen per liter of blood that has just gone through the lungs

v = quantity of oxygen per liter of blood that is about to enter the lungs

In one minute,

$$\text{Amount of oxygen used} = \text{Amount of oxygen per liter} \times \text{Liters of blood pumped.}$$

If C is the number of liters of blood pumped through the lungs in one minute, then

$$b = (a - v) \cdot C \qquad \text{or} \qquad C = \frac{b}{a - v}.$$

Source: Anaesthesia UK.

a. Find the number of liters of blood pumped through the lungs in one minute if $a = 160$, $b = 200$, and $v = 125$.

b. Find the approximate change in C when a changes from 160 to 161, $b = 200$, and $v = 125$.

c. Find the approximate change in C when $a = 160$, b changes from 200 to 201, and $v = 125$.

d. Find the approximate change in C when $a = 160$, $b = 200$, and v changes from 125 to 126.

e. A change of 1 unit in which quantity of oxygen produces the greatest change in the liters of blood pumped?

57. Health A weight-loss counselor has prepared a program of diet and exercise for a client. If the client sticks to the program, the weight loss that can be expected (in pounds per week) is given by

$$\text{Weight loss} = f(n, c) = \frac{1}{8}n^2 - \frac{1}{5}c + \frac{1937}{8},$$

where c is the average daily calorie intake for the week and n is the number of 40-minute aerobic workouts per week.

a. How many pounds can the client expect to lose by eating an average of 1200 cal per day and participating in four 40-minute workouts in a week?

b. Find and interpret $\partial f/\partial n$.

c. The client currently averages 1100 cal per day and does three 40-minute workouts each week. What would be the approximate impact on weekly weight loss of adding a fourth workout per week?

58. Health The body mass index is a number that can be calculated for any individual as follows: Multiply a person's weight by 703 and divide by the person's height squared. That is,

$$B = \frac{703w}{h^2},$$

where w is in pounds and h is in inches. The National Heart, Lung and Blood Institute uses the body mass index to determine whether a person is "overweight" $(25 \le B < 30)$ or "obese" $(B \ge 30)$. ***Source: The National Institutes of Health.***

a. Calculate the body mass index for Miami Dolphins offensive tackle Jake Long, who weighs 317 lb and is 6′7″ tall.

b. Calculate $\dfrac{\partial B}{\partial w}$ and $\dfrac{\partial B}{\partial h}$ and interpret.

c. Using the fact that 1 in. = 0.0254 m and 1 lb. = 0.4536 kg. transform this formula to handle metric units.

59. Drug Reaction The reaction to x units of a drug t hours after it was administered is given by

$$R(x, t) = x^2(a - x)t^2e^{-t},$$

for $0 \le x \le a$ (where a is a constant). Find the following.

a. $\dfrac{\partial R}{\partial x}$ **b.** $\dfrac{\partial R}{\partial t}$ **c.** $\dfrac{\partial^2 R}{\partial x^2}$ **d.** $\dfrac{\partial^2 R}{\partial x \partial t}$

e. Interpret your answers to parts a and b.

60. Scuba Diving In 1908, J. Haldane constructed diving tables that provide a relationship between the water pressure on body tissues for various water depths and dive times. The tables were successfully used by divers to virtually eliminate decompression sickness. The pressure in atmospheres for a no-stop dive is given by the following formula:*

$$p(l, t) = 1 + \frac{l}{33}(1 - 2^{-t/5}),$$

where t is in minutes, l is in feet, and p is in atmospheres (atm). ***Source: The UMAP Journal.***

a. Find the pressure at 33 ft for a 10-minute dive.

b. Find $p_l(33, 10)$ and $p_t(33, 10)$ and interpret. (*Hint:* $D_t(a^t) = \ln(a)a^t$.)

c. Haldane estimated that decompression sickness for no-stop dives could be avoided if the diver's tissue pressure did not exceed 2.15 atm. Find the maximum amount of time that a diver could stay down (time includes going down and coming back up) if he or she wants to dive to a depth of 66 ft.

61. Wind Chill In 1941, explorers Paul Siple and Charles Passel discovered that the amount of heat lost when an object is exposed to cold air depends on both the temperature of the air and the velocity of the wind. They developed the *Wind Chill Index* as a way to measure the danger of frostbite while doing outdoor activities. The wind chill can be calculated as follows:

$$W(V, T) = 91.4 - \frac{(10.45 + 6.69\sqrt{V} - 0.447V)(91.4 - T)}{22}$$

where V is the wind speed in miles per hour and T is the temperature in Fahrenheit for wind speeds between 4 and 45 mph. ***Source: The UMAP Journal.***

a. Find the wind chill for a wind speed of 20 mph and 10°F.

b. If a weather report indicates that the wind chill is −25°F and the actual outdoor temperature is 5°F, use a graphing calculator to find the corresponding wind speed to the nearest mile per hour.

c. Find $W_V(20, 10)$ and $W_T(20, 10)$ and interpret.

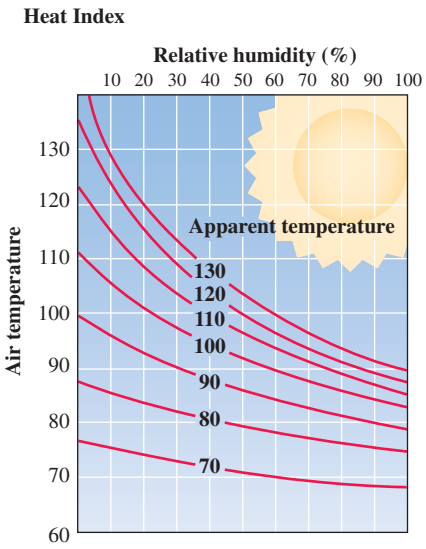
d. Using the table command on a graphing calculator or a spreadsheet, develop a wind chill chart for various wind speeds and temperatures.

62. Heat Index The chart on the next page shows the heat index, which combines the effects of temperature with humidity to give a measure of the apparent temperature, or how hot it feels to

*These estimates are conservative. Please consult modern dive tables before making a dive.

the body. ***Source: The Weather Channel.*** For example, when the outside temperature is 90°F and the relative humidity is 40%, then the apparent temperature is approximately 93°F. Let $I = f(T, H)$ give the heat index, I, as a function of the temperature T (in degrees Fahrenheit) and the percent humidity H. Estimate the following.

a. $f(90, 30)$ **b.** $f(90, 75)$ **c.** $f(80, 75)$



Estimate the following by approximating the partial derivative using a value of $h = 5$ in the difference quotient.

d. $f_T(90, 30)$ **e.** $f_H(90, 30)$ **f.** $f_T(90, 75)$

g. $f_H(90, 75)$

h. Describe in words what your answers in parts d–g mean.

63. Breath Volume The table at the bottom of this page accompanies the Voldyne® 5000 Volumetric Exerciser. The table gives the typical lung capacity (in milliliters) for women of various ages and heights. Based on the chart, it is possible to conclude that the partial derivative of the lung capacity with respect to age and with respect to height has constant values. What are those values?

Social Sciences

64. Education A developmental mathematics instructor at a large university has determined that a student's probability of success in the university's pass/fail remedial algebra course is a function of s, n, and a, where s is the student's score on the departmental placement exam, n is the number of semesters of mathematics passed in high school, and a is the student's mathematics SAT score. She estimates that p, the probability of passing the course (in percent), will be

$$p = f(s, n, a) = 0.05a + 6(sn)^{1/2}$$

for $200 \le a \le 800$, $0 \le s \le 10$, and $0 \le n \le 8$. Assuming that the above model has some merit, find the following.

a. If a student scores 6 on the placement exam, has taken 4 semesters of high school math, and has an SAT score of 460, what is the probability of passing the course?

b. Find p for a student with 5 semesters of high school mathematics, a placement score of 4, and an SAT score of 300.

c. Find and interpret $f_n(4, 5, 480)$ and $f_a(4, 5, 480)$.

Physical Sciences

65. Gravitational Attraction The gravitational attraction F on a body a distance r from the center of Earth, where r is greater than the radius of Earth, is a function of its mass m and the distance r as follows:

$$F = \frac{mgR^2}{r^2},$$

	Height (in.)	58″	60″	62″	64″	66″	68″	70″	72″	74″
	20	1900	2100	2300	2500	2700	2900	3100	3300	3500
A	25	1850	2050	2250	2450	2650	2850	3050	3250	3450
G	30	1800	2000	2200	2400	2600	2800	3000	3200	3400
E	35	1750	1950	2150	2350	2550	2750	2950	3150	3350
	40	1700	1900	2100	2300	2500	2700	2900	3100	3300
I	45	1650	1850	2050	2250	2450	2650	2850	3050	3250
N	50	1600	1800	2000	2200	2400	2600	2800	3000	3200
	55	1550	1750	1950	2150	2350	2550	2750	2950	3150
Y	60	1500	1700	1900	2100	2300	2500	2700	2900	3100
E	65	1450	1650	1850	2050	2250	2450	2650	2850	3050
A	70	1400	1600	1800	2000	2200	2400	2600	2800	3000
R	75	1350	1550	1750	1950	2150	2350	2550	2750	2950
S	80	1300	1500	1700	1900	2100	2300	2500	2700	2900

where R is the radius of Earth and g is the force of gravity—about 32 feet per second per second (ft per sec^2).

a. Find and interpret F_m and F_r.

b. Show that $F_m > 0$ and $F_r < 0$. Why is this reasonable?

66. Velocity In 1931, Albert Einstein developed the following formula for the sum of two velocities, x and y:

$$w(x, y) = \frac{x + y}{1 + \frac{xy}{c^2}},$$

where x and y are in miles per second and c represents the speed of light, 186,282 miles per second. ***Source: The Mathematics Teacher***.

a. Suppose that, relative to a stationary observer, a new super space shuttle is capable of traveling at 50,000 miles per second and that, while traveling at this speed, it launches a rocket that travels at 150,000 miles per second. How fast is the rocket traveling relative to the stationary observer?

b. What is the instantaneous rate of change of w with respect to the speed of the space shuttle, x, when the space shuttle is traveling at 50,000 miles per second and the rocket is traveling at 150,000 miles per second?

c. Hypothetically, if a person is driving at the speed of light, c, and she turns on the headlights, what is the velocity of the light coming from the headlights relative to a stationary observer?

67. Movement Time Fitts' law is used to estimate the amount of time it takes for a person, using his or her arm, to pick up a light object, move it, and then place it in a designated target area. Mathematically, Fitts' law for a particular individual is given by

$$T(s, w) = 105 + 265 \log_2 \left(\frac{2s}{w}\right),$$

where s is the distance (in feet) the object is moved, w is the width of the area in which the object is being placed, and T is the time (in msec). ***Source: Human Factors in Engineering Design***.

a. Calculate $T(3, 0.5)$.

b. Find $T_s(3, 0.5)$ and $T_w(3, 0.5)$ and interpret these values. (*Hint:* $\log_2 x = \ln x / \ln 2$.)

YOUR TURN ANSWERS

1. $f_x(x,y) = 4xy^3 + 30x^4y^4$; $f_y(x,y) = 6x^2y^2 + 24x^5y^3$

2. $f_x(x,y) = 6xye^{3x^2y}$; $f_y(x,y) = 3x^2e^{3x^2y}$

3. $9e^5$, $8e^5$

4. $f_{xx}(x,y) = 2e^{7y} + 12x^2y^5$;
$f_{yy}(x,y) = 49x^2e^{7y} + 20x^4y^3$;
$f_{xy}(x,y) = f_{yx}(x, y) = 14xe^{7y} + 20x^3y^4$

17.3 Maxima and Minima

APPLY IT

What amounts of sugar and flavoring produce the minimum cost per batch of a soft drink? What is the minimum cost?

In this section we will learn how to answer questions such as this one, which is answered in Example 4.

FOR REVIEW

It may be helpful to review Section 13.2 on relative extrema at this point. The concepts presented there are basic to what will be done in this section.

One of the most important applications of calculus is finding maxima and minima of functions. Earlier, we studied this idea extensively for functions of a single independent variable; now we will see that extrema can be found for functions of two variables. In particular, an extension of the second derivative test can be defined and used to identify maxima or minima. We begin with the definitions of relative maxima and minima.

Relative Maxima and Minima

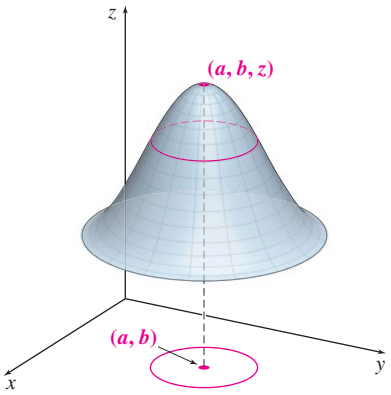
Let (a, b) be the center of a circular region contained in the xy-plane. Then, for a function $z = f(x, y)$ defined for every (x, y) in the region, $f(a, b)$ is a **relative (or local) maximum** if

$$f(a, b) \geq f(x, y)$$

for all points (x, y) in the circular region, and $f(a, b)$ is a **relative (or local) minimum** if

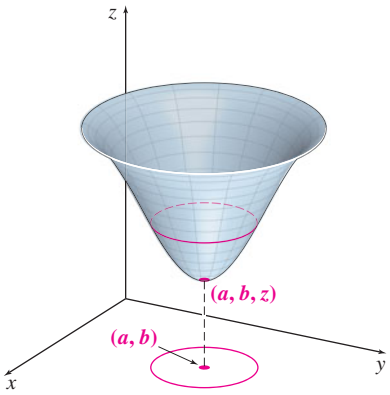
$$f(a, b) \leq f(x, y)$$

for all points (x, y) in the circular region.



Relative maximum at (a, b)

FIGURE 17



Relative minimum at (a, b)

FIGURE 18

As before, the word *extremum* is used for either a relative maximum or a relative minimum. Examples of a relative maximum and a relative minimum are given in Figures 17 and 18.

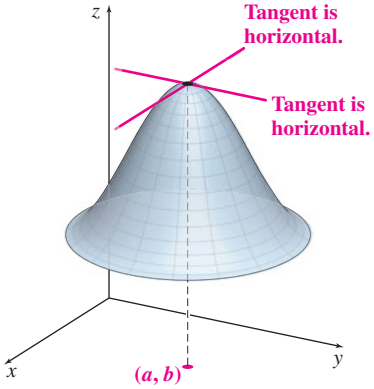


FIGURE 19

NOTE When functions of a single variable were discussed, a distinction was made between relative extrema and absolute extrema. The methods for finding absolute extrema are quite involved for functions of two variables, so we will discuss only relative extrema here. In many practical applications the relative extrema coincide with the absolute extrema. In this brief discussion of extrema for multivariable functions, we omit cases where an extremum occurs on a boundary of the domain.

As suggested by Figure 19, at a relative maximum the tangent line parallel to the xz-plane has a slope of 0, as does the tangent line parallel to the yz-plane. (Notice the similarity to functions of one variable.) That is, if the function $z = f(x, y)$ has a relative extremum at (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$, as stated in the next theorem.

Location of Extrema

Let a function $z = f(x, y)$ have a relative maximum or relative minimum at the point (a, b). Let $f_x(a, b)$ and $f_y(a, b)$ both exist. Then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

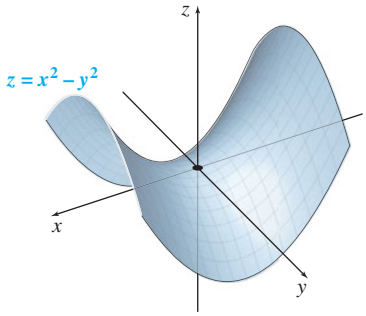


FIGURE 20

Just as with functions of one variable, the fact that the slopes of the tangent lines are 0 is no guarantee that a relative extremum has been located. For example, Figure 20 shows the graph of $z = f(x, y) = x^2 - y^2$. Both $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, and yet $(0, 0)$ leads to neither a relative maximum nor a relative minimum for the function. The point $(0, 0, 0)$ on the graph of this function is called a **saddle point**; it is a minimum when approached from one direction but a maximum when approached from another direction. A saddle point is neither a maximum nor a minimum.

The theorem on location of extrema suggests a useful strategy for finding extrema. First, locate all points (a, b) where $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Then test each of these points separately, using the test given after the next example. For a function $f(x, y)$, the points (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ are called **critical points**.

NOTE When we discussed functions of a single variable, we allowed critical points to include points from the domain where the derivative does not exist. For functions of more than one variable, to avoid complications, we will only consider cases in which the function is differentiable.

EXAMPLE 1 Critical Points

Find all critical points for

$$f(x, y) = 6x^2 + 6y^2 + 6xy + 36x - 5.$$

SOLUTION Find all points (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Here

$$f_x(x, y) = 12x + 6y + 36 \quad \text{and} \quad f_y(x, y) = 12y + 6x.$$

Set each of these two partial derivatives equal to 0.

$$12x + 6y + 36 = 0 \quad \text{and} \quad 12y + 6x = 0$$

These two equations make up a system of linear equations. We can use the substitution method to solve this system. First, rewrite $12y + 6x = 0$ as follows:

$$\begin{aligned} 12y + 6x &= 0 \\ 6x &= -12y \\ x &= -2y. \end{aligned}$$

Now substitute $-2y$ for x in the other equation and solve for y.

$$\begin{aligned} 12x + 6y + 36 &= 0 \\ 12(-2y) + 6y + 36 &= 0 \\ -24y + 6y + 36 &= 0 \\ -18y + 36 &= 0 \\ -18y &= -36 \\ y &= 2 \end{aligned}$$

From the equation $x = -2y$, $x = -2(2) = -4$. The solution of the system of equations is $(-4, 2)$. Since this is the only solution of the system, $(-4, 2)$ is the only critical point for the given function. By the theorem above, if the function has a relative extremum, it will occur at $(-4, 2)$. TRY YOUR TURN 1

YOUR TURN 1 Find all critical points for $f(x,y) = 4x^3 + 3xy + 4y^3$.

The results of the next theorem can be used to decide whether $(-4, 2)$ in Example 1 leads to a relative maximum, a relative minimum, or neither.

Test for Relative Extrema

For a function $z = f(x, y)$, let f_{xx}, f_{yy}, and f_{xy} all exist in a circular region contained in the xy-plane with center (a, b). Further, let

$$\boldsymbol{f_x(a, b) = 0} \quad \text{and} \quad \boldsymbol{f_y(a, b) = 0}.$$

Define the number D, known as **the discriminant**, by

$$\boldsymbol{D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2}.$$

Then

a. $f(a, b)$ is a relative maximum if $D > 0$ and $f_{xx}(a, b) < 0$;

b. $f(a, b)$ is a relative minimum if $D > 0$ and $f_{xx}(a, b) > 0$;

c. $f(a, b)$ is a saddle point (neither a maximum nor a minimum) if $D < 0$;

d. if $D = 0$, the test gives no information.

This test is comparable to the second derivative test for extrema of functions of one independent variable. The following table summarizes the conclusions of the theorem.

	$f_{xx}(a,b) < 0$	$f_{xx}(a,b) > 0$
$D > 0$	Relative maximum	Relative minimum
$D = 0$	No information	
$D < 0$	Saddle point	

Notice that in parts a and b of the test for relative extrema, it is only necessary to test the second partial $f_{xx}(a,b)$ and not $f_{yy}(a,b)$. This is because if $D > 0$, $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must have the same sign.

EXAMPLE 2 Relative Extrema

The previous example showed that the only critical point for the function

$$f(x,y) = 6x^2 + 6y^2 + 6xy + 36x - 5$$

is $(-4, 2)$. Does $(-4, 2)$ lead to a relative maximum, a relative minimum, or neither?

SOLUTION Find out by using the test above. From Example 1,

$$f_x(-4,2) = 0 \quad \text{and} \quad f_y(-4,2) = 0.$$

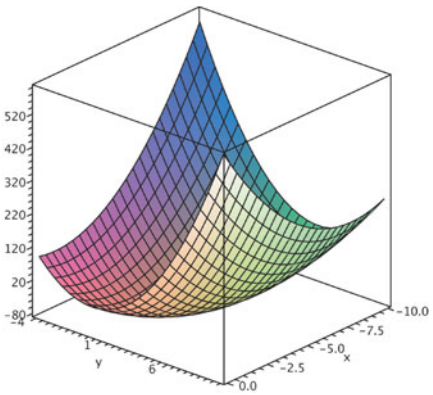


FIGURE 21

Now find the various second partial derivatives used in finding D. From $f_x(x,y) = 12x + 6y + 36$ and $f_y(x,y) = 12y + 6x$,

$$f_{xx}(x,y) = 12, \quad f_{yy}(x,y) = 12, \quad \text{and} \quad f_{xy}(x,y) = 6.$$

(If these second-order partial derivatives had not all been constants, they would have had to be evaluated at the point $(-4, 2)$.) Now

$$D = f_{xx}(-4,2) \cdot f_{yy}(-4,2) - [f_{xy}(-4,2)]^2 = 12 \cdot 12 - 6^2 = 108.$$

Since $D > 0$ and $f_{xx}(-4,2) = 12 > 0$, part b of the theorem applies, showing that $f(x,y) = 6x^2 + 6y^2 + 6xy + 36x - 5$ has a relative minimum at $(-4, 2)$. This relative minimum is $f(-4,2) = -77$. A graph of this surface drawn by the computer program Maple™ is shown in Figure 21.

EXAMPLE 3 Saddle Point

Find all points where the function

$$f(x,y) = 9xy - x^3 - y^3 - 6$$

has any relative maxima or relative minima.

SOLUTION First find any critical points. Here

$$f_x(x,y) = 9y - 3x^2 \quad \text{and} \quad f_y(x,y) = 9x - 3y^2.$$

Set each of these partial derivatives equal to 0.

$$\begin{aligned} f_x(x,y) &= 0 & f_y(x,y) &= 0 \\ 9y - 3x^2 &= 0 & 9x - 3y^2 &= 0 \\ 9y &= 3x^2 & 9x &= 3y^2 \\ 3y &= x^2 & 3x &= y^2 \end{aligned}$$

The substitution method can be used again to solve the system of equations

$$\begin{aligned} 3y &= x^2 \\ 3x &= y^2. \end{aligned}$$

The first equation, $3y = x^2$, can be rewritten as $y = x^2/3$. Substitute this into the second equation to get

$$3x = y^2 = \left(\frac{x^2}{3}\right)^2 = \frac{x^4}{9}.$$

Solve this equation as follows.

$$\begin{aligned} 27x &= x^4 && \text{Multiply both sides by 9.} \\ x^4 - 27x &= 0 \\ x(x^3 - 27) &= 0 && \text{Factor.} \\ x = 0 \quad \text{or} \quad x^3 - 27 &= 0 && \text{Set each factor equal to 0.} \\ x = 0 \quad \text{or} \quad x^3 &= 27 \\ x = 0 \quad \text{or} \quad x &= 3 && \text{Take the cube root on both sides.} \end{aligned}$$

Use these values of x, along with the equation $3y = x^2$, rewritten as $y = x^2/3$, to find y. If $x = 0$, $y = 0^2/3 = 0$. If $x = 3$, $y = 3^2/3 = 3$. The critical points are $(0, 0)$ and $(3, 3)$. To identify any extrema, use the test. Here

$$f_{xx}(x, y) = -6x, \qquad f_{yy}(x, y) = -6y, \qquad \text{and} \qquad f_{xy}(x, y) = 9.$$

Test each of the possible critical points.

For $(0, 0)$:	For $(3, 3)$:
$f_{xx}(0, 0) = -6(0) = 0$	$f_{xx}(3, 3) = -6(3) = -18$
$f_{yy}(0, 0) = -6(0) = 0$	$f_{yy}(3, 3) = -6(3) = -18$
$f_{xy}(0, 0) = 9$	$f_{xy}(3, 3) = 9$
$D = 0 \cdot 0 - 9^2 = -81.$	$D = -18(-18) - 9^2 = 243.$
Since $D < 0$, there is a saddle point at $(0, 0)$.	Here $D > 0$ and $f_{xx}(3, 3) = -18 < 0$; there is a relative maximum at $(3, 3)$.

Notice that these values are in accordance with the graph generated by the computer program Maple™ shown in Figure 22. TRY YOUR TURN 2

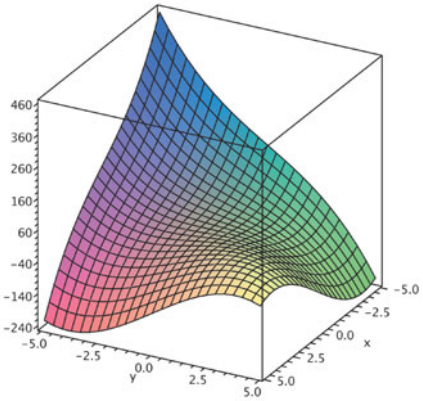


FIGURE 22

YOUR TURN 2 Identify each of the critical points in YOUR TURN 1 as a relative maximum, relative minimum, or saddle point.

EXAMPLE 4 Production Costs

A company is developing a new soft drink. The cost in dollars to produce a batch of the drink is approximated by

$$C(x, y) = 2200 + 27x^3 - 72xy + 8y^2,$$

where x is the number of kilograms of sugar per batch and y is the number of grams of flavoring per batch. Find the amounts of sugar and flavoring that result in the minimum cost per batch. What is the minimum cost?

APPLY IT

SOLUTION

Method 1 Calculating by Hand

Start with the following partial derivatives.

$$C_x(x, y) = 81x^2 - 72y \qquad \text{and} \qquad C_y(x, y) = -72x + 16y$$

Set each of these equal to 0 and solve for y.

$$\begin{aligned} 81x^2 - 72y &= 0 & \qquad -72x + 16y &= 0 \\ -72y &= -81x^2 & \qquad 16y &= 72x \\ y &= \frac{9}{8}x^2 & \qquad y &= \frac{9}{2}x \end{aligned}$$

Since $(9/8)x^2$ and $(9/2)x$ both equal y, they are equal to each other. Set them equal, and solve the resulting equation for x.

$$\frac{9}{8}x^2 = \frac{9}{2}x$$
$$9x^2 = 36x$$
$$9x^2 - 36x = 0 \qquad \text{Subtract } 36x \text{ from both sides.}$$
$$9x(x - 4) = 0 \qquad \text{Factor.}$$
$$9x = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{Set each factor equal to 0.}$$

The equation $9x = 0$ leads to $x = 0$ and $y = 0$, which cannot be a minimizer of $C(x, y)$ since, for example, $C(1, 1) < C(0, 0)$. This fact can also be verified by the test for relative extrema. Substitute $x = 4$, the solution of $x - 4 = 0$, into $y = (9/2)x$ to find y.

$$y = \frac{9}{2}x = \frac{9}{2}(4) = 18$$

Now check to see whether the critical point $(4, 18)$ leads to a relative minimum. Here

$$C_{xx}(x, y) = 162x, \quad C_{yy}(x, y) = 16, \quad \text{and} \quad C_{xy}(x, y) = -72.$$

For $(4, 18)$,

$$C_{xx}(4, 18) = 162(4) = 648, \quad C_{yy}(4, 18) = 16, \quad \text{and} \quad C_{xy}(4, 18) = -72,$$

so that

$$D = (648)(16) - (-72)^2 = 5184.$$

Since $D > 0$ and $C_{xx}(4, 18) > 0$, the cost at $(4, 18)$ is a minimum.

To find the minimum cost, go back to the cost function and evaluate $C(4, 18)$.

$$C(x, y) = 2200 + 27x^3 - 72xy + 8y^2$$
$$C(4, 18) = 2200 + 27(4)^3 - 72(4)(18) + 8(18)^2 = 1336$$

The minimum cost for a batch of soft drink is \$1336.00. A graph of this surface drawn by the computer program Maple™ is shown in Figure 23.

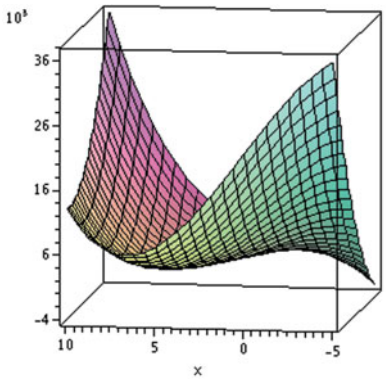


FIGURE 23

Method 2 Spreadsheets

Finding the maximum or minimum of a function of one or more variables can be done using a spreadsheet. The Solver included with Excel is located in the Tools menu and requires that cells be identified ahead of time for each variable in the problem. (On some versions of Excel, the Solver must be installed from an outside source. For details, see the *Graphing Calculator and Excel Spreadsheet Manual* available with this text.) It also requires that another cell be identified where the function, in terms of the variable cells, is placed. For example, to solve the above problem, we could identify cells A1 and B1 to represent the variables x and y, respectively. The Solver requires that we place a guess for the answer in these cells. Thus, our initial value or guess will be to place the number 5 in each of these cells. An expression for the function must be placed in another cell, with x and y replaced by A1 and B1. If we choose cell A3 to represent the function, in cell A3 we would type "`= 2200 + 27*A1^3 - 72*A1*B1 + 8*B1^2`."

We now click on the Tools menu and choose Solver. This solver will attempt to find a solution that either maximizes or minimizes the value of cell A3. Figure 24 illustrates the Solver box and the items placed in it.

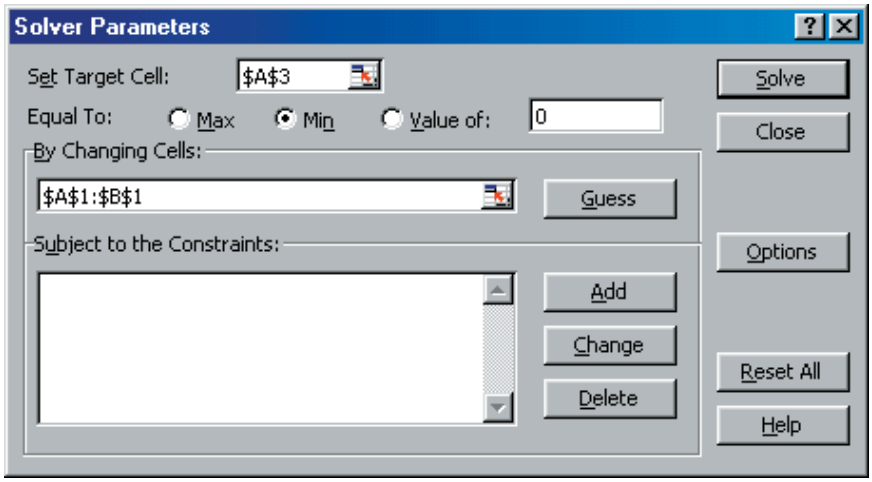


FIGURE 24

To obtain a solution, click on Solve. The rounded solution $x = 4$ and $y = 18$ is located in cells A1 and B1. The minimum cost $C(4, 18) = 1336$ is located in cell A3.

CAUTION One must be careful when using Solver because it will not find a maximizer or minimizer of a function if the initial guess is the exact place in which a saddle point occurs. For example, in the problem above, if our initial guess was $(0, 0)$, the Solver would have returned the value of $(0, 0)$ as the place where a minimum occurs. But $(0, 0)$ is a saddle point. Thus, it is always a good idea to run the Solver for two different initial values and compare the solutions.

17.3 EXERCISES

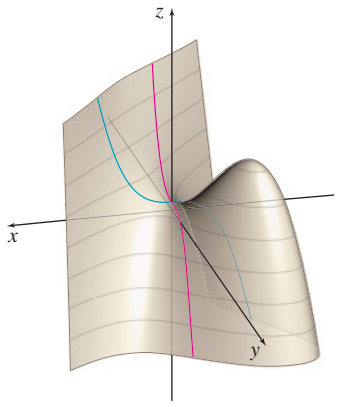
Find all points where the functions have any relative extrema. Identify any saddle points.

1. $f(x, y) = xy + y - 2x$
2. $f(x, y) = 3xy + 6y - 5x$
3. $f(x, y) = 3x^2 - 4xy + 2y^2 + 6x - 10$
4. $f(x, y) = x^2 + xy + y^2 - 6x - 3$
5. $f(x, y) = x^2 - xy + y^2 + 2x + 2y + 6$
6. $f(x, y) = 2x^2 + 3xy + 2y^2 - 5x + 5y$
7. $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y$
8. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y - 4$
9. $f(x, y) = 4xy - 10x^2 - 4y^2 + 8x + 8y + 9$
10. $f(x, y) = 4y^2 + 2xy + 6x + 4y - 8$
11. $f(x, y) = x^2 + xy - 2x - 2y + 2$
12. $f(x, y) = x^2 + xy + y^2 - 3x - 5$
13. $f(x, y) = 3x^2 + 2y^3 - 18xy + 42$
14. $f(x, y) = 7x^3 + 3y^2 - 126xy - 63$
15. $f(x, y) = x^2 + 4y^3 - 6xy - 1$
16. $f(x, y) = 3x^2 + 7y^3 - 42xy + 5$
17. $f(x, y) = e^{x(y+1)}$
18. $f(x, y) = y^2 + 2e^x$
19. Describe the procedure for finding critical points of a function in two independent variables.
20. How are second-order partial derivatives used in finding extrema?

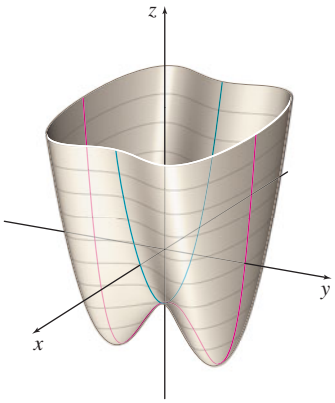
Figures a–f show the graphs of the functions defined in Exercises 21–26. Find all relative extrema for each function, and then match the equation to its graph.

21. $z = -3xy + x^3 - y^3 + \frac{1}{8}$
22. $z = \frac{3}{2}y - \frac{1}{2}y^3 - x^2y + \frac{1}{16}$
23. $z = y^4 - 2y^2 + x^2 - \frac{17}{16}$
24. $z = -2x^3 - 3y^4 + 6xy^2 + \frac{1}{16}$
25. $z = -x^4 + y^4 + 2x^2 - 2y^2 + \frac{1}{16}$
26. $z = -y^4 + 4xy - 2x^2 + \frac{1}{16}$

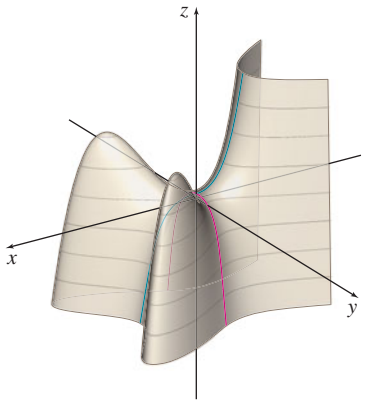
a.



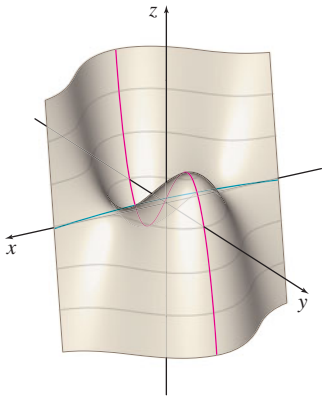
b.



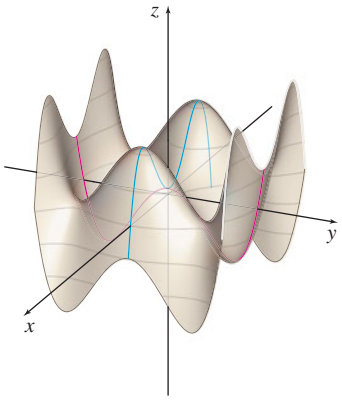
c.



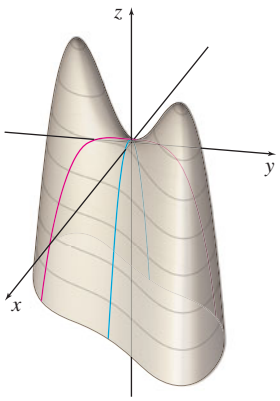
d.



e.



f.



27. Show that $f(x, y) = 1 - x^4 - y^4$ has a relative maximum, even though D in the theorem is 0.

28. Show that $D = 0$ for $f(x, y) = x^3 + (x - y)^2$ and that the function has no relative extrema.

29. A friend taking calculus is puzzled. She remembers that for a function of one variable, if the first derivative is zero at a point and the second derivative is positive, then there must be a relative minimum at the point. She doesn't understand why that isn't true for a function of two variables—that is, why $f_x(x, y) = 0$ and $f_{xx}(x, y) > 0$ doesn't guarantee a relative minimum. Provide an explanation.

30. Let $f(x, y) = y^2 - 2x^2y + 4x^3 + 20x^2$. The only critical points are $(-2, 4)$, $(0, 0)$, and $(5, 25)$. Which of the following correctly describes the behavior of f at these points? ***Source: Society of Actuaries***.

a. $(-2, 4)$: local (relative) minimum
$(0, 0)$: local (relative) minimum
$(5, 25)$: local (relative) maximum

b. $(-2, 4)$: local (relative) minimum
$(0, 0)$: local (relative) maximum
$(5, 25)$: local (relative) maximum

c. $(-2, 4)$: neither a local (relative) minimum nor a local (relative) maximum
$(0, 0)$: local (relative) maximum
$(5, 25)$: local (relative) minimum

d. $(-2, 4)$: local (relative) maximum
$(0, 0)$: neither a local (relative) minimum nor a local (relative) maximum
$(5, 25)$: local (relative) minimum

e. $(-2, 4)$: neither a local (relative) minimum nor a local (relative) maximum
$(0, 0)$: local (relative) minimum
$(5, 25)$: neither a local (relative) minimum nor a local (relative) maximum

31. Consider the function $f(x, y) = x^2(y + 1)^2 + k(x + 1)^2y^2$.

a. For what values of k is the point $(x, y) = (0, 0)$ a critical point?

b. For what values of k is the point $(x, y) = (0, 0)$ a relative minimum of the function?

32. In Exercise 5 of Section 1.3, we found the least squares line through a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by

choosing the slope of the line m and the y-intercept b to minimize the quantity

$$S(m, b) = \sum (mx + b - y)^2,$$

where the summation symbol Σ means that we sum over all the data points. Minimize S by setting $S_m(m, b) = 0$ and $S_b(m, b) = 0$, and then rearrange the results to derive the equations from Section 1.3

$$\begin{aligned} \left(\sum x\right)b + \left(\sum x^2\right)m &= \sum xy \\ nb + \left(\sum x\right)m &= \sum y. \end{aligned}$$

33. Suppose a function $z = f(x, y)$ satisfies the criteria for the test for relative extrema at a point (a, b), and $f_{xx}(a, b) > 0$, while $f_{yy}(a, b) < 0$. What does this tell you about $f(a, b)$? Based on the sign of $f_{xx}(a, b)$ and $f_{yy}(a, b)$, why does this seem intuitively plausible?

APPLICATIONS

Business and Economics

34. Profit Suppose that the profit (in hundreds of dollars) of a certain firm is approximated by

$$P(x, y) = 1500 + 36x - 1.5x^2 + 120y - 2y^2,$$

where x is the cost of a unit of labor and y is the cost of a unit of goods. Find values of x and y that maximize profit. Find the maximum profit.

35. Labor Costs Suppose the labor cost (in dollars) for manufacturing a precision camera can be approximated by

$$L(x, y) = \frac{3}{2}x^2 + y^2 - 2x - 2y - 2xy + 68,$$

where x is the number of hours required by a skilled craftsperson and y is the number of hours required by a semiskilled person. Find values of x and y that minimize the labor cost. Find the minimum labor cost.

36. Cost The total cost (in dollars) to produce x units of electrical tape and y units of packing tape is given by

$$C(x, y) = 2x^2 + 2y^2 - 3xy + 4x - 94y + 4200.$$

Find the number of units of each kind of tape that should be produced so that the total cost is a minimum. Find the minimum total cost.

37. Revenue The total revenue (in hundreds of dollars) from the sale of x spas and y solar heaters is approximated by

$$R(x, y) = 15 + 169x + 182y - 5x^2 - 7y^2 - 7xy.$$

Find the number of each that should be sold to produce maximum revenue. Find the maximum revenue.

38. Profit The profit (in thousands of dollars) that Aunt Mildred's Metalworks earns from producing x tons of steel and y tons of aluminum can be approximated by

$$P(x, y) = 36xy - x^3 - 8y^3.$$

Find the amounts of steel and aluminum that maximize the profit, and find the value of the maximum profit.

39. Time The time (in hours) that a branch of Amalgamated Entities needs to spend to meet the quota set by the main office can be approximated by

$$T(x, y) = x^4 + 16y^4 - 32xy + 40,$$

where x represents how many thousands of dollars the factory spends on quality control and y represents how many thousands of dollars they spend on consulting. Find the amount of money they should spend on quality control and on consulting to minimize the time spent, and find the minimum number of hours.

Social Sciences

40. Political Science The probability that a three-person jury will make a correct decision is given by

$$\begin{aligned} P(\alpha, r, s) = {} & \alpha[3r^2(1 - r) + r^3] \\ & + (1 - \alpha)[3s^2(1 - s) + s^3], \end{aligned}$$

where $0 < \alpha < 1$ is the probability that the person is guilty of the crime, r is the probability that a given jury member will vote "guilty" when the defendant is indeed guilty of the crime, and s is the probability that a given jury member will vote "innocent" when the defendant is indeed innocent. ***Source: Frontiers of Economics***.

a. Calculate $P(0.9, 0.5, 0.6)$ and $P(0.1, 0.8, 0.4)$ and interpret your answers.

b. Using common sense and without using calculus, what value of r and s would maximize the jury's probability of making the correct verdict? Do these values depend on α in this problem? Should they? What is the maximum probability?

c. Verify your answer for part b using calculus. (*Hint:* There are two critical points. Argue that the maximum value occurs at one of these points.)

Physical Sciences

41. Computer Chips The table on the following page, which illustrates the dramatic increase in the number of transistors in personal computers since 1985, was given in the chapter on Nonlinear Functions, Section 10.4, Exercise 54.

a. To fit the data to a function of the form $y = ab^t$, where t is the number of years since 1985 and y is the number of transistors (in millions), we could take natural logarithms of both sides of the equation to get $\ln y = \ln a + t \ln b$. We could then let $w = \ln y$, $r = \ln a$, and $s = \ln b$ to form $w = r + st$. Using linear regression, find values for r and s that will fit the data. Then find the function $y = ab^t$. (*Hint:* Take the natural logarithm of the values in the transistors column and then use linear regression to find values of r and s that fit the data. Once you know r and s, you can determine the values of a and b by calculating $a = e^r$ and $b = e^s$.)

Year (since 1985)	Chip	Transistors (in millions)
0	386	0.275
4	486	1.2
8	Pentium	3.1
12	Pentium II	7.5
14	Pentium III	9.5
15	Pentium 4	42
20	Pentium D	291
22	Penryn	820
24	Nehalem	1900

b. Use the solver capability of a spreadsheet to find a function of the form $y = ab^t$ that fits the data above. (*Hint:* Using the ideas from part a, find values for a and b that minimize the function

$$\begin{aligned} f(a,b) = {} & [\ln(0.275) - 0\ln b - \ln a]^2 \\ & + [\ln(1.2) - 4\ln b - \ln a]^2 \\ & + [\ln(3.1) - 8\ln b - \ln a]^2 \\ & + [\ln(7.5) - 12\ln b - \ln a]^2 \\ & + [\ln(9.5) - 14\ln b - \ln a]^2 \\ & + [\ln(42) - 15\ln b - \ln a]^2 \\ & + [\ln(291) - 20\ln b - \ln a]^2 \\ & + [\ln(820) - 22\ln b - \ln a]^2 \\ & + [\ln(1900) - 24\ln b - \ln a]^2.) \end{aligned}$$

c. Compare your answer to this problem with the one found with a graphing calculator in the chapter on Nonlinear Functions, Section 10.4, Exercise 54.

General Interest

42. Food Frying The process of frying food changes its quality, texture, and color. According to research done at the University of Saskatchewan, the total change in color E (which is measured in the form of energy as kJ/mol) of blanched potato strips can be estimated by the function

$$E(t,T) = 436.16 - 10.57t - 5.46T - 0.02t^2 + 0.02T^2 + 0.08Tt,$$

where T is the temperature (in °C) and t is the frying time (in min). ***Source: Critical Reviews in Food Science and Nutrition.***

a. What is the value of E prior to cooking? (Assume that $T = 0$.)

b. Use this function to estimate the total change in color of a potato strip that has been cooked for 10 minutes at 180°C.

c. Determine the critical point of this function and determine if a maximum, minimum, or saddle point occurs at that point. Describe what may be happening at this point.

YOUR TURN ANSWERS

1. (0, 0) and (−1/4, −1/4)
2. Saddle point at (0, 0); relative maximum at (−1/4, −1/4)

17.4 Lagrange Multipliers

APPLY IT **What dimensions for a new building will maximize the floor space at a fixed cost?**

Using Lagrange multipliers, we will answer this question in Example 2.

In Section 14.2 on Applications of Extrema, it was possible to express problems involving two variables as equivalent problems requiring only a single variable. This method works well, provided that it is possible to use algebra to express the one variable in terms of the other. It is not always possible to do this, however, and most real applications require more than two variables and one or more additional restrictions, called **constraints**.

An approach that works well when there is a constraint in the problem uses an additional variable, called the **Lagrange multiplier**. For example, in the opening question, suppose a builder wants to maximize the floor space in a new building while keeping the costs fixed at $500,000. The building will be 40 ft high, with a rectangular floor plan and three stories. The costs, which depend on the dimensions of the rectangular floor plan, are given by

$$\text{Costs} = xy + 20y + 20x + 474{,}000,$$

where x is the width and y the length of the rectangle. Thus, the builder wishes to maximize the area $A(x, y) = xy$ and satisfy the condition

$$xy + 20y + 20x + 474{,}000 = 500{,}000.$$

In addition to maximizing area, then, the builder must keep costs at (or below) $500,000. We will see how to solve this problem in Example 2 of this section.

Problems with constraints are often solved by the method of Lagrange multipliers, named for the French mathematician Joseph Louis Lagrange (1736–1813). The method of Lagrange multipliers is used for problems of the form:

$$\text{Find the relative extrema for } z = f(x, y),$$
$$\text{subject to } g(x, y) = 0.$$

We state the method only for functions of two independent variables, but it is valid for any number of variables.

Lagrange Multipliers

All relative extrema of the function $z = f(x, y)$, subject to a constraint $g(x, y) = 0$, will be found among those points (x, y) for which there exists a value of λ such that

$$F_x(x, y, \lambda) = 0, \qquad F_y(x, y, \lambda) = 0, \qquad F_\lambda(x, y, \lambda) = 0,$$

where

$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y),$$

and all indicated partial derivatives exist.

In the theorem, the function $F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$ is called the Lagrange function; λ, the Greek letter *lambda*, is the *Lagrange multiplier*.

CAUTION If the constraint is not of the form $g(x, y) = 0$, it must be put in that form before using the method of Lagrange multipliers. For example, if the constraint is $x^2 + y^3 = 5$, subtract 5 from both sides to get $g(x, y) = x^2 + y^3 - 5 = 0$.

EXAMPLE 1 Lagrange Multipliers

Find the minimum value of

$$f(x, y) = 5x^2 + 6y^2 - xy,$$

subject to the constraint $x + 2y = 24$.

SOLUTION Go through the following steps.

Step 1 Rewrite the constraint in the form $g(x, y) = 0$.

In this example, the constraint $x + 2y = 24$ becomes

$$x + 2y - 24 = 0,$$

with

$$g(x, y) = x + 2y - 24.$$

Step 2 Form the Lagrange function $F(x, y, \lambda)$, the difference of the function $f(x, y)$ and the product of λ and $g(x, y)$.

Here,

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) - \lambda \cdot g(x, y) \\ &= 5x^2 + 6y^2 - xy - \lambda(x + 2y - 24) \\ &= 5x^2 + 6y^2 - xy - \lambda x - 2\lambda y + 24\lambda. \end{aligned}$$

Step 3 Find $F_x(x, y, \lambda)$, $F_y(x, y, \lambda)$, and $F_\lambda(x, y, \lambda)$.

$$F_x(x, y, \lambda) = 10x - y - \lambda$$
$$F_y(x, y, \lambda) = 12y - x - 2\lambda$$
$$F_\lambda(x, y, \lambda) = -x - 2y + 24$$

Step 4 Form the system of equations $F_x(x, y, \lambda) = 0$, $F_y(x, y, \lambda) = 0$, and $F_\lambda(x, y, \lambda) = 0$.

$$10x - y - \lambda = 0 \quad (1)$$
$$12y - x - 2\lambda = 0 \quad (2)$$
$$-x - 2y + 24 = 0 \quad (3)$$

Step 5 Solve the system of equations from Step 4 for x, y, and λ.

One way to solve this system is to begin by solving each of the first two equations for λ, then set the two results equal and simplify, as follows.

$$10x - y - \lambda = 0 \quad \text{becomes} \quad \lambda = 10x - y$$
$$12y - x - 2\lambda = 0 \quad \text{becomes} \quad \lambda = \frac{-x + 12y}{2}$$

$$10x - y = \frac{-x + 12y}{2} \qquad \text{Set the expressions for } \lambda \text{ equal.}$$
$$20x - 2y = -x + 12y$$
$$21x = 14y$$
$$x = \frac{14y}{21} = \frac{2y}{3}$$

Now substitute $2y/3$ for x in Equation (3).

$$-x - 2y + 24 = 0$$
$$-\frac{2y}{3} - 2y + 24 = 0 \qquad \text{Let } x = \frac{2y}{3}.$$
$$2y + 6y - 72 = 0 \qquad \text{Multiply by } -3.$$
$$8y = 72$$
$$y = \frac{72}{8} = 9$$

Since $x = 2y/3$ and $y = 9$, $x = 6$. It is not necessary to find the value of λ.

Thus, if $f(x, y) = 5x^2 + 6y^2 - xy$ has a minimum value subject to the constraint $x + 2y = 24$, it is at the point $(6, 9)$. The value of $f(6, 9)$ is 612.

We need to convince ourselves that $f(6, 9) = 612$ is indeed a minimum for the function. How can we tell that it is not a maximum? The second derivative test from the previous section does not apply to the solutions found by Lagrange multipliers. (See Exercise 21.) We could gain some insight by trying a point very close to $(6, 9)$ that also satisfies the constraint $x + 2y = 24$. For example, let $y = 9.1$, so $x = 24 - 2y = 24 - 2(9.1) = 5.8$. Then $f(5.8, 9.1) = 5(5.8)^2 + 6(9.1)^2 - (5.8)(9.1) = 612.28$, which is greater than 612. Because a nearby point has a value larger than 612, the value 612 is probably not a maximum. Another method would be to use a computer to sketch the graph of the function and see that it has a minimum but not a maximum. In practical problems, such as Example 2, it is often obvious whether a function has a minimum or a maximum. TRY YOUR TURN 1

YOUR TURN 1 Find the minimum value of $f(x, y) = x^2 + 2x + 9y^2 + 3y + 6xy$ subject to the constraint $2x + 3y = 12$.

NOTE In Example 1, we solved the system of equations by solving each equation with λ in it for λ. We then set these expressions for λ equal and solved for one of the original variables. This is a good general approach to use in solving these systems of equations, since we are usually not interested in the value of λ.

CAUTION Lagrange multipliers give only the relative extrema, not the absolute extrema. In many applications, the relative extrema will be the absolute extrema, but this is not guaranteed. In some cases in which the method of Lagrange multipliers finds a solution, there may not even be any absolute extrema. For example, see Exercises 18 and 19 at the end of this section.

Before looking at applications of Lagrange multipliers, let us summarize the steps involved in solving a problem by this method.

Using Lagrange Multipliers

1. Write the constraint in the form $g(x, y) = 0$.
2. Form the Lagrange function
$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y).$$
3. Find $F_x(x, y, \lambda)$, $F_y(x, y, \lambda)$, and $F_\lambda(x, y, \lambda)$.
4. Form the system of equations
$$F_x(x, y, \lambda) = 0, \qquad F_y(x, y, \lambda) = 0, \qquad F_\lambda(x, y, \lambda) = 0.$$
5. Solve the system in Step 4; the relative extrema for f are among the solutions of the system.

The proof of this method is complicated and is not given here, but we can explain why the method is plausible. Consider the curve formed by points in the xy-plane that satisfy $F_\lambda(x, y, \lambda) = -g(x, y) = 0$ (or just $g(x, y) = 0$). Figure 25 shows how such a curve might look. Crossing this region are curves $f(x, y) = k$ for various values of k. Notice that at the points where the curve $f(x, y) = k$ is tangent to the curve $g(x, y) = 0$, the largest and smallest meaningful values of f occur. It can be shown that this is equivalent to $f_x(x, y) = \lambda g_x(x, y)$ and $f_y(x, y) = \lambda g_y(x, y)$ for some constant λ. In Exercise 20, you are asked to show that this is equivalent to the system of equations found in Step 4 above.

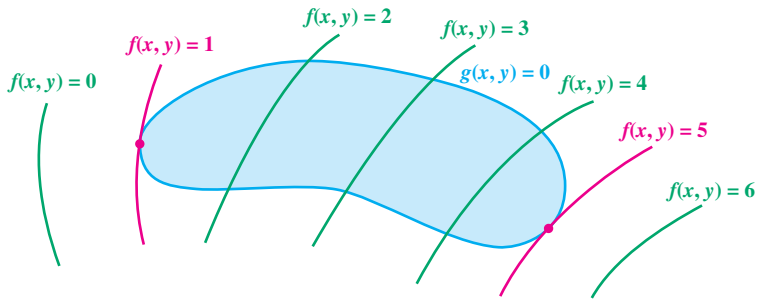


FIGURE 25

Lagrange multipliers are widely used in economics, where a frequent goal is to maximize a utility function, which measures how well consumption satisfies the consumers' desires, subject to constraints on income or time.

EXAMPLE 2 Lagrange Multipliers

Complete the solution of the problem given in the introduction to this section. Maximize the area, $A(x, y) = xy$, subject to the cost constraint

$$xy + 20y + 20x + 474{,}000 = 500{,}000.$$

APPLY IT

SOLUTION Go through the five steps presented earlier.

Step 1 $g(x, y) = xy + 20y + 20x - 26{,}000 = 0$

Step 2 $F(x, y, \lambda) = xy - \lambda(xy + 20y + 20x - 26{,}000)$

Step 3 $F_x(x, y, \lambda) = y - \lambda y - 20\lambda$

$F_y(x, y, \lambda) = x - \lambda x - 20\lambda$

$F_\lambda(x, y, \lambda) = -xy - 20y - 20x + 26{,}000$

Step 4 $y - \lambda y - 20\lambda = 0$ (4)

$x - \lambda x - 20\lambda = 0$ (5)

$-xy - 20y - 20x + 26{,}000 = 0$ (6)

Step 5 Solving Equations (4) and (5) for λ gives

$$\lambda = \frac{y}{y + 20} \quad \text{and} \quad \lambda = \frac{x}{x + 20}$$

$$\frac{y}{y + 20} = \frac{x}{x + 20}$$

$$y(x + 20) = x(y + 20)$$

$$xy + 20y = xy + 20x$$

$$x = y.$$

Now substitute y for x in Equation (6) to get

$$-y^2 - 20y - 20y + 26{,}000 = 0$$

$$-y^2 - 40y + 26{,}000 = 0.$$

Use the quadratic formula to find $y \approx -182.5$ or $y \approx 142.5$. We eliminate the negative value because length cannot be negative. Since $x = y$, we know that $x \approx 142.5$.

The maximum area of $(142.5)^2 \approx 20{,}306$ ft^2 will be achieved if the floor plan is a square with a side of 142.5 ft. You can verify that this answer is a maximum using the method at the end of Example 1.

As mentioned earlier, the method of Lagrange multipliers works for more than two independent variables. The next example shows how to find extrema for a function of three independent variables.

EXAMPLE 3 Volume of a Box

Find the dimensions of the closed rectangular box of maximum volume that can be produced from 6 ft^2 of material.

Method I Lagrange Multipliers

SOLUTION In Chapter 14 on Applications of the Derivative, we were able to solve problems such as this by adding an extra constraint, such as requiring the bottom of the box to be square. Here we have no such constraint. Let x, y, and z represent the dimensions of the box, as shown in Figure 26 on the next page. The volume of the box is given by

$$f(x, y, z) = xyz.$$

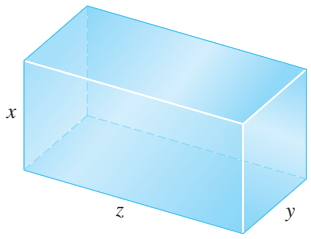


FIGURE 26

As shown in Figure 26, the total amount of material required for the two ends of the box is $2xy$, the total needed for the sides is $2xz$, and the total needed for the top and bottom is $2yz$. Since 6 ft^2 of material is available,

$$2xy + 2xz + 2yz = 6 \qquad \text{or} \qquad xy + xz + yz = 3.$$

In summary, $f(x, y, z) = xyz$ is to be maximized subject to the constraint $xy + xz + yz = 3$. Go through the steps that were given.

Step 1 $g(x, y, z) = xy + xz + yz - 3 = 0$

Step 2 $F(x, y, z, \lambda) = xyz - \lambda(xy + xz + yz - 3)$

Step 3 $F_x(x, y, z, \lambda) = yz - \lambda y - \lambda z$

$F_y(x, y, z, \lambda) = xz - \lambda x - \lambda z$

$F_z(x, y, z, \lambda) = xy - \lambda x - \lambda y$

$F_\lambda(x, y, z, \lambda) = -xy - xz - yz + 3$

Step 4 $yz - \lambda y - \lambda z = 0$

$xz - \lambda x - \lambda z = 0$

$xy - \lambda x - \lambda y = 0$

$-xy - xz - yz + 3 = 0$

Step 5 Solve each of the first three equations for λ. You should get

$$\lambda = \frac{yz}{y + z}, \qquad \lambda = \frac{xz}{x + z}, \qquad \text{and} \qquad \lambda = \frac{xy}{x + y}.$$

Set these expressions for λ equal, and simplify as follows. Notice in the second and last steps that since none of the dimensions of the box can be 0, we can divide both sides of each equation by x or z.

$$\frac{yz}{y + z} = \frac{xz}{x + z} \qquad \text{and} \qquad \frac{xz}{x + z} = \frac{xy}{x + y}$$

$$\frac{y}{y + z} = \frac{x}{x + z} \qquad\qquad \frac{z}{x + z} = \frac{y}{x + y}$$

$$xy + yz = xy + xz \qquad\qquad zx + zy = yx + yz$$

$$yz = xz \qquad\qquad zx = yx$$

$$y = x \qquad\qquad z = y$$

(Setting the first and third expressions equal gives no additional information.) Thus $x = y = z$. From the fourth equation in Step 4, with $x = y$ and $z = y$,

$$\begin{aligned} -xy - xz - yz + 3 &= 0 \\ -y^2 - y^2 - y^2 + 3 &= 0 \\ -3y^2 &= -3 \\ y^2 &= 1 \\ y &= \pm 1. \end{aligned}$$

The negative solution is not applicable, so the solution of the system of equations is $x = 1$, $y = 1$, $z = 1$. In other words, the box with maximum volume under the constraint is a cube that measures 1 ft on each side. As in the previous examples, verify that this is a maximum.

Method 2 Spreadsheets

Finding extrema of a constrained function of one or more variables can be done using a spreadsheet. In addition to the requirements stated in the last section, the constraint must also be input into the Excel Solver. To do this, we need to input the left-hand or variable part of the constraint into a designated cell. If A5 is the designated cell, then in cell A5 we would type "`=A1*B1 + A1*C1 + B1*C1`."

We now click on the Tools menu and choose Solver. This solver will attempt to find a solution that either maximizes or minimizes the value of cell A3, depending on which option we choose. Figure 27 illustrates the Solver box and the items placed in it.

To obtain a solution, click on Solve. The solution $x = 1$ and $y = 1$ and $z = 1$ is located in cells A1, B1, and C1, respectively. The maximum volume $f(1, 1, 1) = 1$ is located in cell A3.

YOUR TURN 2 Solve Example 3 with the box changed so that the front and the top are missing.

TRY YOUR TURN 2

CAUTION One must be careful when using Solver because the solution may depend on the initial value. Thus, it is always a good idea to run the Solver for two different initial values and compare the solutions.

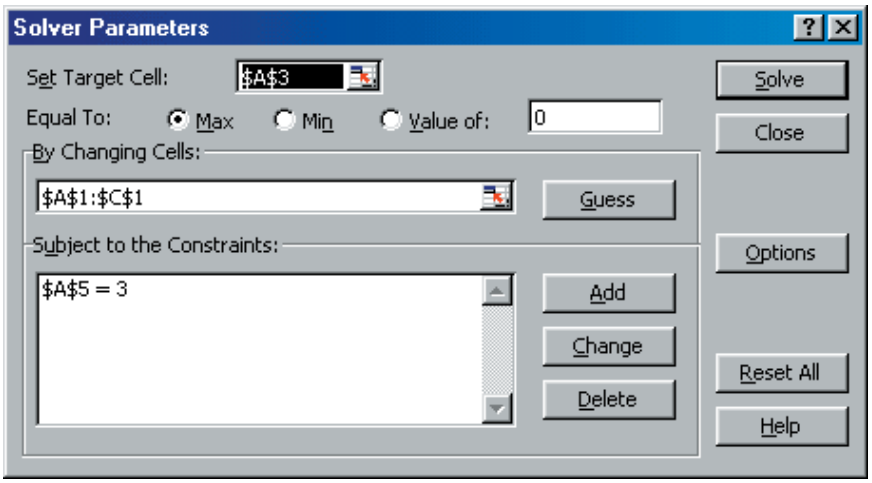


FIGURE 27

Utility Functions

A **utility function** of two variables is a function $z = f(x, y)$ in which x and y give the quantity of items that a consumer might value, such as cereal and milk, and z is a measure of the value that the consumer places on the combination of items represented by the point (x, y). (Naturally, this definition can be extended to any number of variables, but for simplicity we will restrict this discussion to two variables.) For example, if $z = f(x, y) = x^2y^4$, where x represents the number of quarts of milk and y represents the number of pounds of cereal, then 4 quarts of milk and 3 pounds of cereal has a utility of $4^2 \cdot 3^4 = 1296$. The consumer would value this combination as much as 36 quarts of milk and 1 pound of cereal, since this combination also has a utility of $36^2 \cdot 1^4 = 1296$, but less so than 3 quarts of milk and 4 pounds of cereal, which has a utility of $3^2 \cdot 4^4 = 2304$. The set of points $c = f(x, y)$ form an **indifference curve**, since the consumer considers all points on this curve to be equally desirable.

Now suppose a quart of milk costs \$2 and a pound of cereal costs \$3, so that the combination of milk and cereal represented by the point (x, y) costs $2x + 3y$. Suppose further that the consumer has \$90 to spend. A natural question would be how much of each quantity to buy to maximize the consumer's utility. In other words, the consumer wishes to maximize $f(x, y) = x^2y^4$ subject to the constraint that $2x + 3y = 90$, or $g(x, y) = 2x + 3y - 90 = 0$. This is exactly the type of problem that Lagrange multipliers were designed to solve. Use Lagrange multipliers to verify that the function $f(x, y) = x^2y^4$ subject to the constraint $g(x, y) = 2x + 3y - 90 = 0$ has a maximum value of 36,000,000 when $x = 15$ and $y = 20$, so the consumer should purchase 15 quarts of milk and 20 pounds of cereal.

17.4 EXERCISES

Find the relative maxima or minima in Exercises 1–10.

1. Maximum of $f(x, y) = 4xy$, subject to $x + y = 16$
2. Maximum of $f(x, y) = 2xy + 4$, subject to $x + y = 20$
3. Maximum of $f(x, y) = xy^2$, subject to $x + 2y = 15$
4. Maximum of $f(x, y) = 8x^2y$, subject to $3x - y = 9$
5. Minimum of $f(x, y) = x^2 + 2y^2 - xy$, subject to $x + y = 8$
6. Minimum of $f(x, y) = 3x^2 + 4y^2 - xy - 2$, subject to $2x + y = 21$
7. Maximum of $f(x, y) = x^2 - 10y^2$, subject to $x - y = 18$
8. Maximum of $f(x, y) = 12xy - x^2 - 3y^2$, subject to $x + y = 16$
9. Maximum of $f(x, y, z) = xyz^2$, subject to $x + y + z = 6$
10. Maximum of $f(x, y, z) = xy + 2xz + 2yz$, subject to $xyz = 32$
11. Find positive numbers x and y such that $x + y = 24$ and $3xy^2$ is maximized.
12. Find positive numbers x and y such that $x + y = 48$ and $5x^2y + 10$ is maximized.
13. Find three positive numbers whose sum is 90 and whose product is a maximum.
14. Find three positive numbers whose sum is 240 and whose product is a maximum.
15. Find the maximum and minimum values of $f(x,y) = x^3 + 2xy + 4y^2$ subject to $x + 2y = 12$. Be sure to use the method at the end of Example 1 to determine whether each solution is a maximum or a minimum.
16. Explain the difference between the two methods we used in Sections 3 and 4 to solve extrema problems.
17. Why is it unnecessary to find the value of λ when using the method explained in this section?
18. Show that the function $f(x, y) = xy^2$ in Exercise 3, subject to $x + 2y = 15$, does not have an absolute minimum or maximum. (*Hint*: Solve the constraint for x and substitute into f.)
19. Show that the function $f(x, y) = 8x^2y$ in Exercise 4, subject to $3x - y = 9$, does not have an absolute minimum or maximum. (*Hint*: Solve the constraint for y and substitute into f.)
20. Show that the three equations in Step 4 of the box "Using Lagrange Multipliers" are equivalent to the three equations

$$f_x(x, y) = \lambda g_x(x, y),\ f_y(x, y) = \lambda g_y(x, y),\ g(x, y) = 0.$$

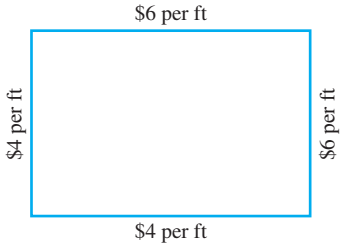
21. Consider the problem of minimizing $f(x,y) = x^2 + 2x + 9y^2 + 4y + 8xy$ subject to $x + y = 1$.
    - **a.** Find the solution using the method of Lagrange multipliers.
    - **b.** Experiment with points very near the point from part a to convince yourself that the point from part a actually gives a minimum. (*Hint:* See the last paragraph of Example 1.)
    - **c.** Solve $x + y = 1$ for y and substitute the expression for y into $f(x,y)$. Then explain why the resulting expression in x has a minimum but no maximum.
    - **d.** Suppose you erroneously applied the method of finding the discriminant D from the previous section to determine whether the point found in part a is a minimum. What does the test erroneously tell you about the point?
22. Discuss the advantages and disadvantages of the method of Lagrange multipliers compared with solving the equation $g(x, y) = 0$ for y (or x), substituting that expression into f and then minimizing or maximizing f as a function of one variable. You might want to try some examples both ways and consider what happens when there are more than two variables.

APPLICATIONS

Business and Economics

Utility **Maximize each of the following utility functions, with the cost of each commodity and total amount available to spend given.**

23. $f(x,y) = xy^2$, cost of a unit of x is \$1, cost of a unit of y is \$2, and \$60 is available.
24. $f(x,y) = x^2y^3$, cost of a unit of x is \$2, cost of a unit of y is \$1, and \$80 is available.
25. $f(x,y) = x^4y^2$, cost of a unit of x is \$2, cost of a unit of y is \$4, and \$60 is available.
26. $f(x,y) = x^3y^4$, cost of a unit of x is \$3, cost of a unit of y is \$3, and \$42 is available.
27. **Maximum Area for Fixed Expenditure** Because of terrain difficulties, two sides of a fence can be built for \$6 per ft, while the other two sides cost \$4 per ft. (See the sketch.) Find the field of maximum area that can be enclosed for \$1200.



28. **Maximum Area for Fixed Expenditure** To enclose a yard, a fence is built against a large building, so that fencing material is used only on three sides. Material for the ends costs \$15 per ft; material for the side opposite the building costs \$25 per ft. Find the dimensions of the yard of maximum area that can be enclosed for \$2400.
29. **Cost** The total cost to produce x large jewelry-making kits and y small ones is given by

$$C(x, y) = 2x^2 + 6y^2 + 4xy + 10.$$

If a total of ten kits must be made, how should production be allocated so that total cost is minimized?

30. Profit The profit from the sale of x units of radiators for automobiles and y units of radiators for generators is given by

$$P(x, y) = -x^2 - y^2 + 4x + 8y.$$

Find values of x and y that lead to a maximum profit if the firm must produce a total of 6 units of radiators.

31. Production A manufacturing firm estimates that its total production of automobile batteries in thousands of units is

$$f(x, y) = 3x^{1/3}y^{2/3},$$

where x is the number of units of labor and y is the number of units of capital utilized. Labor costs are \$80 per unit, and capital costs are \$150 per unit. How many units each of labor and capital will maximize production, if the firm can spend \$40,000 for these costs?

32. Production For another product, the manufacturing firm in Exercise 31 estimates that production is a function of labor x and capital y as follows:

$$f(x, y) = 12x^{3/4}y^{1/4}.$$

If \$25,200 is available for labor and capital, and if the firm's costs are \$100 and \$180 per unit, respectively, how many units of labor and capital will give maximum production?

33. Area A farmer has 500 m of fencing. Find the dimensions of the rectangular field of maximum area that can be enclosed by this amount of fencing.

34. Area Find the area of the largest rectangular field that can be enclosed with 600 m of fencing. Assume that no fencing is needed along one side of the field.

35. Surface Area A cylindrical can is to be made that will hold 250π in^3 of candy. Find the dimensions of the can with minimum surface area.

36. Surface Area An ordinary 12-oz beer or soda pop can holds about 25 in^3. Find the dimensions of a can with minimum surface area. Measure a can and see how close its dimensions are to the results you found.

37. Volume A rectangular box with no top is to be built from 500 m^2 of material. Find the dimensions of such a box that will enclose the maximum volume.

38. Surface Area A 1-lb soda cracker box has a volume of 185 in^3. The end of the box is square. Find the dimensions of such a box that has minimum surface area.

39. Cost A rectangular closed box is to be built at minimum cost to hold 125 m^3. Since the cost will depend on the surface area, find the dimensions that will minimize the surface area of the box.

40. Cost Find the dimensions that will minimize the surface area (and hence the cost) of a rectangular fish aquarium, open on top, with a volume of 32 ft^3.

41. Container Construction A company needs to construct a box with an open top that will be used to transport 400 yd^3 of material, in several trips, from one place to another. Two of the sides and bottom of the box can be made of a free, lightweight material, but only 4 yd^2 of the material is available. Because of the nature of the material to be transported, the two ends of the box must be made from a heavyweight material that costs \$20 per yd^2. Each trip costs 10 cents. ***Source: Geometric Programming***.

a. Let x, y, and z denote the length, width, and height of the box, respectively. If we want to use all of the free material, show that the total cost in dollars is given by the function

$$f(x, y, z) = \frac{40}{xyz} + 40yz,$$

subject to the constraint $2xz + xy = 4$.

b. Use the Solver feature on a spreadsheet to find the dimensions of the box that minimize the transportation cost, subject to the constraint.

Social Sciences

42. Political Science The probability that the majority of a three-person jury will convict a guilty person is given by the formula:

$$P(r, s, t) = rs(1 - t) + (1 - r)st + r(1 - s)t + rst$$

subject to the constraint that

$$r + s + t = \alpha,$$

where r, s, and t represent each of the three jury members' probability of reaching a guilty verdict and α is some fixed constant that is generally less than or equal to the number of jurors. ***Source: Mathematical Social Sciences***.

a. Form the Lagrange function.

b. Find the values of r, s, and t that maximize the probability of convicting a guilty person when $\alpha = 0.75$.

c. Find the values of r, s, and t that maximize the probability of convicting a guilty person when $\alpha = 3$.

YOUR TURN ANSWERS

1. $f(12,-4) = 12$

2. The box should be 2 ft wide and 1 ft high and long.

17.5 Total Differentials and Approximations

APPLY IT **How do errors in measuring the length and radius of a blood vessel affect the calculation of its volume?**

In Example 3 in this section, we will see how to answer this question using a total differential.

In the second section of this chapter we used partial derivatives to find the marginal productivity of labor and of capital for a production function. The marginal productivity approximates the change of production for a 1-unit change in labor or capital. To estimate

the change in productivity for a small change in both labor and capital, we can extend the concept of differential, introduced in an earlier chapter for functions of one variable, to the concept of *total differential*.

Total Differential for Two Variables

Let $z = f(x, y)$ be a function of x and y. Let dx and dy be real numbers. Then the **total differential** of z is

$$dz = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy.$$

(Sometimes dz is written df.)

Recall that the differential for a function of one variable $y = f(x)$ is used to approximate the function by its tangent line. This works because a differentiable function appears very much like a line when viewed closely. Similarly, the differential for a function of two variables $z = f(x, y)$ is used to approximate a function by its tangent plane. A differentiable function of two variables looks like a plane when viewed closely, which is why the earth looks flat when you are standing on it.

FOR REVIEW

In Chapter 14 on Applications of the Derivative, we introduced the differential. Recall that the differential of a function defined by $y = f(x)$ is

$$dy = f'(x) \cdot dx,$$

where dx, the differential of x, is any real number (usually small). We saw that the differential dy is often a good approximation of Δy, where $\Delta y = f(x + \Delta x) - f(x)$ and $\Delta x = dx$.

EXAMPLE 1 Total Differentials

Consider the function $z = f(x, y) = 9x^3 - 8x^2y + 4y^3$.

(a) Find dz.

SOLUTION First find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = 27x^2 - 16xy \quad \text{and} \quad f_y(x, y) = -8x^2 + 12y^2$$

By the definition,

$$dz = (27x^2 - 16xy)\,dx + (-8x^2 + 12y^2)\,dy.$$

(b) Evaluate dz when $x = 1$, $y = 3$, $dx = 0.01$, and $dy = -0.02$.

SOLUTION Putting these values into the result from part (a) gives

$$\begin{aligned} dz &= [27(1)^2 - 16(1)(3)](0.01) + [-8(1)^2 + 12(3)^2](-0.02) \\ &= (-21)(0.01) + (100)(-0.02) \\ &= -2.21. \end{aligned}$$

This result indicates that an increase of 0.01 in x and a decrease of 0.02 in y, when $x = 1$ and $y = 3$, will produce an approximate *decrease* of 2.21 in $f(x, y)$.

TRY YOUR TURN 1

YOUR TURN 1 For the function $f(x,y) = 3x^2y^4 + 6\sqrt{x^2 - 7y^2}$, find **(a)** dz, and **(b)** the value of dz when $x = 4$, $y = 1$, $dx = 0.02$, and $dy = -0.03$.

Approximations

Recall that with a function of one variable, $y = f(x)$, the differential dy approximates the change in y, Δy, corresponding to a change in x, Δx or dx. The approximation for a function of two variables is similar.

Approximations

For small values of dx and dy,

$$dz \approx \Delta z,$$

where $\Delta z = f(x + dx, y + dy) - f(x, y)$.

EXAMPLE 2 Approximations

Approximate $\sqrt{2.98^2 + 4.01^2}$.

SOLUTION Notice that $2.98 \approx 3$ and $4.01 \approx 4$, and we know that $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$. We, therefore, let $f(x, y) = \sqrt{x^2 + y^2}$, $x = 3$, $dx = -0.02$, $y = 4$, and $dy = 0.01$. We then use dz to approximate $\Delta z = \sqrt{2.98^2 + 4.01^2} - \sqrt{3^2 + 4^2}$.

$$\begin{aligned} dz &= f_x(x, y) \cdot dx + f_y(x, y) \cdot dy \\ &= \left(\frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x\right) dx + \left(\frac{1}{2\sqrt{x^2 + y^2}} \cdot 2y\right) dy \\ &= \left(\frac{x}{\sqrt{x^2 + y^2}}\right) dx + \left(\frac{y}{\sqrt{x^2 + y^2}}\right) dy \\ &= \frac{3}{5}(-0.02) + \frac{4}{5}(0.01) \\ &= -0.004 \end{aligned}$$

Thus, $\sqrt{2.98^2 + 4.01^2} \approx 5 + (-0.004) = 4.996$. A calculator gives $\sqrt{2.98^2 + 4.01^2} \approx 4.996048$. The error is approximately 0.000048. TRY YOUR TURN 2

YOUR TURN 2
Approximate $\sqrt{5.03^2 + 11.99^2}$.

For small values of dx and dy, the values of Δz and dz are approximately equal. Since $\Delta z = f(x + dx, y + dy) - f(x, y)$,

$$f(x + dx, y + dy) = f(x, y) + \Delta z$$

or

$$f(x + dx, y + dy) \approx f(x, y) + dz.$$

Replacing dz with the expression for the total differential gives the following result.

Approximations by Differentials

For a function f having all indicated partial derivatives, and for small values of dx and dy,

$$f(x + dx, y + dy) \approx f(x, y) + dz,$$

or

$$f(x + dx, y + dy) \approx f(x, y) + f_x(x, y) \cdot dx + f_y(x, y) \cdot dy.$$

The idea of a total differential can be extended to include functions of three or more independent variables.

Total Differential for Three Variables

If $w = f(x, y, z)$, then the total differential dw is

$$dw = f_x(x, y, z)\, dx + f_y(x, y, z)\, dy + f_z(x, y, z)\, dz,$$

provided all indicated partial derivatives exist.

EXAMPLE 3 Blood Vessels

A short length of blood vessel is in the shape of a right circular cylinder (see Figure 28).

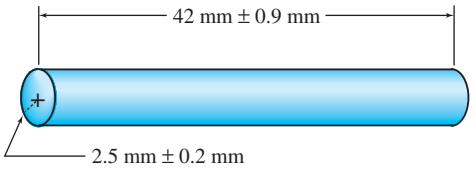


FIGURE 28

(a) The length of the vessel is measured as 42 mm, and the radius is measured as 2.5 mm. Suppose the maximum error in the measurement of the length is 0.9 mm, with an error of no more than 0.2 mm in the measurement of the radius. Find the maximum possible error in calculating the volume of the blood vessel.

APPLY IT

SOLUTION The volume of a right circular cylinder is given by $V = \pi r^2 h$. To approximate the error in the volume, find the total differential, dV.

$$dV = (2\pi rh) \cdot dr + (\pi r^2) \cdot dh$$

Here, $r = 2.5$, $h = 42$, $dr = 0.2$, and $dh = 0.9$. Substitution gives

$$dV = [(2\pi)(2.5)(42)](0.2) + [\pi(2.5)^2](0.9) \approx 149.6.$$

The maximum possible error in calculating the volume is approximately 149.6 mm^3.

(b) Suppose that the errors in measuring the radius and length of the vessel are at most 1% and 3%, respectively. Estimate the maximum percent error in calculating the volume.

SOLUTION To find the percent error, calculate dV/V.

$$\frac{dV}{V} = \frac{(2\pi rh)dr + (\pi r^2)dh}{\pi r^2 h} = 2\frac{dr}{r} + \frac{dh}{h}$$

Because $dr/r = 0.01$ and $dh/h = 0.03$,

$$\frac{dV}{V} = 2(0.01) + 0.03 = 0.05.$$

The maximum percent error in calculating the volume is approximately 5%.

YOUR TURN 3 In Example 3, estimate the maximum percent error in calculating the volume if the errors in measuring the radius and length of the vessel are at most 4% and 2%, respectively.

TRY YOUR TURN 3

EXAMPLE 4 Volume of a Can of Beer

The formula for the volume of a cylinder given in Example 3 also applies to cans of beer, for which $r \approx 1.5$ in. and $h \approx 5$ in. How sensitive is the volume to changes in the radius compared with changes in the height?

SOLUTION Using the formula for dV from the previous example with $r = 1.5$ and $h = 5$ gives

$$dV = (2\pi)(1.5)(5)dr + \pi(1.5)^2 dh = \pi(15dr + 2.25dh).$$

The factor of 15 in front of dr in this equation, compared with the factor of 2.25 in front of dh, shows that a small change in the radius has almost 7 times the effect on the volume as a small change in the height. One author argues that this is the reason that beer cans are so tall and thin. ***Source: The College Mathematics Journal***. The brewers can reduce the radius by a tiny amount and compensate by making the can taller. The resulting can appears larger in volume than the shorter, wider can. (Others have argued that a shorter, wider can does not fit as easily in the hand.)

17.5 EXERCISES

Evaluate dz using the given information.

1. $z = 2x^2 + 4xy + y^2$; $x = 5, y = -1, dx = 0.03, dy = -0.02$

2. $z = 5x^3 + 2xy^2 - 4y$; $x = 1, y = 3, dx = 0.01, dy = 0.02$

3. $z = \dfrac{y^2 + 3x}{y^2 - x}$; $x = 4, y = -4, dx = 0.01, dy = 0.03$

4. $z = \ln(x^2 + y^2)$; $x = 2, y = 3, dx = 0.02, dy = -0.03$

Evaluate dw using the given information.

5. $w = \dfrac{5x^2 + y^2}{z + 1}$; $x = -2, y = 1, z = 1, dx = 0.02, dy = -0.03, dz = 0.02$

6. $w = x \ln(yz) - y \ln \dfrac{x}{z}$; $x = 2, y = 1, z = 4, dx = 0.03, dy = 0.02, dz = -0.01$

Use the total differential to approximate each quantity. Then use a calculator to approximate the quantity, and give the absolute value of the difference in the two results to 4 decimal places.

7. $\sqrt{8.05^2 + 5.97^2}$

8. $\sqrt{4.96^2 + 12.06^2}$

9. $(1.92^2 + 2.1^2)^{1/3}$

10. $(2.93^2 - 0.94^2)^{1/3}$

11. $1.03e^{0.04}$

12. $0.98e^{-0.04}$

13. $0.99 \ln 0.98$

14. $2.03 \ln 1.02$

APPLICATIONS

Business and Economics

15. Manufacturing Approximate the volume of aluminum needed for a beverage can of radius 2.5 cm and height 14 cm. Assume the walls of the can are 0.08 cm thick.

16. Manufacturing Approximate the volume of material needed to make a water tumbler of diameter 3 cm and height 9 cm. Assume the walls of the tumbler are 0.2 cm thick.

17. Volume of a Coating An industrial coating 0.1 in. thick is applied to all sides of a box of dimensions 10 in. by 9 in. by 18 in. Estimate the volume of the coating used.

18. Manufacturing Cost The manufacturing cost of a smartphone is approximated by

$$M(x, y) = 45x^2 + 40y^2 - 20xy + 50,$$

where x is the cost of the parts and y is the cost of labor. Right now, the company spends \$8 on parts and \$14 on labor. Use differentials to approximate the change in cost if the company spends \$8.25 on parts and \$13.75 on labor.

19. Production The production function for one country is

$$z = x^{0.65}y^{0.35},$$

where x stands for units of labor and y for units of capital. At present, 50 units of labor and 29 units of capital are available. Use differentials to estimate the change in production if the number of units of labor is increased to 52 and capital is decreased to 27 units.

20. Production The production function for another country is

$$z = x^{0.8}y^{0.2},$$

where x stands for units of labor and y for units of capital. At present, 20 units of labor and 18 units of capital are being provided. Use differentials to estimate the change in production if an additional unit of labor is provided and if capital is decreased to 16 units.

Life Sciences

21. Bone Preservative Volume A piece of bone in the shape of a right circular cylinder is 7 cm long and has a radius of 1.4 cm. It is coated with a layer of preservative 0.09 cm thick. Estimate the volume of preservative used.

22. Blood Vessel Volume A portion of a blood vessel is measured as having length 7.9 cm and radius 0.8 cm. If each measurement could be off by as much as 0.15 cm, estimate the maximum possible error in calculating the volume of the vessel.

23. Blood Volume In Exercise 56 of Section 2 in this chapter, we found that the number of liters of blood pumped through the lungs in one minute is given by

$$C = \frac{b}{a - v}.$$

Suppose $a = 160$, $b = 200$, and $v = 125$. Estimate the change in C if a becomes 145, b becomes 190, and v changes to 130.

24. Heat Loss In Exercise 54 of Section 2 of this chapter, we found that the rate of heat loss (in watts) in harbor seal pups could be approximated by

$$H(m, T, A) = \frac{15.2m^{0.67}(T - A)}{10.23 \ln m - 10.74},$$

where m is the body mass of the pup (in kg), and T and A are the body core temperature and ambient water temperature, respectively (in °C). Suppose m is 25 kg, T is 36.0°, and A is 12.0°C. Approximate the change in H if m changes to 26 kg, T to 36.5°, and A to 10.0°C.

25. Dialysis A model that estimates the concentration of urea in the body for a particular dialysis patient, following a dialysis session, is given by

$$C(t, g) = 0.6(0.96)^{(210t/1500)-1} + \frac{gt}{126t - 900}[1 - (0.96)^{(210t/1500)-1}],$$

where t represents the number of minutes of the dialysis session and g represents the rate at which the body generates urea in mg per minute. ***Source: Clinical Dialysis***.

a. Find $C(180, 8)$.

b. Using the total differential, estimate the urea concentration if the dialysis session of part a was cut short by 10 minutes and the urea generation rate was 9 mg per minute. Compare this with the actual concentration. (*Hint:* First, replace the variable g with the number 8, thus reducing the function to one variable. Then use your graphing calculator to calculate the partial derivative $C_t(180, 8)$. A similar procedure can be done for $C_g(180, 8)$.)

26. Horn Volume The volume of the horns from bighorn sheep was estimated by researchers using the equation

$$V = \frac{h\pi}{3}(r_1^2 + r_1r_2 + r_2^2),$$

where h is the length of a horn segment (in centimeters) and r_1 and r_2 are the radii of the two ends of the horn segment (in centimeters). ***Source: Conservation Biology***.

a. Determine the volume of a segment of horn that is 40 cm long with radii of 5 cm and 3 cm, respectively.

b. Use the total differential to estimate the volume of the segment of horn if the horn segment from part a was actually 42 cm long with radii of 5.1 cm and 2.9 cm, respectively. Compare this with the actual volume.

27. Eastern Hemlock Ring shake, which is the separation of the wood between growth rings, is a serious problem in hemlock trees. Researchers have developed the following function that estimates the probability P that a given hemlock tree has ring shake.

$$P(A, B, D) = \frac{1}{1 + e^{3.68 - 0.016A - 0.77B - 0.12D}},$$

where A is the age of the tree (yr), B is 1 if bird pecking is present and 0 otherwise, and D is the diameter (in.) of the tree at breast height. ***Source: Forest Products Journal***.

a. Estimate the probability that a 150-year-old tree, with bird pecking present and a breast height diameter of 20 in., will have ring shake.

b. Estimate the probability that a 150-year-old tree, with no presence of bird pecking and a breast height diameter of 20 in., will have ring shake.

c. Develop a statement about what can be said about the influence that the three variables have on the probability of ring shake.

d. Using the total differential, estimate the probability if the actual age of the tree was 160 years and the diameter at breast height was 25 in. Assume that no bird pecking was present. Compare your answer to the actual value. (*Hint*: Assume that $B = 0$ and exclude that variable from your calculations.)

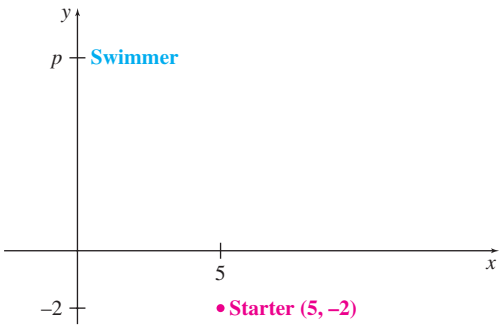
e. Comment on the practicality of using differentials in part d.

Physical Sciences

28. Swimming The amount of time in seconds it takes for a swimmer to hear a single, hand-held, starting signal is given by the formula

$$t(x, y, p, C) = \frac{\sqrt{x^2 + (y - p)^2}}{331.45 + 0.6C},$$

where (x, y) is the location of the starter (in meters), $(0, p)$ is the location of the swimmer (in meters), and C is the air temperature (in degrees Celsius). ***Source: COMAP***. Assume that the starter is located at the point $(x, y) = (5, -2)$. See the diagram.



a. Calculate $t(5, -2, 20, 20)$ and $t(5, -2, 10, 20)$. Could the difference in time change the outcome of a race?

b. Calculate the total differential for t if the starter remains stationary, the swimmer moves from 20 m to 20.5 m away from the starter in the y direction, and the temperature decreases from 20°C to 15°C. Interpret your answer.

General Interest

29. Estimating Area The height of a triangle is measured as 37.5 cm, with the base measured as 15.8 cm. The measurement of the height can be off by as much as 0.8 cm and that of the base by no more than 1.1 cm. Estimate the maximum possible error in calculating the area of the triangle.

30. Estimating Volume The height of a cone is measured as 9.3 cm and the radius as 3.2 cm. Each measurement could be off by as much as 0.1 cm. Estimate the maximum possible error in calculating the volume of the cone.

31. Estimating Volume Suppose that in measuring the length, width, and height of a box, there is a maximum 1% error in each measurement. Estimate the maximum error in calculating the volume of the box.

32. Estimating Volume Suppose there is a maximum error of $a\%$ in measuring the radius of a cone and a maximum error of $b\%$ in measuring the height. Estimate the maximum percent error in calculating the volume of the cone, and compare this value with the maximum percent error in calculating the volume of a cylinder.

33. Ice Cream Cone An ice cream cone has a radius of approximately 1 in. and a height of approximately 4 in. By what factor does a change in the radius affect the volume compared with a change in the height?

34. Hose A hose has a radius of approximately 0.5 in. and a length of approximately 20 ft. By what factor does a change in the radius affect the volume compared with a change in the length?

YOUR TURN ANSWERS

1. (a) $dz = (6xy^4 + 6x/\sqrt{x^2 - 7y^2})dx + (12x^2y^3 - 42y/\sqrt{x^2 - 7y^2})dy$ **(b)** -4.7

2. 13.0023 **3.** 10%

17.6 Double Integrals

APPLY IT **How can we find the volume of a bottle with curved sides?**

We will answer this question in Example 6 using a double integral, the key idea in this section.

In an earlier chapter, we saw how integrals of functions with one variable may be used to find area. In this section, this idea is extended and used to find volume. We found partial derivatives of functions of two or more variables at the beginning of this chapter by holding

FOR REVIEW

You may wish to review the key ideas of indefinite and definite integrals from Chapter 15 on Integration before continuing with this section. See the review problems at the end of that chapter.

constant all variables except one. A similar process is used in this section to find antiderivatives of functions of two or more variables. For example, in

$$\int (5x^3y^4 - 6x^2y + 2)\,dy$$

the notation dy indicates integration with respect to y, so we treat y as the variable and x as a constant. Using the rules for antiderivatives gives

$$\int (5x^3y^4 - 6x^2y + 2)\,dy = x^3y^5 - 3x^2y^2 + 2y + C(x).$$

The constant C used earlier must be replaced with $C(x)$ to show that the "constant of integration" here can be any function involving only the variable x. Just as before, check this work by taking the derivative (actually the partial derivative) of the answer:

$$\frac{\partial}{\partial y}[x^3y^5 - 3x^2y^2 + 2y + C(x)] = 5x^3y^4 - 6x^2y + 2 + 0,$$

which shows that the antiderivative is correct.

We can use this antiderivative to evaluate a definite integral.

EXAMPLE 1 Definite Integral

Evaluate $\int_1^2 (5x^3y^4 - 6x^2y + 2)\,dy$.

SOLUTION

$$\begin{aligned}\int_1^2 (5x^3y^4 - 6x^2y + 2)\,dy &= [x^3y^5 - 3x^2y^2 + 2y + C(x)]\Big|_1^2 && \text{Use the indefinite integral previously found.}\\ &= x^3 2^5 - 3x^2 2^2 + 2\cdot 2 + C(x)\\ &\quad - [x^3 1^5 - 3x^2 1^2 + 2\cdot 1 + C(x)]\\ &= 32x^3 - 12x^2 + 4 + C(x)\\ &\quad - [x^3 - 3x^2 + 2 + C(x)]\\ &= 31x^3 - 9x^2 + 2 && \text{Simplify.}\end{aligned}$$

In the second step, we substituted $y = 2$ and $y = 1$ and subtracted, according to the Fundamental Theorem of Calculus. Notice that $C(x)$ does not appear in the final answer, just as the constant does not appear in a regular definite integral. Therefore, from now on we will not include $C(x)$ when we find the antiderivative for a definite integral with respect to y.

YOUR TURN 1 Evaluate $\int_1^3 (6x^2y^2 + 4xy + 8x^3 + 10y^4 + 3)dy$.

TRY YOUR TURN 1

By integrating the result from Example 1 with respect to x, we can evaluate a double integral.

EXAMPLE 2 Definite Integral

Evaluate $\int_0^3 \left[\int_1^2 (5x^3y^4 - 6x^2y + 2)\,dy\right] dx$.

SOLUTION

$$\begin{aligned}\int_0^3 \left[\int_1^2 (5x^3y^4 - 6x^2y + 2)\,dy\right] dx &= \int_0^3 [31x^3 - 9x^2 + 2]dx && \text{Use the result from Example1.}\\ &= \frac{31}{4}x^4 - 3x^3 + 2x\Big|_0^3 && \text{Use the Fundamental Theorem of Calculus.}\\ &= \frac{31}{4}\cdot 3^4 - 3\cdot 3^3 + 2\cdot 3 - \left(\frac{31}{4}\cdot 0^4 - 3\cdot 0^3 + 2\cdot 0\right)\\ &= \frac{2211}{4}\end{aligned}$$

We can integrate the inner integral with respect to y and the outer integral with respect to x, as in Example 2, or in the reverse order. The next example shows the same integral done both ways.

EXAMPLE 3 Definite Integrals

Evaluate each integral.

(a) $\int_1^2 \left[\int_3^5 (6xy^2 + 12x^2y + 4y)\, dx \right] dy$

SOLUTION

$$\int_1^2 \left[\int_3^5 (6xy^2 + 12x^2y + 4y)dx \right] dy = \int_1^2 \left[(3x^2y^2 + 4x^3y + 4xy) \Big|_3^5 \right] dy$$ Integrate with respect to x.

$$= \int_1^2 [(3 \cdot 5^2 \cdot y^2 + 4 \cdot 5^3 \cdot y + 4 \cdot 5 \cdot y) - (3 \cdot 3^2 \cdot y^2 + 4 \cdot 3^3 \cdot y + 4 \cdot 3 \cdot y)]dy$$

$$= \int_1^2 [(75y^2 + 500y + 20y) - (27y^2 + 108y + 12y)]dy$$

$$= \int_1^2 (48y^2 + 400y)dy$$

$$= (16y^3 + 200y^2) \Big|_1^2$$ Integrate with respect to y.

$$= 16 \cdot 2^3 + 200 \cdot 2^2 - (16 \cdot 1^3 + 200 \cdot 1^2)$$

$$= 128 + 800 - (16 + 200)$$

$$= 712.$$

(b) $\int_3^5 \left[\int_1^2 (6xy^2 + 12x^2y + 4y)\, dy \right] dx$

SOLUTION (This is the same integrand with the same limits of integration as in part (a), but the order of integration is reversed.)

$$\int_3^5 \left[\int_1^2 (6xy^2 + 12x^2y + 4y)dy \right] dx = \int_3^5 \left[(2xy^3 + 6x^2y^2 + 2y^2) \Big|_1^2 \right] dx$$ Integrate with respect to y.

$$= \int_3^5 [(2x \cdot 2^3 + 6x^2 \cdot 2^2 + 2 \cdot 2^2) - (2x \cdot 1^3 \cdot + 6x^2 \cdot 1^2 + 2 \cdot 1^2)]dx$$

$$= \int_3^5 [(16x + 24x^2 + 8) - (2x + 6x^2 + 2)]dx$$

$$= \int_3^5 (14x + 18x^2 + 6)dx$$ Integrate with respect to x.

$$= (7x^2 + 6x^3 + 6x) \Big|_3^5$$

$$= 7 \cdot 5^2 + 6 \cdot 5^3 + 6 \cdot 5 - (7 \cdot 3^2 + 6 \cdot 3^3 + 6 \cdot 3)$$

$$= 175 + 750 + 30 - (63 + 162 + 18) = 712$$

YOUR TURN 2 Evaluate $\int_0^2 \left[\int_1^3 (6x^2y^2 + 4xy + 8x^3 + 10y^4 + 3)\, dy \right] dx$, and then integrate with the order of integration changed.

TRY YOUR TURN 2

NOTE In the second step of Example 3 (a), it might help you avoid confusion as to whether to put the limits of 3 and 5 into x or y by writing the integral as

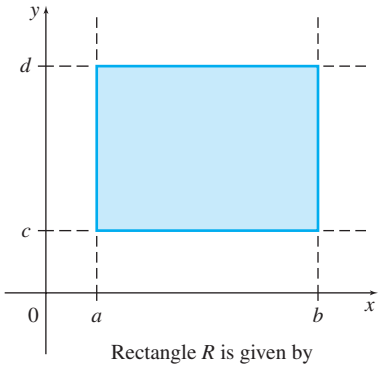
$$\int_1^2 \left[(3x^2y^2 + 4x^3y + 4xy) \Big|_{x=3}^{x=5} \right] dy$$

The brackets we have used for the inner integral in Example 3 are not essential because the order of integration is indicated by the order of $dx\,dy$ or $dy\,dx$. For example, if the integral is written as

$$\int_1^2 \int_3^5 (6xy^2 + 12x^2y + 4y)\, dx\, dy,$$

we first integrate with respect to x, letting x vary from 3 to 5, and then with respect to y, letting y vary from 1 to 2, as in Example 3(a).

The answers in the two parts of Example 3 are equal. It can be proved that for a large class of functions, including most functions that occur in applications, the following equation holds true.



Rectangle R is given by $a \le x \le b, c \le y \le d$.

FIGURE 29

Fubini's Theorem

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Either of these integrals is called an **iterated integral** since it is evaluated by integrating twice, first using one variable and then using the other. The fact that the iterated integrals above are equal makes it possible to define a *double integral*. First, the set of points (x, y), with $a \le x \le b$ and $c \le y \le d$, defines a rectangular region R in the plane, as shown in Figure 29. Then, the *double integral over R* is defined as follows.

Double Integral

The **double integral** of $f(x, y)$ over a rectangular region R is written

$$\iint_R f(x, y)\, dy\, dx \quad \text{or} \quad \iint_R f(x, y)\, dx\, dy,$$

and equals either

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \quad \text{or} \quad \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Extending earlier definitions, $f(x, y)$ is the **integrand** and R is the **region of integration**.

EXAMPLE 4 Double Integrals

Find $\displaystyle\iint_R \frac{3\sqrt{x} \cdot y}{y^2 + 1}\, dx\, dy$ over the rectangular region R defined by $0 \le x \le 4$ and $0 \le y \le 2$.

SOLUTION Integrate first with respect to x; then integrate the result with respect to y.

$$\iint_R \frac{3\sqrt{x}\cdot y}{y^2+1}\,dx\,dy = \int_0^2 \int_0^4 \frac{3\sqrt{x}\cdot y}{y^2+1}\,dx\,dy$$

$$= \int_0^2 \frac{2x^{3/2}\cdot y}{y^2+1}\Big|_0^4 dy \qquad \text{Use the power rule with } x^{1/2}.$$

$$= \int_0^2 \left(\frac{2(4)^{3/2}\cdot y}{y^2+1} - \frac{2(0)^{3/2}\cdot y}{y^2+1}\right) dy$$

$$= 8\int_0^2 \frac{2y}{y^2+1}\,dy \qquad \text{Factor out } 4^{3/2} = 8.$$

$$= 8\int_1^5 \frac{du}{u} \qquad \text{Let } u = y^2+1. \text{ Change limits of integration.}$$

$$= 8 \ln u\Big|_1^5$$

$$= 8 \ln 5 - 8 \ln 1 = 8 \ln 5$$

As a check, integrate with respect to y first. The answer should be the same.

TRY YOUR TURN 3

YOUR TURN 3 Find

$$\iint_R \frac{1}{\sqrt{x+y+3}}\,dx\,dy$$

over the rectangular region R defined by $0 \le x \le 5$ and $1 \le y \le 6$.

Volume

As shown earlier, the definite integral $\int_a^b f(x)\,dx$ can be used to find the area under a curve. In a similar manner, double integrals are used to find the *volume under a surface*. Figure 30 shows that portion of a surface $f(x, y)$ directly over a rectangle R in the xy-plane. Just as areas were approximated by a large number of small rectangles, volume could be approximated by adding the volumes of a large number of properly drawn small boxes. The height of a typical box would be $f(x, y)$ with the length and width given by dx and dy. The formula for the volume of a box would then suggest the following result.

Volume

Let $z = f(x, y)$ be a function that is never negative on the rectangular region R defined by $a \le x \le b$, $c \le y \le d$. The volume of the solid under the graph of f and over the region R is

$$\iint_R f(x, y)\,dx\,dy.$$

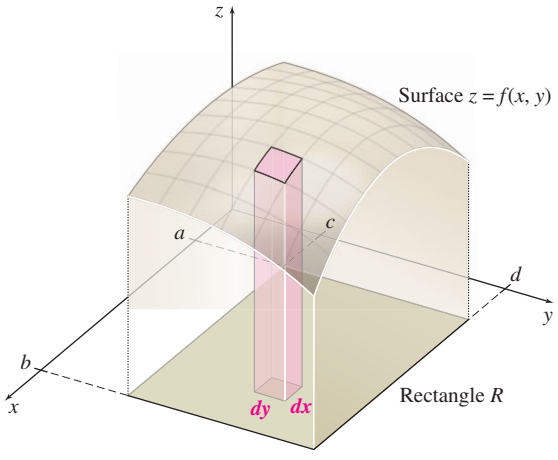


FIGURE 30

EXAMPLE 5 Volume

Find the volume under the surface $z = x^2 + y^2$ shown in Figure 31.

SOLUTION By the equation just given, the volume is

$$\iint_R f(x, y)\, dx\, dy,$$

where $f(x, y) = x^2 + y^2$ and R is the region $0 \le x \le 4$, $0 \le y \le 4$. By definition,

$$\begin{aligned}\iint_R f(x, y)\, dx\, dy &= \int_0^4 \int_0^4 (x^2 + y^2)\, dx\, dy \\ &= \int_0^4 \left(\frac{1}{3}x^3 + xy^2\right)\Big|_0^4 dy \\ &= \int_0^4 \left(\frac{64}{3} + 4y^2\right) dy = \left(\frac{64}{3}y + \frac{4}{3}y^3\right)\Big|_0^4 \\ &= \frac{64}{3} \cdot 4 + \frac{4}{3} \cdot 4^3 - 0 = \frac{512}{3}.\end{aligned}$$

YOUR TURN 4 Find the volume under the surface $z = 4 - x^3 - y^3$ over the rectangular region $0 \le x \le 1$, $0 \le y \le 1$.

TRY YOUR TURN 4

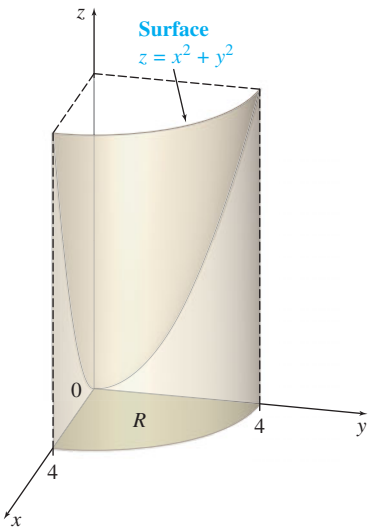


FIGURE 31

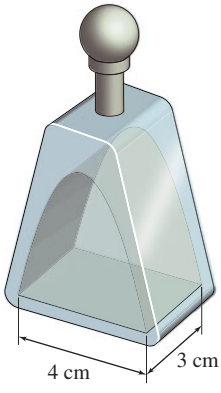


FIGURE 32

EXAMPLE 6 Perfume Bottle

A product design consultant for a cosmetics company has been asked to design a bottle for the company's newest perfume. The thickness of the glass is to vary so that the outside of the bottle has straight sides and the inside has curved sides, with flat ends shaped like parabolas on the 4-cm sides, as shown in Figure 32. Before presenting the design to management, the consultant needs to make a reasonably accurate estimate of the amount each bottle will hold. If the base of the bottle is to be 4 cm by 3 cm, and if a cross section of its interior is to be a parabola of the form $z = -y^2 + 4y$, what is its internal volume?

APPLY IT

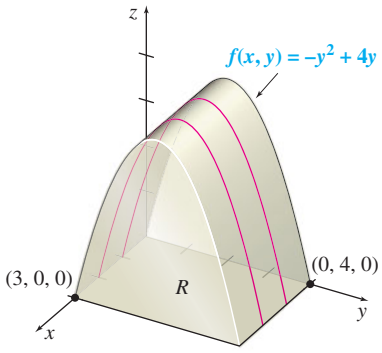


FIGURE 33

SOLUTION The interior of the bottle can be graphed in three-dimensional space, as shown in Figure 33, where $z = 0$ corresponds to the base of the bottle. Its volume is simply the volume above the region R in the xy-plane and below the graph of $f(x, y) = -y^2 + 4y$. This volume is given by the double integral

$$\begin{aligned}\int_0^3 \int_0^4 (-y^2 + 4y)\, dy\, dx &= \int_0^3 \left(\frac{-y^3}{3} + \frac{4y^2}{2} \right) \Big|_0^4 dx \\ &= \int_0^3 \left(\frac{-64}{3} + 32 - 0 \right) dx \\ &= \frac{32}{3} x \Big|_0^3 \\ &= 32 - 0 = 32.\end{aligned}$$

The bottle holds 32 cm^3.

Double Integrals Over Other Regions

In this section, we found double integrals over rectangular regions by evaluating iterated integrals with constant limits of integration. We can also evaluate iterated integrals with *variable* limits of integration. (Notice in the following examples that the variable limits always go on the *inner* integral sign.)

The use of variable limits of integration permits evaluation of double integrals over the types of regions shown in Figure 34. Double integrals over more complicated regions are discussed in more advanced books. Integration over regions such as those in Figure 34 is done with the results of the following theorem.

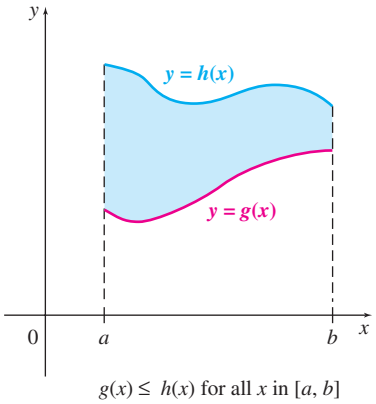
Double Integrals Over Variable Regions

Let $z = f(x, y)$ be a function of two variables. If R is the region (in Figure 34(a)) defined by $a \le x \le b$ and $g(x) \le y \le h(x)$, then

$$\iint_R f(x, y)\, dy\, dx = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)\, dy \right] dx.$$

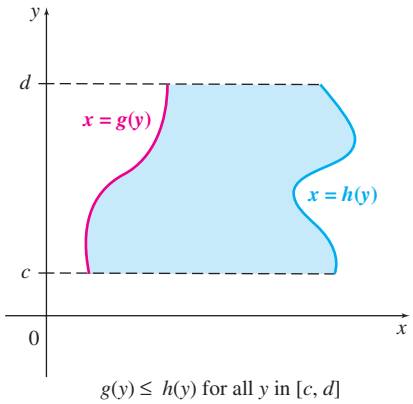
If R is the region (in Figure 34(b)) defined by $g(y) \le x \le h(y)$ and $c \le y \le d$, then

$$\iint_R f(x, y)\, dx\, dy = \int_c^d \left[\int_{g(y)}^{h(y)} f(x, y)\, dx \right] dy.$$



$g(x) \le h(x)$ for all x in $[a, b]$

(a)



$g(y) \le h(y)$ for all y in $[c, d]$

(b)

FIGURE 34

EXAMPLE 7 Double Integrals

Evaluate $\int_1^2 \int_y^{y^2} xy \, dx \, dy$.

SOLUTION The region of integration is shown in Figure 35. Integrate first with respect to x, then with respect to y.

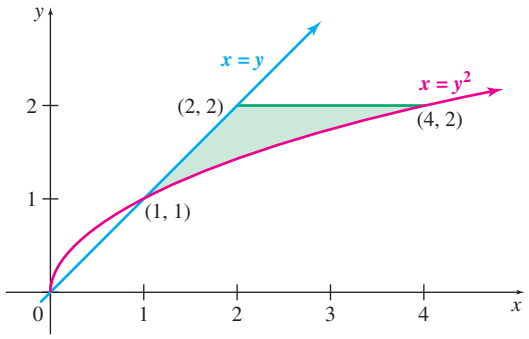


FIGURE 35

$$\int_1^2 \int_y^{y^2} xy \, dx \, dy = \int_1^2 \left(\int_y^{y^2} xy \, dx \right) dy = \int_1^2 \left(\frac{1}{2} x^2 y \right) \Big|_y^{y^2} dy$$

Replace x first with y^2 and then with y, and subtract.

$$\begin{aligned} \int_1^2 \int_y^{y^2} xy \, dx \, dy &= \int_1^2 \left[\frac{1}{2}(y^2)^2 y - \frac{1}{2}(y)^2 y \right] dy \\ &= \int_1^2 \left(\frac{1}{2} y^5 - \frac{1}{2} y^3 \right) dy = \left(\frac{1}{12} y^6 - \frac{1}{8} y^4 \right) \Big|_1^2 \\ &= \left(\frac{1}{12} \cdot 2^6 - \frac{1}{8} \cdot 2^4 \right) - \left(\frac{1}{12} \cdot 1^6 - \frac{1}{8} \cdot 1^4 \right) \\ &= \frac{64}{12} - \frac{16}{8} - \frac{1}{12} + \frac{1}{8} = \frac{27}{8} \end{aligned}$$

YOUR TURN 5 Find $\iint_R (x^3 + 4y) \, dy \, dx$ over the region bounded by $y = 4x$ and $y = x^3$ for $0 \le x \le 2$.

TRY YOUR TURN 5

EXAMPLE 8 Double Integrals

Let R be the shaded region in Figure 36, and evaluate

$$\iint_R (x + 2y) \, dy \, dx.$$

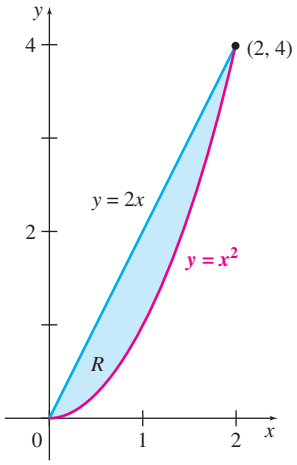


FIGURE 36

SOLUTION Region R is bounded by $h(x) = 2x$ and $g(x) = x^2$, with $0 \le x \le 2$. By the first result in the previous theorem,

$$\begin{aligned} \iint_R (x + 2y) \, dy \, dx &= \int_0^2 \int_{x^2}^{2x} (x + 2y) \, dy \, dx \\ &= \int_0^2 (xy + y^2) \Big|_{x^2}^{2x} dx \\ &= \int_0^2 [x(2x) + (2x)^2 - [x \cdot x^2 + (x^2)^2]] \, dx \end{aligned}$$

$$= \int_0^2 [2x^2 + 4x^2 - (x^3 + x^4)]\,dx$$

$$= \int_0^2 (6x^2 - x^3 - x^4)\,dx$$

$$= \left(2x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5\right)\Big|_0^2$$

$$= 2 \cdot 2^3 - \frac{1}{4} \cdot 2^4 - \frac{1}{5} \cdot 2^5 - 0$$

$$= 16 - 4 - \frac{32}{5} = \frac{28}{5}.$$

Interchanging Limits of Integration

Sometimes it is easier to integrate first with respect to x and then y, while with other integrals the reverse process is easier. The limits of integration can be reversed whenever the region R is like the region in Figure 36, which has the property that it can be viewed as either type of region shown in Figure 34. In practice, this means that all boundaries can be written in terms of y as a function of x, or by solving for x as a function of y.

For instance, in Example 8, the same result would be found if we evaluated the double integral first with respect to x and then with respect to y. In that case, we would need to define the equations of the boundaries in terms of y rather than x, so R would be defined by $y/2 \le x \le \sqrt{y}$, $0 \le y \le 4$. The resulting integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (x + 2y)\,dx\,dy = \int_0^4 \left(\frac{x^2}{2} + 2xy\right)\Big|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \left[\left(\frac{y}{2} + 2y\sqrt{y}\right) - \left(\frac{y^2}{8} + 2\left(\frac{y}{2}\right)y\right)\right] dy$$

$$= \int_0^4 \left(\frac{y}{2} + 2y^{3/2} - \frac{9}{8}y^2\right) dy$$

$$= \left(\frac{y^2}{4} + \frac{4}{5}y^{5/2} - \frac{3}{8}y^3\right)\Big|_0^4$$

$$= 4 + \frac{4}{5} \cdot 4^{5/2} - 24$$

$$= \frac{28}{5}.$$

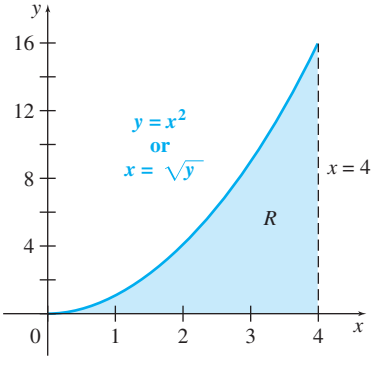


FIGURE 37

EXAMPLE 9 Interchanging Limits of Integration

Evaluate

$$\int_0^{16} \int_{\sqrt{y}}^4 \sqrt{x^3 + 4}\,dx\,dy.$$

SOLUTION Notice that it is impossible to first integrate this function with respect to x. Thus, we attempt to interchange the limits of integration.

For this integral, region R is given by $\sqrt{y} \le x \le 4$, $0 \le y \le 16$. A graph of R is shown in Figure 37.

The same region R can be written in an alternate way. As Figure 37 shows, one boundary of R is $x = \sqrt{y}$. Solving for y gives $y = x^2$. Also, Figure 37 shows that $0 \le x \le 4$. Since R can be written as $0 \le y \le x^2$, $0 \le x \le 4$, the double integral above can be written

$$\begin{aligned}
\int_0^4 \int_0^{x^2} \sqrt{x^3 + 4}\, dy\, dx &= \int_0^4 y\sqrt{x^3 + 4}\Big|_0^{x^2} dx \\
&= \int_0^4 x^2\sqrt{x^3 + 4}\, dx \\
&= \frac{1}{3}\int_0^4 3x^2\sqrt{x^3 + 4}\, dx \qquad \text{Let } u = x^3 + 4. \text{ Change limits of integration.} \\
&= \frac{1}{3}\int_4^{68} u^{1/2} du \\
&= \frac{2}{9}u^{3/2}\Big|_4^{68} \\
&= \frac{2}{9}[68^{3/2} - 4^{3/2}] \\
&\approx 122.83.
\end{aligned}$$

CAUTION Fubini's Theorem cannot be used to interchange the order of integration when the limits contain variables, as in Example 9. Notice in Example 9 that after the order of integration was changed, the new limits were completely different. It would be a serious error to rewrite the integral in Example 9 as $\int_{\sqrt{y}}^4 \int_0^{16} \sqrt{x^3 + 4}\, dy\, dx$.

17.6 EXERCISES

Evaluate each integral.

1. $\int_0^5 (x^4y + y)\, dx$
2. $\int_1^2 (xy^3 - x)\, dy$
3. $\int_4^5 x\sqrt{x^2 + 3y}\, dy$
4. $\int_3^6 x\sqrt{x^2 + 3y}\, dx$
5. $\int_4^9 \frac{3 + 5y}{\sqrt{x}}\, dx$
6. $\int_2^7 \frac{3 + 5y}{\sqrt{x}}\, dy$
7. $\int_2^6 e^{2x+3y}\, dx$
8. $\int_{-1}^1 e^{2x+3y}\, dy$
9. $\int_0^3 ye^{4x+y^2}\, dy$
10. $\int_1^5 ye^{4x+y^2}\, dx$

Evaluate each iterated integral. (Many of these use results from Exercises 1–10.)

11. $\int_1^2 \int_0^5 (x^4y + y)\, dx\, dy$
12. $\int_0^3 \int_1^2 (xy^3 - x)\, dy\, dx$
13. $\int_0^1 \int_3^6 x\sqrt{x^2 + 3y}\, dx\, dy$
14. $\int_0^3 \int_4^5 x\sqrt{x^2 + 3y}\, dy\, dx$
15. $\int_1^2 \int_4^9 \frac{3 + 5y}{\sqrt{x}}\, dx\, dy$
16. $\int_{16}^{25} \int_2^7 \frac{3 + 5y}{\sqrt{x}}\, dy\, dx$
17. $\int_1^3 \int_1^3 \frac{1}{xy}\, dy\, dx$
18. $\int_1^5 \int_2^4 \frac{1}{y}\, dx\, dy$
19. $\int_2^4 \int_3^5 \left(\frac{x}{y} + \frac{y}{3}\right) dx\, dy$
20. $\int_3^4 \int_1^2 \left(\frac{6x}{5} + \frac{y}{x}\right) dx\, dy$

Find each double integral over the rectangular region R with the given boundaries.

21. $\iint_R (3x^2 + 4y)\, dx\, dy; \quad 0 \le x \le 3, 1 \le y \le 4$
22. $\iint_R (x^2 + 4y^3)\, dy\, dx; \quad 1 \le x \le 2, 0 \le y \le 3$
23. $\iint_R \sqrt{x + y}\, dy\, dx; \quad 1 \le x \le 3, 0 \le y \le 1$
24. $\iint_R x^2\sqrt{x^3 + 2y}\, dx\, dy; \quad 0 \le x \le 2, 0 \le y \le 3$
25. $\iint_R \frac{3}{(x + y)^2}\, dy\, dx; \quad 2 \le x \le 4, 1 \le y \le 6$

26. $\iint_R \frac{y}{\sqrt{2x + 5y^2}}\,dx\,dy; \quad 0 \le x \le 2, 1 \le y \le 3$

27. $\iint_R ye^{x+y^2}\,dx\,dy; \quad 2 \le x \le 3, 0 \le y \le 2$

28. $\iint_R x^2e^{x^3+2y}\,dx\,dy; \quad 1 \le x \le 2, 1 \le y \le 3$

Find the volume under the given surface $z = f(x, y)$ and above the rectangle with the given boundaries.

29. $z = 8x + 4y + 10; \ -1 \le x \le 1, 0 \le y \le 3$

30. $z = 3x + 10y + 20; \quad 0 \le x \le 3, -2 \le y \le 1$

31. $z = x^2; \quad 0 \le x \le 2, 0 \le y \le 5$

32. $z = \sqrt{y}; \ 0 \le x \le 4, 0 \le y \le 9$

33. $z = x\sqrt{x^2 + y}; \quad 0 \le x \le 1, 0 \le y \le 1$

34. $z = yx\sqrt{x^2 + y^2}; \quad 0 \le x \le 4, 0 \le y \le 1$

35. $z = \frac{xy}{(x^2 + y^2)^2}; \quad 1 \le x \le 2, 1 \le y \le 4$

36. $z = e^{x+y}; \quad 0 \le x \le 1, 0 \le y \le 1$

Although it is often true that a double integral can be evaluated by using either *dx* or *dy* first, sometimes one choice over the other makes the work easier. Evaluate the double integrals in Exercises 37 and 38 in the easiest way possible.

37. $\iint_R xe^{xy}\,dx\,dy; \quad 0 \le x \le 2, 0 \le y \le 1$

38. $\iint_R 2x^3e^{x^2y}\,dx\,dy; \quad 0 \le x \le 1, 0 \le y \le 1$

Evaluate each double integral.

39. $\int_2^4 \int_2^{x^2} (x^2 + y^2)\,dy\,dx$

40. $\int_0^2 \int_0^{3y} (x^2 + y)\,dx\,dy$

41. $\int_0^4 \int_0^x \sqrt{xy}\,dy\,dx$

42. $\int_1^4 \int_0^x \sqrt{x + y}\,dy\,dx$

43. $\int_2^6 \int_{2y}^{4y} \frac{1}{x}\,dx\,dy$

44. $\int_1^4 \int_x^{x^2} \frac{1}{y}\,dy\,dx$

45. $\int_0^4 \int_1^{e^x} \frac{x}{y}\,dy\,dx$

46. $\int_0^1 \int_{2x}^{4x} e^{x+y}\,dy\,dx$

Use the region *R* with the indicated boundaries to evaluate each double integral.

47. $\iint_R (5x + 8y)\,dy\,dx; \quad 1 \le x \le 3, 0 \le y \le x - 1$

48. $\iint_R (2x + 6y)\,dy\,dx; \quad 2 \le x \le 4, 2 \le y \le 3x$

49. $\iint_R (4 - 4x^2)\,dy\,dx; \quad 0 \le x \le 1, 0 \le y \le 2 - 2x$

50. $\iint_R \frac{1}{x}\,dy\,dx; \quad 1 \le x \le 2, 0 \le y \le x - 1$

51. $\iint_R e^{x/y^2}\,dx\,dy; \quad 1 \le y \le 2, 0 \le x \le y^2$

52. $\iint_R (x^2 - y)\,dy\,dx; \quad -1 \le x \le 1, -x^2 \le y \le x^2$

53. $\iint_R x^3y\,dy\,dx; \quad R$ bounded by $y = x^2, y = 2x$

54. $\iint_R x^2y^2\,dx\,dy; \quad R$ bounded by $y = x, y = 2x, x = 1$

55. $\iint_R \frac{1}{y}\,dy\,dx; \quad R$ bounded by $y = x, y = \frac{1}{x}, x = 2$

56. $\iint_R e^{2y/x}\,dy\,dx; \quad R$ bounded by $y = x^2, y = 0, x = 2$

Evaluate each double integral. If the function seems too difficult to integrate, try interchanging the limits of integration, as in Exercises 37 and 38.

57. $\int_0^{\ln 2} \int_{e^y}^2 \frac{1}{\ln x}\,dx\,dy$

58. $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$

59. Recall from the Volume and Average Value section in the previous chapter that volume could be found with a single integral. In this section volume is found using a double integral. Explain when volume can be found with a single integral and when a double integral is needed.

60. Give an example of a region that cannot be expressed by either of the forms shown in Figure 34. (One example is the disk with a hole in the middle between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ in Figure 10.)

The idea of the average value of a function, discussed earlier for functions of the form $y = f(x)$, can be extended to functions of more than one independent variable. For a function $z = f(x, y)$, the average value of *f* over a region *R* is defined as

$$\frac{1}{A}\iint_R f(x, y)\,dx\,dy,$$

where *A* is the area of the region *R*. Find the average value for each function over the regions *R* having the given boundaries.

61. $f(x, y) = 6xy + 2x; \quad 2 \le x \le 5, 1 \le y \le 3$

62. $f(x, y) = x^2 + y^2; \quad 0 \le x \le 2, 0 \le y \le 3$

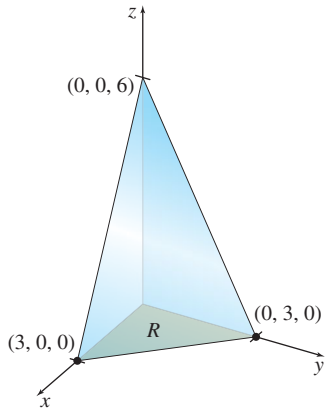
63. $f(x, y) = e^{-5y+3x}; \quad 0 \le x \le 2, 0 \le y \le 2$

64. $f(x, y) = e^{2x+y}; \quad 1 \le x \le 2, 2 \le y \le 3$

APPLICATIONS

Business and Economics

65. Packaging The manufacturer of a fruit juice drink has decided to try innovative packaging in order to revitalize sagging sales. The fruit juice drink is to be packaged in containers in the shape of tetrahedra in which three edges are perpendicular, as shown in the figure on the next page. Two of the perpendicular edges will be 3 in. long, and the third edge will be 6 in. long. Find the volume of the container. (*Hint:* The equation of the plane shown in the figure is $z = f(x, y) = 6 - 2x - 2y$.)



66. Average Cost A company's total cost for operating its two warehouses is

$$C(x, y) = \frac{1}{9}x^2 + 2x + y^2 + 5y + 100$$

dollars, where x represents the number of units stored at the first warehouse and y represents the number of units stored at the second. Find the average cost to store a unit if the first warehouse has between 40 and 80 units, and the second has between 30 and 70 units. (*Hint:* Refer to Exercises 61–64.)

67. Average Production A production function is given by

$$P(x, y) = 500x^{0.2}y^{0.8},$$

where x is the number of units of labor and y is the number of units of capital. Find the average production level if x varies from 10 to 50 and y from 20 to 40. (*Hint:* Refer to Exercises 61–64.)

68. Average Profit The profit (in dollars) from selling x units of one product and y units of a second product is

$$P = -(x - 100)^2 - (y - 50)^2 + 2000.$$

The weekly sales for the first product vary from 100 units to 150 units, and the weekly sales for the second product vary from 40 units to 80 units. Estimate average weekly profit for these two products. (*Hint:* Refer to Exercises 61–64.)

69. Average Revenue A company sells two products. The demand functions of the products are given by

$$q_1 = 300 - 2p_1 \quad \text{and} \quad q_2 = 500 - 1.2p_2,$$

where q_1 units of the first product are demanded at price p_1 and q_2 units of the second product are demanded at price p_2. The total revenue will be given by

$$R = q_1p_1 + q_2p_2.$$

Find the average revenue if the price p_1 varies from \$25 to \$50 and the price p_2 varies from \$50 to \$75. (*Hint:* Refer to Exercises 61–64.)

70. Time In an exercise earlier in this chapter, we saw that the time (in hours) that a branch of Amalgamated Entities needs to spend to meet the quota set by the main office can be approximated by

$$T(x, y) = x^4 + 16y^4 - 32xy + 40,$$

where x represents how many thousands of dollars the factory spends on quality control and y represents how many thousands of dollars they spend on consulting. Find the average time if the amount spent on quality control varies from \$0 to \$4000 and the amount spent on consulting varies from \$0 to \$2000. (*Hint:* Refer to Exercises 61–64.)

71. Profit In an exercise earlier in this chapter, we saw that the profit (in thousands of dollars) that Aunt Mildred's Metalworks earns from producing x tons of steel and y tons of aluminum can be approximated by

$$P(x, y) = 36xy - x^3 - 8y^3.$$

Find the average profit if the amount of steel produced varies from 0 to 8 tons, and the amount of aluminum produced varies from 0 to 4 tons. (*Hint:* Refer to Exercises 61–64.)

YOUR TURN ANSWERS

1. $52x^2 + 16x + 16x^3 + 490$ **2.** $3644/3$

3. $(56\sqrt{14} - 184)/3$ **4.** $7/2$ **5.** $5888/105$

17 CHAPTER REVIEW

SUMMARY

In this chapter, we extended our study of calculus to include functions of several variables. We saw that it is possible to produce three-dimensional graphs of functions of two variables and that the process is greatly enhanced using level curves. Level curves are formed by determining the values of x and y that produce a particular functional value. We also saw the graphs of several surfaces, including the

- paraboloid, whose equation is $z = x^2 + y^2$,
- ellipsoid, whose general equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$,
- hyperbolic paraboloid, whose equation is $z = x^2 - y^2$, and
- hyperboloid of two sheets, whose equation is $-x^2 - y^2 + z^2 = 1$.

Level curves are also important in economics and are used to indicate combinations of the values of x and y that produce the same value of production z. This procedure was used to analyze the Cobb-Douglas production function, which has the general form

$$z = P(x, y) = Ax^ay^{1-a}, \text{ where } A \text{ is constant and } 0 < a < 1.$$

Partial derivatives are the extension of the concept of differentiation with respect to one of the variables while the other variables are held constant. Partial derivatives were used to identify extrema of a function of several variables. In particular, we identified all points where the partial with respect to x and the partial with respect to y are both zero, which we called critical points. We then

classified each critical point as a relative maximum, a relative minimum, or a saddle point. Recall that a saddle point is a minimum when approached from one direction but a maximum when approached from another direction. We introduced the method of Lagrange multipliers to determine extrema in problems with constraints. Differentials, introduced earlier for functions of one variable, were generalized to define the total differential. We saw that total differentials can be used to approximate the value of a function using its tangent plane. We concluded the chapter by introducing double integrals, which are simply two iterated integrals, one for each variable. Double integrals were then used to find volume.

Function of Two Variables

The expression $z = f(x, y)$ is a function of two variables if a unique value of z is obtained from each ordered pair of real numbers (x, y). The variables x and y are independent variables, and z is the dependent variable. The set of all ordered pairs of real numbers (x, y) such that $f(x, y)$ exists is the domain of f; the set of all values of $f(x, y)$ is the range.

Plane

The graph of $ax + by + cz = d$ is a plane if a, b, and c are not all 0.

Partial Derivatives (Informal Definition)

The partial derivative of f with respect to x is the derivative of f obtained by treating x as a variable and y as a constant.

The partial derivative of f with respect to y is the derivative of f obtained by treating y as a variable and x as a constant.

Partial Derivatives (Formal Definition)

Let $z = f(x, y)$ be a function of two independent variables. Let all indicated limits exist. Then the partial derivative of f with respect to x is

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

and the partial derivative of f with respect to y is

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

If the indicated limits do not exist, then the partial derivatives do not exist.

Second-Order Partial Derivatives

For a function $z = f(x, y)$, if the partial derivative exists, then

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = z_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = z_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = z_{xy} \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = z_{yx}$$

Relative Extrema

Let (a, b) be the center of a circular region contained in the xy-plane. Then, for a function $z = f(x, y)$ defined for every (x, y) in the region, $f(a, b)$ is a relative maximum if

$$f(a, b) \geq f(x, y)$$

for all points (x, y) in the circular region, and $f(a, b)$ is a relative minimum if

$$f(a, b) \leq f(x, y)$$

for all points (x, y) in the circular region.

Location of Extrema

Let a function $z = f(x, y)$ have a relative maximum or relative minimum at the point (a, b). Let $f_x(a, b)$ and $f_y(a, b)$ both exist. Then

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

Test for Relative Extrema

For a function $z = f(x, y)$, let f_{xx}, f_{yy}, and f_{xy} all exist in a circular region contained in the xy-plane with center (a, b). Further, let

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

Define D, known as the discriminant, by

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

Then

a. $f(a, b)$ is a relative maximum if $D > 0$ and $f_{xx}(a, b) < 0$;

b. $f(a, b)$ is a relative minimum if $D > 0$ and $f_{xx}(a, b) > 0$;

c. $f(a, b)$ is a saddle point (neither a maximum nor a minimum) if $D < 0$;

d. if $D = 0$, the test gives no information.

Lagrange Multipliers

All relative extrema of the function $z = f(x, y)$, subject to the constraint $g(x, y) = 0$, will be found among those points (x, y) for which there exists a value of λ such that

$$F_x(x, y, \lambda) = 0, F_y(x, y, \lambda) = 0, \text{ and } F_\lambda(x, y, \lambda) = 0,$$

where

$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y),$$

and all indicated partial derivatives exist.

Using Lagrange Multipliers

1. Write the constraint in the form $g(x, y) = 0$.
2. Form the Lagrange function
$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y).$$
3. Find $F_x(x, y, \lambda)$, $F_y(x, y, \lambda)$, and $F_\lambda(x, y, \lambda)$.
4. Form the system of equations
$$F_x(x, y, \lambda) = 0, F_y(x, y, \lambda) = 0, \text{ and } F_\lambda(x, y, \lambda) = 0.$$
5. Solve the system in Step 4; the relative extrema for f are among the solutions of the system.

Total Differential for Two Variables

Let $z = f(x, y)$ be a function of x and y. Let dx and dy be real numbers. Then the total differential of z is

$$dz = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy.$$

(Sometimes dz is written df.)

Approximations

For small values of dx and dy,

$$dz \approx \Delta z$$

where $\Delta z = f(x + dx, y + dy) - f(x, y)$.

Approximations by Differentials

For a function f having all indicated partial derivatives, and for small values of dx and dy,

$$f(x + dx, y + dy) \approx f(x, y) + dz,$$

or

$$f(x + dx, y + dy) \approx f(x, y) + f_x(x, y) \cdot dx + f_y(x, y) \cdot dy.$$

Total Differential for Three Variables

If $w = f(x, y, z)$, then the total differential dw is

$$dw = f_x(x, y, z) \cdot dx + f_y(x, y, z) \cdot dy + f_z(x, y, z) \cdot dz,$$

provided all indicated partial derivatives exist.

Double Integral

The double integral of $f(x, y)$ over a rectangular region R defined by $a \le x \le b$, $c \le y \le d$ is written

$$\iint_R f(x, y)\, dy\, dx \text{ or } \iint_R f(x, y)\, dx\, dy,$$

and equals either

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \text{ or } \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Volume

Let $z = f(x, y)$ be a function that is never negative on the rectangular region R defined by $a \le x \le b$, $c \le y \le d$. The volume of the solid under the graph of f and over the region R is

$$\iint_R f(x, y)\, dy\, dx.$$

Double Integrals over Variable Regions

Let $z = f(x, y)$ be a function of two variables. If R is the region defined by $a \le x \le b$ and $g(x) \le y \le h(x)$, then

$$\iint_R f(x, y)\, dy\, dx = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)\, dy \right] dx.$$

If R is the region defined by $g(y) \le x \le h(y)$ and $c \le y \le d$, then

$$\iint_R f(x, y)\, dx\, dy = \int_c^d \left[\int_{g(y)}^{h(y)} f(x, y)\, dx \right] dy.$$

KEY TERMS

17.1
function of two variables
independent variable
dependent variable
domain
range
ordered triple
first octant
plane
surface
trace
level curves
paraboloid
production function
Cobb-Douglas production function
level surface
ellipsoid
hyperbolic paraboloid
hyperboloid of two sheets

17.2
partial derivative
second-order partial derivative

17.3
relative maximum
relative minimum
saddle point
critical point
discriminant

17.4
constraints
Lagrange multiplier
utility function
indifference curve

17.5
total differential

17.6
Fubini's Theorem
iterated integral
double integral
integrand
region of integration

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. The graph of $6x - 2y + 7z = 14$ is a plane.

2. The graph of $2x + 4y = 10$ is a plane that is parallel to the z-axis.

3. A level curve for a paraboloid could be a single point.

4. If the partial derivatives with respect to x and y at some point are both 0, the tangent plane to the function at that point is horizontal.

5. If $f(x, y) = 3x^2 + 2xy + y^2$, then $f(x + h, y) = 3(x + h)^2 + 2xy + h + y^2$.

6. For a function $z = f(x, y)$, suppose that the point (a, b) has been identified such that $f_x(a, b) = f_y(a, b) = 0$. We can conclude that a relative maximum or a relative minimum must exist at (a, b).

7. A saddle point can be a relative maximum or a relative minimum.

8. A function of two variables may have both a relative maximum and an absolute maximum at the same point.

9. The method of Lagrange multipliers tells us whether a point identified by the method is a maximum or minimum.

10. $\displaystyle\int_2^4\int_1^5 (3x + 4y)\,dy\,dx = \int_2^4\int_1^5 (3x + 4y)\,dx\,dy$

11. $\displaystyle\int_0^1\int_{-2}^2 xe^y\,dy\,dx = \int_{-2}^2\int_0^1 xe^y\,dx\,dy$

12. $\displaystyle\int_0^4\int_1^x (x + xy^2)\,dy\,dx = \int_1^x\int_0^4 (x + xy^2)\,dx\,dy$

PRACTICE AND EXPLORATIONS

13. Describe in words how to take a partial derivative.

14. Describe what a partial derivative means geometrically.

15. Describe what a total differential is and how it is useful.

16. Suppose you are walking through the region of New York state shown in the topographical map in Figure 11 in the first section of this chapter. Assume you are heading north, toward the top of the map, over the western side of the mountain at the left, but not directly over the peak. Explain why you reach your highest point when you are going in the same direction as a contour line. Explain how this relates to Lagrange multipliers. (*Hint:* See Figure 25.)

Find $f(-1, 2)$ and $f(6, -3)$ for the following.

17. $f(x, y) = -4x^2 + 6xy - 3$

18. $f(x, y) = 2x^2y^2 - 7x + 4y$

19. $f(x, y) = \dfrac{x - 2y}{x + 5y}$ **20.** $f(x, y) = \dfrac{\sqrt{x^2 + y^2}}{x - y}$

Graph the first-octant portion of each plane.

21. $x + y + z = 4$ **22.** $x + 2y + 6z = 6$

23. $5x + 2y = 10$ **24.** $4x + 3z = 12$

25. $x = 3$ **26.** $y = 4$

27. Let $z = f(x, y) = 3x^3 + 4x^2y - 2y^2$. Find the following.

a. $\dfrac{\partial z}{\partial x}$ **b.** $\dfrac{\partial z}{\partial y}(-1, 4)$ **c.** $f_{xy}(2, -1)$

28. Let $z = f(x, y) = \dfrac{x + y^2}{x - y^2}$. Find the following.

a. $\dfrac{\partial z}{\partial y}$ **b.** $\dfrac{\partial z}{\partial x}(0, 2)$ **c.** $f_{xx}(-1, 0)$

Find $f_x(x, y)$ and $f_y(x, y)$.

29. $f(x, y) = 6x^2y^3 - 4y$ **30.** $f(x, y) = 5x^4y^3 - 6x^5y$

31. $f(x, y) = \sqrt{4x^2 + y^2}$ **32.** $f(x, y) = \dfrac{2x + 5y^2}{3x^2 + y^2}$

33. $f(x, y) = x^3e^{3y}$ **34.** $f(x, y) = (y - 2)^2e^{x+2y}$

35. $f(x, y) = \ln|2x^2 + y^2|$ **36.** $f(x, y) = \ln|2 - x^2y^3|$

Find $f_{xx}(x, y)$ and $f_{xy}(x, y)$.

37. $f(x, y) = 5x^3y - 6xy^2$

38. $f(x, y) = -3x^2y^3 + x^3y$

39. $f(x, y) = \dfrac{3x}{2x - y}$

40. $f(x, y) = \dfrac{3x + y}{x - 1}$

41. $f(x, y) = 4x^2e^{2y}$

42. $f(x, y) = ye^{x^2}$

43. $f(x, y) = \ln|2 - x^2y|$

44. $f(x, y) = \ln|1 + 3xy^2|$

Find all points where the functions defined below have any relative extrema. Find any saddle points.

45. $z = 2x^2 - 3y^2 + 12y$

46. $z = x^2 + y^2 + 9x - 8y + 1$

47. $f(x, y) = x^2 + 3xy - 7x + 5y^2 - 16y$

48. $z = x^3 - 8y^2 + 6xy + 4$

49. $z = \dfrac{1}{2}x^2 + \dfrac{1}{2}y^2 + 2xy - 5x - 7y + 10$

50. $f(x, y) = 2x^2 + 4xy + 4y^2 - 3x + 5y - 15$

51. $z = x^3 + y^2 + 2xy - 4x - 3y - 2$

52. $f(x, y) = 7x^2 + y^2 - 3x + 6y - 5xy$

53. Describe the different types of points that might occur when $f_x(x, y) = f_y(x, y) = 0$.

Use Lagrange multipliers to find the extrema of the functions defined in Exercises 54 and 55.

54. $f(x, y) = x^2y; \quad x + y = 4$

55. $f(x, y) = x^2 + y^2; \quad x = y - 6$

56. Find positive numbers x and y, whose sum is 80, such that x^2y is maximized.

57. Find positive numbers x and y, whose sum is 75, such that xy^2 is maximized.

58. Notice in the previous two exercises that we specified that x and y must be positive numbers. Does a maximum exist without this requirement? Explain why or why not.

Evaluate dz using the given information.

59. $z = 6x^2 - 7y^2 + 4xy; \quad x = 3, y = -1, dx = 0.03, dy = 0.01$

60. $z = \dfrac{x + 5y}{x - 2y}; \quad x = 1, y = -2, dx = -0.04, dy = 0.02$

Use the total differential to approximate each quantity. Then use a calculator to approximate the quantity, and give the absolute value of the difference in the two results to 4 decimal places.

61. $\sqrt{5.1^2 + 12.05^2}$

62. $\sqrt{4.06}\, e^{0.04}$

Evaluate the following.

63. $\displaystyle\int_1^4 \frac{4y - 3}{\sqrt{x}}\, dx$

64. $\displaystyle\int_1^5 e^{3x+5y}\, dx$

65. $\displaystyle\int_0^5 \frac{6x}{\sqrt{4x^2 + 2y^2}}\, dx$

66. $\displaystyle\int_1^3 \frac{y^2}{\sqrt{7x + 11y^3}}\, dy$

Evaluate each iterated integral.

67. $\displaystyle\int_0^2 \int_0^4 (x^2y^2 + 5x)\, dx\, dy$

68. $\displaystyle\int_0^3 \int_0^5 (2x + 6y + y^2)\, dy\, dx$

69. $\displaystyle\int_3^4 \int_2^5 \sqrt{6x + 3y}\, dx\, dy$

70. $\displaystyle\int_1^2 \int_3^5 e^{2x-7y}\, dx\, dy$

71. $\displaystyle\int_2^4 \int_2^4 \frac{1}{y}\, dx\, dy$

72. $\displaystyle\int_1^2 \int_1^2 \frac{1}{x}\, dx\, dy$

Find each double integral over the region R with boundaries as indicated.

73. $\displaystyle\iint_R (x^2 + 2y^2)\, dx\, dy; \quad 0 \le x \le 5, 0 \le y \le 2$

74. $\displaystyle\iint_R \sqrt{2x + y}\, dx\, dy; \quad 1 \le x \le 3, 2 \le y \le 5$

75. $\displaystyle\iint_R \sqrt{y + x}\, dx\, dy; \quad 0 \le x \le 7, 1 \le y \le 9$

76. $\displaystyle\iint_R ye^{y^2+x}\, dx\, dy; \quad 0 \le x \le 1, 0 \le y \le 1$

Find the volume under the given surface $z = f(x, y)$ and above the given rectangle.

77. $z = x + 8y + 4; \quad 0 \le x \le 3, 1 \le y \le 2$

78. $z = x^2 + y^2; \quad 3 \le x \le 5, 2 \le y \le 4$

Evaluate each double integral. If the function seems too difficult to integrate, try interchanging the limits of integration.

79. $\displaystyle\int_0^1 \int_0^{2x} xy\, dy\, dx$

80. $\displaystyle\int_1^2 \int_2^{2x^2} y\, dy\, dx$

81. $\displaystyle\int_0^1 \int_{x^2}^{x} x^3y\, dy\, dx$

82. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} x\, dx\, dy$

83. $\displaystyle\int_0^2 \int_{x/2}^1 \frac{1}{y^2 + 1}\, dy\, dx$

84. $\displaystyle\int_0^8 \int_{x/2}^4 \sqrt{y^2 + 4}\, dy\, dx$

Use the region R, with boundaries as indicated, to evaluate the given double integral.

85. $\displaystyle\iint_R (2x + 3y)\, dx\, dy; \quad 0 \le y \le 1, y \le x \le 2 - y$

86. $\displaystyle\iint_R (2 - x^2 - y^2)\, dy\, dx; \quad 0 \le x \le 1, x^2 \le y \le x$

APPLICATIONS

Business and Economics

87. Charge for Auto Painting The charge (in dollars) for painting a sports car is given by

$$C(x, y) = 4x^2 + 5y^2 - 4xy + \sqrt{x},$$

where x is the number of hours of labor needed and y is the number of gallons of paint and sealant used. Find the following.

a. The charge for 10 hours and 5 gal of paint and sealant

b. The charge for 15 hours and 10 gal of paint and sealant

c. The charge for 20 hours and 20 gal of paint and sealant

88. Manufacturing Costs The manufacturing cost (in dollars) for a certain computer is given by

$$c(x, y) = 2x + y^2 + 4xy + 25,$$

where x is the memory capacity of the computer in gigabytes (GB) and y is the number of hours of labor required. For 640 GB and 6 hours of labor, find the following.

a. The approximate change in cost for an additional 1 GB of memory

b. The approximate change in cost for an additional hour of labor

89. Productivity The production function z for one country is

$$z = x^{0.7}y^{0.3},$$

where x represents the amount of labor and y the amount of capital. Find the marginal productivity of the following.

a. Labor **b.** Capital

90. Cost The cost (in dollars) to manufacture x solar cells and y solar collectors is

$$c(x, y) = x^2 + 5y^2 + 4xy - 70x - 164y + 1800.$$

a. Find values of x and y that produce minimum total cost.

b. Find the minimum total cost.

Utility Maximize each of the following utility functions, with the cost of each commodity and total amount available to spend given.

91. $f(x,y) = xy^3$, cost of a unit of x is \$2, cost of a unit of y is \$4, and \$80 is available.

92. $f(x,y) = x^5y^2$, cost of a unit of x is \$10, cost of a unit of y is \$6, and \$42 is available.

93. Cost The cost (in dollars) to produce x satellite receiving dishes and y transmitters is given by

$$C(x, y) = 100 \ln(x^2 + y) + e^{xy/20}.$$

Production schedules now call for 15 receiving dishes and 9 transmitters. Use differentials to approximate the change in costs if 1 more dish and 1 fewer transmitter are made.

94. Production Materials Approximate the volume of material needed to manufacture a cone of radius 2 cm, height 8 cm, and wall thickness 0.21 cm.

95. Production Materials A sphere of radius 2 ft is to receive an insulating coating 1 in. thick. Approximate the volume of the coating needed.

96. Production Error The height of a sample cone from a production line is measured as 11.4 cm, while the radius is measured as 2.9 cm. Each of these measurements could be off by 0.2 cm. Approximate the maximum possible error in the volume of the cone.

97. Profit The total profit from 1 acre of a certain crop depends on the amount spent on fertilizer, x, and on hybrid seed, y, according to the model

$$P(x, y) = 0.01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600).$$

The budget for fertilizer and seed is limited to \$280.

a. Use the budget constraint to express one variable in terms of the other. Then substitute into the profit function to get a function with one independent variable. Use the method shown in Chapter 14 on Applications of the Derivative to find the amounts spent on fertilizer and seed that will maximize profit. What is the maximum profit per acre? (*Hint:* Throughout this exercise you may ignore the coefficient of 0.01 until you need to find the maximum profit.)

b. Find the amounts spent on fertilizer and seed that will maximize profit using the first method shown in this chapter. (*Hint:* You will not need to use the budget constraint.)

c. Use the Lagrange multiplier method to solve the original problem.

d. Look for the relationships among these methods.

Life Sciences

98. Blood Vessel Volume A length of blood vessel is measured as 2.7 cm, with the radius measured as 0.7 cm. If each of these measurements could be off by 0.1 cm, estimate the maximum possible error in the volume of the vessel.

99. Total Body Water Accurate prediction of total body water is critical in determining adequate dialysis doses for patients with renal disease. For African American males, total body water can be estimated by the function

$$T(A, M, S) = -18.37 - 0.09A + 0.34M + 0.25S,$$

where T is the total body water (in liters), A is age (in years), M is mass (in kilograms), and S is height (in centimeters). ***Source: Kidney International.***

a. Find $T(65, 85, 180)$.

b. Find and interpret $T_A(A, M, S)$, $T_M(A, M, S)$, and $T_S(A, M, S)$.

100. Brown Trout Researchers from New Zealand have determined that the length of a brown trout depends on both its mass and age and that the length can be estimated by

$$L(m, t) = (0.00082t + 0.0955)e^{(\ln m + 10.49)/2.842},$$

where $L(m, t)$ is the length of the trout (in centimeters), m is the mass of the trout (in grams), and t is the age of the trout (in years). ***Source: Transactions of the American Fisheries Society.***

a. Find $L(450, 4)$.

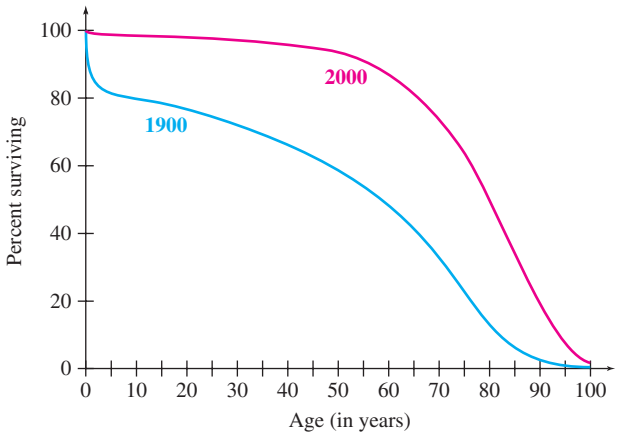
b. Find $L_m(450, 7)$ and $L_t(450, 7)$ and interpret.

101. Survival Curves The following figure shows survival curves (percent surviving as a function of age) for people in the United States in 1900 and 2000. ***Source: National Vital Statistics Report.*** Let $f(x, y)$ give the proportion surviving at

age x in year y. Use the graph to estimate the following. Interpret each answer in words.

a. $f(60, 1900)$ **b.** $f(70, 2000)$

c. $f_x(60, 1900)$ **d.** $f_x(70, 2000)$



General Interest

102. Area The bottom of a planter is to be made in the shape of an isosceles triangle, with the two equal sides 3 ft long and the third side 2 ft long. The area of an isosceles triangle with two equal sides of length a and third side of length b is

$$f(a, b) = \frac{1}{4}b\sqrt{4a^2 - b^2}.$$

a. Find the area of the bottom of the planter.

b. The manufacturer is considering changing the shape so that the third side is 2.5 ft long. What would be the approximate effect on the area?

103. Surface Area A closed box with square ends must have a volume of 125 in^3. Use Lagrange multipliers to find the dimensions of such a box that has minimum surface area.

104. Area Use Lagrange multipliers to find the maximum rectangular area that can be enclosed with 400 ft of fencing, if no fencing is needed along one side.

EXTENDED APPLICATION

USING MULTIVARIABLE FITTING TO CREATE A RESPONSE SURFACE DESIGN

Suppose you are designing a flavored drink with orange and banana flavors. You want to find the ideal concentrations of orange and banana flavoring agents, but since the concentrations could range from 0% to 100%, you can't try every possibility. A common design technique in the food industry is to make up several test drinks using different combinations of flavorings and have them rated for taste appeal by a panel of tasters. Such ratings are called *hedonic responses* and are often recorded on a 10-point scale from 0 (worst) to 9 (best). One combination will most likely get the highest average score, but since you have only tried a few of the infinite number of flavor combinations, the winning combination on the taste test might be far from the mix that would be the most popular in the market. How can you use the information from your test to locate the best point on the *flavor plane*?

One approach to this problem uses *response surfaces*, three-dimensional surfaces that approximate the data points from your flavor test.* For your test, you might choose mixtures that are spread out over the flavor plane. For example, you could combine low, medium, and high orange with low, medium, and high banana to get 9 different flavors. If you had 15 tasters and used intensities of 20, 50, and 80 for each fruit, the test data might look like the table.

Average Hedonic Scores ($n = 15$) Banana Intensity (0 to 100)				
		20	*50*	*80*
Orange	20	3.2	4.9	2.8
Intensity	50	6.0	7.2	5.1
0 to 100	80	4.5	5.5	4.8

For example, the table shows that the drink with orange intensity 20 and banana intensity 80 got an average flavor rating of 2.8 from the test panel (they didn't like it).

Your test results are points in space, where you can think of the x-axis as the orange axis, the y-axis as banana, and the z-axis as taste score. A three-dimensional bar chart is a common way of displaying data of this kind. Figure 38 on the next page is a bar chart of the flavor test results.

Looking at the bar chart, we can guess that the best flavor mix will be somewhere near the middle. We'd like to "drape" a smooth surface over the bars and see where that surface has a maximum. But as with any sample, our tasters are not perfectly representative

*For a brief introduction to response surfaces, see Devore, Jay L. and Nicholas R. Farnum, *Applied Statistics for Engineers and Scientists*, Duxbury, 2004.

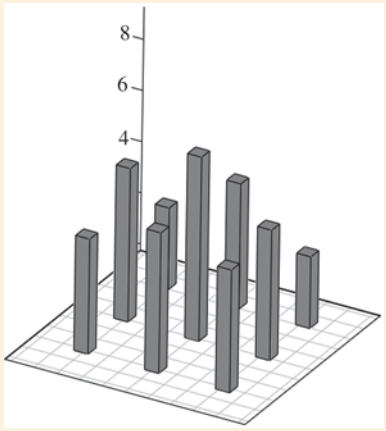


FIGURE 38

of the whole population: Our test results give the general shape of the true population response, but each bar includes an error that results from our small sample size. The solution is to fit a *smooth* surface to the data points.

A simple type of function for modeling such data sets is a quadratic function. You've seen many quadratic functions of two variables in the examples and exercises for this chapter, and you know that they can have maxima, minima, and saddle points. We don't know in advance which quadratic shape will give us the best fit, so we'll use the most general quadratic,

$$G(x, y) = Ax^2 + By^2 + Cxy + Dx + Ey + F.$$

Our job is to find the six coefficients, A through F, that give the best fit to our nine data points. As with the least squares line formula you used in Section 1.3, there are formulas for these six coefficients. Most statistical software packages will generate them directly from your data set, and here is the best-fitting quadratic found by one such program:

$$G(x, y) = -0.00202x^2 - 0.00163y^2 + 0.000194xy + 0.21380x + 0.14768y - 2.36204.$$

In this case the response surface shows how the dependent variable, taste rating, *responds* to the two independent variables, orange and banana intensity. Figures 39 and 40 are two views of

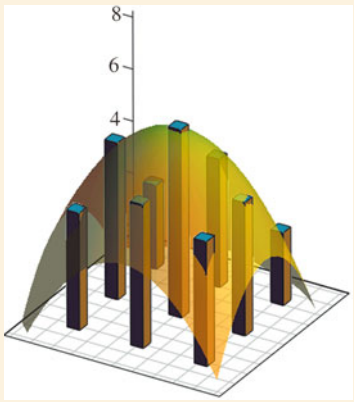


FIGURE 39

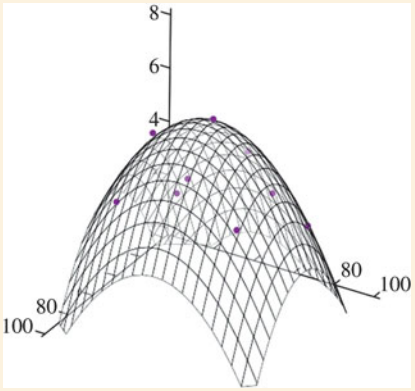


FIGURE 40

the surface together with the data: a surface superimposed on the bar chart, and the same surface with the data shown as points in space.

In research papers, response surface models are often reported using level curves. A contour map for the surface we have found looks like Figure 41, with orange increasing from left to right and banana from bottom to top.

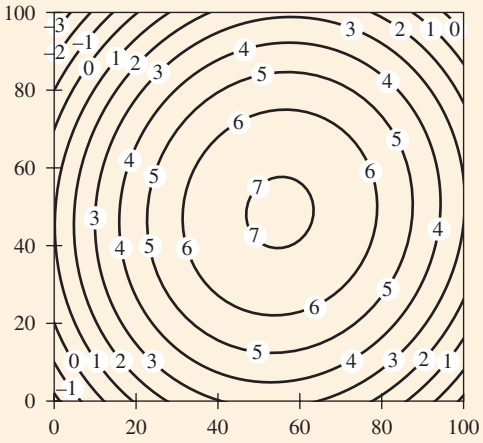


FIGURE 41

It's quite easy to estimate the location of the maximum by marking a point in the middle of the central ellipse and finding its coordinates on the two axes (try it!). You can also use the techniques you learned in the section on Maxima and Minima, computing partial derivatives of G with respect to x and y and solving the resulting linear system. The numbers are awkward, but with some help from a calculator you'll find that the maximum occurs at approximately $(55.3, 48.6)$. So the quadratic model predicts that the most popular drink will have an orange concentration of 55.3 and a banana concentration of 48.6. The model also predicts the public's flavor rating for this drink: We would expect it to be $G(55.3, 48.6)$, which turns

out to be about 7.14. When food technologists design a new food, this kind of modeling is often a first step. The next step might be to make a new set of test drinks with concentrations clustered around the point $(55.3, 48.6)$, and use further tests to explore this region of the "flavor plane" in greater detail.

Response surfaces are also helpful for constrained optimization. In the section on Lagrange Multipliers, you saw how Lagrange multipliers could solve problems of the form:

Find the relative extrema for $z = f(x, y)$,
subject to the constraint $g(x, y) = 0$.

Sometimes the constraints have a different form: You may have *several* dependent variables that respond to the same inputs, and the design goal is to keep each variable *within a given range*. Here's an example based on the data in U.S. Patent No. 4,276,316, which is titled *Process for Treating Nuts*. ***Source: United States Patent and Trademark Office***. The patent granted to researcher Shri C. Sharma and assigned to CPC International, Inc. covers a method for preparing nuts for blanching (that is, having their skins removed). The patent summary reads in part:

> *The nuts are heated with a gas at a temperature of* 125° *to* 175°C *for 30 to 180 seconds and then immediately cooled to below* 35°C *within 5 minutes prior to blanching. This provides improved blanching, sorting and other steps in a process for producing products ranging from nuts per se to peanut butters or spreads.*

In support of the effectiveness of the method, the patent offers data that describe the effects of nine different combinations of air temperature and treatment time on three variables of interest for blanched peanuts: blanching efficiency, roasted peanut flavor, and overall flavor. Efficiency is given in percent, and the two hedonic variables were rated by tasters on a scale of 0 to 9.

The time variable has been converted into a natural logarithm because treatment time effects typically scale with the log of the time. The problem is now to pick a temperature and time range that give the optimum combination of efficiency, roasted flavor, and overall flavor. Each of these three dependent variables responds to the inputs in a different way, and the patent documentation includes quadratic response surfaces for each variable. The lighter shading in Figures 42–44 indicates higher values, which are more

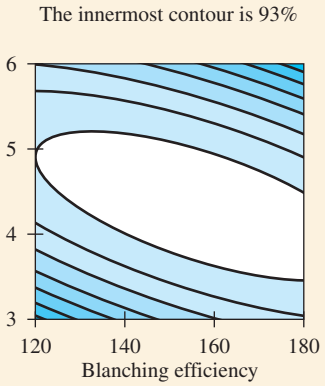


FIGURE 42

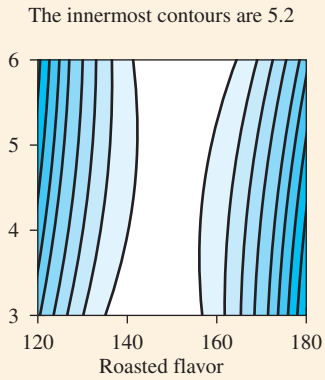


FIGURE 43

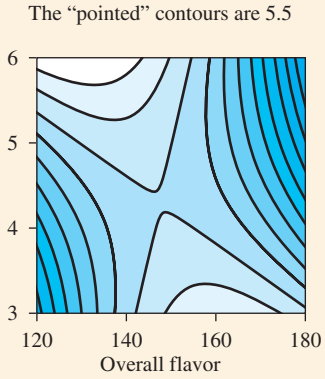


FIGURE 44

Air Temperature, °C	Treatment Time, Seconds	Log of Treatment Time	Blanching Efficiency	Roasted Peanut Flavor	Overall Flavor
138	45	3.807	93.18	4.94	5.51
160	120	4.787	94.99	5.24	5.37
149	75	4.317	98.43	5.27	5.10
138	120	4.787	96.42	5.05	5.71
160	45	3.807	96.48	5.17	5.62
127	75	4.317	93.56	4.64	5.04
149	180	5.193	94.99	5.24	5.37
149	30	3.401	87.30	5.43	5.44
171	45	3.807	94.40	4.37	5.18

desirable. Temperature is plotted across the bottom, and the log of treatment time increases from bottom to top.

Sometimes process designers faced with this kind of problem will combine the dependent variables into a single function by taking a weighted average of their values, and then use a single response surface to optimize this function. Here we look at a different scenario. Suppose we set the following process goals: We want blanching efficiency of at least 93%, a roasted flavor rating of at least 5.2, and an overall flavor rating of at least 5.5. Is there a combination of time and temperature that meets these criteria? If so, what is it?

The first step is to identify the "successful" area on each response surface, which we can do by shading the corresponding region in the contour plot, shown in Figures 45–47.

Now the strategy is clear: We want to stack the three plots on top of each other and see if the shaded regions overlap. Figure 48 is the result.

So we can see that there are two regions on the temperature– time plane that will work. For example, the upper area of overlap suggests a processing temperature of 140°C to 150°C, with a processing time between 90 and 150 seconds (remember that the numbers on the vertical axis are *natural logarithms* of the time in seconds).

Response surfaces are a standard tool in designing everything from food to machine parts, and we have touched on only a small part of the theory here. Frequently a process depends on more than two independent variables. For example, a soft-drink formula might include three flavorings, an acidifying agent, and a sweetener. The response "surface" now lives in six dimensions and we can no longer draw nice pictures, but the same multivariable mathematics that generated our quadratic response surfaces will lead us to the optimal combination of variables.

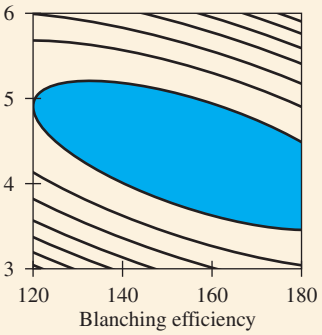


FIGURE 45

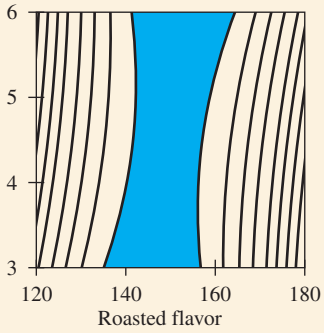


FIGURE 46

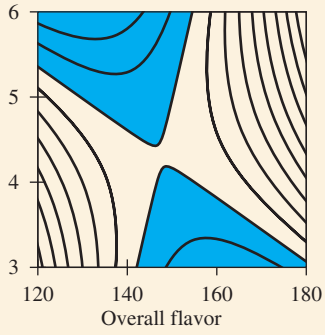


FIGURE 47

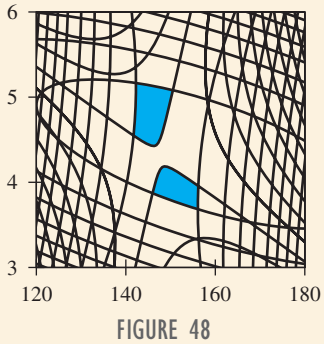


FIGURE 48

EXERCISES

1. The general quadratic function of two variables has six terms. How many terms are in the general cubic function of two variables?
2. Use the contour plot of orange-banana flavor to estimate the "flavor coordinates" of the best-tasting drink.
3. Find the maximum on the flavor response surface by finding the critical point of the function $G(x, y)$.
4. Without shading or numbers on the contours, how would you know that the point you found in Exercises 2 and 3 represents the best flavor rather than the worst flavor? (*Hint:* Compute the discriminant D as described in the section on Maxima and Minima.)
5. Our best drink has a predicted flavor rating of 7.14, but one of our test drinks got a *higher* rating, 7.2. What's going on?
6. Blanching efficiency has a maximum near the center of the temperature–time plane. What is going on near the center of the plane for the roasted flavor and overall flavor response surfaces? Within the domain plotted, where does overall flavor reach a maximum?
7. In the overall flavor contour plot, if we move one contour toward higher flavor from the "pointed" 5.5 contours, we find curved contours that represent an overall flavor rating of about 5.6. If instead of requiring an overall flavor rating of 5.5 we decided to require a rating of 5.6, what would happen to our process design?
8. Use the last figure to describe the other region in the temperature–time plane that delivers a successful process for preparing nuts for blanching.
9. At the website WolframAlpha.com, you can enter "maximize" followed by an expression $f(x,y)$, such as $4 - x^2 - y^2$, and you will be told the maximum value of the expression, as well as the point where the maximum occurs. You will also be shown a graph of the surface $z = f(x,y)$ and a contour plot. Try this for the function $G(x,y)$ given in this Extended Application, and compare with your answers to Exercises 2 and 3.

DIRECTIONS FOR GROUP PROJECT

Perform an experiment that is similar to the flavored drink example from the text on some other product. For example, you could perform an experiment where you develop hedonic responses for various levels of salt and butter on popcorn. Using technology, to the extent that it is available to you, carry out the analysis of your experiment to determine an optimal mixture of each ingredient.

18 Probability and Calculus

18.1 Continuous Probability Models

18.2 Expected Value and Variance of Continuous Random Variables

18.3 Special Probability Density Functions

Chapter 18 Review

Extended Application: Exponential Waiting Times

Though earthquakes may appear to strike at random, the times between quakes can be modeled with an exponential density function. Such *continuous probability models* have many applications in science, engineering, and medicine. In an exercise in Section 1 of this chapter we'll use an exponential density function to describe the times between major earthquakes in Southern California, and in Section 3 we will compute the mean and standard deviation for this distribution.

In recent years, probability has become increasingly useful in fields ranging from manufacturing to medicine, as well as in all types of research. The foundations of probability were laid in the seventeenth century by Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665), who investigated *the problem of the points*. This problem dealt with the fair distribution of winnings in an interrupted game of chance between two equally matched players whose scores were known at the time of the interruption.

Probability has advanced from a study of gambling to a well-developed, deductive mathematical system. In this chapter we give a brief introduction to the use of calculus in probability.

18.1 Continuous Probability Models

APPLY IT **What is the probability that there is a bird's nest within 0.5 kilometers of a given point?**

In Example 3, we will answer the question posed above.

In this section, we show how calculus is used to find the probability of certain events. Before discussing probability, however, we need to introduce some new terminology.

Suppose that a bank is studying the transaction times of its tellers. The lengths of time spent on observed transactions, rounded to the nearest minute, are shown in the following table.

Frequency of Transaction Times											
Time	1	2	3	4	5	6	7	8	9	10	
Frequency	3	5	9	12	15	11	10	6	3	1	(Total: 75)

The table shows, for example, that 9 of the 75 transactions in the study took 3 minutes, 15 transactions took 5 minutes, and 1 transaction took 10 minutes. Because the time for any particular transaction is a random event, the number of minutes for a transaction is called a **random variable**. The frequencies can be converted to probabilities by dividing each frequency by the total number of transactions (75) to get the results shown in the next table.*

Probability of Transaction Times										
Time	1	2	3	4	5	6	7	8	9	10
Probability	0.04	0.07	0.12	0.16	0.20	0.15	0.13	0.08	0.04	0.01

Because each value of the random variable is associated with just one probability, this table defines a function. Such a function is called a **probability function**, and it has the following special properties.

*One definition of the *probability of an event* is the number of outcomes that favor the event divided by the total number of equally likely outcomes in an experiment.

Probability Function of a Random Variable

If the function f is a probability function with domain $\{x_1, x_2, \ldots, x_n\}$, and $f(x_i)$ is the probability that event x_i occurs, then for $1 \le i \le n$,

$$\mathbf{0 \le f(x_i) \le 1},$$

and

$$\mathbf{f(x_1) + f(x_2) + \cdots + f(x_n) = 1}.$$

Note that $f(x_i) = 0$ implies that event x_i will not occur and $f(x_i) = 1$ implies that event x_i will occur.

The information in the second table can be displayed graphically with a special kind of bar graph called a **histogram**. The bars of a histogram have the same width, and their heights are determined by the probabilities of the various values of the random variable. See Figure 1.

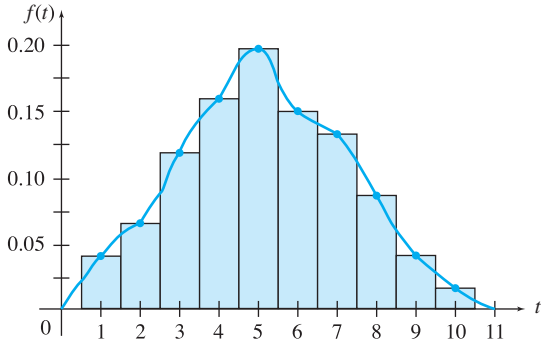


FIGURE 1

The probability function in the second table is a **discrete probability function** because it has a finite domain—the integers from 1 to 10, inclusive. A discrete probability function has a finite domain or an infinite domain that can be listed. For example, if we flip a coin until we get heads, and let the random variable be the number of flips, then the domain is $1, 2, 3, 4, \ldots$. On the other hand, the distribution of heights (in inches) of college women includes infinitely many possible measurements, such as 53, 54.2, 66.5, $72.\overline{3}$, and so on, *within some real number interval*. Probability functions with such domains are called *continuous probability distributions*.

Continuous Probability Distribution

A **continuous random variable** can take on any value in some interval of real numbers. The distribution of this random variable is called a **continuous probability distribution**.

Some probability functions are inherently discrete. For example, the number of houses that a real estate agent sells in a year must be an integer, such as 0, 1, or 2, and could never take on any value in between. But the bank example discussed earlier is different, because you could think of it as a simplification of a continuous distribution. It would be possible to time the teller transactions with greater precision—to the nearest tenth of a minute, or even to the nearest 1/60 of a minute if desired. Theoretically, at least, t could take on any positive real-number value between, say, 0 and 11 minutes. The graph of the probabilities $f(t)$ of these transaction times can be thought of as the continuous curve shown in Figure 1. As indicated in Figure 1, the curve was derived from our table by connecting the points at the tops of the bars in the corresponding histogram and smoothing the resulting polygon into a curve.

To clarify some concepts in probability, we will follow the common convention of using capital letters to indicate random variables and lower case letters to indicate the values that the random variables take on. For example, to indicate the probability that a random variable takes on the value 2, we will write $P(X = 2)$. To indicate the probability that a random variable takes on the arbitrary value x, we will write $P(X = x)$.

For a discrete probability function, the area of each bar (or rectangle) gives the probability of a particular transaction time. Thus, by considering the possible transaction times T as all the real numbers between 0 and 11, the area under the curve of Figure 2 between any two values of T can be interpreted as the probability that a transaction time will be between those two numbers. For example, the shaded region in Figure 2 corresponds to the probability that T is between a and b, written $P(a \le T \le b)$.

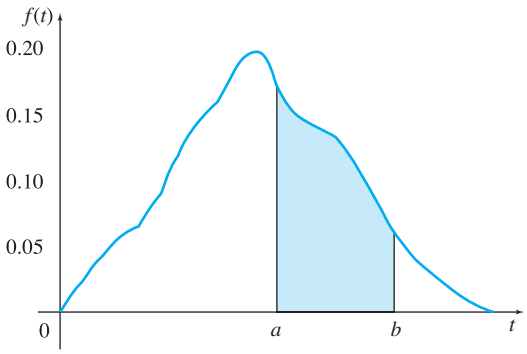


FIGURE 2

FOR REVIEW

The connection between area and the definite integral is discussed in Chapter 15 on Integration. For example, in that chapter we solved such problems as the following:

Find the area between the x-axis and the graph of $f(x) = x^2$ from $x = 1$ to $x = 4$.

Answer: $\int_1^4 x^2\,dx = 21$

It was shown earlier that the definite integral of a continuous function f, where $f(x) \ge 0$, gives the area under the graph of $f(x)$ from $x = a$ to $x = b$. If a function f can be found to describe a continuous probability distribution, then the definite integral can be used to find the area under the curve from a to b that represents the probability that x will be between a and b.

If X is a continuous random variable whose distribution is described by the function f on $[a, b]$, then

$$P(a \le X \le b) = \int_a^b f(x)\,dx.$$

Probability Density Functions

A function f that describes a continuous probability distribution is called a *probability density function*. Such a function must satisfy the following conditions.

Probability Density Function

The function f is a **probability density function** of a random variable X in the interval $[a, b]$ if

1. $f(x) \ge 0$ for all x in the interval $[a, b]$; and
2. $\int_b^a f(x)\,dx = 1$.

Intuitively, Condition 1 says that the probability of a particular event can never be negative. Condition 2 says that the total probability for the interval must be 1; *something* must happen.

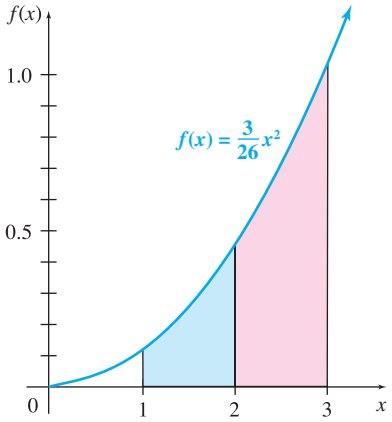


FIGURE 3

YOUR TURN 1 Repeat Example 1(a) for the function $f(x) = 2/x^2$ on $[1, 2]$. What is the probability that X will be between $3/2$ and 2?

EXAMPLE 1 Probability Density Function

(a) Show that the function defined by $f(x) = (3/26)x^2$ is a probability density function for the interval $[1, 3]$.

SOLUTION First, note that Condition 1 holds; that is, $f(x) \geq 0$ for the interval $[1, 3]$. Next show that Condition 2 holds.

$$\int_1^3 \frac{3}{26}x^2 dx = \frac{3}{26}\left(\frac{x^3}{3}\right)\Big|_1^3 = \frac{3}{26}\left(9 - \frac{1}{3}\right) = 1$$

Since both conditions hold, $f(x)$ is a probability density function.

(b) Find the probability that X will be between 1 and 2.

SOLUTION The desired probability is given by the area under the graph of $f(x)$ between $x = 1$ and $x = 2$, as shown in blue in Figure 3. The area is found by using a definite integral.

$$P(1 \leq X \leq 2) = \int_1^2 \frac{3}{26}x^2\, dx = \frac{3}{26}\left(\frac{x^3}{3}\right)\Big|_1^2 = \frac{7}{26}$$ TRY YOUR TURN 1

Earlier, we noted that determining a suitable function is the most difficult part of applying mathematics to actual situations. Sometimes a function appears to model an application well but does not satisfy the requirements for a probability density function. In such cases, we may be able to change the function into a probability density function by multiplying it by a suitable constant, as shown in the next example.

EXAMPLE 2 Probability Density Function

Is there a constant k such that $f(x) = kx^2$ is a probability density function for the interval $[0, 4]$?

SOLUTION First,

$$\int_0^4 kx^2\, dx = \frac{kx^3}{3}\Big|_0^4 = \frac{64k}{3}.$$

The integral must be equal to 1 for the function to be a probability density function. To convert it to one, let $k = 3/64$. The function defined by $(3/64)x^2$ for $[0, 4]$ will be a probability density function, since $(3/64)x^2 \geq 0$ for all x in $[0, 4]$ and

$$\int_0^4 \frac{3}{64}x^2 = 1.$$ TRY YOUR TURN 2

YOUR TURN 2 Repeat Example 2 for the function $f(x) = kx^3$ on the interval $[0, 4]$.

An important distinction is made between a discrete probability function and a probability density function (which is continuous). In a discrete distribution, the probability that the random variable, X, will assume a specific value is given in the distribution for every possible value of X. In a probability density function, however, the probability that X equals a specific value, say, c, is

$$P(X = c) = \int_c^c f(x)\, dx = 0.$$

For a probability density function, only probabilities of *intervals* can be found. For example, suppose the random variable is the annual rainfall for a given region. The amount of rainfall in one year can take on any value within some continuous interval that depends on the region; however, the probability that the rainfall in a given year will be some specific amount, say 33.25 in., is actually zero.

The definition of a probability density function is extended to intervals such as $(-\infty, b]$, $(-\infty, b)$, $[a, \infty)$, (a, ∞), or $(-\infty, \infty)$ by using improper integrals, as follows.

Probability Density Functions on $(-\infty, \infty)$

If f is a probability density function for a continuous random variable X on $(-\infty, \infty)$, then

$$P(X \le b) = P(X < b) = \int_{-\infty}^{b} f(x)\,dx,$$

$$P(X \ge a) = P(X > a) = \int_{a}^{\infty} f(x)\,dx,$$

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

The total area under the graph of a probability density function of this type must still equal 1.

EXAMPLE 3 Location of a Bird's Nest

Suppose the random variable X is the distance (in kilometers) from a given point to the nearest bird's nest, with the probability density function of the distribution given by $f(x) = 2xe^{-x^2}$ for $x \ge 0$.

FOR REVIEW

Improper integrals, those with one or two infinite limits, were discussed in Chapter 16 on Further Techniques and Applications of Integration. The type of improper integral we shall need was defined as

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

For example,

$$\begin{aligned}\int_1^{\infty} x^{-2}\,dx &= \lim_{b\to\infty} \int_1^b x^{-2}\,dx \\ &= \lim_{b\to\infty} \left(-\frac{1}{x}\Big|_1^b\right) \\ &= \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1}\right) \\ &= 0 + 1 = 1.\end{aligned}$$

(a) Show that $f(x)$ is a probability density function.

SOLUTION Since $e^{-x^2} = 1/e^{x^2}$ is always positive, and $x \ge 0$,

$$f(x) = 2xe^{-x^2} \ge 0,$$

and Condition 1 holds.

Use substitution to evaluate the definite integral $\int_0^{\infty} 2xe^{-x^2}\,dx$. Let $u = -x^2$, so that $du = -2x\,dx$, and

$$\begin{aligned}\int 2xe^{-x^2}\,dx &= -\int e^{-x^2}(-2x\,dx) \\ &= -\int e^u\,du = -e^u = -e^{-x^2}.\end{aligned}$$

Then

$$\begin{aligned}\int_0^{\infty} 2xe^{-x^2}\,dx &= \lim_{b\to\infty} \int_0^b 2xe^{-x^2}\,dx = \lim_{b\to\infty} \left(-e^{-x^2}\right)\Big|_0^b \\ &= \lim_{b\to\infty} \left(-\frac{1}{e^{b^2}} + e^0\right) = 0 + 1 = 1,\end{aligned}$$

and Condition 2 holds.

The function defined by $f(x) = 2xe^{-x^2}$ satisfies the two conditions required of a probability density function.

APPLY IT

(b) Find the probability that there is a bird's nest within 0.5 km of the given point.

SOLUTION Find $P(X \le 0.5)$ where $X \ge 0$. This probability is given by

$$P(0 \le X \le 0.5) = \int_0^{0.5} 2xe^{-x^2}\,dx.$$

Now evaluate the integral. The indefinite integral was found in part (a).

$$P(0 \le X \le 0.5) = \int_0^{0.5} 2xe^{-x^2}\,dx = \left(-e^{-x^2}\right)\Big|_0^{0.5}$$
$$= -e^{-(0.5)^2} - \left(-e^0\right) = -e^{-0.25} + 1$$
$$\approx -0.7788 + 1 = 0.2212$$

The probability that a bird's nest will be found within 0.5 km of the given point is about 0.22.

TRY YOUR TURN 3

YOUR TURN 3 Using the probability density function of Example 3, find the probability that there is a bird's nest within 1 km of the given point.

TECHNOLOGY

EXAMPLE 4 Computing Mortality

According to the National Center for Health Statistics, if we start with 100,000 people who are 50 years old, we can expect a certain number of them to die within each 5-year interval, as indicated by the following table.* *Source: National Vital Statistics Reports*.

Years from Age 50	Life Table Midpoint of Interval	Number Dying in Each Interval
0–5	2.5	2565
5–10	7.5	3659
10–15	12.5	5441
15–20	17.5	7622
20–25	22.5	10,498
25–30	27.5	13,858
30–35	32.5	16,833
35–40	37.5	16,720
40–45	42.5	13,211
45–50	47.5	7068
50–55	52.5	2525

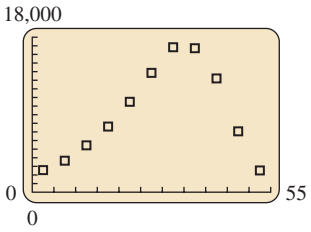


FIGURE 4

(a) Plot the data.

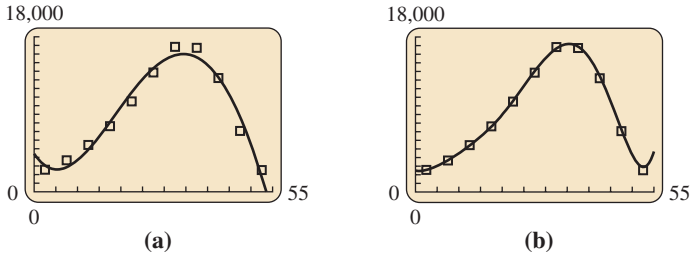
SOLUTION Figure 4 shows that the plot appears to have the shape of a polynomial.

(b) Find a polynomial equation that models the number of deaths, $N(t)$, as a function of the number of years, t, since age 50. Use the midpoints and the number of deaths in each interval from the table above.

SOLUTION The highest degree polynomial that the regression feature on a TI-84 Plus calculator can find is fourth degree. As Figure 5(a) shows, this roughly captures the behavior of the data, but it has two drawbacks. For one, it doesn't reach the highest data points. Also, it's decreasing in the beginning when it should be increasing. Higher degree polynomials can be fit using Excel or using the Multiple Regression tool on the Statistics with List Editor application for the TI-89. We were thus able to find that the function

$$N(t) = 5.03958 \times 10^{-5}t^6 - 0.006603t^5 + 0.2992t^4 - 6.0507t^3 + 67.867t^2 - 110.3t + 2485.1$$

fits the data quite well, as shown in Figure 5(b).



(a) **(b)**

FIGURE 5

*For simplicity, we have placed all those who lived past 100 in the class of those who lived from 100 to 105.

(c) Use the answer from part (b) to find a probability density function for the random variable T representing the number of additional years that a 50-year-old person lives.

SOLUTION We will construct a density function $S(t) = kN(t)$ by finding a suitable constant k, as we did in Example 2. The graph of the function turns up after $t = 52.5$, which is unlikely for the actual mortality function, so we will restrict the domain of the density function to the interval [0, 52.5], even though this ignores those who live more than 102.5 years. Using the integration feature on our calculator, we find that

$$\int_0^{52.5} S(t)dt = k\int_0^{52.5} N(t)dt = 497{,}703k.$$

Notice that this number is close to the product of 5 years (interval length) and 100,000 (the total number of people). This is not a coincidence! We set the above integral equal to 1 to get $k = 1/497{,}703$. The function defined by

$$\begin{aligned} S(t) &= \frac{1}{497{,}703}N(t) \\ &= \frac{1}{497{,}703}(5.03958 \times 10^{-5}t^6 - 0.006603t^5 + 0.2992t^4 \\ &\quad - 6.0507t^3 + 67.867t^2 - 110.3t + 2485.1) \end{aligned}$$

is a probability density function for [0, 52.5] because

$$\int_0^{52.5} S(t)\,dt = 1, \quad \text{and } S(t) \geq 0 \text{ for all } t \text{ in } [0, 52.5].$$

(d) Find the probability that a randomly chosen 50-year-old person will live at least until age 70.

SOLUTION Again using the integration feature on our calculator,

$$P(T \geq 20) = \int_{20}^{52.5} S(t)dt \approx 0.8054.$$

Thus a 50-year-old person has a 80.54% chance of living at least until age 70.

Notice that this value could also be estimated from the table by finding the number of people who have not died by age 70 and then dividing this number by 100,000. Thus, according to our table, there are 80,713 people still alive at age 70, representing 80.7% of the original population. As you can see, our estimate agrees quite well with the actual number.

Another important concept in probability is the *cumulative distribution function*, which gives the probability that a random variable X is less than or equal to an arbitrary value x.

Cumulative Distribution Function

If f is a probability density function of a random variable in the interval $[a, b]$, then the **cumulative distribution function** is defined as

$$F(x) = P(X \leq x) = \int_a^x f(t)dt$$

for $x \geq a$. Also, $F(x) = 0$ for $x < a$.

NOTE

1. We integrate with respect to the variable t in the integral, rather than x, because we are already using x for the upper limit on the integral. It doesn't matter what variable of integration is used in a definite integral, since that variable doesn't appear in the final answer. We just need to use a variable that's not being used for another purpose.
2. If the random variable is defined on the interval $(-\infty, \infty)$, simply replace a with $-\infty$ in the above definition.

EXAMPLE 5 Cumulative Distribution Function

Consider the random variable X defined in Example 3, giving the distance (in kilometers) from a given point to the nearest bird's nest, with probability density function $f(x) = 2xe^{-x^2}$ for $x \geq 0$.

(a) Find the cumulative distribution function for this random variable.

SOLUTION The cumulative distribution function is given by

$$F(x) = P(X \leq x) = \int_0^x 2te^{-t^2}\,dt$$ Use the density function with t as the variable

$$= -e^{-t^2}\Big|_0^x$$ Use the antiderivative found in Example 3.

$$= -e^{-x^2} + 1$$

for $x \geq 0$. The cumulative distribution function can be written as $F(x) = 1 - e^{-x^2}$ for $x \geq 0$. Note that for $x < 0$, $F(x) = 0$.

YOUR TURN 4 Use part (a) of Example 5 to calculate the probability that there is a bird's nest within 1 km of the given point.

(b) Use the solution to part (a) to calculate the probability that there is a bird's nest within 0.5 km of the given point.

SOLUTION To find $P(X \leq 0.5)$, calculate $F(0.5) = 1 - e^{-0.5^2} \approx 0.2212$. Notice that this is the same answer that we found in Example 3(b). TRY YOUR TURN 4

18.1 EXERCISES

Decide whether the functions defined as follows are probability density functions on the indicated intervals. If not, tell why.

1. $f(x) = \frac{1}{9}x - \frac{1}{18};\ [2, 5]$

2. $f(x) = \frac{1}{3}x - \frac{1}{6};\ [3, 4]$

3. $f(x) = \frac{x^2}{21};\ [1, 4]$

4. $f(x) = \frac{3}{98}x^2;\ [3, 5]$

5. $f(x) = 4x^3;\ [0, 3]$

6. $f(x) = \frac{x^3}{81};\ [0, 3]$

7. $f(x) = \frac{x^2}{16};\ [-2, 2]$

8. $f(x) = 2x^2;\ [-1, 1]$

9. $f(x) = \frac{5}{3}x^2 - \frac{5}{90};\ [-1, 1]$

10. $f(x) = \frac{3}{13}x^2 - \frac{12}{13}x + \frac{45}{52};\ [0, 4]$

Find a value of k that will make f a probability density function on the indicated interval.

11. $f(x) = kx^{1/2};\ [1, 4]$

12. $f(x) = kx^{3/2};\ [4, 9]$

13. $f(x) = kx^2;\ [0, 5]$

14. $f(x) = kx^2;\ [-1, 2]$

15. $f(x) = kx;\ [0, 3]$

16. $f(x) = kx;\ [2, 3]$

17. $f(x) = kx;\ [1, 5]$

18. $f(x) = kx^3;\ [2, 4]$

Find the cumulative distribution function for the probability density function in each of the following exercises.

19. Exercise 1

20. Exercise 2

21. Exercise 3

22. Exercise 4

23. Exercise 11

24. Exercise 12

25. The total area under the graph of a probability density function always equals _____.

26. In your own words, define a random variable.

27. What is the difference between a discrete probability function and a probability density function?

28. Why is $P(X = c) = 0$ for any number c in the domain of a probability density function?

Show that each function defined as follows is a probability density function on the given interval; then find the indicated probabilities.

29. $f(x) = \frac{1}{2}(1 + x)^{-3/2}$; $[0, \infty)$

a. $P(0 \le X \le 2)$ **b.** $P(1 \le X \le 3)$

c. $P(X \ge 5)$

30. $f(x) = e^{-x}$; $[0, \infty)$

a. $P(0 \le X \le 1)$ **b.** $P(1 \le X \le 2)$

c. $P(X \le 2)$

31. $f(x) = (1/2)e^{-x/2}$; $[0, \infty)$

a. $P(0 \le X \le 1)$ **b.** $P(1 \le X \le 3)$

c. $P(X \ge 2)$

32. $f(x) = \frac{20}{(x + 20)^2}$; $[0, \infty)$

a. $P(0 \le X \le 1)$ **b.** $P(1 \le X \le 5)$

c. $P(X \ge 5)$

33. $f(x) = \begin{cases} \frac{x^3}{12} & \text{if } 0 \le x \le 2 \\ \frac{16}{3x^3} & \text{if } x > 2 \end{cases}$

a. $P(0 \le X \le 2)$ **b.** $P(X \ge 2)$

c. $P(1 \le X \le 3)$

34. $f(x) = \begin{cases} \frac{20x^4}{9} & \text{if } 0 \le x \le 1 \\ \frac{20}{9x^5} & \text{if } x > 1 \end{cases}$

a. $P(0 \le X \le 1)$ **b.** $P(X \ge 1)$

c. $P(0 \le X \le 2)$

APPLICATIONS

Business and Economics

35. Life Span of a Computer Part The life (in months) of a certain electronic computer part has a probability density function defined by

$$f(t) = \frac{1}{2}e^{-t/2} \quad \text{for } t \text{ in } [0, \infty).$$

Find the probability that a randomly selected component will last the following lengths of time.

a. At most 12 months

b. Between 12 and 20 months

c. Find the cumulative distribution function for this random variable.

d. Use the answer to part c to find the probability that a randomly selected component will last at most 6 months.

36. Machine Life A machine has a useful life of 4 to 9 years, and its life (in years) has a probability density function defined by

$$f(t) = \frac{1}{11}\left(1 + \frac{3}{\sqrt{t}}\right).$$

Find the probabilities that the useful life of such a machine selected at random will be the following.

a. Longer than 6 years

b. Less than 5 years

c. Between 4 and 7 years

d. Find the cumulative distribution function for this random variable.

e. Use the answer to part d to find the probability that a randomly selected machine has a useful life of at most 8 years.

37. Machine Part The lifetime of a machine part has a continuous distribution on the interval (0, 40) with probability density function f, where $f(x)$ is proportional to $(10 + x)^{-2}$. Calculate the probability that the lifetime of the machine part is less than 6. Choose one of the following. ***Source: Society of Actuaries.***

a. 0.04 **b.** 0.15 **c.** 0.47 **d.** 0.53 **e.** 0.94

38. Insurance An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64. Calculate C. Choose one of the following. (*Hint:* The payment is 0 unless the loss is greater than the deductible, in which case the payment is the loss minus the deductible.) ***Source: Society of Actuaries.***

a. 0.1 **b.** 0.3 **c.** 0.4 **d.** 0.6 **e.** 0.8

Life Sciences

39. Petal Length The length of a petal on a certain flower varies from 1 cm to 4 cm and has a probability density function defined by

$$f(x) = \frac{1}{2\sqrt{x}}.$$

Find the probabilities that the length of a randomly selected petal will be as follows.

a. Greater than or equal to 3 cm

b. Less than or equal to 2 cm

c. Between 2 cm and 3 cm

40. Clotting Time of Blood The clotting time of blood is a random variable t with values from 1 second to 20 seconds and probability density function defined by

$$f(t) = \frac{1}{(\ln 20)t}.$$

Find the following probabilities for a person selected at random.

a. The probability that the clotting time is between 1 and 5 seconds

b. The probability that the clotting time is greater than 10 seconds

41. Flour Beetles Researchers who study the abundance of the flour beetle, *Tribolium castaneum*, have developed a probability density function that can be used to estimate the abundance of the beetle in a population. The density function, which is a member of the gamma distribution, is

$$f(x) = 1.185 \times 10^{-9} x^{4.5222} e^{-0.049846x},$$

where x is the size of the population. ***Source: Ecology***.

a. Estimate the probability that a randomly selected flour beetle population is between 0 and 150.

b. Estimate the probability that a randomly selected flour beetle population is between 100 and 200.

42. Flea Beetles The mobility of an insect is an important part of its survival. Researchers have determined that the probability that a marked flea beetle, *Phyllotreta cruciferae* and *Phyllotreta striolata*, will be recaptured within a certain distance and time after release can be calculated from the probability density function

$$p(x, t) = \frac{e^{-x^2/(4Dt)}}{\int_0^L e^{-u^2/(4Dt)}\, du},$$

where t is the time after release (in hours), x is the distance (in meters) from the release point that recaptures occur, L is the maximum distance from the release point that recaptures can occur, and D is the diffusion coefficient. ***Source: Ecology Monographs***.

a. If $t = 12$, $L = 6$, and $D = 38.3$, find the probability that a flea beetle will be recaptured within 3 m of the release point.

b. Using the same values for t, L, and D, find the probability that a flea beetle will be recaptured between 1 and 5 m of the release point.

Social Sciences

43. Social Network The number of U.S. users (in millions) on Facebook, a computer social network, in 2009 is given in the table below. ***Source: Inside Facebook***.

Age Interval (years)	Midpoint of Interval (year)	Number of Users in Each Interval (millions)
13–17	15	6.049
18–25	21.5	19.461
26–34	30	13.423
35–44	39.5	9.701
45–54	49.5	4.582
55–65	60	2.849
Total		56.065

a. Plot the data. What type of function appears to best match this data?

b. Use the regression feature on your graphing calculator to find a quartic equation that models the number of years, t, since birth and the number of Facebook users, $N(t)$. Use the midpoint value to estimate the point in each interval for the age of the Facebook user. Graph the function with the plot of the data. Does the function resemble the data?

c. By finding an appropriate constant k, find a function $S(t) = kN(t)$ that is a probability density function describing the probability of the age of a Facebook user. (*Hint:* Because the function in part b is negative for values less than 13.4 and greater than 62.0, restrict the domain of the density function to the interval [13.4, 62.0]. That is, integrate the function you found in part b from 13.4 to 62.0.)

d. For a randomly chosen person who uses Facebook, find the probabilities that the person was at least 35 but less than 45 years old, at least 18 but less than 35 years old, and at least 45 years old. Compare these with the actual probabilities.

44. Time to Learn a Task The time required for a person to learn a certain task is a random variable with probability density function defined by

$$f(t) = \frac{8}{7(t-2)^2}.$$

The time required to learn the task is between 3 and 10 minutes. Find the probabilities that a randomly selected person will learn the task in the following lengths of time.

a. Less than 4 minutes

b. More than 5 minutes

Physical Sciences

45. Annual Rainfall The annual rainfall in a remote Middle Eastern country varies from 0 to 5 in. and is a random variable with probability density function defined by

$$f(x) = \frac{5.5 - x}{15}.$$

Find the following probabilities for the annual rainfall in a randomly selected year.

a. The probability that the annual rainfall is greater than 3 in.

b. The probability that the annual rainfall is less than 2 in.

c. The probability that the annual rainfall is between 1 in. and 4 in.

46. Earthquakes The time between major earthquakes in the Southern California region is a random variable with probability density function

$$f(t) = \frac{1}{960} e^{-t/960},$$

where t is measured in days. ***Source: Journal of Seismology***.

a. Find the probability that the time between a major earthquake and the next one is less than 365 days.

b. Find the probability that the time between a major earthquake and the next one is more than 960 days.

47. Earthquakes The time between major earthquakes in the Taiwan region is a random variable with probability density function

$$f(t) = \frac{1}{3650.1} e^{-t/3650.1},$$

where t is measured in days. ***Source: Journal of Seismology***.

a. Find the probability that the time between a major earthquake and the next one is more than 1 year but less than 3 years.

b. Find the probability that the time between a major earthquake and the next one is more than 7300 days.

General Interest

48. Drunk Drivers The frequency of alcohol-related traffic fatalities has dropped in recent years but is still high among young people. Based on data from the National Highway Traffic Safety Administration, the age of a randomly selected, alcohol-impaired driver in a fatal car crash is a random variable with probability density function given by

$$f(t) = \frac{4.045}{t^{1.532}} \quad \text{for } t \text{ in } [16, 80].$$

Find the following probabilities of the age of such a driver. ***Source: Traffic Safety Facts***.

a. Less than or equal to 25

b. Greater than or equal to 35

c. Between 21 and 30

d. Find the cumulative distribution function for this random variable.

e. Use the answer to part d to find the probability that a randomly selected alcohol-impaired driver in a fatal car crash is at most 21 years old.

49. Driving Fatalities We saw in a review exercise in Chapter 12 on Calculating the Derivative that driver fatality rates were highest for the youngest and oldest drivers. When adjusted for the number of miles driven by people in each age group, the number of drivers in fatal crashes goes down with age, and the age of a randomly selected driver in a fatal car crash is a random variable with probability density function given by

$$f(t) = 0.06049e^{-0.03211t} \quad \text{for } t \text{ in } [16, 84].$$

Find the following probabilities of the age of such a driver. ***Source: National Highway Traffic Safety Administration***.

a. Less than or equal to 25

b. Greater than or equal to 35

c. Between 21 and 30

d. Find the cumulative distribution function for this random variable.

e. Use the answer to part d to find the probability that a randomly selected driver in a fatal crash is at most 21 years old.

50. Length of a Telephone Call The length of a telephone call (in minutes), t, for a certain town is a continuous random variable with probability density function defined by

$$f(t) = 3t^{-4}, \quad \text{for } t \text{ in } [1, \infty).$$

Find the probabilities for the following situations.

a. The call lasts between 1 and 2 minutes.

b. The call lasts between 3 and 5 minutes.

c. The call lasts longer than 3 minutes.

51. Time of Traffic Fatality The National Highway Traffic Safety Administration records the time of day of fatal crashes. The following table gives the time of day (in hours since midnight) and the frequency of fatal crashes. ***Source: The National Highway Traffic Safety Administration***.

Time of Day	Midpoint of Interval (hours)	Frequency
0–3	1.5	4486
3–6	4.5	2774
6–9	7.5	3236
9–12	10.5	3285
12–15	13.5	4356
15–18	16.5	5325
18–21	19.5	5342
21–24	22.5	4952
Total		33,756

a. Plot the data. What type of function appears to best match this data?

b. Use the regression feature on your graphing calculator to find a cubic equation that models the time of day, t, and the number of traffic fatalities, $T(t)$. Use the midpoint value to estimate the time in each interval. Graph the function with the plot of the data. Does the graph fit the data?

c. By finding an appropriate constant k, find a function $S(t) = kT(t)$ that is a probability density function describing the probability of a traffic fatality at a particular time of day.

d. For a randomly chosen traffic fatality, find the probabilities that the accident occurred between 12 am and 2 am ($t = 0$ to $t = 2$) and between 4 pm and 5:30 pm ($t = 16$ to $t = 17.5$).

YOUR TURN ANSWERS

1. $P\left(\frac{3}{2} \le X \le 2\right) = \frac{1}{3}$ **2.** $k = 1/64$

3. 0.6321 **4.** 0.6321

18.2 Expected Value and Variance of Continuous Random Variables

APPLY IT **What is the average age of a drunk driver in a fatal car crash?**
You will be asked to answer this question in Exercise 40.

It often is useful to have a single number, a typical or "average" number, that represents a random variable. The *mean* or *expected value* for a discrete random variable is found by multiplying each value of the random variable by its corresponding probability, as follows.

Expected Value

Suppose the random variable X can take on the n values, $x_1, x_2, x_3, \ldots, x_n$. Also, suppose the probabilities that each of these values occurs are, respectively, $p_1, p_2, p_3, \ldots, p_n$. Then the **mean**, or **expected value**, of the random variable is

$$\mu = x_1p_1 + x_2p_2 + x_3p_3 + \cdots + x_np_n = \sum_{i=1}^{n} x_ip_i.$$

For the banking example in the previous section, the expected value is given by

$$\begin{aligned}\mu &= 1(0.04) + 2(0.07) + 3(0.12) + 4(0.16) + 5(0.20) + 6(0.15) \\ &\quad + 7(0.13) + 8(0.08) + 9(0.04) + 10(0.01) \\ &= 5.09.\end{aligned}$$

Thus, the average time a person can expect to spend with the bank teller is 5.09 minutes.

This definition can be extended to continuous random variables by using definite integrals. Suppose a continuous random variable has probability density function f on $[a, b]$. We can divide the interval from a to b into n subintervals of length Δx, where $\Delta x = (b - a)/n$. In the ith subinterval, the probability that the random variable takes a value close to x_i is approximately $f(x_i)\,\Delta x$, and so

$$\mu \approx \sum_{i=1}^{n} x_i f(x_i)\,\Delta x.$$

As $n \to \infty$, the limit of this sum gives the expected value

$$\mu = \int_a^b x f(x)\,dx.$$

The **variance** of a probability distribution is a measure of the *spread* of the values of the distribution. For a discrete distribution, the variance is found by taking the expected value of the squares of the differences of the values of the random variable and the mean. If the random variable X takes the values $x_1, x_2, x_3, \ldots, x_n$, with respective probabilities $p_1, p_2, p_3, \ldots, p_n$ and mean μ, then the variance of X is

$$\text{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i.$$

Think of the variance as the expected value of $(X - \mu)^2$, which measures how far X is from the mean μ. The **standard deviation** of X is defined as

$$\sigma = \sqrt{\text{Var}(X)}.$$

For the banking example in the previous section, the variance and standard deviation are

$$\begin{aligned}\text{Var}(X) &= (1-5.09)^2(0.04) + (2-5.09)^2(0.07) + (3-5.09)^2(0.12)\\ &\quad + (4-5.09)^2(0.16) + (5-5.09)^2(0.20) + (6-5.09)^2(0.15)\\ &\quad + (7-5.09)^2(0.13) + (8-5.09)^2(0.08) + (9-5.09)^2(0.04)\\ &\quad + (10-5.09)^2(0.01)\\ &= 4.1819\end{aligned}$$

and

$$\sigma = \sqrt{\text{Var}(X)} \approx 2.0450.$$

Like the mean or expected value, the variance of a continuous random variable is an integral.

$$\text{Var}(X) = \int_a^b (x-\mu)^2 f(x)\,dx$$

To find the standard deviation of a continuous probability distribution, like that of a discrete distribution, we find the square root of the variance. The formulas for the expected value, variance, and standard deviation of a continuous probability distribution are summarized here.

Expected Value, Variance, and Standard Deviation

If X is a continuous random variable with probability density function f on $[a, b]$, then the expected value of X is

$$\boldsymbol{E(X) = \mu = \int_a^b xf(x)\,dx.}$$

The variance of X is

$$\boldsymbol{\text{Var}(X) = \int_a^b (x-\mu)^2 f(x)\,dx,}$$

and the standard deviation of X is

$$\boldsymbol{\sigma = \sqrt{\text{Var}(X)}.}$$

NOTE In the formulas for expected value, variance, and standard deviation, and all other formulas in this section, it is possible that $a = -\infty$ or $b = \infty$, in which case the density function f is defined on $[a, \infty)$, $(-\infty, b]$, or $(-\infty, \infty)$. In this case, the integrals in these formulas become improper integrals, which are handled according to the procedure described in Section 16.4 on Improper Integrals. Example 2 will illustrate this procedure.

Geometrically, the expected value (or mean) of a probability distribution represents the balancing point of the distribution. If a fulcrum were placed at μ on the x-axis, the figure would be in balance. See Figure 6.

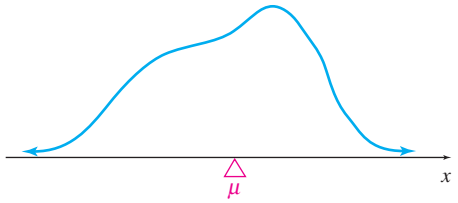


FIGURE 6

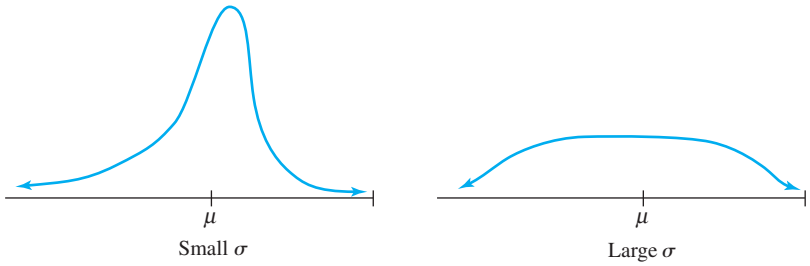


FIGURE 7

The variance and standard deviation of a probability distribution indicate how closely the values of the distribution cluster about the mean. These measures are most useful for comparing different distributions, as in Figure 7.

EXAMPLE 1 Expected Value and Variance

Find the expected value and variance of the random variable X with probability density function defined by $f(x) = (3/26)x^2$ on $[1, 3]$.

SOLUTION By the definition of expected value just given,

$$\begin{aligned}\mu &= \int_1^3 xf(x)dx \\ &= \int_1^3 x\left(\frac{3}{26}x^2\right)dx \\ &= \frac{3}{26}\int_1^3 x^3\,dx \\ &= \frac{3}{26}\left(\frac{x^4}{4}\right)\Big|_1^3 = \frac{3}{104}(81-1) = \frac{30}{13},\end{aligned}$$

or about 2.3077.

The variance is

$$\begin{aligned}\text{Var}(X) &= \int_1^3 \left(x - \frac{30}{13}\right)^2\left(\frac{3}{26}x^2\right)dx \\ &= \int_1^3 \left(x^2 - \frac{60}{13}x + \frac{900}{169}\right)\left(\frac{3}{26}x^2\right)dx && \text{Square } \left(x - \tfrac{30}{13}\right). \\ &= \frac{3}{26}\int_1^3 \left(x^4 - \frac{60}{13}x^3 + \frac{900}{169}x^2\right)dx && \text{Multiply.} \\ &= \frac{3}{26}\left(\frac{x^5}{5} - \frac{60}{13}\cdot\frac{x^4}{4} + \frac{900}{169}\cdot\frac{x^3}{3}\right)\Big|_1^3 && \text{Integrate.} \\ &= \frac{3}{26}\left[\left(\frac{243}{5} - \frac{60(81)}{52} + \frac{900(27)}{169(3)}\right) - \left(\frac{1}{5} - \frac{60}{52} + \frac{300}{169}\right)\right] \\ &\approx 0.2592.\end{aligned}$$

From the variance, the standard deviation is $\sigma \approx \sqrt{0.2592} \approx 0.5091$. The expected value and standard deviation are shown on the graph of the probability density function in Figure 8.

TRY YOUR TURN 1

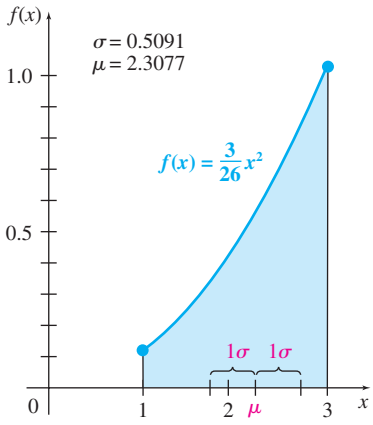


FIGURE 8

YOUR TURN 1 Repeat Example 1 for the probability density function $f(x) = \dfrac{8}{3x^3}$ on $[1, 2]$.

Calculating the variance in the last example was a messy job. An alternative version of the formula for the variance is easier to compute. This alternative formula is derived as follows.

$$\begin{aligned}\text{Var}(X) &= \int_a^b (x-\mu)^2 f(x)\,dx \\ &= \int_a^b (x^2 - 2\mu x + \mu^2) f(x)\,dx \\ &= \int_a^b x^2 f(x)\,dx - 2\mu\int_a^b x f(x)\,dx + \mu^2\int_a^b f(x)\,dx \end{aligned} \tag{1}$$

By definition,

$$\int_a^b x f(x)\,dx = \mu,$$

and, since $f(x)$ is a probability density function,

$$\int_a^b f(x)\,dx = 1.$$

Substitute back into Equation (1) to get the alternative formula,

$$\text{Var}(X) = \int_a^b x^2 f(x)\,dx - 2\mu^2 + \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2.$$

Alternative Formula for Variance

If X is a random variable with probability density function f on $[a, b]$, and if $E(X) = \mu$, then

$$\textbf{Var}(X) = \int_a^b x^2 f(x)\,dx - \mu^2.$$

CAUTION Notice that the term μ^2 comes *after* the dx and so is *not* integrated.

EXAMPLE 2 Variance

Use the alternative formula for variance to compute the variance of the random variable X with probability density function defined by $f(x) = 3/x^4$ for $x \geq 1$.

SOLUTION To find the variance, first find the expected value:

$$\begin{aligned}\mu &= \int_1^\infty x f(x)\,dx = \int_1^\infty x \cdot \frac{3}{x^4}\,dx = \int_1^\infty \frac{3}{x^3}\,dx \\ &= \lim_{b\to\infty}\int_1^b \frac{3}{x^3}\,dx = \lim_{b\to\infty}\left(\frac{3}{-2x^2}\right)\Bigg|_1^b = \frac{3}{2},\end{aligned}$$

or 1.5. Now find the variance by the alternative formula for variance:

$$\begin{aligned}\text{Var}(X) &= \int_1^\infty x^2\left(\frac{3}{x^4}\right)dx - \left(\frac{3}{2}\right)^2 \\ &= \int_1^\infty \frac{3}{x^2}\,dx - \frac{9}{4} \\ &= \lim_{b\to\infty}\int_1^b \frac{3}{x^2}\,dx - \frac{9}{4} \\ &= \lim_{b\to\infty}\left(\frac{-3}{x}\right)\Bigg|_1^b - \frac{9}{4} \\ &= 3 - \frac{9}{4} = \frac{3}{4}, \quad \text{or } 0.75.\end{aligned}$$

YOUR TURN 2 Repeat Example 2 for the probability density function $f(x) = 4/x^5$ for $x \geq 1$.

TRY YOUR TURN 2

EXAMPLE 3 Passenger Arrival

A recent study has shown that airline passengers arrive at the gate with the amount of time (in hours) before the scheduled flight time given by the probability density function $f(t) = (3/4)(2t - t^2)$ for $0 \le t \le 2$.

(a) Find and interpret the expected value for this distribution.

SOLUTION The expected value is

$$\mu = \int_0^2 t\left(\frac{3}{4}\right)(2t - t^2)dt = \int_0^2 \left(\frac{3}{4}\right)(2t^2 - t^3)dt$$

$$= \left(\frac{3}{4}\right)\left(\frac{2t^3}{3} - \frac{t^4}{4}\right)\Big|_0^2 = \left(\frac{3}{4}\right)\left(\frac{16}{3} - 4\right) = 1.$$

This result indicates that passengers arrive at the gate an average of 1 hour before the scheduled flight time.

(b) Compute the standard deviation.

SOLUTION First compute the variance. We use the alternative formula.

$$\text{Var}(T) = \int_0^2 t^2\left(\frac{3}{4}\right)(2t - t^2)dt - 1^2$$

$$= \int_0^2 \left(\frac{3}{4}\right)(2t^3 - t^4)dt - 1$$

$$= \left(\frac{3}{4}\right)\left(\frac{t^4}{2} - \frac{t^5}{5}\right)\Big|_0^2 - 1$$

$$= \left(\frac{3}{4}\right)\left(8 - \frac{32}{5}\right) - 1 = \frac{6}{5} - 1 = \frac{1}{5}$$

The standard deviation is $\sigma = \sqrt{1/5} \approx 0.45$.

(c) Calculate the probability that passengers will arrive at the gate within one standard deviation of the mean.

SOLUTION Since the mean, or expected value, is 1 and the standard deviation is approximately 0.45, we are calculating the probability that passengers will arrive between

$$\mu - \sigma = 1 - 0.45 = 0.55 \text{ hours}$$

and

$$\mu + \sigma = 1 + 0.45 = 1.45 \text{ hours}$$

before the scheduled flight time. The probability is given by

$$P(0.55 \le T \le 1.45) = \int_{0.55}^{1.45} \frac{3}{4}(2t - t^2)dt.$$

Evaluating the integral gives

$$\int_{0.55}^{1.45} \frac{3}{4}(2t - t^2)dt = \frac{3}{4}\left(t^2 - \frac{t^3}{3}\right)\Big|_{0.55}^{1.45} \approx 0.6294.$$

The probability that passengers will arrive within one standard deviation of the mean is about 0.63.

TECHNOLOGY

EXAMPLE 4 Life Expectancy

In the previous section of this chapter we used statistics compiled by the National Center for Health Statistics to determine a probability density function that can be used to study the proportion of all 50-year-olds who will be alive in t years. The function is given by

$$S(t) = \frac{1}{497{,}703}(5.03958 \times 10^{-5}t^6 - 0.006603t^5 + 0.2992t^4 - 6.0507t^3 + 67.867t^2 - 110.3t + 2485.1)$$

for $0 \le t \le 52.5$.

(a) Find the life expectancy of a 50-year-old person.

SOLUTION Since this is a complicated function that is tedious to integrate analytically, we will employ the integration feature on a TI-84 Plus calculator to calculate

$$\mu = \int_0^{52.5} tS(t)dt \approx 30.38 \text{ years.}$$

According to life tables, the life expectancy of a person between the ages of 50 and 55 is 30.6 years. Our estimate is remarkably accurate given the limited number of data points and the function used in our original analysis. Life expectancy is generally calculated with techniques from life table analysis. ***Source: National Center for Health Statistics.***

(b) Find the standard deviation of this probability function.

SOLUTION Using the alternate formula, we first calculate the variance.

$$\text{Var}(T) = \int_0^{52.5} t^2S(t)dt - \mu^2 = 1057.7195 - (30.38)^2 \approx 134.775$$

Thus, $\sigma = \sqrt{\text{Var}(T)} \approx 11.61$ years.

As we mentioned earlier, the expected value is also referred to as the mean of the random variable. It is a type of average. There is another type of average, known as the *median*, that is often used. It is the value of the random variable for which there is a 50% probability of being larger and a 50% probability of being smaller. The precise definition is as follows.

Median

If X is a random variable with probability density function f on $[a, b]$, then the **median** of X is the number m such that

$$\int_a^m f(x)\,dx = \frac{1}{2}.$$

The median is particularly useful when the random variable is not distributed symmetrically about the mean. An example of this would be a random variable representing the price of homes in a city. There is a small probability that a home will be much more expensive than most of the homes in the city, and this tends to make the mean abnormally high. The median price is a better representation of the average price of a home.

EXAMPLE 5 Median

Find the median for the random variable described in Example 2, with density function defined by $f(x) = 3/x^4$ for $x \ge 1$.

SOLUTION According to the formula,

$$\int_1^m \frac{3}{x^4}\,dx = \frac{1}{2}.$$

Evaluating the integral on the left, we have

$$-\frac{1}{x^3}\Big|_1^m = -\frac{1}{m^3} + 1.$$

Set this equal to $1/2$.

$$-\frac{1}{m^3} + 1 = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{m^3} \qquad \text{Subtract 1/2 from both sides, and add } 1/m^3.$$

$$m^3 = 2 \qquad \text{Cross multiply.}$$

$$m = \sqrt[3]{2}$$

The median value is, therefore, $\sqrt[3]{2} \approx 1.2599$. Notice that this is smaller than the mean of 1.5 found in Example 2. This is because the random variable can take on arbitrarily large values, which pulls up the mean but doesn't affect the median. TRY YOUR TURN 3

YOUR TURN 3 Repeat Example 5 for the probability density function $f(x) = 4/x^5$ for $x \geq 1$.

Using the notion of cumulative distribution function from the previous section, we can say that the median m is the value for which the cumulative distribution function is 0.5; that is, $F(m) = 0.5$.

18.2 EXERCISES

In Exercises 1–8, a probability density function of a random variable is defined. Find the expected value, the variance, and the standard deviation. Round answers to the nearest hundredth.

1. $f(x) = \frac{1}{4}$; $[3, 7]$

2. $f(x) = \frac{1}{10}$; $[0, 10]$

3. $f(x) = \frac{x}{8} - \frac{1}{4}$; $[2, 6]$

4. $f(x) = 2(1 - x)$; $[0, 1]$

5. $f(x) = 1 - \frac{1}{\sqrt{x}}$; $[1, 4]$

6. $f(x) = \frac{1}{11}\left(1 + \frac{3}{\sqrt{x}}\right)$; $[4, 9]$

7. $f(x) = 4x^{-5}$; $[1, \infty)$

8. $f(x) = 3x^{-4}$; $[1, \infty)$

9. What information does the mean (expected value) of a continuous random variable give?

10. Suppose two random variables have standard deviations of 0.10 and 0.23, respectively. What does this tell you about their distributions?

In Exercises 11–14, the probability density function of a random variable is defined.

a. Find the expected value to the nearest hundredth.

b. Find the variance to the nearest hundredth.

c. Find the standard deviation. Round to the nearest hundredth.

d. Find the probability that the random variable has a value greater than the mean.

e. Find the probability that the value of the random variable is within 1 standard deviation of the mean. Use the value of the standard deviation to the accuracy of your calculator.

11. $f(x) = \frac{\sqrt{x}}{18}$; $[0, 9]$

12. $f(x) = \frac{x^{-1/3}}{6}$; $[0, 8]$

13. $f(x) = \frac{1}{4}x^3$, $[0, 2]$

14. $f(x) = \frac{3}{16}(4 - x^2)$; $[0, 2]$

For Exercises 15–20, (a) find the median of the random variable with the probability density function given, and (b) find the probability that the random variable is between the expected value (mean) and the median. The expected value for each of these functions was found in Exercises 1–8.

15. $f(x) = \frac{1}{4}$; $[3, 7]$

16. $f(x) = \frac{1}{10}$; $[0, 10]$

17. $f(x) = \frac{x}{8} - \frac{1}{4}$; $[2, 6]$

18. $f(x) = 2(1 - x)$; $[0, 1]$

19. $f(x) = 4x^{-5}$; $[1, \infty)$

20. $f(x) = 3x^{-4}$; $[1, \infty)$

Find the expected value, the variance, and the standard deviation, when they exist, for each probability density function.

21. $f(x) = \begin{cases} \dfrac{x^3}{12} & \text{if } 0 \le x \le 2 \\ \dfrac{16}{3x^3} & \text{if } x > 2 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{20x^4}{9} & \text{if } 0 \le x \le 1 \\ \dfrac{20}{9x^5} & \text{if } x > 1 \end{cases}$

23. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \dfrac{|x|}{10} & \text{for } -2 \le x \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected value of X. ***Source: Society of Actuaries***. Choose one of the following.

a. 1/5 **b.** 3/5 **c.** 1 **d.** 28/15 **e.** 12/5

APPLICATIONS

Business and Economics

24. Life of a Light Bulb The life (in hours) of a certain kind of light bulb is a random variable with probability density function defined by

$$f(t) = \frac{1}{58\sqrt{t}} \quad \text{for } t \text{ in } [1{,}900].$$

a. What is the expected life of such a bulb?

b. Find σ.

c. Find the probability that one of these bulbs lasts longer than 1 standard deviation above the mean.

d. Find the median life of these bulbs.

25. Machine Life The life (in years) of a certain machine is a random variable with probability density function defined by

$$f(t) = \frac{1}{11}\left(1 + \frac{3}{\sqrt{t}}\right) \quad \text{for } t \text{ in } [4, 9].$$

a. Find the mean life of this machine.

b. Find the standard deviation of the distribution.

c. Find the probability that a particular machine of this kind will last longer than the mean number of years.

26. Life of an Automobile Part The life span of a certain automobile part (in months) is a random variable with probability density function defined by

$$f(t) = \frac{1}{2}e^{-t/2} \quad \text{for } t \text{ in } [0, \infty).$$

a. Find the expected life of this part.

b. Find the standard deviation of the distribution.

c. Find the probability that one of these parts lasts less than the mean number of months.

d. Find the median life of these parts.

27. Losses After Deductible A manufacturer's annual losses follow a distribution with density function

$$f(x) = \begin{cases} \dfrac{2.5(0.6)^{2.5}}{x^{3.5}} & \text{for } x > 0.6 \\ 0 & \text{otherwise.} \end{cases}$$

To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2. What is the mean of the manufacturer's annual losses not paid by the insurance policy? Choose one of the following. (*Hint:* The loss not paid by the insurance policy will equal the actual loss if the actual loss is less than the deductible. Otherwise it will equal the deductible.) ***Source: Society of Actuaries***.

a. 0.84 **b.** 0.88 **c.** 0.93 **d.** 0.95 **e.** 1.00

28. Insurance Reimbursement An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function:

$$f(y) = \begin{cases} \dfrac{2}{y^3} & \text{for } y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy? Choose one of the following. (*Hint:* The benefit paid will be equal to the actual loss if the actual loss is less than the limit. Otherwise it will equal the limit.) ***Source: Society of Actuaries***.

a. 1.0 **b.** 1.3 **c.** 1.8 **d.** 1.9 **e.** 2.0

29. Insurance Claims An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1 + x)^{-4}$, where $0 < x < \infty$. Determine the company's expected monthly claims. Choose one of the following. ***Source: Society of Actuaries***.

a. 1/6 **b.** 1/3 **c.** 1/2 **d.** 1 **e.** 3

30. Dental Insurance An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is

$$f(x) = \begin{cases} ce^{-0.004x} & \text{for } x \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant. Calculate the median benefit for this policy. Choose one of the following. (*Hint:* As long as the expenses are less than 250, the expenses and the benefit are equal.) ***Source: Society of Actuaries***.

a. 161 **b.** 165 **c.** 173 **d.** 182 **e.** 250

Life Sciences

31. Blood Clotting Time The clotting time of blood (in seconds) is a random variable with probability density function defined by

$$f(t) = \frac{1}{(\ln 20)t} \quad \text{for } t \text{ in } [1, 20].$$

a. Find the mean clotting time.

b. Find the standard deviation.

c. Find the probability that a person's blood clotting time is within 1 standard deviation of the mean.

d. Find the median clotting time.

32. Length of a Leaf The length of a leaf on a tree is a random variable with probability density function defined by

$$f(x) = \frac{3}{32}(4x - x^2) \quad \text{for } x \text{ in } [0, 4].$$

a. What is the expected leaf length?

b. Find σ for this distribution.

c. Find the probability that the length of a given leaf is within 1 standard deviation of the expected value.

33. Petal Length The length (in centimeters) of a petal on a certain flower is a random variable with probability density function defined by

$$f(x) = \frac{1}{2\sqrt{x}} \quad \text{for } x \text{ in } [1, 4].$$

a. Find the expected petal length.

b. Find the standard deviation.

c. Find the probability that a petal selected at random has a length more than 2 standard deviations above the mean.

d. Find the median petal length.

34. Flea Beetles As we saw in Exercise 42 of the previous section, the probability that a marked flea beetle, *Phyllotreta cruciferae* and *Phyllotreta striolata*, will be recaptured within a certain distance and time after release can be calculated from the probability density function

$$p(x, t) = \frac{e^{-x^2/(4Dt)}}{\int_0^L e^{-u^2/(4Dt)}\, du},$$

where t is the time (in hours) after release, x is the distance (in meters) from the release point that recaptures occur, L is the maximum distance from the release point that recaptures can occur, and D is the diffusion coefficient. If $t = 12$, $L = 6$, and $D = 38.3$, find the expected recapture distance. ***Source: Ecology Monographs***.

35. Flour Beetles As we saw in Exercise 41 of the previous section, a probability density function has been developed to estimate the abundance of the flour beetle, *Tribolium castaneum*. The density function, which is a member of the gamma distribution, is

$$f(x) = 1.185 \times 10^{-9} x^{4.5222} e^{-0.049846x},$$

where x is the size of the population. Calculate the expected size of a flour beetle population. (*Hint:* Use 1000 as the upper limit of integration.) ***Source: Ecology***.

Social Sciences

36. Time to Learn a Task In Exercise 44 of the previous section, the probability density function for the time required for a person to learn a certain task was given by

$$f(t) = \frac{8}{7(t - 2)^2},$$

for $3 \le t \le 10$ minutes. Find the median time for a person to learn the task.

37. Social Network In Exercise 43 of the previous section, the probability density function for the number of U.S. users of Facebook, a computer social network, was found to be

$$S(t) = \frac{1}{466.26}(-0.00007445t^4 + 0.01243t^3 - 0.7419t^2 + 18.18t - 137.5)$$

where t was the number of years since birth on [13.4, 62.0]. Calculate the expected age of a Facebook user, as well as the standard deviation. ***Source: Inside Facebook***.

Physical Sciences

38. Earthquakes The time between major earthquakes in the Southern California region is a random variable with probability density function defined by

$$f(t) = \frac{1}{960} e^{-t/960},$$

where t is measured in days. ***Source: Journal of Seismology***. Find the expected value and the standard deviation of this probability density function.

39. Annual Rainfall The annual rainfall in a remote Middle Eastern country is a random variable with probability density function defined by

$$f(x) = \frac{5.5 - x}{15}, \quad \text{for } x \text{ in } [0, 5].$$

a. Find the mean annual rainfall.

b. Find the standard deviation.

c. Find the probability of a year with rainfall less than 1 standard deviation below the mean.

General Interest

40. Drunk Drivers In the last section, we saw that the age of a randomly selected, alcohol-impaired driver in a fatal car crash is a random variable with probability density function given by

$$f(t) = \frac{4.045}{t^{1.532}} \quad \text{for } t \text{ in } [16, 80].$$

Source: Traffic Safety Facts.

a. **APPLY IT** Find the expected age of a drunk driver in a fatal car crash.

b. Find the standard deviation of the distribution.

c. Find the probability that such a driver will be younger than 1 standard deviation below the mean.

d. Find the median age of a drunk driver in a fatal car crash.

41. Driving Fatalities In the last section, we saw that the age of a randomly selected driver in a fatal car crash is a random variable with probability density function given by

$$f(t) = 0.06049e^{-0.03211t} \quad \text{for } t \text{ in } [16, 84].$$

Source: National Highway Traffic Safety Administration.

a. Find the expected age of a driver in a fatal car crash.

b. Find the standard deviation of the distribution.

c. Find the probability that such a driver will be younger than 1 standard deviation below the mean.

d. Find the median age of a driver in a fatal car crash.

42. Length of a Telephone Call The length of a telephone call (in minutes), t, for a certain town is a continuous random variable with probability density function defined by

$$f(t) = 3t^{-4}$$

for t in $[1, \infty)$. Find the expected length of a phone call.

43. Time of Traffic Fatality In Exercise 51 of the previous section, the probability density function for the number of fatal traffic accidents was found to be

$$S(t) = \frac{1}{101{,}370}(-2.564t^3 + 99.11t^2 - 964.6t + 5631)$$

where t is the number of hours since midnight on [0, 24]. Calculate the expected time of day at which a fatal accident will occur. ***Source: The National Highway Traffic Safety Administration.***

YOUR TURN ANSWERS

1. 4/3, 0.0706 **2.** 2/9 **3.** $\sqrt[4]{2} \approx 1.1892$

18.3 Special Probability Density Functions

APPLY IT **What is the probability that the maximum outdoor temperature will be higher than 24°C? What is the probability that a flashlight battery will last longer than 40 hours?**

These questions, presented in Examples 1 and 2, can be answered if the probability density function for the maximum temperature and for the life of the battery are known.

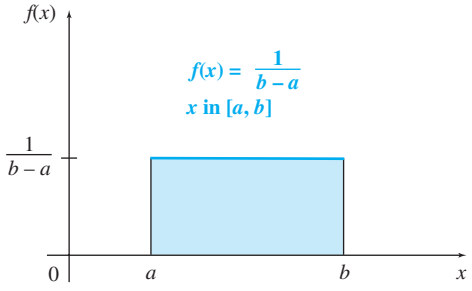
In practice, it is not feasible to construct a probability density function for every experiment. Instead, a researcher uses one of several probability density functions that are well known, matching the shape of the experimental distribution to one of the known distributions. In this section we discuss some of the most commonly used probability distributions.

Uniform Distribution

The simplest probability distribution occurs when the probability density function of a random variable remains constant over the sample space. In this case, the random variable is said to be *uniformly distributed* over the sample space. The probability density function for the **uniform distribution** is defined by

$$f(x) = \frac{1}{b-a} \quad \text{for } x \text{ in } [a, b],$$

where a and b are constant real numbers. The graph of $f(x)$ is shown in Figure 9.



Uniform distribution

FIGURE 9

Since $b - a$ is positive, $f(x) \geq 0$, and

$$\int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}x\Big|_a^b = \frac{1}{b-a}(b-a) = 1.$$

Therefore, the function is a probability density function.

The expected value for the uniform distribution is

$$\mu = \int_a^b \left(\frac{1}{b-a}\right)x\,dx = \left(\frac{1}{b-a}\right)\frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{2(b-a)}(b^2 - a^2) = \frac{1}{2}(b+a). \qquad b^2 - a^2 = (b-a)(b+a)$$

The variance is given by

$$\text{Var}(X) = \int_a^b \left(\frac{1}{b-a}\right)x^2\,dx - \left(\frac{b+a}{2}\right)^2$$

$$= \left(\frac{1}{b-a}\right)\frac{x^3}{3}\Big|_a^b - \frac{(b+a)^2}{4}$$

$$= \frac{1}{3(b-a)}(b^3 - a^3) - \frac{1}{4}(b+a)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4} \qquad b^3 - a^3 = (b-a)(b^2+ab+a^2)$$

$$= \frac{b^2-2ab+a^2}{12}. \qquad \text{Get a common denominator; subtract.}$$

Thus

$$\text{Var}(X) = \frac{1}{12}(b-a)^2, \qquad \text{Factor.}$$

and

$$\sigma = \frac{1}{\sqrt{12}}(b-a).$$

These properties of the uniform distribution are summarized below.

Uniform Distribution

If X is a random variable with probability density function

$$f(x) = \frac{1}{b-a} \quad \text{for } x \text{ in } [a, b],$$

then

$$\mu = \frac{1}{2}(b+a) \quad \text{and} \quad \sigma = \frac{1}{\sqrt{12}}(b-a).$$

EXAMPLE 1 Daily Temperature

A couple is planning to vacation in San Francisco. They have been told that the maximum daily temperature during the time they plan to be there ranges from 15°C to 27°C. Assume that the probability of any temperature between 15°C and 27°C is equally likely for any given day during the specified time period.

(a) What is the probability that the maximum temperature on the day they arrive will be higher than 24°C?

APPLY IT

SOLUTION If the random variable T represents the maximum temperature on a given day, then the uniform probability density function for T is defined by $f(t) = 1/12$ for the interval $[15, 27]$. By definition,

$$P(T > 24) = \int_{24}^{27} \frac{1}{12}\,dt = \frac{1}{12}t\Big|_{24}^{27} = \frac{1}{4}.$$

(b) What average maximum temperature can they expect?

SOLUTION The expected maximum temperature is

$$\mu = \frac{1}{2}(27 + 15) = 21,$$

or 21°C.

(c) What is the probability that the maximum temperature on a given day will be one standard deviation or more below the mean?

SOLUTION First find σ.

$$\sigma = \frac{1}{\sqrt{12}}(27 - 15) = \frac{12}{\sqrt{12}} = \sqrt{12} = 2\sqrt{3} \approx 3.464.$$

One standard deviation below the mean indicates a temperature of $21 - 3.464 = 17.536$°C.

$$P(T \le 17.536) = \int_{15}^{17.536} \frac{1}{12}\,dt = \frac{1}{12}t\Big|_{15}^{17.536} \approx 0.2113$$

The probability is about 0.21 that the temperature will not exceed 17.5°C.

YOUR TURN 1 The next vacation for the couple in Example 1 is to a desert with a maximum daily temperature that uniformly ranges from 27˚C to 42˚C. Find the expected maximum temperature and the probability that the maximum temperature will be within one standard deviation of the mean.

TRY YOUR TURN 1

Exponential Distribution

The next distribution is very important in reliability and survival analysis. When manufactured items and living things have a constant failure rate over a period of time, the exponential distribution is used to describe their probability of failure. In this case, the random variable is said to be *exponentially distributed* over the sample space. The probability density function for the **exponential distribution** is defined by

$$f(x) = ae^{-ax} \qquad \text{for } x \text{ in } [0, \infty),$$

where a is a positive constant. The graph of $f(x)$ is shown in Figure 10.

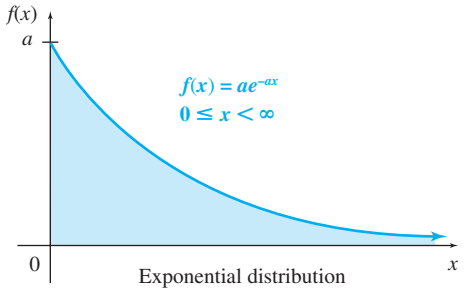


FIGURE 10

Here $f(x) \ge 0$, since e^{-ax} and a are both positive for all values of x. Also,

$$\int_0^\infty ae^{-ax}\,dx = \lim_{b\to\infty}\int_0^b ae^{-ax}\,dx$$

$$= \lim_{b\to\infty}(-e^{-ax})\Big|_0^b = \lim_{b\to\infty}\left(\frac{-1}{e^{ab}} + \frac{1}{e^0}\right) = 1,$$

so the function is a probability density function.

The expected value and the standard deviation of the exponential distribution can be found using integration by parts. The results are given below. (See Exercise 20 at the end of this section.)

Exponential Distribution

If X is a random variable with probability density function

$$f(x) = ae^{-ax} \quad \text{for } x \text{ in } [0, \infty),$$

then

$$\mu = \frac{1}{a} \quad \text{and} \quad \sigma = \frac{1}{a}.$$

EXAMPLE 2 Flashlight Battery

Suppose the useful life (in hours) of a flashlight battery is the random variable T, with probability density function given by the exponential distribution

$$f(t) = \frac{1}{20}e^{-t/20} \quad \text{for } t \geq 0.$$

(a) Find the probability that a particular battery, selected at random, has a useful life of less than 100 hours.

SOLUTION The probability is given by

$$\begin{aligned} P(T \leq 100) &= \int_0^{100} \frac{1}{20}e^{-t/20}\,dt = \frac{1}{20}\left(-20e^{-t/20}\right)\Big|_0^{100} \\ &= -(e^{-100/20} - e^0) = -(e^{-5} - 1) \\ &\approx 1 - 0.0067 = 0.9933. \end{aligned}$$

(b) Find the expected value and standard deviation of the distribution.

SOLUTION Use the formulas given above. Both μ and σ equal $1/a$, and since $a = 1/20$ here,

$$\mu = 20 \quad \text{and} \quad \sigma = 20.$$

This means that the average life of a battery is 20 hours, and no battery lasts less than 1 standard deviation below the mean.

APPLY IT

YOUR TURN 2 Repeat Example 2 for a flashlight battery with a useful life given by the probability density function $f(t) = \frac{1}{25}e^{-t/25}$ for $t \geq 0$.

(c) What is the probability that a battery will last longer than 40 hours?

SOLUTION The probability is given by

$$P(T > 40) = \int_{40}^{\infty} \frac{1}{20}e^{-t/20}\,dt = \lim_{b\to\infty}\left(-e^{-t/20}\right)\Big|_{40}^{b} = \frac{1}{e^2} \approx 0.1353,$$

or about 14%.

TRY YOUR TURN 2

Normal Distribution

The **normal distribution**, with its well-known bell-shaped graph, is undoubtedly the most important probability density function. It is widely used in various applications of statistics. The random variables associated with these applications are said to be normally distributed. The probability density function for the normal distribution has the following characteristics.

Normal Distribution

If μ and σ are real numbers, $\sigma > 0$, and if X is a random variable with probability density function defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \quad \text{for } x \text{ in } (-\infty, \infty),$$

then

$$E(X) = \mu \qquad \text{and} \qquad \text{Var}(X) = \sigma^2, \quad \text{with standard deviation } \sigma.$$

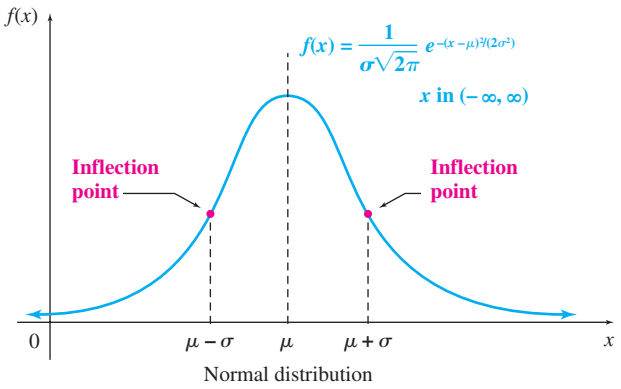
Notice that the definition of the probability density function includes σ, which is the standard deviation of the distribution.

Advanced techniques can be used to show that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\, dx = 1.$$

Deriving the expected value and standard deviation for the normal distribution also requires techniques beyond the scope of this text.

Each normal probability distribution has associated with it a bell-shaped curve, called a **normal curve**, such as the one in Figure 11. Each normal curve is symmetric about a vertical line through the mean, μ. Vertical lines at points $+1\sigma$ and -1σ from the mean show the inflection points of the graph. (See Exercise 22 at the end of this section.) A normal curve never touches the x-axis; it extends indefinitely in both directions.



Normal distribution

FIGURE 11

The development of the normal curve is credited to the Frenchman Abraham De Moivre (1667–1754). Three of his publications dealt with probability and associated topics: *Annuities upon Lives* (which contributed to the development of actuarial studies), *Doctrine of Chances*, and *Miscellanea Analytica*.

Many different normal curves have the same mean. In such cases, a larger value of σ produces a "flatter" normal curve, while smaller values of σ produce more values near the mean, resulting in a "taller" normal curve. See Figure 12 on the next page.

It would be far too much work to calculate values for the normal probability distribution for various values of μ and σ. Instead, values are calculated for the **standard normal**

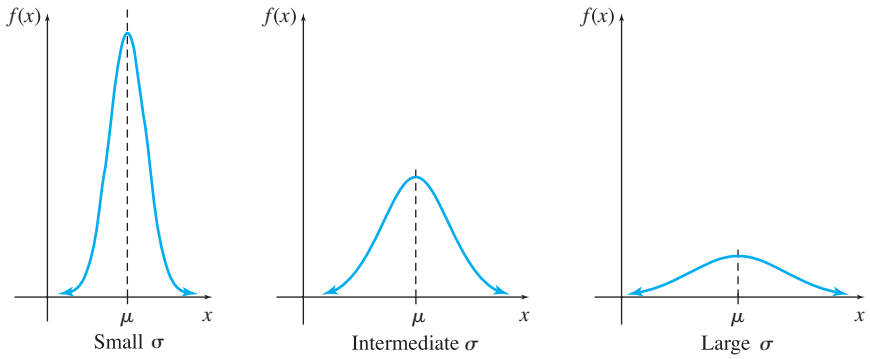
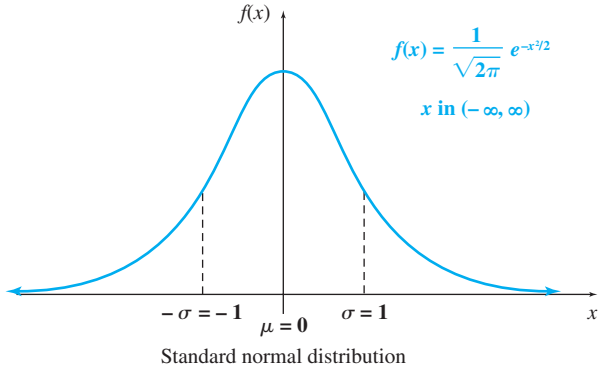


FIGURE 12

distribution, which has $\mu = 0$ and $\sigma = 1$. The graph of the standard normal distribution is shown in Figure 13.



Standard normal distribution

FIGURE 13

Probabilities for the standard normal distribution come from the definite integral

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

Since $f(x) = e^{-x^2/2}$ does not have an antiderivative that can be expressed in terms of functions used in this course, numerical methods are used to find values of this definite integral. A table in the appendix of this book gives areas under the standard normal curve, along with a sketch of the curve. Each value in this table is the total area under the standard normal curve to the left of the number z.

If a normal distribution does not have $\mu = 0$ and $\sigma = 1$, we use the following theorem, which is proved in Exercise 21.

z-Scores Theorem

Suppose a normal distribution has mean μ and standard deviation σ. The area under the associated normal curve that is to the left of the value x is exactly the same as the area to the left of

$$z = \frac{x - \mu}{\sigma}$$

for the standard normal curve.

Using this result, the table can be used for *any* normal distribution, regardless of the values of μ and σ. The number z in the theorem is called a **z-score**.

EXAMPLE 3 Life Spans

According to actuarial tables, life spans in the United States are approximately normally distributed with a mean of about 75 years and a standard deviation of about 16 years. By computing the areas under the associated normal curve, find the following probabilities. ***Source: Psychological Science***.

(a) Find the probability that a randomly selected person lives less than 88 years.

SOLUTION Let T represent the life span of a random individual. To find $P(T < 88)$, we calculate the corresponding z-score using $t = 88$, $\mu = 75$, and $\sigma = 16$. Round to the nearest hundredths, since this is the extent of our normal curve table.

$$z = \frac{88 - 75}{16} = \frac{13}{16} \approx 0.81$$

Look up 0.81 in the normal curve table in the Appendix. The corresponding area is 0.7910. Thus, the shaded area shown in Figure 14 is 0.7910. This means that the probability of a randomly selected person living less than 88 years is $P(T < 88) = P(Z < 0.81) = 0.7910$, or about 79%.

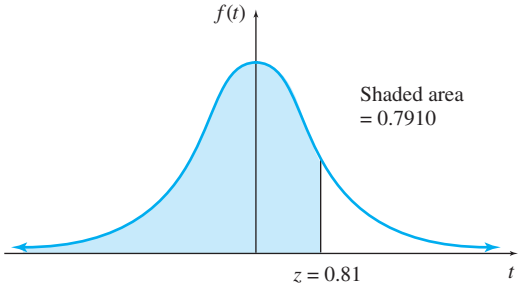


FIGURE 14

(b) Find the probability that a randomly selected person lives more than 67 years.

SOLUTION To calculate $P(T > 67)$, first find the corresponding z-score.

$$z = \frac{67 - 75}{16} = -0.5$$

From the normal curve table, the area to the *left* of $z = -0.5$ is 0.3085. Therefore, the area to the *right* is $P(T > 67) = P(Z > -0.5) = 1 - 0.3085 = 0.6915$. See Figure 15. Thus, the probability of a randomly selected person living more than 67 years is 0.6915, or about 69%.

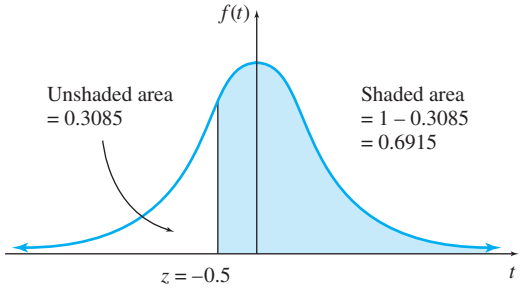


FIGURE 15

(c) Find the probability that a randomly selected person lives between 61 and 70 years.

SOLUTION Find z-scores for both values.

$$z = \frac{61 - 75}{16} = -0.88 \quad \text{and} \quad z = \frac{70 - 75}{16} = -0.31$$

YOUR TURN 3 Using the information provided in Example 3, find the probability that a randomly selected person lives (a) more than 79 years and (b) between 67 and 83 years.

Start with the area to the left of $z = -0.31$ and subtract the area to the left of $z = -0.88$. Thus,

$$P(61 \le T \le 70) = P(-0.88 \le Z \le -0.31) = 0.3783 - 0.1894 = 0.1889.$$

The required area is shaded in Figure 16. The probability of a randomly selected person living between 61 and 70 years is about 19%. TRY YOUR TURN 3

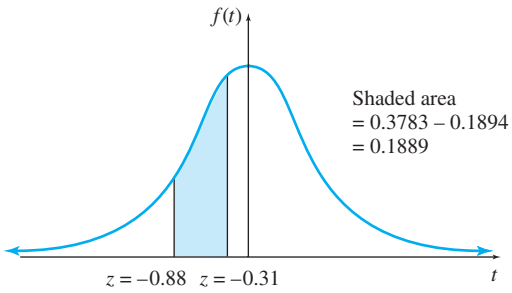


FIGURE 16

It's worth noting that there is always some error in approximating a discrete distribution with a continuous distribution. For example, when T is the life span of a randomly selected person, then $P(T = 65)$ is clearly positive, but if T is a normal random variable, then $P(T = 65) = 0$, since it represents no area. The problem is that a person's age jumps from 65 to 66, but a continuous random variable takes on all real numbers in between. If we were to measure a person's age to the nearest nanosecond, the probability that someone's age is exactly 65 years and 0 nanoseconds would be virtually 0.

Furthermore, a bit of thought shows us that the approximation of life spans by a normal distribution can't be perfect. After all, three standard deviations to the left and right of 75 give $75 - 3 \times 16 = 27$ and $75 + 3 \times 16 = 123$. Because of the symmetry of the normal distribution, $P(T < 27)$ and $P(T > 123)$ should be equal. Yet there are people who die before the age of 27, and no human has been verified to live beyond the age of 123.

TECHNOLOGY NOTE

As an alternative to using the normal curve table, we can use a graphing calculator. Enter the formula for the normal distribution into the calculator, using $\mu = 75$ and $\sigma = 16$. Plot the function on a window that contains at least four standard deviations to the left and right of μ; for Example 3(a), we will let $0 \le t \le 140$. Then use the integration feature (under CALC on a TI-84 Plus) to find the area under the curve to the left of 88.

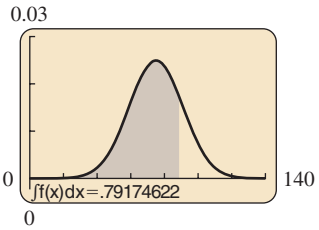


FIGURE 17

The result is shown in Figure 17. In place of $-\infty$, we have used $t = 0$ as the left endpoint. This is far enough to the left of $\mu = 75$ that it can be considered as $-\infty$ for all practical purposes. It also makes sense in this application, since life span can't be a negative number. You can verify that choosing a slightly different lower limit makes little difference in the answer. In fact, the answer of 0.79174622 is more accurate than the answer of 0.7910 that we found in Example 3(a), where we needed to round 13/16 = 0.8125 to 0.81 in order to use the table.

We could get the answer on a TI-84 Plus without generating a graph using the command `fnInt` and entering $\int_0^{88}(Y_1)dx$, where Y_1 is the formula for the normal distribution with $\mu = 75$ and $\sigma = 16$.

The numerical integration method works with any probability density function. In addition, many graphing calculators are programmed with information about specific density functions, such as the normal. We can solve the first part of Example 3 on the TI-84 Plus by entering `normalcdf(-1E99,88,75,16)`. The calculator responds with 0.7917476687. (`-1E99` stands for -1×10^{99}, which the calculator uses for $-\infty$.) If you use this method in the exercises, your answers will differ slightly from those in the back of the book, which were generated using the normal curve table in the Appendix.

The z-scores are actually standard deviation multiples; that is, a z-score of 2.5 corresponds to a value 2.5 standard deviations above the mean. For example, looking up $z = 1.00$ and $z = -1.00$ in the table shows that

$$0.8413 - 0.1587 = 0.6826,$$

so that 68.26% of the area under a normal curve is within 1 standard deviation of the mean. Also, using $z = 2.00$ and $z = -2.00$,

$$0.9772 - 0.0228 = 0.9544,$$

meaning 95.44% of the area is within 2 standard deviations of the mean. These results, summarized in Figure 18, can be used to get a quick estimate of results when working with normal curves.

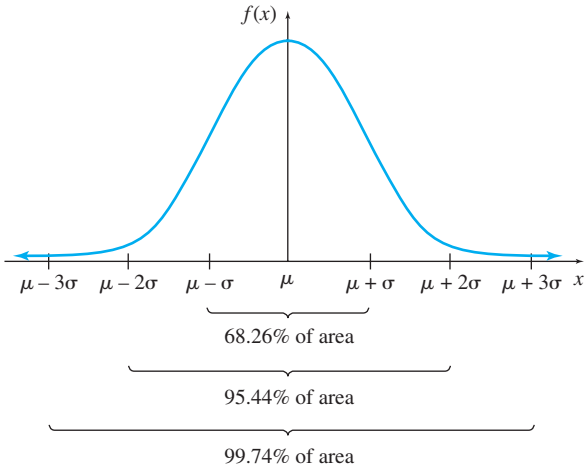
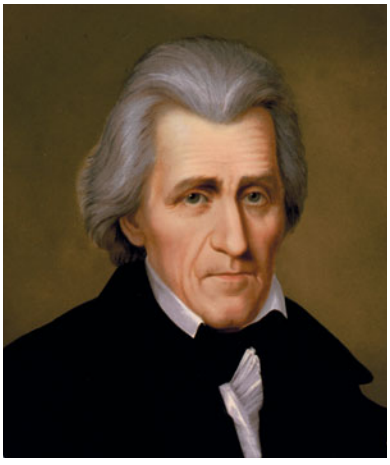


FIGURE 18

Manufacturers make use of the fact that a normal random variable is almost always within 3 standard deviations of the mean to design control charts. When a sample of items produced by a machine has a mean farther than 3 standard deviations from the desired specification, the machine is assumed to be out of control, and adjustments are made to ensure that the items produced meet the tolerance required.

EXAMPLE 4 Lead Poisoning



Historians and biographers have collected evidence suggesting that President Andrew Jackson suffered from lead poisoning. Recently, researchers measured the amount of lead in samples of Jackson's hair from 1815. The results of this experiment showed that Jackson had a mean lead level of 130.5 ppm. ***Source: JAMA***.

(a) Levels of lead in hair samples from that time period follow a normal distribution with mean 93 and standard deviation 16. ***Source: Science***. Find the probability that a randomly selected person from this time period would have a lead level of 130.5 ppm or higher. Does this provide evidence that Jackson suffered from lead poisoning during this time period?

SOLUTION $P(X \geq 130.5) = P\left(Z \geq \dfrac{130.5 - 93}{16}\right) = P(Z \geq 2.34) = 0.0096$

Since this probability is so low, it is likely that Jackson suffered from lead poisoning during this time period.*

*Although this provides evidence that Andrew Jackson had elevated lead levels, the authors of the paper concluded that Andrew Jackson did not die from lead poisoning.

(b) Today's normal lead levels follow a normal distribution with approximate mean of 10 ppm and standard deviation of 5 ppm. ***Source: Clinical Chemistry***. By today's standards, calculate the probability that a randomly selected person from today would have a lead level of 130.5 ppm or higher. From this, can we conclude that Andrew Jackson had lead poisoning?

SOLUTION $P(X \geq 130.5) = P\left(Z \geq \dfrac{130.5 - 10}{5}\right) = P(Z \geq 24.1) \approx 0$

By today's standards, which may not be valid for this experiment, Jackson certainly suffered from lead poisoning.

18.3 EXERCISES

Find (a) the mean of the distribution, (b) the standard deviation of the distribution, and (c) the probability that the random variable is between the mean and 1 standard deviation above the mean.

1. The length (in centimeters) of the leaf of a certain plant is a continuous random variable with probability density function defined by

$$f(x) = \frac{5}{7} \quad \text{for } x \text{ in } [3, 4.4].$$

2. The price of an item (in hundreds of dollars) is a continuous random variable with probability density function defined by

$$f(x) = 4 \quad \text{for } x \text{ in } [2.75, 3].$$

3. The length of time (in years) until a particular radioactive particle decays is a random variable t with probability density function defined by

$$f(t) = 4e^{-4t} \quad \text{for } t \text{ in } [0, \infty).$$

4. The length of time (in years) that a seedling tree survives is a random variable t with probability density function defined by

$$f(t) = 0.05e^{-0.05t} \quad \text{for } t \text{ in } [0, \infty).$$

5. The length of time (in days) required to learn a certain task is a random variable t with probability density function defined by

$$f(t) = \frac{e^{-t/3}}{3} \quad \text{for } t \text{ in } [0, \infty).$$

6. The distance (in meters) that seeds are dispersed from a certain kind of plant is a random variable x with probability density function defined by

$$f(x) = 0.1e^{-0.1x} \quad \text{for } x \text{ in } [0, \infty).$$

Find the proportion of observations of a standard normal distribution that are between the mean and the given number of standard deviations above the mean.

7. 3.50 **8.** 1.68

Find the proportion of observations of a standard normal distribution that are between the given z-scores.

9. 1.28 and 2.05 **10.** -2.13 and -0.04

Find a z-score satisfying the conditions given in Exercises 11–14. (*Hint:* Use the table backwards.)

11. 10% of the total area is to the left of z.

12. 2% of the total area is to the left of z.

13. 18% of the total area is to the right of z.

14. 22% of the total area is to the right of z.

15. Describe the standard normal distribution. What are its characteristics?

16. What is a z-score? How is it used?

17. Describe the shape of the graph of each probability distribution.

a. Uniform **b.** Exponential **c.** Normal

In the second section of this chapter, we defined the median of a probability distribution as an integral. The median also can be defined as the number m such that $P(X \leq m) = P(X \geq m)$.

18. Find an expression for the median of the uniform distribution.

19. Find an expression for the median of the exponential distribution.

20. Verify the expected value and standard deviation of the exponential distribution given in the text.

21. Prove the z-scores theorem. (*Hint:* Write the formula for the normal distribution with mean μ and standard deviation σ, using t instead of x as the variable. Then write the integral representing the area to the left of the value x, and make the substitution $u = (t - \mu)/\sigma$.)

22. Show that a normal random variable has inflection points at $x = \mu - \sigma$ and $x = \mu + \sigma$.

23. Use Simpson's rule with $n = 140$, or use the integration feature on a graphing calculator, to approximate the following integrals.

a. $\displaystyle\int_0^{35} 0.5e^{-0.5x}\,dx$ **b.** $\displaystyle\int_0^{35} 0.5xe^{-0.5x}\,dx$

c. $\displaystyle\int_0^{35} 0.5x^2e^{-0.5x}\,dx$

24. Use your results from Exercise 23 to verify that, for the exponential distribution with $a = 0.5$, the total probability is 1, and both the mean and the standard deviation are equal to $1/a$.

25. Use Simpson's rule with $n = 40$, or the integration feature on a graphing calculator, to approximate the following for the standard normal probability distribution. Use limits of -6 and 6 in place of $-\infty$ and ∞.

a. The mean **b.** The standard deviation

26. A very important distribution for analyzing the reliability of manufactured goods is the Weibull distribution, whose probability density function is defined by

$$f(x) = abx^{b-1}e^{-ax^b} \quad \text{for } x \text{ in } [0, \infty),$$

where a and b are constants. Notice that when $b = 1$, this reduces to the exponential distribution. The Weibull distribution is more general than the exponential, because it applies even when the failure rate is not constant. Use Simpson's rule with $n = 100$, or the integration feature on a graphing calculator, to approximate the following for the Weibull distribution with $a = 4$ and $b = 1.5$. Use a limit of 3 in place of ∞.

a. The mean **b.** The standard deviation

27. Determine the cumulative distribution function for the uniform distribution.

28. Determine the cumulative distribution function for the exponential distribution.

APPLICATIONS

Business and Economics

29. Insurance Sales The amount of insurance (in thousands of dollars) sold in a day by a particular agent is uniformly distributed over the interval $[10, 85]$.

a. What amount of insurance does the agent sell on an average day?

b. Find the probability that the agent sells more than \$50,000 of insurance on a particular day.

30. Fast-Food Outlets The number of new fast-food outlets opening during June in a certain city is exponentially distributed, with a mean of 5.

a. Give the probability density function for this distribution.

b. What is the probability that the number of outlets opening is between 2 and 6?

31. Sales Expense A salesperson's monthly expenses (in thousands of dollars) are exponentially distributed, with an average of 4.25 (thousand dollars).

a. Give the probability density function for the expenses.

b. Find the probability that the expenses are more than \$10,000.

In Exercises 32–34, assume a normal distribution.

32. Machine Accuracy A machine that fills quart bottles with apple juice averages 32.8 oz per bottle, with a standard deviation of 1.1 oz. What are the probabilities that the amount of juice in a bottle is as follows?

a. Less than 1 qt

b. At least 1 oz more than 1 qt

33. Machine Accuracy A machine produces screws with a mean length of 2.5 cm and a standard deviation of 0.2 cm. Find the probabilities that a screw produced by this machine has lengths as follows.

a. Greater than 2.7 cm

b. Within 1.2 standard deviations of the mean

34. Customer Expenditures Customers at a certain pharmacy spend an average of \$54.40, with a standard deviation of \$13.50. What are the largest and smallest amounts spent by the middle 50% of these customers?

35. Insured Loss An insurance policy is written to cover a loss, X, where X has a uniform distribution on [0, 1000]. At what level must a deductible be set in order for the expected payment to be 25% of what it would be with no deductible? Choose one of the following. (*Hint:* Use a variable, such as D, for the deductible. The payment is 0 if the loss is less than D, and the loss minus D if the loss is greater than D.) ***Source: Society of Actuaries***.

a. 250 **b.** 375 **c.** 500 **d.** 625 **e.** 750

36. High-Risk Drivers The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year. What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year? Choose one of the following. ***Source: Society of Actuaries***.

a. 0.15 **b.** 0.34 **c.** 0.43 **d.** 0.57 **e.** 0.66

37. Printer Failure The lifetime of a printer costing \$200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds? Choose one of the following. ***Source: Society of Actuaries***.

a. 6321 **b.** 7358 **c.** 7869 **d.** 10,256 **e.** 12,642

38. Electronic Device The time to failure of a component in an electronic device has an exponential distribution with a median of four hours. Calculate the probability that the component will work without failing for at least five hours. Choose one of the following. ***Source: Society of Actuaries***.

a. 0.07 **b.** 0.29 **c.** 0.38 **d.** 0.42 **e.** 0.57

Life Sciences

39. Insect Life Span The life span of a certain insect (in days) is uniformly distributed over the interval $[20, 36]$.

a. What is the expected life of this insect?

b. Find the probability that one of these insects, randomly selected, lives longer than 30 days.

40. Location of a Bee Swarm A swarm of bees is released from a certain point. The proportion of the swarm located at least 2 m from the point of release after 1 hour is a random variable that is exponentially distributed with $a = 2$.

a. Find the expected proportion under the given conditions.

b. Find the probability that fewer than $1/3$ of the bees are located at least 2 m from the release point after 1 hour.

41. Digestion Time The digestion time (in hours) of a fixed amount of food is exponentially distributed with $a = 1$.

a. Find the mean digestion time.

b. Find the probability that the digestion time is less than 30 minutes.

42. Pygmy Heights The average height of a member of a certain tribe of pygmies is 3.2 ft, with a standard deviation of 0.2 ft. If the heights are normally distributed, what are the largest and smallest heights of the middle 50% of this population?

43. Finding Prey H. R. Pulliam found that the time (in minutes) required by a predator to find a prey is a random variable that is exponentially distributed, with $\mu = 25$. ***Source: American Naturalist***.

a. According to this distribution, what is the longest time within which the predator will be 90% certain of finding a prey?

b. What is the probability that the predator will have to spend more than 1 hour looking for a prey?

44. Life Expectancy According to the National Center for Health Statistics, the life expectancy for a 55-year-old African American female is 26.1 years. Assuming that from age 55, the survival of African American females follows an exponential distribution, determine the following probabilities. ***Source: National Vital Statistics Report***.

a. The probability that a randomly selected 55-year-old African American female will live beyond 80 years of age (at least 25 more years)

b. The probability that a randomly selected 55-year-old African American female will live less than 20 more years

45. Life Expectancy According to the National Center for Health Statistics, life expectancy for a 70-year-old African American male is 12.3 years. Assuming that from age 70, the survival of African American males follows an exponential distribution, determine the following probabilities. ***Source: National Vital Statistics Report***.

a. The probability that a randomly selected 70-year-old African American male will live beyond 90 years of age

b. The probability that a randomly selected 70-year-old African American male will live between 10 and 20 more years

46. Mercury Poisoning Historians and biographers have collected evidence that suggests that President Andrew Jackson suffered from mercury poisoning. Recently, researchers measured the amount of mercury in samples of Jackson's hair from 1815. The results of this experiment showed that Jackson had a mean mercury level of 6.0 ppm. ***Source: JAMA***.

a. Levels of mercury in hair samples from that time period followed a normal distribution with mean 6.9 and standard deviation 4.6. ***Source: Science of the Total Environment***. Find the probability that a randomly selected person from that time period would have a mercury level of 6.0 ppm or higher. Discuss whether this provides evidence that Jackson suffered from mercury poisoning during this time period.

b. Today's accepted normal mercury levels follow a normal distribution with approximate mean 0.6 ppm and standard deviation 0.3 ppm. ***Source: Clinical Chemistry***. By today's standards, how likely is it that a randomly selected person from today would have a mercury level of 6.0 ppm or higher? Discuss whether we can conclude from this that Andrew Jackson suffered from mercury poisoning.

Social Sciences

47. Dating a Language Over time, the number of original basic words in a language tends to decrease as words become obsolete or are replaced with new words. In 1950, C. Feng and M. Swadesh established that of the original 210 basic ancient Chinese words from 950 A.D., 167 were still being used. The proportion of words that remain after t millennia is a random variable that is exponentially distributed with $a = 0.229$. ***Source: The UMAP Journal***.

a. Find the life expectancy and standard deviation of a Chinese word.

b. Calculate the probability that a randomly chosen Chinese word will remain after 2000 years.

Physical Sciences

48. Rainfall The rainfall (in inches) in a certain region is uniformly distributed over the interval $[32, 44]$.

a. What is the expected number of inches of rainfall?

b. What is the probability that the rainfall will be between 38 and 40 in.?

49. Dry Length Days Researchers have shown that the number of successive dry days that occur after a rainstorm for particular regions of Catalonia, Spain, is a random variable that is distributed exponentially with a mean of 8 days. ***Source: International Journal of Climatology***.

a. Find the probability that 10 or more successive dry days occur after a rainstorm.

b. Find the probability that fewer than 2 dry days occur after a rainstorm.

50. Earthquakes The proportion of the times (in days) between major earthquakes in the north-south seismic belt of China is a random variable that is exponentially distributed, with $a = 1/609.5$. ***Source: Journal of Seismology***.

a. Find the expected number of days and the standard deviation between major earthquakes for this region.

b. Find the probability that the time between a major earthquake and the next one is more than 1 year.

General Interest

51. Soccer The time between goals (in minutes) for the Wolves soccer team in the English Premier League during a recent season can be approximated by an exponential distribution with $a = 1/90$. ***Source: The Mathematical Spectrum***.

a. The Wolves scored their first goal of the season 71 minutes into their first game. Find the probability that the time for a goal is no more than 71 minutes.

b. It was 499 minutes later (in game time) before the Wolves scored their next goal. Find the probability that the time for a goal is 499 minutes or more.

52. Football The margin of victory over the point spread (defined as the number of points scored by the favored team minus the number of points scored by the underdog minus the point spread, which is the difference between the previous two, as predicted by oddsmakers) in National Football League games has been found to be normally distributed with mean 0 and standard deviation 13.861. Suppose New England is favored over Miami by 3 points. What is the probability that New England wins? (*Hint:* Calculate the probability that the margin of victory over the point spread is greater than −3.) ***Source: The American Statistician***.

YOUR TURN ANSWERS

1. 34.5°C, 0.5774
2. (a) 0.9817, (b) 25 and 25, (c) 0.2019
3. (a) 0.4013 (b) 0.3830

18 CHAPTER REVIEW

SUMMARY

In this chapter, we gave a brief introduction to the use of calculus in the study of probability. In particular, the idea of a random variable and its connection to a probability density function and a cumulative distribution function were given. We explored four important concepts:

- expected value (the average value of a random variable that we would expect in the long run),
- variance (a measure of the spread of the values of a distribution),
- standard deviation (the square root of the variance), and
- median (the value of a random variable for which there is a 50% probability of being larger and a 50% probability of being smaller).

Integration techniques were used to determine probabilities, expected value, and variance of continuous random variables. Three probability density functions that have a wide range of applications were studied in detail:

- uniform (when the probability density function remains constant over the sample space),
- exponential (for items that have a constant failure rate over time), and
- normal (for random variables with a bell-shaped distribution).

Probability Density Function on [*a*, *b*]

1. $f(x) \geq 0$ for all x in the interval $[a, b]$.
2. $\displaystyle\int_a^b f(x)\,dx = 1.$
3. $P(c \leq X \leq d) = \displaystyle\int_c^d f(x)dx$ for c, d in $[a, b]$.

Cumulative Distribution Function

$$F(x) = P(X \leq x) = \int_a^x f(t)\,dt$$

Expected Value for a Density Function on [*a*, *b*]

$$E(X) = \mu = \int_a^b xf(x)\,dx$$

Variance for a Density Function on [*a*, *b*]

$$\text{Var}(X) = \int_a^b (x-\mu)^2 f(x)\,dx$$

Alternative Formula for Variance

$$\text{Var}(X) = \int_a^b x^2 f(x)\,dx - \mu^2$$

Standard Deviation

$$\sigma = \sqrt{\text{Var}(X)}$$

Median

The value m such that $\displaystyle\int_a^m f(x)\,dx = \frac{1}{2}$.

Uniform Distribution $f(x) = \dfrac{1}{b-a}$ on $[a, b]$

$$\mu = \frac{1}{2}(b + a) \quad \text{and} \quad \sigma = \frac{1}{\sqrt{12}}(b - a)$$

Exponential Distribution $f(x) = ae^{-ax}$ on $[0, \infty)$

$$\mu = \frac{1}{a} \quad \text{and} \quad \sigma = \frac{1}{a}$$

Normal Distribution $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ on $(-\infty, \infty)$

$$E(X) = \mu \text{ and } \text{Var}(X) = \sigma^2$$

***z*-Scores Theorem** For a normal curve with mean μ and standard deviation σ, the area to the left of x is the same as the area to the left of

$$z = \frac{x - \mu}{\sigma}$$

for the standard normal curve.

KEY TERMS

18.1
random variable
probability function
histogram
discrete probability function
continuous random variable
continuous probability distribution
probability density function
cumulative distribution function

18.2
mean
expected value
variance
standard deviation
median

18.3
uniform distribution
exponential distribution
normal distribution
normal curve
standard normal distribution
z-score

REVIEW EXERCISES

CONCEPT CHECK

Determine whether each of the following statements is true or false, and explain why.

1. A continuous random variable can take on values greater than 1.

2. A probability density function can take on values greater than 1.

3. A continuous random variable can take on values less than 0.

4. A probability density function can take on values less than 0.

5. The expected value of a random variable must always be at least 0.

6. The variance of a random variable must always be at least 0.

7. The expected value of a uniform random variable is the average of the endpoints of the interval over which the density function is positive.

8. For an exponential random variable, the expected value and standard deviation are always equal.

9. The normal distribution and the exponential distribution have approximately the same shape.

10. In the standard normal distribution, the expected value is 1 and the standard deviation is 0.

PRACTICE AND EXPLORATIONS

11. In a probability function, the y-values (or function values) represent _____.

12. Define a continuous random variable.

13. Give the two conditions that a probability density function for $[a, b]$ must satisfy.

14. In a probability density function, the probability that X equals a specific value, $P(X = c)$, is _____.

Decide whether each function defined as follows is a probability density function for the given interval.

15. $f(x) = \sqrt{x}$; $[4, 9]$

16. $f(x) = \dfrac{1}{27}(2x + 4)$; $[1, 4]$

17. $f(x) = 0.7e^{-0.7x}$; $[0, \infty)$

18. $f(x) = 0.4$; $[4, 6.5]$

In Exercises 19 and 20, find a value of k that will make $f(x)$ define a probability density function for the indicated interval.

19. $f(x) = kx^2$; $[1, 4]$ **20.** $f(x) = k\sqrt{x}$; $[4, 9]$

21. The probability density function of a random variable X is defined by

$$f(x) = \frac{1}{10} \quad \text{for } x \text{ in } [10, 20].$$

Find the following probabilities.

a. $P(X \le 12)$ **b.** $P(X \ge 31/2)$ **c.** $P(10.8 \le X \le 16.2)$

22. The probability density function of a random variable X is defined by

$$f(x) = 1 - \frac{1}{\sqrt{x-1}} \quad \text{for } x \text{ in } [2, 5].$$

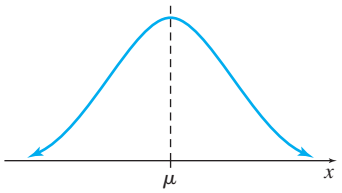
Find the following probabilities.

a. $P(X \ge 3)$ **b.** $P(X \le 4)$ **c.** $P(3 \le X \le 4)$

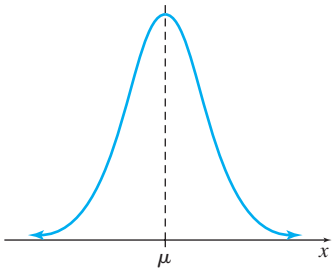
23. Describe what the expected value or mean of a probability distribution represents geometrically.

24. The probability density functions shown in the graphs have the same mean. Which has the smallest standard deviation?

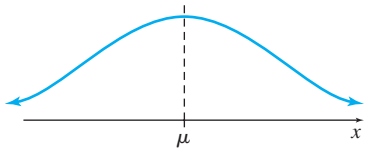
a.



b.



c.



For the probability density functions defined in Exercises 25–28, find (a) the expected value, (b) the variance, (c) the standard deviation, (d) the median, and (e) the cumulative distribution function.

25. $f(x) = \frac{2}{9}(x - 2)$; $[2, 5]$ **26.** $f(x) = \frac{1}{5}$; $[4, 9]$

27. $f(x) = 5x^{-6}$; $[1, \infty)$

28. $f(x) = \frac{1}{20}\left(1 + \frac{3}{\sqrt{x}}\right)$; $[1, 9]$

29. The probability density function of a random variable is defined by $f(x) = 4x - 3x^2$ for x in $[0, 1]$. Find the following for the distribution.

a. The mean **b.** The standard deviation

c. The probability that the value of the random variable will be less than the mean

d. The probability that the value of the random variable will be within 1 standard deviation of the mean

30. Find the median of the random variable of Exercise 29. Then find the probability that the value of the random variable will lie between the median and the mean of the distribution.

For Exercises 31 and 32, find (a) the mean of the distribution, (b) the standard deviation of the distribution, and (c) the probability that the value of the random variable is within 1 standard deviation of the mean.

31. $f(x) = 0.01e^{-0.01x}$ for x in $[0, \infty)$

32. $f(x) = \frac{5}{112}(1 - x^{-3/2})$ for x in $[1, 25]$

In Exercises 33–40, find the proportion of observations of a standard normal distribution for each region.

33. The region to the left of $z = -0.43$

34. The region to the right of $z = 1.62$

35. The region between $z = -1.17$ and $z = -0.09$

36. The region between $z = -1.39$ and $z = 1.28$

37. The region that is 1.2 standard deviations or more below the mean

38. The region that is up to 2.5 standard deviations above the mean

39. Find a z-score so that 52% of the area under the normal curve is to the right of z.

40. Find a z-score so that 21% of the area under the normal curve is to the left of z.

The topics in this short chapter involved much of the material studied earlier in this book, including functions, domain and range, exponential functions, area and integration, improper integrals, integration by parts, and numerical integration. For the following special probability density functions, give

a. the type of distribution;

b. the domain and range;

c. the graph;

d. the mean and standard deviation;

e. $P(\mu - \sigma \le X \le \mu + \sigma)$.

41. $f(x) = 0.05$ for x in $[10, 30]$

42. $f(x) = e^{-x}$ for x in $[0, \infty)$

43. $f(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$ for x in $(-\infty, \infty)$ (*Hint:* $\sigma = 1/\sqrt{2}$.)

44. The chi-square distribution is important in statistics for testing whether data comes from a specified distribution and for testing the independence of two characteristics of a set of data.

When a quantity called the *degrees of freedom* is equal to 4, the probability density function is given by

$$f(x) = \frac{xe^{-x/2}}{4} \text{ for } x \text{ in } [0, \infty).$$

a. Verify that this is a probability density function by noting that $f(x) \geq 0$ and by finding $P(0 \leq X < \infty)$.

b. Find $P(0 \leq X \leq 3)$.

45. When the degrees of freedom in the chi-square distribution (see the previous exercise) is 1, the probability density function is given by

$$f(x) = \frac{x^{-1/2}e^{-x/2}}{\sqrt{2\pi}} \text{ for } x \text{ in } (0, \infty).$$

Calculating probabilities is now complicated by the fact that the density function cannot be antidifferentiated. Numerical integration is complicated because the density function becomes unbounded as x approaches 0.

a. Show that one application of integration by parts (or column integration with just two rows, similar to Example 2 in Section 16.1 on Integration by Parts) allows $P(0 < X \leq b)$ to be rewritten as

$$\frac{1}{\sqrt{2\pi}}\left[2x^{1/2}e^{-x/2}\Big|_0^b + \int_0^b x^{1/2}e^{-x/2}\,dx\right].$$

b. Using Simpson's rule with $n = 12$ in the result from part a, approximate $P(0 < X \leq 1)$.

c. Using Simpson's rule with $n = 12$ in the result from part a, approximate $P(0 < X \leq 10)$.

d. What should be the limit as $b \to \infty$ of the expression in part a? Do the results from parts b and c support this?

APPLICATIONS

Business and Economics

46. Mutual Funds The price per share (in dollars) of a particular mutual fund is a random variable x with probability density function defined by

$$f(x) = \frac{3}{4}(x^2 - 16x + 65) \quad \text{for } x \text{ in } [8, 9].$$

a. Find the probability that the price will be less than \$8.50.

b. Find the expected value of the price.

c. Find the standard deviation.

47. Machine Repairs The time (in years) until a certain machine requires repairs is a random variable t with probability density function defined by

$$f(t) = \frac{5}{112}(1 - t^{-3/2}) \quad \text{for } t \text{ in } [1, 25].$$

a. Find the probability that no repairs are required in the first three years by finding the probability that a repair will be needed in years 4 through 25.

b. Find the expected value for the number of years before the machine requires repairs.

c. Find the standard deviation.

48. Retail Outlets The number of new outlets for a clothing manufacturer is an exponential distribution with probability density function defined by

$$f(x) = \frac{1}{6}e^{-x/6} \quad \text{for } x \text{ in } [0, \infty).$$

Find the following for this distribution.

a. The mean

b. The standard deviation

c. The probability that the number of new outlets will be greater than the mean

49. Product Repairs The number of repairs required by a new product each month is exponentially distributed with an average of 8.

a. What is the probability density function for this distribution?

b. Find the expected number of repairs per month.

c. Find the standard deviation.

d. What is the probability that the number of repairs per month will be between 5 and 10?

50. Useful Life of an Appliance Part The useful life of a certain appliance part (in hundreds of hours) is 46.2, with a standard deviation of 15.8. Find the probability that one such part would last for at least 6000 (60 hundred) hours. Assume a normal distribution.

51. Equipment Insurance A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made. At what level must x be set if the expected payment made under this insurance is to be 1000? *Source: Society of Actuaries*. Choose one of the following.

a. 3858 **b.** 4449 **c.** 5382 **d.** 5644 **e.** 7235

Life Sciences

52. Weight Gain of Rats The weight gain (in grams) of rats fed a certain vitamin supplement is a continuous random variable with probability density function defined by

$$f(x) = \frac{8}{7}x^{-2} \quad \text{for } x \text{ in } [1, 8].$$

a. Find the mean of the distribution.

b. Find the standard deviation of the distribution.

c. Find the probability that the value of the random variable is within 1 standard deviation of the mean.

53. Movement of a Released Animal The distance (in meters) that a certain animal moves away from a release point is exponentially distributed, with a mean of 100 m. Find the probability that the animal will move no farther than 100 m away.

54. Snowfall The snowfall (in inches) in a certain area is uniformly distributed over the interval $[2, 30]$.

a. What is the expected snowfall?

b. What is the probability of getting more than 20 inches of snow?

55. Body Temperature of a Bird The body temperature (in degrees Celsius) of a particular species of bird is a continuous random variable with probability density function defined by

$$f(x) = \frac{3}{19{,}696}(x^2 + x) \quad \text{for } x \text{ in } [38, 42].$$

a. What is the expected body temperature of this species?

b. Find the probability of a body temperature below the mean.

56. Average Birth Weight The average birth weight of infants in the United States is 7.8 lb, with a standard deviation of 1.1 lb. Assuming a normal distribution, what is the probability that a newborn will weigh more than 9 lb?

57. Heart Muscle Tension In a pilot study on tension of the heart muscle in dogs, the mean tension was 2.2 g, with a standard deviation of 0.4 g. Find the probability of a tension of less than 1.9 g. Assume a normal distribution.

58. Life Expectancy According to the National Center for Health Statistics, the life expectancy for a 65-year-old American male is 17.0 years. Assuming that from age 65, the survival of American males follows an exponential distribution, determine the following probabilities. ***Source: National Vital Statistics Report***.

a. The probability that a randomly selected 65-year-old American male will live beyond 80 years of age (at least 15 more years)

b. The probability that a randomly selected 65-year-old American male will live less than 10 more years

59. Life Expectancy According to the National Center for Health Statistics, the life expectancy for a 50-year-old American female is 32.5 years. Assuming that from age 50, the survival of American females follows an exponential distribution, determine the following probabilities. ***Source: National Vital Statistics Report***.

a. The probability that a randomly selected 50-year-old American female will live beyond 90 years of age (at least 40 more years)

b. The probability that a randomly selected 50-year-old American female will live between 30 and 50 more years

Social Sciences

60. Assaults The number of deaths in the United States caused by assault (murder) for each age group is given in the following table. ***Source: National Vital Statistics***.

Age Interval (years)	Midpoint of Interval (year)	Number Dying in Each Interval
0–14	7	1096
15–24	19.5	5729
25–34	29.5	4729
35–44	39.5	3013
45–54	49.5	2207
55–64	59.5	1011
65–74	69.5	397
75–84	79.5	274
85 +	89.5 (est)	32
Total		18,488

a. Plot the data. What type of function appears to best match this data?

b. Use the regression feature on your graphing calculator to find a quartic equation that models the number of years, t, since birth and the number of deaths caused by assault, $N(t)$. Use the midpoint value to estimate the point in each interval when the person died. Graph the function with the plot of the data. Does the function resemble the data?

c. By finding an appropriate constant k, find a function $S(t) = kN(t)$ that is a probability density function describing the probability of death by assault. (*Hint:* Because the function in part b is negative for values less than 5.2 and greater than 88.9, restrict the domain of the density function to the interval $[5.2, 88.9]$. That is, integrate the function you found in part b from 5.2 to 88.9.)

d. For a randomly chosen person who was killed by assault, find the probabilities that the person killed was less than 25 years old, at least 45 but less than 65 years old, and at least 75 years old, and compare these with the actual probabilities.

e. Estimate the expected age at which a person will die by assault.

f. Find the standard deviation of this distribution.

Physical Sciences

61. Earthquakes The time between major earthquakes in the Taiwan region is a random variable with probability density function defined by

$$f(t) = \frac{1}{3650.1} e^{-t/3650.1},$$

where t is measured in days. Find the expected value and standard deviation of this probability density function. ***Source: Journal of Seismology***.

General Interest

62. State-Run Lotteries The average state "take" on lotteries is 40%, with a standard deviation of 13%. Assuming a normal distribution, what is the probability that a state-run lottery will have a "take" of more than 50%?

EXTENDED APPLICATION

EXPONENTIAL WAITING TIMES

We have seen in this chapter how probabilities that are spread out over continuous time intervals can be modeled by continuous probability density functions. The exponential distribution you met in the last section of this chapter is often used to model *waiting times,* the gaps between events that are randomly distributed in time, such as decays of a radioactive nucleus or arrivals of customers in the waiting line at a bank. In this application we investigate some properties of the exponential family of distributions.

Suppose that in a badly run subway system, the times between arrivals of subway trains at your station are exponentially distributed with a mean of 10 minutes. Sometimes trains arrive very close together, sometimes far apart, but if you keep track over many days, you'll find that the *average* time between trains is 10 minutes. According to the last section of this chapter, the exponential distribution with density function $f(t) = ae^{-at}$ has mean $1/a$, so the probability density function for our interarrival times is

$$f(t) = \frac{1}{10}e^{-t/10}.$$

First let's see what these waiting times look like. We have used a random-number generator from a statistical software package to draw 25 waiting times from this distribution. Figure 19 shows cumulative arrival times, which is what you would observe if you recorded the arrival time of each train measured in minutes from an arbitrary 0 point.

You can see that 25 trains arrive in a span of about 260 minutes, so the average interarrival time was indeed close to 10 minutes. You may also notice that there are some large gaps and some cases where trains arrived very close together.

To get a better feeling for the distribution of long and short interarrival times, look at the following list, which gives the 25 interarrival times in minutes, sorted from smallest to largest.

0.016	4.398	15.659
0.226	4.573	15.954
0.457	5.415	16.403
0.989	9.570	18.978
1.576	10.413	20.736
1.988	10.916	33.013
2.738	13.109	39.073
3.133	13.317	
3.895	14.622	

You can see that there were some very short waits. (In fact, the shortest time between trains is only 1 second, which means our model needs to be adjusted somehow to allow for the time trains spend stopped in the station.) The longest time between trains was 39 minutes, almost four times as long as the average! Although the exponential model exaggerates the irregularities of typical subway service, the problem of pile-ups and long gaps is very real for public transportation, especially for bus routes that are subject to unpredictable traffic delays. Anyone who works at a customer service job is also familiar with this behavior: The waiting line at a bank may be empty for minutes at a stretch, and then several customers walk in at nearly the same time. In this case, the customer interarrival times are exponentially distributed.

Planners who are involved with scheduling need to understand this "clumping" behavior. One way to explore it is to find probabilities for ranges of interarrival times. Here integrals are the natural tool. For example, if we want to estimate the fraction of interarrival times that will be less than 2 minutes, we compute

$$\frac{1}{10}\int_0^2 e^{-t/10}\,dt = 1 - e^{-1/5} \approx 0.1813.$$

So on average, 18% of the interarrival times will be less than 2 minutes, which indicates that clustering of trains will be a problem in our system. (If you have ridden a system like the one in New York City, you may have boarded a train that was ordered to "stand by" for several minutes to spread out a cluster of trains.) We can also compute the probability of a gap of 30 minutes or longer. It will be

$$\frac{1}{10}\int_{30}^{\infty} e^{-t/10}\,dt = e^{-3} \approx 0.0498.$$

So in a random sample of 25 interarrival times we might expect one or two long waits, and our simulation, which includes times of 33 and 39 minutes, is not a fluke. Of course, the rider's experience depends on when she arrives at the station, which is another random input to our model. If she arrives in the middle of a cluster, she'll get a train right away, but if she arrives at the beginning of a long gap she may have a half-hour wait. So we would also like to model the rider's *waiting time,* the time between the rider's arrival at the station and the arrival of the next train.

A remarkable fact about the exponential distribution is that if our passenger arrives at the station at a random time, the distribution of the rider's waiting times is *the same* as the distribution of

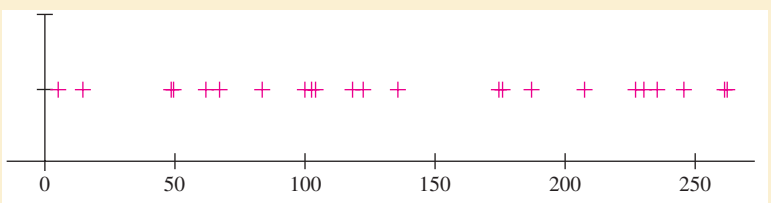


FIGURE 19

interarrival times (that is, exponential with mean 10 minutes). At first this seems paradoxical; since she usually arrives between trains, she should wait less, on average, than the average time between trains. But remember that she's more likely to arrive at the station in one of those long gaps. In our simulation, 72 out of 260 minutes is taken up with long gaps, and even if the rider arrives at the middle of such a gap she'll still wait longer than 15 minutes. Because of this feature the exponential distribution is often called *memoryless:* If you dip into the process at random, it is as if you were starting all over. If you arrive at the station just as a train leaves, your waiting time for the next one still has an exponential distribution with mean 10 minutes. The next train doesn't "know" anything about the one that just left.*

Because the riders' waiting times are exponential, the calculations we have already made tell us what riders will experience: A wait of less than 2 minutes has probability about 0.18. The average wait is 10 minutes, but long waits of more than 30 minutes are not all that rare (probability about 0.05).

Customers waiting for service care about the average wait, but they may care even more about the *predictability* of the wait. In this chapter we stated that the standard deviation for an exponential distribution is the same as the mean, so in our model the standard deviation of riders' waiting times will be 10 minutes. This indicates that a wait of twice the average length is not a rare event. (See Exercise 3.)

Let's compare the experience of riders on our exponential subway with the experience of riders of a perfectly regulated service in which trains arrive *exactly* 10 minutes apart. We'll still assume that the passenger arrives at random. But now the waiting time is uniformly distributed on the time interval $[0 \text{ minutes}, 10 \text{ minutes}]$. This uniform distribution has density function

$$f(t) = \begin{cases} \dfrac{1}{10} & \text{for } 0 \le t \le 10 \\ 0 & \text{otherwise} \end{cases}$$

The mean waiting time is

$$\int_0^{10} \frac{1}{10} \cdot t\, dt = 5 \text{ minutes}$$

and the standard deviation of the waiting times is

$$\sqrt{\int_0^{10} \frac{(t-5)^2}{10}\, dt} = \sqrt{\frac{25}{3}} \approx 2.89 \text{ minutes.}$$

Clearly the rider has a better experience on this system. Even though the same average number of trains is running per hour as in the exponential subway, the average wait for the uniform subway is only 5 minutes with a standard deviation of 2.89 minutes, and no one ever waits longer than 10 minutes!

*See Chapter 1 in Volume 2 of Feller, William, *An Introduction to Probability Theory and Its Applications,* 2nd ed., New York: Wiley, 1971.

Any subway run is subject to unpredictable accidents and variations, and this random input is always pushing the riders' waiting times toward the exponential model. Indeed, even with uniform scheduling of trains, there will be service bottlenecks because the exponential distribution is also a reasonable model (over a short time period) for interarrival times of *passengers* entering the station. The goal of schedulers is to move passengers efficiently in spite of random train delays and random input of passengers. One proposed solution, the PRT or personal rapid transit system, uses small vehicles holding just a few passengers that can be scheduled to match a fluctuating demand.

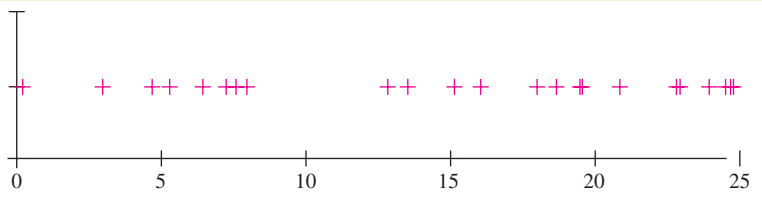
The subway scheduling problem is part of a branch of statistics called *queueing theory*, the study of any process in which inputs arrive at a service point and wait in a line or queue to be served. Examples include telephone calls arriving at a customer service center, our passengers entering the subway station, packets of information traveling through the Internet, and even pieces of code waiting for a processor in a multiprocessor computer. The following Web sites provide a small sampling of work in this very active research area.

- *http://web2.uwindsor.ca/math/hlynka/queue.html* (A collection of information on queueing theory)
- *http://faculty.washington.edu/jbs/itrans/ingsim.htm* (an article on scheduling a PRT)

EXERCISES

1. If X is a continuous random variable, $P(a \le X \le b)$ is the same as $P(a < X < b)$. Since these are different events, how can they have the same probability?
2. Someone who rides the subway back and forth to work each weekday makes about 40 trips a month. On the exponential subway, how many times a month can this commuter expect a wait longer than half an hour?
3. Find the probability that a rider of the exponential subway waits more than 20 minutes for a train; that is, find the probability of a wait more than twice as long as the average.
4. On the exponential subway, what is the probability that a randomly arriving passenger has a wait of between 9 and 10 minutes? What is the corresponding probability on the uniform subway?
5. If our system is aiming for an average interarrival time of 10 minutes, we might set a tolerance of plus or minus 2 minutes and try to keep the interarrival times between 8 and 12 minutes. Under the exponential model, what fraction of interarrival times fall in this range? How about under the uniform model?

6. Most mathematical software includes routines for generating "pseudo-random" numbers (that is, numbers that behave randomly even though they are generated by arithmetic). That's what we used to simulate the exponential waiting times for our subway system. But a source on the Internet (http://www.fourmilab.ch/hotbits/) delivers random numbers based on the times between decay events in a sample of Krypton-85. As noted above, the waiting times between decay events have an exponential distribution, so we can see what nature's random numbers look like. Here's a short sample:



Actually, this source builds its random numbers from random bits, that is, 0's and 1's that occur with equal probability. See if you can think of a way of turning a sequence of exponential waiting times into a random sequence of 0's and 1's.

DIRECTIONS FOR GROUP PROJECT

Find a situation in which you and your group can gather actual wait times, such as a bus stop, doctor's office, teller line at a bank, or check-out line at a grocery store. Collect data on interarrival/service times and determine the mean service time. Using this average, determine whether the data appears to follow an exponential distribution. Develop a table that lists the percentage of the time that particular waiting times occur using both the data and the exponential function. Construct a poster that could be placed near the location where people wait that estimates the waiting time for service.

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Appendix A

Solutions to Prerequisite Skills Diagnostic Test *(with references to Ch. R)*

For more practice on the material in questions 1–4, see *Beginning and Intermediate Algebra* (5th ed.) by Margaret L. Lial, John Hornsby, and Terry McGinnis, Pearson, 2012.

1. $10/50 = 0.20 = 20\%$

2.
$$\begin{aligned} \frac{13}{7} - \frac{2}{5} &= \frac{13}{7}\cdot\frac{5}{5} - \frac{2}{5}\cdot\frac{7}{7} && \text{Get a common denominator.} \\ &= \frac{65}{35} - \frac{14}{35} \\ &= \frac{51}{35} \end{aligned}$$

3. The total number of apples and oranges is $x + y$, so $x + y = 75$.

4. The sentence can be rephrased as "The number of students is at least four times the number of professors," or $s \geq 4p$.

5.
$$\begin{aligned} 7k + 8 &= -4(3 - k) \\ 7k + 8 &= -12 + 4k && \text{Multiply out.} \\ 7k - 4k + 8 - 8 &= -12 - 8 + 4k - 4k && \text{Subtract 8 and } 4k \text{ from both sides.} \\ 3k &= -20 && \text{Simplify.} \\ k &= -20/3 && \text{Divide both sides by 3.} \end{aligned}$$

For more practice, see Sec. R.4.

6.
$$\begin{aligned} \frac{5}{8}x + \frac{1}{16}x &= \frac{11}{16} + x \\ \frac{5}{8}x + \frac{1}{16}x - x &= \frac{11}{16} && \text{Subtract } x \text{ from both sides.} \\ \frac{5x}{8}\cdot\frac{2}{2} + \frac{x}{16} - x\cdot\frac{16}{16} &= \frac{11}{16} && \text{Get a common denominator.} \\ \frac{10x}{16} + \frac{x}{16} - \frac{16x}{16} &= \frac{11}{16} && \text{Simplify.} \\ \frac{-5x}{16} &= \frac{11}{16} && \text{Simplify.} \\ x &= \frac{11}{16}\cdot\frac{16}{-5} = -\frac{11}{5} && \text{Multiply both sides by the reciprocal of } -5/16. \end{aligned}$$

For more practice, see Sec. R.4.

7. The interval $-2 < x \leq 5$ is written as $(-2, 5]$. For more practice, see Sec. R.5.

8. The interval $(-\infty, -3]$ is written as $x \leq -3$. For more practice, see Sec. R.5.

9.
$$\begin{aligned} 5(y - 2) + 1 &\leq 7y + 8 \\ 5y - 9 &\leq 7y + 8 && \text{Multiply out and simplify.} \\ 5y - 7y - 9 + 9 &\leq 7y - 7y + 8 + 9 && \text{Subtract } 7y \text{ from both sides and add 9.} \\ -2y &\leq 17 && \text{Simplify.} \\ y &\geq -17/2 && \text{Divide both sides by 3.} \end{aligned}$$

10.
$$\begin{aligned} \frac{2}{3}(5p - 3) &> \frac{3}{4}(2p + 1) \\ \frac{10p}{3} - 2 &> \frac{3p}{2} + \frac{3}{4} && \text{Multiply out and simplify.} \\ \frac{10p}{3} - \frac{3p}{2} - 2 + 2 &> \frac{3p}{2} - \frac{3p}{2} + \frac{3}{4} + 2 && \text{Subtract } 3p/2 \text{ from both sides, and add 2.} \\ \frac{10p}{3}\cdot\frac{2}{2} - \frac{3p}{2}\cdot\frac{3}{3} &> \frac{3}{4} + 2\cdot\frac{4}{4} && \text{Simplify and get a common denominator.} \\ \frac{11p}{6} &> \frac{11}{4} && \text{Simplify.} \\ p &> \frac{11}{4}\cdot\frac{6}{11} && \text{Multiply both sides by the reciprocal of } 11/6. \\ p &> \frac{3}{2} && \text{Simplify.} \end{aligned}$$

For more practice, see Sec. R.5.

11. $(5y^2 - 6y - 4) - 2(3y^2 - 5y + 1) = 5y^2 - 6y - 4 - 6y^2 + 10y - 2 = -y^2 + 4y - 6$. For more practice, see Sec. R.1.

12. $(x^2 - 2x + 3)(x + 1) = (x^2 - 2x + 3)x + (x^2 - 2x + 3) = x^3 - 2x^2 + 3x + x^2 - 2x + 3 = x^3 - x^2 + x + 3$. For more practice, see Sec. R.1.

13. $(a - 2b)^2 = a^2 - 4ab + 4b^2$. For more practice, see Sec. R.1.

14. $3pq + 6p^2q + 9pq^2 = 3pq(1 + 2p + 3q)$. For more practice, see Sec. R.2.

15. $3x^2 - x - 10 = (3x + 5)(x - 2)$. For more practice, see Sec. R.2.

16.
$$\frac{a^2 - 6a}{a^2 - 4} \cdot \frac{a - 2}{a} = \frac{a(a - 6)}{(a - 2)(a + 2)} \cdot \frac{a - 2}{a} \qquad \text{Factor.}$$
$$= \frac{a - 6}{a + 2} \qquad \text{Simplify.}$$

For more practice, see Sec. R.3.

17.
$$\frac{x + 3}{x^2 - 1} + \frac{2}{x^2 + x} = \frac{x + 3}{(x - 1)(x + 1)} + \frac{2}{x(x + 1)} \qquad \text{Factor.}$$
$$= \frac{x + 3}{(x - 1)(x + 1)} \cdot \frac{x}{x} + \frac{2}{x(x + 1)} \cdot \frac{x - 1}{x - 1} \qquad \text{Get a common denominator.}$$
$$= \frac{x^2 + 3x}{x(x - 1)(x + 1)} + \frac{2x - 2}{x(x + 1)(x - 1)} \qquad \text{Multiply out.}$$
$$= \frac{x^2 + 5x - 2}{x(x - 1)(x + 1)} \qquad \text{Add fractions.}$$

For more practice, see Sec. R.3.

18.
$$3x^2 + 4x = 1$$
$$3x^2 + 4x - 1 = 0 \qquad \text{Subtract 1 from both sides.}$$
$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 3(-1)}}{2 \cdot 3} \qquad \text{Use the quadratic formula.}$$
$$= \frac{-4 \pm \sqrt{28}}{6} \qquad \text{Simplify.}$$
$$= \frac{-2 \pm \sqrt{7}}{2} \qquad \text{Simplify.}$$

For more practice, see Sec. R.4.

19. First solve the corresponding equation

$$\frac{8x}{z + 3} = 2$$
$$8z = 2(z + 3) \qquad \text{Multiply both sides by } (z + 3).$$
$$8z = 2z + 6 \qquad \text{Multiply out.}$$
$$6z = 6 \qquad \text{Subtract } 2z \text{ from both sides.}$$
$$z = 1 \qquad \text{Divide both sides by 6.}$$

The fraction may also change from being less than 2 to being greater than 2 when the denominator equals 0, namely, at $z = -3$. Testing each of the intervals determined by the numabers -3 and 0 shows that the fraction on the left side of the inequality is less than or equal to 2 on $(-\infty,-3) \cup [1,\infty)$. We do not include $x = -3$ in the solution because that would make the denominator 0. For more practice, see Sec. R.5.

20.
$$\frac{4^{-1}(x^2y^3)^2}{x^{-2}y^5} = \frac{x^2(x^4y^6)}{4y^5} \qquad \text{Simplify.}$$
$$= \frac{x^6y}{4} \qquad \text{Simplify.}$$

For more practice, see Sec. R.6.

21.
$$\frac{4^{1/4}(p^{2/3}q^{-1/3})^{-1}}{4^{-1/4}p^{4/3}q^{4/3}} = \frac{4^{1/2}p^{-2/3}q^{1/3}}{p^{4/3}q^{4/3}}$$ **Simplify.**

$$= \frac{2}{p^2q}$$ **Simplify.**

For more practice, see Sec. R.6.

22.
$$k^{-1} - m^{-1} = \frac{1}{k}\cdot\frac{m}{m} - \frac{1}{m}\cdot\frac{k}{k}$$ **Get a common denominator.**

$$= \frac{m-k}{km}$$ **Simplify.**

For more practice, see Sec. R.6.

23. $(x^2+1)^{-1/2}(x+2) + 3(x^2+1)^{1/2} = (x^2+1)^{-1/2}[x+2+3(x^2+1)] = (x^2+1)^{-1/2}(3x^2+x+5)$. For more practice, see Sec. R.6.

24. $\sqrt[3]{64b^6} = 4b^2$. For more practice, see Sec. R.7.

25. $\dfrac{2}{4-\sqrt{10}}\cdot\dfrac{4+\sqrt{10}}{4+\sqrt{10}} = \dfrac{2(4+\sqrt{10})}{16-10} = \dfrac{4+\sqrt{10}}{3}$. For more practice, see Sec. R.7.

26. $\sqrt{y^2-10y+25} = \sqrt{(y-5)^2} = |y-5|$. For more practice, see Sec. R.7.

Appendix B

Learning Objectives

CHAPTER R: Algebra Reference

R.1: Polynomials

1. Simplify polynomials

R.2: Factoring

1. Factor polynomials

R.3: Rational Expressions

1. Simplify rational expression using properties of rational expressions and order of operations

R.4: Equations

1. Solve linear, quadratic, and rational equations

R.5: Inequalities

1. Solve linear, quadratic, and rational inequalities
2. Graph the solution of linear, quadratic, and rational inequalities
3. Write the solutions of linear, quadratic, and rational inequalities using interval notation

R.6: Exponents

1. Evaluate exponential expressions
2. Simplify exponential expressions

R.7: Radicals

1. Simplify radical expressions
2. Rationalize the denominator in radical expressions
3. Rationalize the numerator in radical expressions

CHAPTER 1: Linear Functions

1.1: Slopes and Equations of Lines

1. Find the slope of a line
2. Find the equation of a line using a point and the slope
3. Find the equation of parallel and perpendicular lines
4. Graph the equation of a line
5. Solve application problems using linear functions

1.2: Linear Functions and Applications

1. Evaluate linear functions
2. Write equations for linear models
3. Solve application problems

1.3: The Least Squares Line

1. Interpret the different value meanings for linear correlation
2. Draw a scatterplot
3. Calculate the correlation coefficient
4. Calculate the least squares line
5. Compute the response variable in a linear model

CHAPTER 2: Systems of Linear Equations and Matrices

2.1: Solution of Linear Systems by the Echelon Method

1. Apply system transformation operations on matrices
2. Solve linear systems using the echelon method
3. Solve application problems

2.2: Solution of Linear Systems by the Gauss-Jordan Method

1. Perform row operations on matrices
2. Solve linear systems by the Gauss-Jordan method
3. Solve application problems

2.3: Addition and Subtraction of Matrices

1. Identify the size of a matrix
2. Add and subtract matrices
3. Solve application problems

2.4: Multiplication of Matrices

1. Multiply a matrix by a constant value
2. Find the product two matrices
3. Simplify matrix expressions
4. Solve application problems

2.5: Matrix Inverses

1. Determine if two matrices are inverses of each other
2. Find the inverse of a matrix (if it exists)
3. Solve a system by using the inverse
4. Solve application problems

2.6: Input-Output Models

1. Find the production matrix for an input-output model
2. Find the demand matrix for an input-output model
3. Solve application problems

CHAPTER 3: Linear Programming: The Graphical Method

3.1: Graphing Linear Inequalities

1. Graph linear inequalities
2. Graph a system of linear inequalities

3. Determine the feasible region for a system of linear inequalities
4. Solve application problems

3.2: Solving Linear Programming Problems Graphically

1. Determine corner points
2. Solve linear programming problems graphically

3.3: Applications of Linear Programming

1. Solve application problems

CHAPTER 4: Linear Programming: The Simplex Method

4.1: Slack Variables and the Pivot

1. Determine the number of slack variables needed for a linear programming problem
2. Add slack variables to a linear programming problem
3. Generate the initial simplex tableau
4. Identify the pivot and find the resulting matrix
5. Solve linear programming problems using the simplex tableau
6. Solve application problems

4.2: Maximization Problems

1. Solve maximization problems using the simplex tableau and simplex method
2. Solve maximization application problems

4.3: Minimization Problems; Duality

1. Find the transpose of a matrix
2. Generate the dual problem
3. Solve minimization problems using the simplex method
4. Solve minimization application problems

4.4: Nonstandard Problems

1. Solve nonstandard linear programming problems

CHAPTER 5: Mathematics of Finance

5.1: Simple and Compound Interest

1. Compute the simple and compound interest
2. Compute the effective rate
3. Solve application problems

5.2: Future Value of an Annuity

1. Find terms of a geometric sequence
2. Find the sum of a geometric sequence
3. Find the future value of an annuity
4. Find the future value of an annuity due
5. Find the payment for a sinking fund
6. Solve application problems

5.3: Present Value of an Annuity; Amortization

1. Find the present value of an annuity
2. Find the payment needed for an amortized load
3. Generate an amortization table
4. Solve application problems

CHAPTER 6: Logic

6.1: Statements

1. Identify statements and compound statements
2. Generate a negation for a statement
3. Convert between statements and symbols
4. Determine the truth value of a compound statement
5. Solve application problems

6.2: Truth Tables and Equivalent Statements

1. Generate a truth table for a compound statement
2. Write the negation for a compound statement
3. Simplify compound statements using De Morgan's law
4. Determine if compound statements are true or false
5. Solve application problems

6.3: The Conditional and Circuits

1. Identify tautologies
2. Identify contradictions
3. Write equivalent statements
4. Simplify circuit problems
5. Draw circuits for statements
6. Solve application problems

6.4: More on the Conditional

1. Generate inverse, converse, and contrapositive statement
2. Solve application problems

6.5: Analyzing Arguments and Proofs

1. Identify valid and invalid arguments
2. Use truth tables to prove rules of logic
3. Solve application problems

6.6: Analyzing Arguments with Quantifiers

1. Use Euler diagrams to test the validity of arguments
2. Construct valid arguments based on Euler diagrams
3. Solve application problems

CHAPTER 7: Sets and Probability

7.1: Sets

1. Understand set notation and terminology
2. Find the union and intersection of sets
3. Solve application problems

7.2: Applications of Venn Diagrams

1. Draw and interpret the Venn diagram for a set
2. Find the number of elements in a set
3. Solve application problems

7.3: Introduction to Probability

1. Generate the sample space for a given experiment
2. Find the probability of an event
3. Identify empirical probabilities
4. Solve application problems

7.4: Basic Concepts of Probability

1. Identify mutually exclusive events
2. Find the probability of an event using probability rules
3. Find the odds of an event
4. Solve application problems

7.5: Conditional Probability; Independent Events

1. Identify independent events
2. Use the product rule to find the probability of an event
3. Find the conditional probability of an event
4. Solve application problems

7.6: Bayes' Theorem

1. Use Bayes' theorem
2. Solve application problems

CHAPTER 8: Counting Principles, Further Probability Topics

8.1: The Multiplication Principle; Permutations

1. Compute the factorial of a number
2. Use the multiplication principle to count outcomes
3. Find the permutation of an outcome
4. Solve application problems

8.2: Combinations

1. Find the combinations of an outcome
2. Solve problems using combinations or permutations
3. Solve application problems

8.3: Probability Applications of Counting Principles

1. Solve probability applications using counting principles

8.4: Binomial Probability

1. Find the probability of binomial experiments
2. Use technology to find binomial probabilities
3. Solve application problems

8.5: Probability Distribution; Expected Value

1. Determine the random variable for a probability distribution
2. Draw a histogram for a probability distribution
3. Generate probability distributions
4. Find the expected value of a probability distribution
5. Solve application problems

CHAPTER 9: Statistics

9.1: Frequency Distributions; Measures of Central Tendency

1. Generate a frequency distribution table
2. Find the mean, median, and mode of a data set
3. Solve application problems

9.2: Measures of Variation

1. Find the range of a data set
2. Find the standard deviation of a data set
3. Solve application problems

9.3: The Normal Distribution

1. Find the area (probability) under the normal curve
2. Understand the standard normal curve
3. Find the z-score associated with a given area (probability) under the normal curve
4. Solve application problems

9.4: Normal Approximation to the Binomial Distribution

1. Use the normal curve distribution to find binomial distribution probabilities
2. Solve application problems

CHAPTER 10: Nonlinear Functions

10.1: Properties of Functions

1. Evaluate nonlinear functions
2. Identify the domain and range of functions
3. Classify functions as odd or even
4. Solve application problems

10.2: Quadratic Functions; Translation and Reflection

1. Complete the square for a given quadratic function
2. Obtain the vertex, y-intercepts and x-intercepts
3. Perform translation and reflection rules to graphs of functions
4. Solve application problems

10.3: Polynomial and Rational Functions

1. Perform translation and reflection rules
2. Identify the degree of a polynomial
3. Find horizontal and vertical asymptotes
4. Solve application problems

10.4: Exponential Functions

1. Evaluate exponential functions
2. Solve exponential equations
3. Graph exponential functions
4. Solve application problems

10.5: Logarithmic Functions

1. Convert expressions between exponential and logarithmic forms
2. Evaluate logarithmic expressions
3. Simplify expressions using logarithmic properties
4. Solve logarithmic equations
5. Identify the domain of a logarithmic function
6. Solve application problems

10.6: Applications: Growth and Decay; Mathematics of Finance

1. Obtain the effective rate
2. Compute future values
3. Solve growth and decay application problems
4. Solve finance application problems

CHAPTER 11: The Derivative

11.1: Limits

1. Determine if the limit exists
2. Evaluate limits by completing tables and graphically
3. Use limit rules to evaluate limits
4. Solve application problems

11.2: Continuity

1. Identify and apply the conditions for continuity
2. Find points of discontinuity
3. Solve application problems

11.3: Rates of Change

1. Find the average rate of change for functions
2. Find the instantaneous rate of change
3. Solve application problems

11.4: Definition of the Derivative

1. Obtain the slope of the tangent line
2. Obtain the equation of the secant line
3. Find the derivative of a function (using the definition of the derivative)
4. Evaluate the derivative
5. Solve application problems

11.5: Graphical Differentiation

1. Interpret graph of a function and its derivative
2. Sketch the graph of the derivative
3. Solve application problems

CHAPTER 12: Calculating the Derivative

12.1: Techniques for Finding Derivative

1. Determine the derivative of given functions
2. Find the slope of the tangent line
3. Obtain horizontal tangent points
4. Solve application problems

12.2: Derivative of Products and Quotients

1. Apply the product rule to find the derivative
2. Apply the quotient rule to find the derivative
3. Solve application problems

12.3: The Chain Rule

1. Find the composition of two functions
2. Find the derivative using the chain rule
3. Solve application problems

12.4: Derivatives of Exponential Functions

1. Find the derivative of exponential functions
2. Solve application problems

12.5: Derivatives of Logarithmic Functions

1. Find the derivative of logarithmic functions
2. Solve application problems

CHAPTER 13: Graphs and the Derivative

13.1: Increasing and Decreasing Functions

1. Identify increasing and decreasing intervals for functions
2. Solve application problems

13.2: Relative Extrema

1. Identify relative extrema points graphically
2. Apply the first derivative test
3. Solve application problems

13.3: Higher Derivatives, Concavity and the Second Derivative Test

1. Find higher order derivatives
2. Identify intervals of upward and downward concavity
3. Identify inflection points
4. Apply the second derivative test
5. Solve application problems

13.4: Curve Sketching

1. Graph a function based on derivative (first and second) information

CHAPTER 14: Applications of the Derivative

14.1: Absolute Extrema

1. Identify absolute extrema locations graphically
2. Find absolute extrema points
3. Solve application problems

14.2: Applications of Extrema

1. Solve maxima/minima application problems

14.3: Further Business Applications: Economic Lot Size; Economic Order Quantity; Elasticity of Demand

1. Solve business and economics application problems

14.4: Implicit Differentiation

1. Differentiate functions of two variables implicitly
2. Find the equation of the tangent line at a given point
3. Solve application problems

14.5: Related Rates

1. Use related rates to solve application problems

14.6: A Differentials: Linear Approximation

1. Obtain differentials of dependent variables
2. Compute approximations using differentials
3. Solve application problems

CHAPTER 15: Integration

15.1: Antiderivative

1. Obtain the antiderivative of a function
2. Solve application problems

15.2: Substitution

1. Find the indefinite integral using the substitution methods
2. Solve application problems

15.3: Area and the Definite Integral

1. Understand the difference between indefinite and definite integrals
2. Approximate the area under the graph of a function using sums
3. Find the area under the graph of a function by integration
4. Solve application problems

15.4: The Fundamental Theorem of Calculus

1. Evaluate the definite integral of a function
2. Find the area covered by the graph of a function over an interval
3. Solve application problems

15.5: The Area Between Two Curves

1. Find the area between two curves
2. Solve application problems

15.6: Numerical Integration

1. Use approximation rules to evaluate the integral numerically
2. Solve application problems

CHAPTER 16: Further Techniques and Applications of Integration

16.1: Integration by Parts

1. Compute the integral using the integration by parts technique
2. Solve application problems

16.2: Volume and Average Value

1. Find the volume of a solid formed by rotation about the x-axis
2. Find the average value of a function
3. Solve application problems

16.3: Continuous Money Flow

1. Compute the present and future value of an annuity (money flow)
2. Solve application problems

16.4: Improper Integrals

1. Determine the convergence or divergence of improper integrals
2. Find the area between the graph of a function over an infinite interval, when possible
3. Solve application problems

16.5: Solutions of Elementary and Separable Differential Equations

1. Find the general solution of a separable differential equations
2. Find the particular solution of a separable differential equations
3. Solve application problems

CHAPTER 17: Multivariable Calculus

17.1: Functions of Several Variables

1. Evaluate functions with several variables
2. Graph functions with several variables
3. Solve application problems

17.2: Partial Derivatives

1. Find the partial derivative of functions with several variables
2. Find the second partial derivative of functions with several variables
3. Solve application problems

17.3: Maxima and Minima

1. Find relative extrema points
2. Solve application problems

17.4: Lagrange Multipliers

1. Use Lagrange multipliers to find relative extrema points
2. Solve application problems

17.5: Total Differentials and Approximations

1. Find the total differential of functions with several variables
2. Solve application problems

17.6: Double Integrals

1. Evaluate double integrals
2. Find the volume under a given surface
3. Solve application problems

CHAPTER 18: Probability and Calculus

18.1: Continuous Probability Models

1. Identify probability density functions
2. Find the cumulative distribution function of probability density functions
3. Solve application problems

18.2: Expected Value and Variance of Continuous Random Variables

1. Find the expected value and variance of probability density functions
2. Find the median of a random variable with a given probability density function
3. Solve application problems

18.3: Special Probability Density Functions

1. Find mean and standard deviation of a probability distribution
2. Compute the z-score
3. Solve application problems

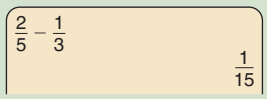
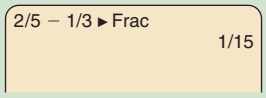
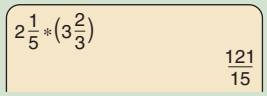
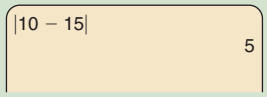
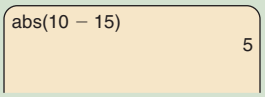
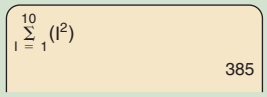
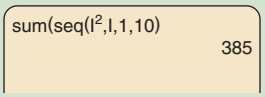
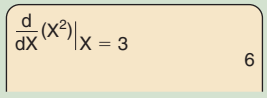
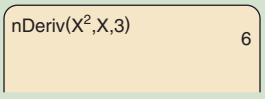
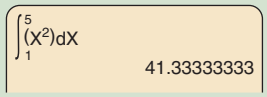
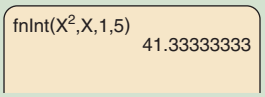
Appendix C

MathPrint Operating System for TI-84 and TI-84 Plus Silver Edition

The graphing calculator screens in this text display math in the format of the TI MathPrint operating system. With MathPrint, the math looks more like that seen in a printed book. You can obtain MathPrint and install it by following the instructions in the *Graphing Calculator and Spreadsheet Manual*. Only the TI-84 family of graphing calculators can be updated with the MathPrint operating system. If you own a TI-83 graphing calculator, you can use this brief appendix to help you "translate" what you see in the Classic mode shown on your calculator.

Translating between MathPrint Mode and Classic Mode

The following table compares displays of several types in MathPrint mode and Classic mode (on a calculator without MathPrint installed).

Feature	MathPrint	Classic (MathPrint not installed)
Improper fractions	 Press ALPHA and F1 and select 1: n/d.	 Enter an expression and press MATH and select 1: ▶Frac.
Mixed fractions	 Press ALPHA and F1 and select 2: Un/d.	Not Supported
Absolute values	 Press ALPHA and F2 and select 1:abs(.	 Press MATH and ▶ and select 1:abs(.
Summation	 Press ALPHA and F2 and select 2:Σ(.	 Press 2ND and LIST and then ▶ twice and select 5:sum(for sum, and press 2ND and LIST and then ▶ and select 5:seq(for sequence.
Numerical derivatives	 Press ALPHA and F2 and select 3:nDeriv(.	 Press MATH and select 8:nDeriv(.
Numerical values of integrals	 Press ALPHA and F2 and select 4:fnInt(.	 Press MATH and select 9:fnInt(.

(*continued*)

Feature	MathPrint	Classic (MathPrint not installed)
Logarithms	$\log_2(32)$ → 5 Press ALPHA and F2 and select 5:logBASE(.	Evaluating logs with bases other than 10 or e cannot be done on a graphing calculator if the MathPrint operating system is not installed. To evaluate $\log_2 32$, use the change-of-base formula: log(32)/log(2) → 5

The Y= Editor

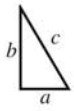
MathPrint features can be accessed from the Y= editor as well as from the home screen. The following table shows examples that illustrate differences between MathPrint in the Y= editor and Classic mode.

Feature	MathPrint	Classic (MathPrint not installed)
Graphing the derivative of $y = x^2$	Plot 1 Plot 2 Plot 3 \Y1 = $\frac{d}{dX}(X^2)\Big\|_{X=X}$ \Y2 =	Plot 1 Plot 2 Plot 3 \Y1 = nDeriv(X^2,X,X) \Y2 =
Graphing an antiderivative of $y = x^2$	Plot 1 Plot 2 Plot 3 \Y1 = $\int_0^X (X^2)dX$ \Y2 =	Plot 1 Plot 2 Plot 3 \Y1 = fnInt(X^2,X,0,X) \Y2 =

Appendix D

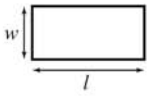
Table I — Formulas from Geometry

PYTHAGOREAN THEOREM
For a right triangle with legs of lengths a and b and hypotenuse of length c, $a^2 + b^2 = c^2$.

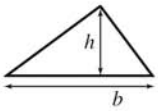


CIRCLE
Area: $A = \pi r^2$
Circumference: $C = 2\pi r$

RECTANGLE
Area: $A = lw$
Perimeter: $P = 2l + 2w$

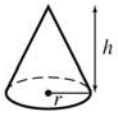


TRIANGLE
Area: $A = \frac{1}{2}bh$

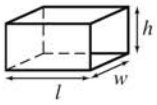


SPHERE
Volume: $V = \frac{4}{3}\pi r^3$
Surface area: $A = 4\pi r^2$

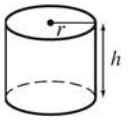
CONE
Volume: $V = \frac{1}{3}\pi r^2 h$



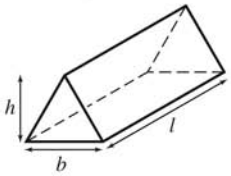
RECTANGULAR BOX
Volume: $V = lwh$
Surface area: $A = 2lh + 2wh + 2lw$



CIRCULAR CYLINDER
Volume: $V = \pi r^2 h$
Surface area: $A = 2\pi r^2 + 2\pi rh$



TRIANGULAR PRISM
Volume: $V = \frac{1}{2}bhl$



GENERAL INFORMATION ON SURFACE AREA
To find the surface area of a figure, break down the total surface area into the individual components and add up the areas of the components. For example, a rectangular box has six sides, each of which is a rectangle. A circular cylinder has two ends, each of which is a circle, plus the side, which forms a rectangle when opened up.

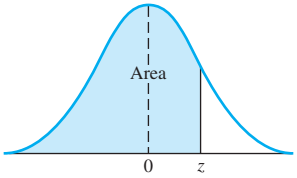


Table 2 — Area Under a Normal Curve to the Left of z, where $z = \dfrac{x - \mu}{\sigma}$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

(*continued*)

Table 2 — Area Under a Normal Curve (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table 3 — Integrals

(*C* is an arbitrary constant.)

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (\text{if } n \neq -1)$
2. $\int e^{kx}\,dx = \dfrac{e^{kx}}{k} + C$
3. $\int \dfrac{a}{x}\,dx = a \ln |x| + C$
4. $\int \ln |ax|\,dx = x(\ln |ax| - 1) + C$
5. $\int \dfrac{1}{\sqrt{x^2 + a^2}}\,dx = \ln \left|x + \sqrt{x^2 + a^2}\right| + C$
6. $\int \dfrac{1}{\sqrt{x^2 - a^2}}\,dx = \ln \left|x + \sqrt{x^2 - a^2}\right| + C$
7. $\int \dfrac{1}{a^2 - x^2}\,dx = \dfrac{1}{2a} \cdot \ln \left|\dfrac{a + x}{a - x}\right| + C \quad (a \neq 0)$
8. $\int \dfrac{1}{x^2 - a^2}\,dx = \dfrac{1}{2a} \cdot \ln \left|\dfrac{x - a}{x + a}\right| + C \quad (a \neq 0)$
9. $\int \dfrac{1}{x\sqrt{a^2 - x^2}}\,dx = -\dfrac{1}{a} \cdot \ln \left|\dfrac{a + \sqrt{a^2 - x^2}}{x}\right| + C \quad (a \neq 0)$
10. $\int \dfrac{1}{x\sqrt{a^2 + x^2}}\,dx = -\dfrac{1}{a} \cdot \ln \left|\dfrac{a + \sqrt{a^2 + x^2}}{x}\right| + C \quad (a \neq 0)$
11. $\int \dfrac{x}{ax + b}\,dx = \dfrac{x}{a} - \dfrac{b}{a^2} \cdot \ln |ax + b| + C \quad (a \neq 0)$
12. $\int \dfrac{x}{(ax + b)^2}\,dx = \dfrac{b}{a^2(ax + b)} + \dfrac{1}{a^2} \cdot \ln |ax + b| + C \quad (a \neq 0)$
13. $\int \dfrac{1}{x(ax + b)}\,dx = \dfrac{1}{b} \cdot \ln \left|\dfrac{x}{ax + b}\right| + C \quad (b \neq 0)$
14. $\int \dfrac{1}{x(ax + b)^2}\,dx = \dfrac{1}{b(ax + b)} + \dfrac{1}{b^2} \cdot \ln \left|\dfrac{x}{ax + b}\right| + C \quad (b \neq 0)$
15. $\int \sqrt{x^2 + a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2} \cdot \ln \left|x + \sqrt{x^2 + a^2}\right| + C$
16. $\int x^n \cdot \ln |x|\,dx = x^{n+1}\left[\dfrac{\ln |x|}{n + 1} - \dfrac{1}{(n + 1)^2}\right] + C \quad (n \neq -1)$
17. $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} \cdot \int x^{n-1} e^{ax}\,dx + C \quad (a \neq 0)$

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Answers to Selected Exercises

Answers to selected writing exercises are provided.

Answers to Prerequisite Skills Diagnostic Test

1. 20% **2.** $51/35$ **3.** $x + y = 75$ **4.** $s \geq 4p$ **5.** $-20/3$ (Sec. R.4) **6.** $-11/5$ (Sec. R.4) **7.** $(-2, 5]$ (Sec. R.5) **8.** $x \leq -3$ (Sec. R.5) **9.** $y \geq -17/2$ (Sec. R.5) **10.** $p > 3/2$ (Sec. R.5) **11.** $-y^2 + 4y - 6$ (Sec. R.1) **12.** $x^3 - x^2 + x + 3$ (Sec. R.1) **13.** $a^2 - 4ab + 4b^2$ (Sec. R.1) **14.** $3pq(1 + 2p + 3q)$ (Sec. R.2) **15.** $(3x + 5)(x - 2)$ (Sec. R.2) **16.** $(a - 6)/(a + 2)$ (Sec. R.3) **17.** $(x^2 + 5x - 2)/[x(x - 1)(x + 1)]$ (Sec. R.3) **18.** $(-2 \pm \sqrt{7})/3$ (Sec. R.4) **19.** $[-\infty, -3) \cup [1, \infty)$ (Sec. R.5) **20.** $x^6y/4$ (Sec. R.6) **21.** $2/(p^2q)$ (Sec. R.6) **22.** $(m - k)/(km)$ (Sec. R.6) **23.** $(x^2 + 1)^{-1/2}(3x^2 + x + 5)$ (Sec. R.6) **24.** $4b^2$ (Sec. R.7) **25.** $(4 + \sqrt{10})/3$ (Sec. R.7) **26.** $|y - 5|$ (Sec. R.7)

Chapter R Algebra Reference

Exercises R.1 (page R-5)

For exercises . . .	1–6	7,8,15–22	9–14	23–26
Refer to example . . .	2	3	4	5

1. $-x^2 + x + 9$ **2.** $-6y^2 + 3y + 10$ **3.** $-16q^2 + 4q + 6$ **4.** $9r^2 - 4r + 19$ **5.** $-0.327x^2 - 2.805x - 1.458$ **6.** $0.8r^2 + 3.6r - 1.5$ **7.** $-18m^3 - 27m^2 + 9m$ **8.** $-12x^4 + 30x^2 + 36x$ **9.** $9t^2 + 9ty - 10y^2$ **10.** $18k^2 - 7kq - q^2$ **11.** $4 - 9x^2$ **12.** $36m^2 - 25$ **13.** $(6/25)y^2 + (11/40)yz + (1/16)z^2$ **14.** $(15/16)r^2 - (7/12)rs - (2/9)s^2$ **15.** $27p^3 - 1$ **16.** $15p^3 + 13p^2 - 10p - 8$ **17.** $8m^3 + 1$ **18.** $12k^4 + 21k^3 - 5k^2 + 3k + 2$ **19.** $3x^2 + xy + 2xz - 2y^2 - 3yz - z^2$ **20.** $2r^2 + 2rs - 5rt - 4s^2 + 8st - 3t^2$ **21.** $x^3 + 6x^2 + 11x + 6$ **22.** $x^3 - 2x^2 - 5x + 6$ **23.** $x^2 + 4x + 4$ **24.** $4a^2 - 16ab + 16b^2$ **25.** $x^3 - 6x^2y + 12xy^2 - 8y^3$ **26.** $27x^3 + 27x^2y + 9xy^2 + y^3$

Exercises R.2 (page R-7)

For exercises . . .	1–4	5–15	16–20	21–32
Refer to example . . .	1	3	2nd CAUTION	4

1. $7a^2(a + 2)$ **2.** $3y(y^2 + 8y + 3)$ **3.** $13p^2q(p^2q - 3p + 2q)$ **4.** $10m^2(6m^2 - 12mn + 5n^2)$ **5.** $(m + 2)(m - 7)$ **6.** $(x + 5)(x - 1)$ **7.** $(z + 4)(z + 5)$ **8.** $(b - 7)(b - 1)$ **9.** $(a - 5b)(a - b)$ **10.** $(s - 5t)(s + 7t)$ **11.** $(y - 7z)(y + 3z)$ **12.** $(3x + 7)(x - 1)$ **13.** $(3a + 7)(a + 1)$ **14.** $(5y + 2)(3y - 1)$ **15.** $(7m + 2n)(3m + n)$ **16.** $6(a - 10)(a + 2)$ **17.** $3m(m + 3)(m + 1)$ **18.** $2(2a + 3)(a + 1)$ **19.** $2a^2(4a - b)(3a + 2b)$ **20.** $12x^2(x - y)(2x + 5y)$ **21.** $(x + 8)(x - 8)$ **22.** $(3m + 5)(3m - 5)$ **23.** $10(x + 4)(x - 4)$ **24.** Prime **25.** $(z + 7y)^2$ **26.** $(s - 5t)^2$ **27.** $(3p - 4)^2$ **28.** $(a - 6)(a^2 + 6a + 36)$ **29.** $(3r - 4s)(9r^2 + 12rs + 16s^2)$ **30.** $3(m + 5)(m^2 - 5m + 25)$ **31.** $(x - y)(x + y)(x^2 + y^2)$ **32.** $(2a - 3b)(2a + 3b)(4a^2 + 9b^2)$

Exercises R.3 (page R-10)

For exercises . . .	1–12	13–38
Refer to example . . .	1	2

1. $v/7$ **2.** $5p/2$ **3.** $8/9$ **4.** $2/(t + 2)$ **5.** $x - 2$ **6.** $4(y + 2)$ **7.** $(m - 2)/(m + 3)$ **8.** $(r + 2)/(r + 4)$ **9.** $3(x - 1)/(x - 2)$ **10.** $(z - 3)/(z + 2)$ **11.** $(m^2 + 4)/4$ **12.** $(2y + 1)/(y + 1)$ **13.** $3k/5$ **14.** $25p^2/9$ **15.** $9/(5c)$ **16.** 2 **17.** $1/4$ **18.** $3/10$ **19.** $2(a + 4)/(a - 3)$ **20.** $2/(r + 2)$ **21.** $(k - 2)/(k + 3)$ **22.** $(m + 6)/(m + 3)$ **23.** $(m - 3)/(2m - 3)$ **24.** $2(2n - 1)/(3n - 5)$ **25.** 1 **26.** $(6 + p)/(2p)$ **27.** $(12 - 15y)/(10y)$ **28.** $137/(30m)$ **29.** $(3m - 2)/[m(m - 1)]$ **30.** $(r - 6)/[r(2r + 3)]$ **31.** $14/[3(a - 1)]$ **32.** $23/[20(k - 2)]$ **33.** $(7x + 1)/[(x - 2)(x + 3)(x + 1)]$ **34.** $(y^2 + 1)/[(y + 3)(y + 1)(y - 1)]$ **35.** $k(k - 13)/[(2k - 1)(k + 2)(k - 3)]$ **36.** $m(3m - 19)/[(3m - 2)(m + 3)(m - 4)]$ **37.** $(4a + 1)/[a(a + 2)]$ **38.** $(5x^2 + 4x - 4)/[x(x - 1)(x + 1)]$

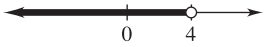
Exercises R.4 (page R-16)

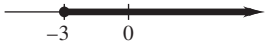
For exercises . . .	1–8	9–26	27–37
Refer to example . . .	2	3–5	6,7

1. -12 **2.** $3/4$ **3.** 12 **4.** $-3/8$ **5.** $-7/8$ **6.** $-6/11$ **7.** 4 **8.** $-10/19$ **9.** $-3, -2$ **10.** $-1, 3$ **11.** 7 **12.** $-2, 5/2$ **13.** $-1/4, 2/3$ **14.** $2, 5$ **15.** $-3, 3$ **16.** $-4, 1/2$ **17.** $0, 4$ **18.** $(5 + \sqrt{13})/6 \approx 1.434$, $(5 - \sqrt{13})/6 \approx 0.232$ **19.** $(2 + \sqrt{10})/2 \approx 2.581$, $(2 - \sqrt{10})/2 \approx -0.581$ **20.** $(-1 + \sqrt{5})/2 \approx 0.618$, $(-1 - \sqrt{5})/2 \approx -1.618$ **21.** $5 + \sqrt{5} \approx 7.236$, $5 - \sqrt{5} \approx 2.764$ **22.** $(4 + \sqrt{6})/5 \approx 1.290$, $(4 - \sqrt{6})/5 \approx 0.310$ **23.** $1, 5/2$ **24.** No real number solutions **25.** $(-1 + \sqrt{73})/6 \approx 1.257$, $(-1 - \sqrt{73})/6 \approx -1.591$ **26.** $-1, 0$ **27.** 3 **28.** 12 **29.** $-59/6$ **30.** 6 **31.** 3 **32.** $-5/2$ **33.** $2/3$ **34.** 1 **35.** 2 **36.** No solution **37.** No solution

Exercises R.5 (page R-21)

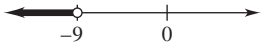
For exercises . . .	1–14	15–26	27–38	39–42	43–54
Refer to example . . .	Figure 1, Example 2	2	3	4	5–7

1. $(-\infty, 4)$ 

2. $[-3, \infty)$ 

3. $[1, 2)$

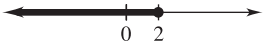
4. $[-2, 3]$

5. $(-\infty, -9)$ 

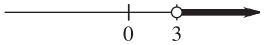
6. $[6, \infty)$

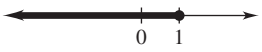
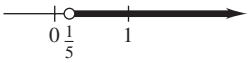
7. $-7 \leq x \leq -3$ **8.** $4 \leq x < 10$

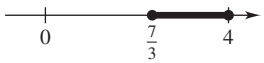
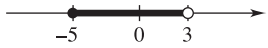
9. $x \leq -1$ **10.** $x > 3$ **11.** $-2 \leq x < 6$ **12.** $0 < x < 8$ **13.** $x \leq -4$ or $x \geq 4$ **14.** $x < 0$ or $x \geq 3$

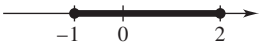
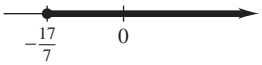
15. $(-\infty, 2]$ 

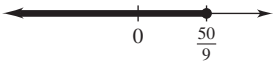
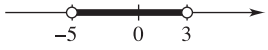
16. $(-\infty, 1)$

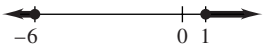
17. $(3, \infty)$ 

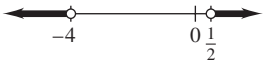
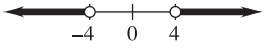
18. $(-\infty, 1]$  **19.** $(1/5, \infty)$  **20.** $(1/3, \infty)$

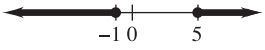
21. $(-4, 6)$ **22.** $[7/3, 4]$  **23.** $[-5, 3)$ 

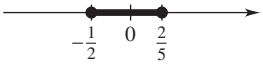
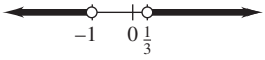
24. $[-1, 2]$  **25.** $[-17/7, \infty)$ 

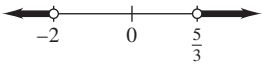
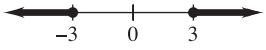
26. $(-\infty, 50/9]$  **27.** $(-5, 3)$ 

28. $(-\infty, -6] \cup [1, \infty)$  **29.** $(1, 2)$

30. $(-\infty, -4) \cup (1/2, \infty)$  **31.** $(-\infty, -4) \cup (4, \infty)$ 

32. $[-3/2, 5]$ **33.** $(-\infty, -1] \cup [5, \infty)$ 

34. $[-1/2, 2/5]$  **35.** $(-\infty, -1) \cup (1/3, \infty)$ 

36. $(-\infty, -2) \cup (5/3, \infty)$  **37.** $(-\infty, -3] \cup [3, \infty)$ 

38. $(-\infty, 0) \cup (16, \infty)$ **39.** $[-2, 0] \cup [2, \infty)$

40. $(-\infty, -4] \cup [-3, 0]$ **41.** $(-\infty, 0) \cup (1, 6)$

42. $(-1, 0) \cup (4, \infty)$ **43.** $(-5, 3]$ **44.** $(-\infty, -1) \cup (1, \infty)$ **45.** $(-\infty, -2)$

46. $(-2, 3/2)$ **47.** $[-8, 5)$ **48.** $(-\infty, -3/2) \cup [-13/9, \infty)$ **49.** $[2, 3)$ **50.** $(-\infty, -1)$ **51.** $(-2, 0] \cup (3, \infty)$

52. $(-4, -2) \cup (0, 2)$ **53.** $(1, 3/2]$ **54.** $(-\infty, -2) \cup (-2, 2) \cup [4, \infty)$

Exercises R.6 (page R-25)

For exercises . . .	1–8	9–26	27–36	37–50	51–56
Refer to example . . .	1	2	3,4	5	6

1. 1/64 **2.** 1/81 **3.** 1 **4.** 1 **5.** $-1/9$ **6.** 1/9 **7.** 36 **8.** 27/64

9. 1/64 **10.** 8^5 **11.** $1/10^8$ **12.** 7 **13.** x^2 **14.** 1 **15.** 2^3k^3 **16.** $1/(3z^7)$ **17.** $x^5/(3y^3)$ **18.** $m^3/5^4$ **19.** a^3b^6

20. $49/(c^6d^4)$ **21.** $(a + b)/(ab)$ **22.** $(1 - ab^2)/b^2$ **23.** $2(m - n)/[mn(m + n^2)]$ **24.** $(3n^2 + 4m)/(mn^2)$

25. $xy/(y - x)$ **26.** $y^4/(xy - 1)^2$ **27.** 11 **28.** 3 **29.** 4 **30.** -25 **31.** 1/2 **32.** 4/3 **33.** 1/16 **34.** 1/5 **35.** 4/3

36. 1000/1331 **37.** 9 **38.** 3 **39.** 64 **40.** 1 **41.** x^4/y^4 **42.** b/a^3 **43.** r **44.** $12^3/y^8$ **45.** $3k^{3/2}/8$ **46.** $1/(2p^2)$

47. $a^{2/3}b^2$ **48.** $y^2/(x^{1/6}z^{5/4})$ **49.** $h^{1/3}t^{1/5}/k^{2/5}$ **50.** m^3p/n **51.** $3x(x^2 + 3x)^2(x^2 - 5)$ **52.** $6x(x^3 + 7)(-2x^3 - 5x + 7)$

53. $5x(x^2 - 1)^{-1/2}(x^2 + 1)$ **54.** $3(6x + 2)^{-1/2}(27x + 5)$ **55.** $(2x + 5)(x^2 - 4)^{-1/2}(4x^2 + 5x - 8)$

56. $(4x^2 + 1)(2x - 1)^{-1/2}(36x^2 - 16x + 1)$

Exercises R.7 (page R-28)

For exercises . . .	1–22	23–26	27–40	41–44
Refer to example . . .	1,2	3	4	5

1. 5 **2.** 6 **3.** -5 **4.** $5\sqrt{2}$ **5.** $20\sqrt{5}$ **6.** $4y^2\sqrt{2y}$ **7.** 9 **8.** 8

9. $7\sqrt{2}$ **10.** $9\sqrt{3}$ **11.** $9\sqrt{7}$ **12.** $-2\sqrt{7}$ **13.** $5\sqrt[3]{2}$ **14.** $3\sqrt[3]{5}$ **15.** $xyz^2\sqrt{2x}$ **16.** $4r^3s^4t^6\sqrt{10rs}$ **17.** $4xy^2z^3\sqrt[3]{2y^2}$

18. $x^2yz^2\sqrt[4]{y^3z^3}$ **19.** $ab\sqrt{ab}(b - 2a^2 + b^3)$ **20.** $p^2\sqrt{pq}(pq - q^4 + p^2)$ **21.** $\sqrt[6]{a^5}$ **22.** $b^2\sqrt[4]{b}$ **23.** $|4 - x|$

24. $|3y + 5|$ **25.** Cannot be simplified **26.** Cannot be simplified **27.** $5\sqrt{7}/7$ **28.** $\sqrt{10}/2$ **29.** $-\sqrt{3}/2$ **30.** $\sqrt{2}$

31. $-3(1 + \sqrt{2})$ **32.** $-5(2 + \sqrt{6})/2$ **33.** $3(2 - \sqrt{2})$ **34.** $(5 - \sqrt{10})/3$ **35.** $(\sqrt{r} + \sqrt{3})/(r - 3)$

36. $5(\sqrt{m} + \sqrt{5})/(m - 5)$ **37.** $\sqrt{y} + \sqrt{5}$ **38.** $(z + \sqrt{5z} - \sqrt{z} - \sqrt{5})/(z - 5)$ **39.** $-2x - 2\sqrt{x(x + 1)} - 1$

40. $[p^2 + p + 2\sqrt{p(p^2 - 1)} - 1]/(-p^2 + p + 1)$ **41.** $-1/[2(1 - \sqrt{2})]$ **42.** $1/(3 + \sqrt{3})$

43. $-1/[2x - 2\sqrt{x(x + 1)} + 1]$ **44.** $2/[p + \sqrt{p(p - 2)}]$

Chapter 1 Linear Functions

Exercises 1.1 (page 13)

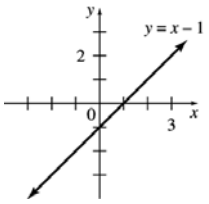
For exercises . . .	1–4	5–8	13,29,30	14,31–34	15–17	18,27	19–24	25,26	28	45–60	61–75
Refer to example . . .	1	3	8	9	4	7	5	2	6	11–13	10,14

1. 3/5 **3.** Not defined **5.** 1

7. 5/9 **9.** Not defined **11.** 0 **13.** 2 **15.** $y = -2x + 5$ **17.** $y = -7$ **19.** $y = -(1/3)x + 10/3$ **21.** $y = 6x - 7/2$

23. $x = -8$ **25.** $x + 2y = -6$ **27.** $x = -6$ **29.** $3x + 2y = 0$ **31.** $x - y = 7$ **33.** $5x - y = -4$ **35.** No **39.** a **41.** -4

45.



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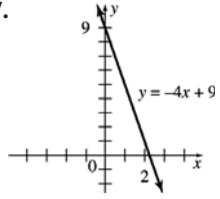

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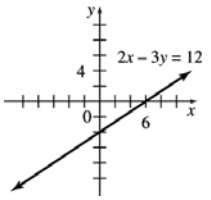

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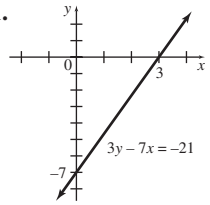

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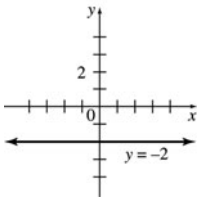

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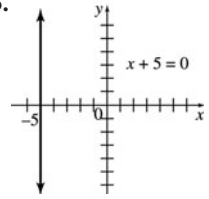
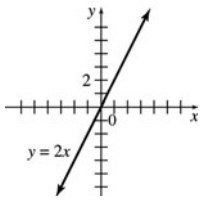
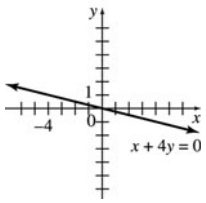


57.

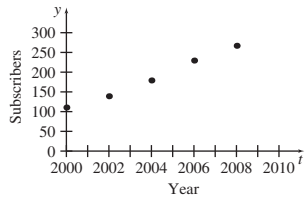


59.



61. a. 12,000 $y = 12{,}000x + 3000$ **b.** 8 years 1 month **63. a.** $y = 4.612t + 86.164$ **b.** 178.4, which is slightly more than the actual CPI. **c.** It is increasing at a rate of approximately 4.6 per year. **65. a.** $u = 0.85(220 - x) = 187 - 0.85x$, $l = 0.7(220 - x) = 154 - 0.7x$ **b.** 140 to 170 beats per minute. **c.** 126 to 153 beats per minute. **d.** The women are 16 and 52. Their pulse is 143 beats per minute. **67.** Approximately 86 yr **69. a.** $y = 0.3444t + 14.1$ **b.** 2022
71. a. $y = 14{,}792.05t - 490{,}416$ **b.** 1,210,670 **73. a.** There appears to be a linear relationship. **b.** $y = 76.9x$ **c.** About 780 megaparsecs (about 1.5×10^{22} mi) **d.** About 12.4 billion yr

75. a.



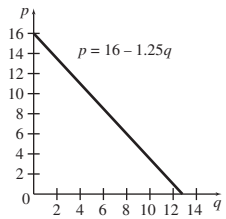
Yes, the data are approximately linear.
b. $y = 1133.4t + 16{,}072$; the slope 1133.4 indicates that tuition and fees have increased approximately \$1133 per year. **c.** The year 2025 is too far in the future to rely on this equation to predict costs; too many other factors may influence these costs by then.

Exercises 1.2 (page 23)

For exercises …	1–10	19–22	23–26	27–32	33–36,48	37–44	45–47
Refer to example …	1	4	5	2,3	5	6	7

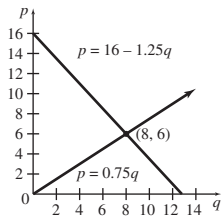
1. -3 **3.** 22 **5.** 0 **7.** -4 **9.** $7 - 5t$ **11.** True
13. True **19.** If $R(x)$ is the cost of renting a snowboard for x hours, then $R(x) = 2.25x + 10$. **21.** If $C(x)$ is the cost of parking a car for x half-hours, then $C(x) = 0.75x + 2$. **23.** $C(x) = 30x + 100$ **25.** $C(x) = 75x + 550$ **27. a.** \$16 **b.** \$11 **c.** \$6
d. 640 watches **e.** 480 watches **f.** 320 watches

g.



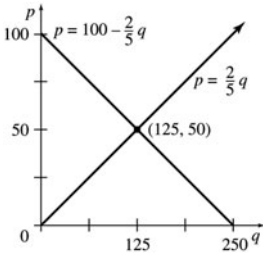
h. 0 watches
i. About 1333 watches
j. About 2667 watches

k.



l. 800 watches, \$6

29. a.

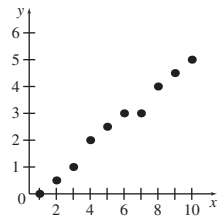


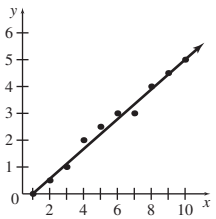
b. 125 tubs, \$50

31. $D(q) = 6.9 - 0.4q$ **33. a.** $C(x) = 3.50x + 90$ **b.** 17 shirts
c. 108 shirts **35. a.** $C(x) = 0.097x + 1.32$ **b.** \$1.32 **c.** \$98.32
d. \$98.42 **e.** 9.7¢ **f.** 9.7¢, the cost of producing one additional cup of coffee would be 9.7¢. **37. a.** 2 units **b.** \$980 **c.** 52 units
39. Break-even quantity is 45 units; don't produce; $P(x) = 20x - 900$
41. Break-even quantity is -50 units; impossible to make a profit when $C(x) > R(x)$ for all positive x; $P(x) = -10x - 500$ (always a loss)
43. 5 **45. a.** 14.4°C **b.** -28.9°C **c.** 122°C **47.** -40°

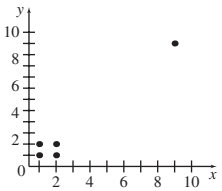
Exercises 1.3 (page 32)

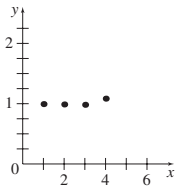
For exercises . . .	3b,4,10d, 11d,12e,13e, 15b,16d,17b, 18d,19c,20b, 22d,24c,25d, 26a	3c,10a,11a, 12a,13a,15c, 16b,17c,18a, 19a,21c,22a, 23a,24b, 25a,26b	3d,10b,11b, 12c,13c,18b, 21d,22bc,23b, 26c	5ab,6ab, 7a,8a, 14ab,21ab	5c,6c, 15e	10c,11c, 12d,13d, 18c,23c
Refer to example . . .	4	1	2	1,4	5	3

3. a. 

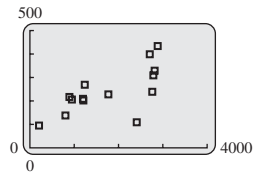
b. 0.993 **c.** $Y = 0.555x - 0.5$ 

d. 5.6

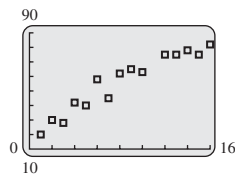
5. a. $Y = 0.9783x + 0.0652, 0.9783$ **b.** $Y = 1.5, 0$ **c.** The point (9,9) is an outlier that has a strong effect on the least squares line and the correlation coefficient. 

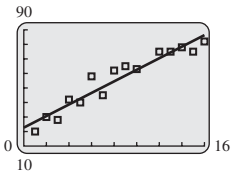
7. a. 0.7746 **b.** 

11. a. $Y = -0.1534x + 11.36$ **b.** About 6760 **c.** 2025 **d.** −0.9890 **13. a.** $Y = 97.73x + 1833.3$ **b.** About \$97.73 billion per year **c.** \$3299 billion **d.** 2023 **e.** 0.9909

15. a. 

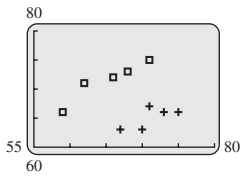
They lie in a linear pattern. **b.** $r = 0.693$; there is a positive correlation between the price and the distance. **c.** $Y = 0.0738x + 111.83$; the marginal cost is 7.38 cents per mile. **d.** In 2000 marginal cost was 2.43 cents per mile; it has increased to 7.38 cents per mile. **e.** Phoenix

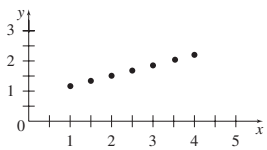
17. a. 

b. 0.959, yes **c.** $Y = 3.98x + 22.7$ 

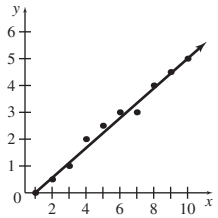
19. a. $Y = -0.1175x + 17.74$ **b.** 14.2 **c.** −0.9326

21. a. $Y = -0.08915x + 74.28$, $r = -0.1035$. The taller the student, the shorter is the ideal partner's height.
b. Females: $Y = 0.6674x + 27.89$, $r = 0.9459$; males: $Y = 0.4348x + 34.04$, $r = 0.7049$

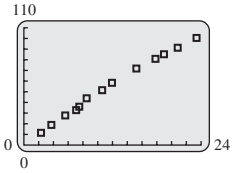


23. a. 

b. $Y = 0.366x + 0.803$; the line seems to fit the data.



c. $r = 0.995$ indicates a good fit, which confirms the conclusion in part b. **25. a.** −0.995; yes **b.** $Y = -0.0769x + 5.91$ **c.** 2.07 points

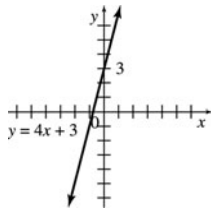
27. a. 4.298 miles per hour **b.** 

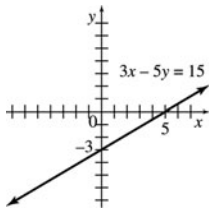
; yes **c.** $Y = 4.317x + 3.419$ **d.** 0.9971; yes **e.** 4.317 miles per hour

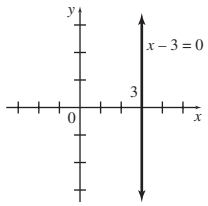
Chapter 1 Review Exercises (page 39)

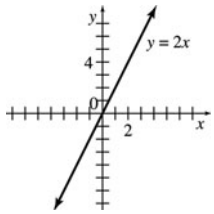
For exercises . . .	1–6,15–44	7–10,13,45–59b,62,63	11,12,14,59c–61,64,65
Refer to section . . .	1	2	3

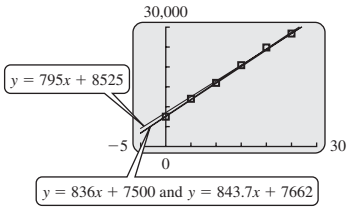
1. False **2.** False **3.** True **4.** False **5.** True **6.** False **7.** True **8.** False **9.** False **10.** False **11.** False **12.** True
15. 1 **17.** $-2/11$ **19.** $-4/3$ **21.** 0 **23.** 5 **25.** $y = (2/3)x - 13/3$ **27.** $y = -x - 3$ **29.** $y = -10$ **31.** $2x - y = 10$
33. $x = -1$ **35.** $y = -5$

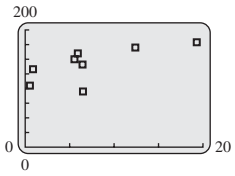
37.


39.


41.


43.


45. a. $E = 352 + 42x$ (where x is in thousands) **b.** $R = 130x$ (where x is in thousands) **c.** More than 4000 chips
47. $S(q) = 0.5q + 10$ **49.** \$41.25, 62.5 diet pills **51.** $C(x) = 180x + 2000$ **53.** $C(x) = 46x + 120$ **55. a.** 40 pounds
b. \$280 **57.** $y = 7.23t + 11.9$ **59. a.** $y = 836x + 7500$ **b.** $y = 795x + 8525$ **c.** $Y = 843.7x + 7662$
d.


f. 0.9995 **61. a.** $Y = 0.9724x + 31.43$ **b.** About 216 **c.** 0.9338 **63.** $y = 1.22t + 48.9$
65. a. 0.6998; yes, but the fit is not very good. **b.**


c. $Y = 3.396x + 117.2$ **d.** \$3396

Chapter 2 Systems of Linear Equations and Matrices

Exercises 2.1 (page 52)

For exercises . . .	1–20	23–26,29–33,40–43	35–39,44–48
Refer to example . . .	1–3	5	4

1. $(3, 2)$ **3.** $(1, 3)$ **5.** $(-2, 0)$ **7.** $(0, 2)$ **9.** $(3, -2)$ **11.** $(2, -2)$
13. No solution **15.** $((2y - 4)/3, y)$ **17.** $(4, 1)$ **19.** $(7, -2)$ **21.** No **23.** 4 **25.** 3 **29.** $(8z - 4, 3 - 5z, z)$
31. $(3 - z, 4 - z, z)$ **35.** \$27 **37.** 400 main floor, 200 balcony **39.** Not possible; inconsistent system **41.** Either 10 buffets, 5 chairs, and no tables, or 11 buffets, 1 chair, and 1 table **43.** $z + 80$ long-sleeve blouses, $260 - 2z$ short-sleeve blouses, and z sleeveless blouses with $0 \le z \le 130$. **45. a.** March 23, March 19 **b.** 1991 **47.** 36 field goals, 28 foul shots

Exercises 2.2 (page 65)

For exercises . . .	1–8,11–42	44–70
Refer to example . . .	1–5	6

1. $\left[\begin{array}{cc|c} 3 & 1 & 6 \\ 2 & 5 & 15 \end{array}\right]$ **3.** $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \\ 1 & 1 & 1 & 2 \end{array}\right]$ **5.** $x = 2, y = 3$ **7.** $x = 4, y = -5, z = 1$ **9.** Row operations

11. $\left[\begin{array}{ccc|c} 3 & 7 & 4 & 10 \\ 0 & 1 & -5 & -8 \\ 0 & 4 & 5 & 11 \end{array}\right]$ **13.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 3 & 2 & 5 \\ 0 & 5 & 3 & 7 \end{array}\right]$ **15.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 5 & 0 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right]$ **17.** $(2, 3)$ **19.** $(1, 6)$ **21.** No solution

23. $((3y + 1)/6, y)$ **25.** $(4, 1, 0)$ **27.** No solution **29.** $(-1, 23, 16)$ **31.** $((-9z + 5)/23, (10z - 3)/23, z)$
33. $((-2z + 62)/35, (3z + 5)/7, z)$ **35.** $((9 - 3y - z)/2, y, z)$ **37.** $(0, 2, -2, 1)$; the answers are given in the order x, y, z, w.
39. $(-w - 3, -4w - 19, -3w - 2, w)$ **41.** $(28.9436, 36.6326, 9.6390, 37.1036)$ **43.** row 1: 3/8, 1/6, 11/24; row 2: 5/12, 1/3, 1/4; row 3: 5/24, 1/2, 7/24 **45.** \$2000 in U.S. Savings bonds, \$4000 in mutual funds, \$4000 in money market **47.** 2000 chairs, 1600 cabinets, and 2500 buffets. **49. a.** 5 trucks, 2 vans, and 3 SUVs. **b.** Use 2 trucks, 6 vans, and 1 SUV, or use 5 trucks, 2 vans, and 3 SUVs. **51. a.** \$12,000 at 8%, \$7000 at 9%, and \$6000 at 10% **b.** The amount borrowed at 10% must be less than or equal to \$9500. If $z =$ \$5000, they borrowed \$11,000 at 8% and \$9000 at 9%. **c.** No solution. **d.** The total annual interest would be \$2220, not \$2190, as specified as one of the conditions. **53.** The first supplier should send 40 units to Roseville and 35 units to Akron. The second supplier should send 0 units to Roseville and 40 units to Akron **55. a.** 26 **b.** 0 two-person, 50 four-person, and 0 six-person tents **c.** 25 two-person, 0 four-person, and 25 six-person tents **57. a.** 400/9 g of group A, 400/3 g of group B, and 2000/9 g of group C. **b.** For any positive number z of grams of group C, there should be z grams less than 800/3 g of group A

and 400/3 g of group B. **c.** No **59.** About 244 fish of species A, 39 fish of species B, and 101 fish of species C. **61. b.** 7,206,360 white cows, 4,893,246 black cows, 3,515,820 spotted cows, and 5,439,213 brown cows **63. a.** $r = 175{,}000$, $b = 375{,}000$ **65.** 8.15, 4.23, 3.15 **67. a.** 24 balls, 57 dolls, and 19 cars **b.** None **c.** 48 **d.** 5 balls, 95 dolls, and 0 cars **e.** 52 balls, 1 doll, and 47 cars **69. a.** (1, 1, 1, 1); the strategy required to turn all the lights out is to push every button one time. **b.** (0, 1, 1, 0); the strategy required to turn all the lights out is to push the button in the first row, second column, and push the button in the second row, first column.

Exercises 2.3 (page 74)

For exercises . . .	1–6,15–20	7–14	21–32	39,40,43	41,48	42	44–46
Refer to example . . .	3	2	4,6	5	1	5,7	7

1. False; not all corresponding elements are equal.
3. True **5.** True **7.** 2×2; square **9.** 3×4 **11.** 2×1; column **13.** Undefined **15.** $x = 4, y = -8, z = 1$
17. $s = 10, t = 0, r = 7$ **19.** $a = 20, b = 5, c = 0, d = 4, f = 1$

21. $\begin{bmatrix} 10 & 4 & -5 & -6 \\ 4 & 5 & 3 & 11 \end{bmatrix}$ **23.** Not possible **25.** $\begin{bmatrix} 1 & 5 & 6 & -9 \\ 5 & 7 & 2 & 1 \\ -7 & 2 & 2 & -7 \end{bmatrix}$ **27.** $\begin{bmatrix} 3 & 4 \\ 4 & 8 \end{bmatrix}$ **29.** $\begin{bmatrix} 10 & -2 \\ 10 & 9 \end{bmatrix}$

31. $\begin{bmatrix} -12x + 8y & -x + y \\ x & 8x - y \end{bmatrix}$ **33.** $\begin{bmatrix} -x & -y \\ -z & -w \end{bmatrix}$ **39. a.** Chicago: $\begin{bmatrix} 4.05 & 7.01 \\ 3.27 & 3.51 \end{bmatrix}$, Seattle: $\begin{bmatrix} 4.40 & 6.90 \\ 3.54 & 3.76 \end{bmatrix}$ **b.** $\begin{bmatrix} 4.42 & 7.43 \\ 3.38 & 3.62 \end{bmatrix}$

41. a. $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 5 & 0 & 7 \\ 0 & 10 & 1 \\ 0 & 15 & 2 \\ 10 & 12 & 8 \end{bmatrix}$ **c.** $\begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$ **43. a.** 8 **b.** 3 **c.** $\begin{bmatrix} 85 & 15 \\ 27 & 73 \end{bmatrix}$ **d.** Yes

45. a. $\begin{bmatrix} 60.0 & 68.3 \\ 63.8 & 72.5 \\ 64.5 & 73.6 \\ 68.3 & 75.2 \end{bmatrix}$ **b.** $\begin{bmatrix} 68.0 & 75.6 \\ 70.7 & 78.1 \\ 72.7 & 79.4 \\ 74.9 & 80.1 \end{bmatrix}$ **c.** $\begin{bmatrix} -8.0 & -7.3 \\ -6.9 & -5.6 \\ -8.2 & -5.8 \\ -6.6 & -4.9 \end{bmatrix}$ **47. a.** $\begin{bmatrix} 51.2 & 7.9 \\ 59.8 & 11.1 \\ 66.2 & 11.3 \\ 73.8 & 13.2 \\ 78.5 & 16.5 \\ 83.0 & 19.6 \end{bmatrix}$ **b.** $\begin{bmatrix} 45.3 & 7.9 \\ 47.9 & 8.5 \\ 50.8 & 9.2 \\ 53.4 & 9.3 \\ 57.0 & 10.6 \\ 62.3 & 13.3 \end{bmatrix}$ **c.** $\begin{bmatrix} 5.9 & 0 \\ 11.9 & 2.6 \\ 15.4 & 2.1 \\ 20.4 & 3.9 \\ 21.5 & 5.9 \\ 20.7 & 6.3 \end{bmatrix}$

Exercises 2.4 (page 83)

For exercises . . .	1–6,49,50	7–12	15–31,51	43–48
Refer to example . . .	1	4	2,3,5	6

1. $\begin{bmatrix} -4 & 8 \\ 0 & 6 \end{bmatrix}$ **3.** $\begin{bmatrix} 12 & -24 \\ 0 & -18 \end{bmatrix}$ **5.** $\begin{bmatrix} -22 & -6 \\ 20 & -12 \end{bmatrix}$ **7.** 2×2; 2×2

9. 3×4; BA does not exist. **11.** AB does not exist; 3×2 **13.** columns; rows **15.** $\begin{bmatrix} 8 \\ -1 \end{bmatrix}$ **17.** $\begin{bmatrix} 14 \\ -23 \end{bmatrix}$ **19.** $\begin{bmatrix} -7 & 2 & 8 \\ 27 & -12 & 12 \end{bmatrix}$

21. $\begin{bmatrix} -2 & 10 \\ 0 & 8 \end{bmatrix}$ **23.** $\begin{bmatrix} 13 & 5 \\ 25 & 15 \end{bmatrix}$ **25.** $\begin{bmatrix} 13 \\ 29 \end{bmatrix}$ **27.** $\begin{bmatrix} 7 \\ -33 \\ 4 \end{bmatrix}$ **29.** $\begin{bmatrix} 22 & -8 \\ 11 & -4 \end{bmatrix}$ **31. a.** $\begin{bmatrix} 16 & 22 \\ 7 & 19 \end{bmatrix}$ **b.** $\begin{bmatrix} 5 & -5 \\ 0 & 30 \end{bmatrix}$ **c.** No **d.** No

39. a. $\begin{bmatrix} 6 & 106 & 158 & 222 & 28 \\ 120 & 139 & 64 & 75 & 115 \\ -146 & -2 & 184 & 144 & -129 \\ 106 & 94 & 24 & 116 & 110 \end{bmatrix}$ **b.** Does not exist **c.** No **41. a.** $\begin{bmatrix} -1 & 5 & 9 & 13 & -1 \\ 7 & 17 & 2 & -10 & 6 \\ 18 & 9 & -12 & 12 & 22 \\ 9 & 4 & 18 & 10 & -3 \\ 1 & 6 & 10 & 28 & 5 \end{bmatrix}$

b. $\begin{bmatrix} -2 & -9 & 90 & 77 \\ -42 & -63 & 127 & 62 \\ 413 & 76 & 180 & -56 \\ -29 & -44 & 198 & 85 \\ 137 & 20 & 162 & 103 \end{bmatrix}$ **c.** $\begin{bmatrix} -56 & -1 & 1 & 45 \\ -156 & -119 & 76 & 122 \\ 315 & 86 & 118 & -91 \\ -17 & -17 & 116 & 51 \\ 118 & 19 & 125 & 77 \end{bmatrix}$ **d.** $\begin{bmatrix} 54 & -8 & 89 & 32 \\ 114 & 56 & 51 & -60 \\ 98 & -10 & 62 & 35 \\ -12 & -27 & 82 & 34 \\ 19 & 1 & 37 & 26 \end{bmatrix}$ **e.** $\begin{bmatrix} -2 & -9 & 90 & 77 \\ -42 & -63 & 127 & 62 \\ 413 & 76 & 180 & -56 \\ -29 & -44 & 198 & 85 \\ 137 & 20 & 162 & 103 \end{bmatrix}$

f. Yes **43. a.**
$$\begin{array}{c} \\ \text{Dept. 1} \\ \text{Dept. 2} \\ \text{Dept. 3} \\ \text{Dept. 4} \end{array}\begin{array}{c}\begin{array}{cc}\text{A} & \text{B}\end{array}\\ \begin{bmatrix}57 & 70\\ 41 & 54\\ 27 & 40\\ 39 & 40\end{bmatrix}\end{array}$$
b. Supplier A: \$164; Supplier B: \$204; Supplier A **45. a.** $\begin{bmatrix}4.24 & 6.95\\ 3.42 & 3.64\end{bmatrix}$ **b.** $\begin{bmatrix}4.41 & 7.17\\ 3.46 & 3.69\end{bmatrix}$

47. a. $\begin{bmatrix}80 & 40 & 120\\ 60 & 30 & 150\end{bmatrix}$ **b.** $\begin{bmatrix}1/2 & 1/5\\ 1/4 & 1/5\\ 1/4 & 3/5\end{bmatrix}$ **c.** $PF = \begin{bmatrix}80 & 96\\ 75 & 108\end{bmatrix}$; the rows give the average price per pair of footwear sold by each store, and the columns give the state. **d.** \$108

49. a. $\begin{bmatrix}20 & 52 & 27\\ 25 & 62 & 35\\ 30 & 72 & 43\end{bmatrix}$; the rows give the amounts of fat, carbohydrates, and protein, respectively, in each of the daily meals.

b. $\begin{bmatrix}75\\ 45\\ 70\\ 168\end{bmatrix}$; the rows give the number of calories in one exchange of each of the food groups. **c.** The rows give the number of calories in each meal.

51. $\begin{bmatrix}66.7 & 74.4\\ 69.6 & 77.2\\ 71.3 & 78.4\\ 73.8 & 79.3\end{bmatrix}$ **53. a.** $\begin{bmatrix}0.0346 & 0.0118\\ 0.0174 & 0.0073\\ 0.0189 & 0.0059\\ 0.0135 & 0.0083\\ 0.0099 & 0.0103\end{bmatrix}$; $\begin{bmatrix}361 & 2038 & 286 & 227 & 460\\ 473 & 2494 & 362 & 252 & 484\\ 627 & 2978 & 443 & 278 & 499\\ 803 & 3435 & 524 & 314 & 511\\ 1013 & 3824 & 591 & 344 & 522\end{bmatrix}$ **b.**
$$\begin{array}{c} \\ 1970 \\ 1980 \\ 1990 \\ 2000 \\ 2010 \end{array}\begin{array}{c}\begin{array}{cc}\textit{Births} & \textit{Deaths}\end{array}\\ \begin{bmatrix}60.98 & 27.45\\ 74.80 & 33.00\\ 90.58 & 39.20\\ 106.75 & 45.51\\ 122.57 & 51.59\end{bmatrix}\end{array}$$

Exercises 2.5 (page 94)

For exercises . . .	1–8	11–26,51–55	27–42,56–58	59–63	64,65
Refer to example . . .	1	2,3	4,6	5	7

1. Yes **3.** No **5.** No **7.** Yes **9.** No; the row of all zeros makes it impossible to get all the 1's in the diagonal of the identity matrix, no matter what matrix is used as an inverse.

11. $\begin{bmatrix}0 & 1/2\\ -1 & 1/2\end{bmatrix}$ **13.** $\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$ **15.** No inverse **17.** $\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ -1 & 0 & 1\end{bmatrix}$ **19.** $\begin{bmatrix}15 & 4 & -5\\ -12 & -3 & 4\\ -4 & -1 & 1\end{bmatrix}$ **21.** No inverse

23. $\begin{bmatrix}-11/2 & -1/2 & 5/2\\ 1/2 & 1/2 & -1/2\\ -5/2 & 1/2 & 1/2\end{bmatrix}$ **25.** $\begin{bmatrix}1/2 & 1/2 & -1/4 & 1/2\\ -1 & 4 & -1/2 & -2\\ -1/2 & 5/2 & -1/4 & -3/2\\ 1/2 & -1/2 & 1/4 & 1/2\end{bmatrix}$ **27.** $(5, 1)$ **29.** $(2, 1)$ **31.** $(15, 21)$

33. No inverse, $(-8y - 12, y)$ **35.** $(-8, 6, 1)$ **37.** $(-36, 8, -8)$ **39.** No inverse, no solution for system

41. $(-7, -34, -19, 7)$

51. $\begin{bmatrix}-0.0447 & -0.0230 & 0.0292 & 0.0895 & -0.0402\\ 0.0921 & 0.0150 & 0.0321 & 0.0209 & -0.0276\\ -0.0678 & 0.0315 & -0.0404 & 0.0326 & 0.0373\\ 0.0171 & -0.0248 & 0.0069 & -0.0003 & 0.0246\\ -0.0208 & 0.0740 & 0.0096 & -0.1018 & 0.0646\end{bmatrix}$

53. $\begin{bmatrix}0.0394 & 0.0880 & 0.0033 & 0.0530 & -0.1499\\ -0.1492 & 0.0289 & 0.0187 & 0.1033 & 0.1668\\ -0.1330 & -0.0543 & 0.0356 & 0.1768 & 0.1055\\ 0.1407 & 0.0175 & -0.0453 & -0.1344 & 0.0655\\ 0.0102 & -0.0653 & 0.0993 & 0.0085 & -0.0388\end{bmatrix}$ **55.** Yes

57. $\begin{bmatrix}1.51482\\ 0.053479\\ -0.637242\\ 0.462629\end{bmatrix}$ **59. a.** $\begin{bmatrix}72\\ 48\\ 60\end{bmatrix}$ **b.** $\begin{bmatrix}2 & 4 & 2\\ 2 & 1 & 2\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}72\\ 48\\ 60\end{bmatrix}$ **c.** 8 type I, 8 type II, and 12 type III

61. a. \$10,000 at 6%, \$10,000 at 6.5%, and \$5000 at 8% **b.** \$14,000 at 6%, \$9000 at 6.5%, and \$7000 at 8% **c.** \$24,000 at 6%, \$4000 at 6.5%, and \$12,000 at 8% **63. a.** 50 Super Vim, 75 Multitab, and 100 Mighty Mix **b.** 75 Super Vim, 50 Multitab, and 60 Mighty Mix **c.** 80 Super Vim, 100 Multitab, and 50 Mighty Mix **65. a.** 262, −161, −12, 186, −103, −22, 264, −168, −9, 208, −134, −5, 224, −152, 5, 92, −50, −3 **b.** $\begin{bmatrix} 1.75 & 2.5 & 3 \\ -0.25 & -0.5 & 0 \\ -0.25 & -0.5 & -1 \end{bmatrix}$ **c.** happy birthday

Exercises 2.6 (page 102)

For exercises . . .	1–6,9–12,17–25	7,8,26–29	13–16
Refer to example . . .	4	5	3,4

1. $\begin{bmatrix} 60 \\ 50 \end{bmatrix}$ **3.** $\begin{bmatrix} 6.43 \\ 26.12 \end{bmatrix}$ **5.** $\begin{bmatrix} 10 \\ 18 \\ 10 \end{bmatrix}$ **7.** 33 : 47 : 23 **9.** $\begin{bmatrix} 7697 \\ 4205 \\ 6345 \\ 4106 \end{bmatrix}$ (rounded) **11.** About 1440 metric tons of wheat and 1938 metric tons of oil. **13.** About 1506 units of agriculture, 1713 units of manufacturing, and 1412 units of transportation. **15.** About 3077 units of agriculture, about 2564 units of manufacturing, and about 3179 units of transportation **17. a.** $7/4$ bushels of yams and $15/8 \approx 2$ pigs **b.** 167.5 bushels of yams and $153.75 \approx 154$ pigs **19.** About 848 units of agriculture, about 516 units of manufacturing, and about 2970 units of households **21.** About 195 million pounds of agriculture, about 26 million pounds of manufacturing, and about 13.6 million pounds of energy **23.** In millions of dollars, the amounts are about 532 for natural resources, about 481 for manufacturing, about 805 for trade and services, and about 1185 for personal consumption.

25. a. $\begin{bmatrix} 1.67 & 0.56 & 0.56 \\ 0.19 & 1.17 & 0.06 \\ 3.15 & 3.27 & 4.38 \end{bmatrix}$ **b.** These multipliers imply that if the demand for one community's output increases by \$1, then the output in the other community will increase by the amount in the row and column of that matrix. For example, if the demand for Hermitage's output increases by \$1, then output from Sharon will increase by \$0.56, Farrell by \$0.06, and Hermitage by \$4.38. **27.** 3 units of coal to every 4 units of steel **29.** 6 units of mining to every 8 units of manufacturing and 5 units of communication

Chapter 2 Review Exercises (page 106)

For exercises . . .	1,3,10,13, 17,23–28, 67,68,74, 75,79,80,82	2,5,29–33, 36,69,73	4,19–22, 66,77,78, 81,83	6,7,18,34, 35,37–43, 76	8,9,11,12 14,15, 44–62,70, 74	16,63,64, 71,72
Refer to section . . .	2	3	1	4	5	6

1. False **2.** False **3.** True **4.** True **5.** False **6.** True **7.** False **8.** False **9.** True **10.** False **11.** False **12.** False **13.** True **14.** True **15.** False **16.** True **19.** $(1, -4)$ **21.** $((34 - 28z)/50, (53 - z)/25, z)$ **23.** $(-9, 3)$ **25.** $(7, -9, -1)$ **27.** $(6 - 7z/3, 1 + z/3, z)$ **29.** 2×2 (square); $a = 2, b = 3, c = 5, q = 9$ **31.** 1×4 (row); $m = 6, k = 3, z = -3, r = -9$

33. $\begin{bmatrix} 9 & 10 \\ -3 & 0 \\ 10 & 16 \end{bmatrix}$ **35.** $\begin{bmatrix} 23 & 20 \\ -7 & 3 \\ 24 & 39 \end{bmatrix}$ **37.** $\begin{bmatrix} -17 & 20 \\ 1 & -21 \\ -8 & -17 \end{bmatrix}$ **39.** Not possible **41.** [9] **43.** $[-14 \ -19]$ **45.** No inverse

47. $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ **49.** No inverse **51.** $\begin{bmatrix} 2/3 & 0 & -1/3 \\ 1/3 & 0 & -2/3 \\ -2/3 & 1 & 1/3 \end{bmatrix}$ **53.** No inverse **55.** $X = \begin{bmatrix} 1 \\ -13 \end{bmatrix}$ **57.** $X = \begin{bmatrix} -22 \\ -18 \\ 15 \end{bmatrix}$

59. $(2, 1)$ **61.** $(-1, 0, 2)$ **63.** $\begin{bmatrix} 218.1 \\ 318.3 \end{bmatrix}$ **65. a.** $(2, 3, -1)$ **b.** $(2, 3, -1)$ **c.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix}$

d. $\begin{bmatrix} 5/22 & 7/22 & 1/22 \\ 7/22 & 1/22 & -3/22 \\ 3/22 & -9/22 & 5/22 \end{bmatrix}$ **e.** $(2, 3, -1)$ **67.** 5 blankets, 3 rugs, 8 skirts **69.** $\begin{bmatrix} 1.33 & 17.6 & 152{,}000 & 26.75 & +1.88 \\ 1.00 & 20.0 & 238{,}200 & 32.36 & -1.50 \\ 0.79 & 25.4 & 39{,}110 & 16.51 & -0.89 \\ 0.27 & 21.2 & 122{,}500 & 28.60 & +0.75 \end{bmatrix}$

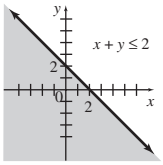
71. a. $\begin{array}{c} \\ \text{c} \\ \text{g} \end{array} \begin{array}{c} \begin{array}{cc} \text{c} & \text{g} \end{array} \\ \begin{bmatrix} 0 & 1/2 \\ 2/3 & 0 \end{bmatrix} \end{array}$ **b.** 1200 units of cheese; 1600 units of goats **73.** $\begin{bmatrix} 8 & 8 & 8 \\ 10 & 5 & 9 \\ 7 & 10 & 7 \\ 8 & 9 & 7 \end{bmatrix}$ **75. a.** No **b. (i)** 0.23, 0.37, 0.42; A is healthy; B and D are tumorous; C is bone. **(ii)** 0.33, 0.27, 0.32; A and C are tumorous, B could be healthy or tumorous, D is bone. **c.** 0.2, 0.4, 0.45, 0.3; A is healthy; B and C are bone; D is tumorous. **d.** One example is to choose beams 1, 2, 3, and 6. **77.** $W_1 = W_2 = 100\sqrt{3}/3 \approx 58$ lb **79. a.** $C = 0.010985t^2 + 0.8803t + 317$ **b.** 2095 **81.** There are y girls and $2500 - 1.5y$ boys, where y is any even integer between 0 and 1666. **83.** A chocolate wafer weighs 4.08 g, and a single layer of vanilla cream weighs 3.17 g.

Chapter 3 Linear Programming: The Graphical Method

Exercises 3.1 (page 118)

For exercises . . .	1–20	21–32	33–36	38–45
Refer to example . . .	1,2	3–5	3	6

1.



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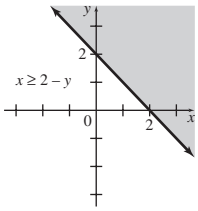

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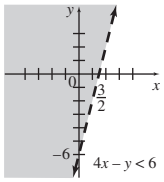

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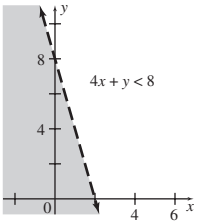

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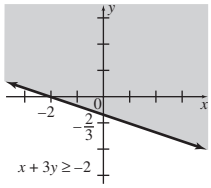

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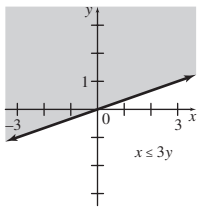
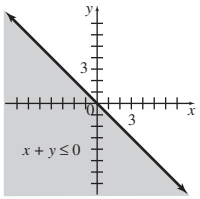


13.



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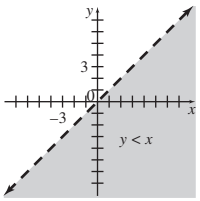

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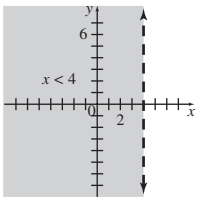

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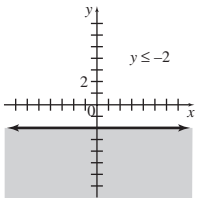

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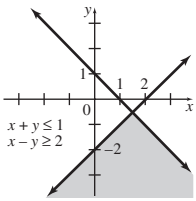
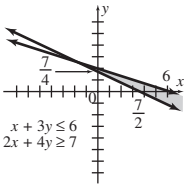


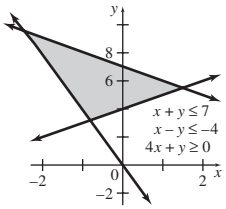
Unbounded

23.



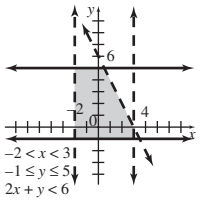
Unbounded

25.



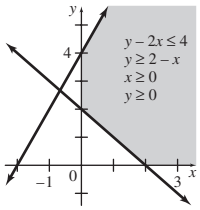
Bounded

27.



Bounded

29.



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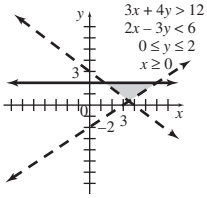
31.



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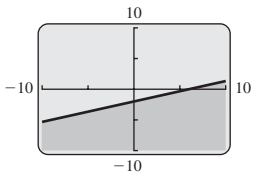
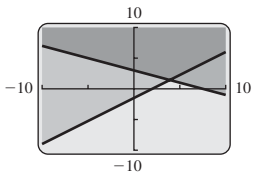
33.



35.

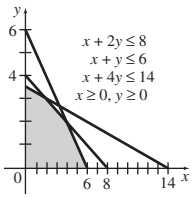


37. **B:** ≤, ≤, ≤; **C:** ≥, ≥, ≤; **D:** ≤, ≥, ≤; **E:** ≤, ≤, ≥; **F:** ≤, ≥, ≥; **G:** ≥, ≥, ≥

39. **a.**

	Shawls	Afghans		Total
Number Made	x	y		
Spinning Time	1	2	≤	8
Dyeing Time	1	1	≤	6
Weaving Time	1	4	≤	14

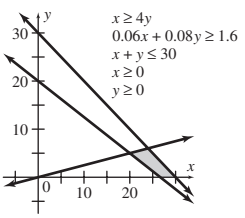
b.



c. Yes; no

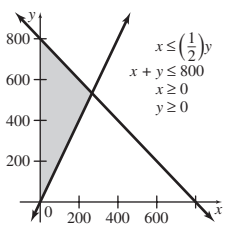
41. **a.** $x \geq 4y$; $0.06x + 0.08y \geq 1.6$; $x + y \leq 30$; $x \geq 0$; $y \geq 0$

b.

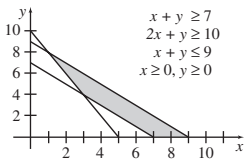


43. **a.** $x \leq (1/2)y$; $x + y \leq 800$; $x \geq 0$; $y \geq 0$

b.



45. a. $x + y \geq 7$; $2x + y \geq 10$; $x + y \leq 9$; $x \geq 0$; $y \geq 0$ **b.**



Exercises 3.2 (page 124)

For exercises . . .	1–16
Refer to example . . .	3

1. a. Maximum of 29 at $(7, 4)$; minimum of 10 at $(0, 5)$ **b.** Maximum of 35 at $(3, 8)$; minimum of 8 at $(4, 1)$ **3. a.** Maximum of 9 at $(0, 12)$; minimum of 0 at $(0, 0)$ **b.** Maximum of 12 at $(8, 0)$; minimum of 0 at $(0, 0)$ **5. a.** No maximum; minimum of 16 at $(0, 8)$ **b.** No maximum; minimum of 18 at $(3, 4)$ **c.** No maximum; minimum of 21 at $(13/2, 2)$ **d.** No maximum; minimum of 12 at $(12, 0)$ **7.** Minimum of 24 when $x = 6$ and $y = 0$ **9.** Maximum of 46 when $x = 6$ and $y = 8$ **11.** Maximum of 1500 when $x = 150$ and $y = 0$, as well as when $x = 50$ and $y = 100$ and all points on the line between **13.** No solution **15. a.** Maximum of 204 when $x = 18$ and $y = 2$ **b.** Maximum of $588/5$ when $x = 12/5$ and $y = 39/5$ **c.** Maximum of 102 when $x = 0$ and $y = 17/2$ **17.** b

Exercises 3.3 (page 131)

For exercises . . .	1–6	7–9,16,21–25	10–15,17–20, 26	27
Refer to example . . .	1–3	3	1,2	2

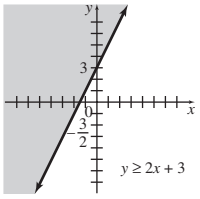
1. Let x be the number of product A produced and y be the number of product B. Then $3x + 5y \leq 60$. **3.** Let x be the number of calcium carbonate supplements and y be the number of calcium citrate supplements. Then $600x + 250y \geq 1500$. **5.** Let x be the number of pounds of \$8 coffee and y be the number of \$10 coffee. Then $x + y \geq 40$. **7.** 51 to plant I and 32 to plant II, for a minimum cost of \$2810 **9. a.** 6 units of policy A and 16 units of policy B, for a minimum premium cost of \$940 **b.** 30 units of policy A and 0 units of policy B, for a minimum premium cost of \$750 **11. a.** 500 type I and 1000 type II **b.** Maximum revenue is \$275. **c.** If the price of the type I bolt exceeds 20¢, then it is more profitable to produce 1050 type I bolts and 450 type II bolts. **13. a.** 120 kg of the half-and-half mix and 120 kg of the other mix, for a maximum revenue of \$1980 **b.** 0 kg of the half-and-half mix and 200 kg of the other mix, for a maximum revenue of \$2200 **15. a.** 40 gal from dairy I and 60 gal from dairy II, for a maximum butterfat of 3.4 gal **b.** 10 gal from dairy I and 20 gal from dairy 2. No. **17.** \$10 million in bonds and \$20 million in mutual funds, or \$5 million in bonds and \$22.5 million in mutual funds (or any solution on the line in between those two points), for a maximum annual interest of \$2 million **19.** a **21. a.** Three of pill 1 and two of pill 2, for a minimum cost of \$1.05 per day **b.** 360 surplus IU of vitamin A. No. **23.** 4 ounces of fruit and 2 ounces of nuts, for a minimum of 140 calories **25.** 0 plants and 18 animals, for a minimum of 270 hours

Chapter 3 Review Exercises (page 135)

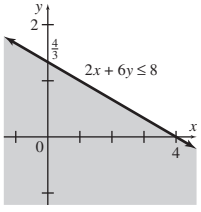
For exercises . . .	1,6,8,10,12–14,21–36	2,7,9,15–20	3,4,5,37–46	11
Refer to Section . . .	2	1	3	2,3

1. False **2.** True **3.** False **4.** False **5.** False
6. False **7.** True **8.** False **9.** False **10.** True **11.** True **12.** True **13.** True

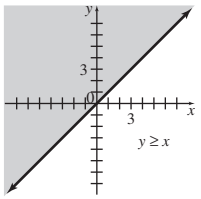
15.



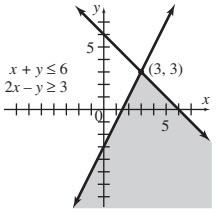
17.



19.



21.



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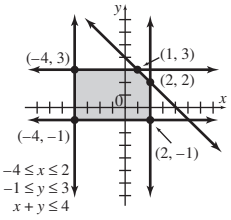
23.



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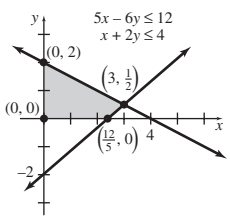
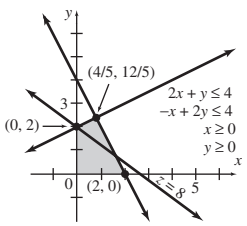
25.



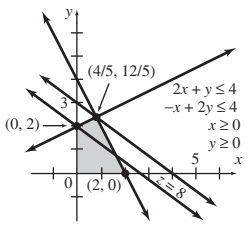
Bounded

27. Maximum of 22 at $(3, 4)$; minimum of 0 at $(0, 0)$
29. Maximum of 24 at $(0, 6)$ **31.** Minimum of 40 at any point on the segment connecting $(0, 20)$ and $(10/3, 40/3)$

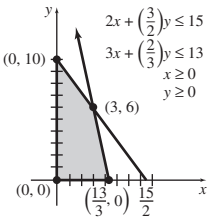
35. a.



b.



37. Let x = number of batches of cakes and y = number of batches of cookies. Then $x \ge 0$, $y \ge 0$, $2x + (3/2)y \le 15$, and $3x + (2/3)y \le 13$.



39. a. 3 batches of cakes and 6 batches of cookies, for a maximum profit of \$210 **b.** If the profit per batch of cookies increases by more than \$2.50 (to above \$22.50), then it will be more profitable to make 10 batches of cookies and no batches of cake. **41.** 7 packages of gardening mixture and 2 packages of potting mixture, for a maximum income of \$31 **43.** Produce no runs of type I and 7 runs of type II, for a minimum cost of \$42,000. **45.** 0 acres for millet and 2 acres for wheat, for a maximum harvest of 1600 lb

Chapter 4 Linear Programming: The Simplex Method

Exercises 4.1 (page 148)

For exercises . . .	1–14	15–18	19–24	27–31
Refer to example . . .	1	3	4	2

1. $x_1 + 2x_2 + s_1 = 6$ **3.** $2.3x_1 + 5.7x_2 + 1.8x_3 + s_1 = 17$ **5. a.** 3
b. s_1, s_2, s_3 **c.** $2x_1 + 3x_2 + s_1 = 15$; $4x_1 + 5x_2 + s_2 = 35$; $x_1 + 6x_2 + s_3 = 20$ **7. a.** 2 **b.** s_1, s_2
c. $7x_1 + 6x_2 + 8x_3 + s_1 = 118$; $4x_1 + 5x_2 + 10x_3 + s_2 = 220$

9.
$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z & \end{matrix} \\ \left[\begin{array}{ccccc|c} 4 & 2 & 1 & 0 & 0 & 5 \\ 1 & 2 & 0 & 1 & 0 & 4 \\ \hline -7 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

11.
$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & s_3 & z & \end{matrix} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 5 & 2 & 0 & 1 & 0 & 0 & 20 \\ 1 & 2 & 0 & 0 & 1 & 0 & 36 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

13.
$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z & \end{matrix} \\ \left[\begin{array}{ccccc|c} 3 & 1 & 1 & 0 & 0 & 12 \\ 1 & 1 & 0 & 1 & 0 & 15 \\ \hline -2 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

15. $x_1 = 0,\ x_2 = 4, x_3 = 0, s_1 = 0,\ s_2 = 8, z = 28$ **17.** $x_1 = 0,\ x_2 = 0, x_3 = 8, s_1 = 0,\ s_2 = 6, s_3 = 7, z = 12$
19. $x_1 = 0,\ x_2 = 20, x_3 = 0, s_1 = 16,\ s_2 = 0,\ z = 60$ **21.** $x_1 = 0,\ x_2 = 0, x_3 = 12, s_1 = 0,\ s_2 = 9, s_3 = 8, z = 36$
23. $x_1 = 0,\ x_2 = 250, x_3 = 0, s_1 = 0,\ s_2 = 50, s_3 = 200, z = 1000$

27. If x_1 is the number of simple figures, x_2 is the number of figures with additions, and x_3 is the number of computer drawn sketches, find $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$ such that $20x_1 + 35x_2 + 60x_3 \le 2200$, $x_1 + x_2 + x_3 \le 400$, $x_3 \le x_1 + x_2$, $x_1 \ge 2x_2$, and $z = 95x_1 + 200x_2 + 325x_3$ is maximized;
$20x_1 + 35x_2 + 60x_3 + s_1 = 2200$, $x_1 + x_2 + x_3 + s_2 = 400$, $-x_1 - x_2 + x_3 + s_3 = 0$, $-x_1 + 2x_2 + s_4 = 0$.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \end{matrix} \\ \left[\begin{array}{cccccccc|c} 20 & 35 & 60 & 1 & 0 & 0 & 0 & 0 & 2200 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 400 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline -95 & -200 & -325 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

29. If x_1 is the number of redwood tables made, x_2 is the number of stained Douglas fir tables made, and x_3 is the number of stained white spruce tables made, find $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$ such that $8x_1 + 7x_2 + 8x_3 \le 720$, $2x_2 + 2x_3 \le 480$, $159x_1 + 138.85x_2 + 129.35x_3 \le 15{,}000$, and $z = x_1 + x_2 + x_3$ is maximized; $8x_1 + 7x_2 + 8x_3 + s_1 = 720$, $2x_2 + 2x_3 + s_2 = 480$, $159x_1 + 138.85x_2 + 129.35x_3 + s_3 = 15{,}000$.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 8 & 7 & 8 & 1 & 0 & 0 & 0 & 720 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & 480 \\ 159 & 138.85 & 129.35 & 0 & 0 & 1 & 0 & 15{,}000 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

31. If x_1 is the number of newspaper ads run, x_2 is the number of Internet banner ads run, and x_3 is the number of TV ads run, find $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$, such that $400x_1 + 20x_2 + 2000x_3 \le 8000$, $x_1 \le 30$, $x_2 \le 60$, $x_3 \le 10$, and $z = 4000x_1 + 3000x_2 + 10{,}000x_3$ is maximized;
$400x_1 + 20x_2 + 2000x_3 + s_1 = 8000$, $x_1 + s_2 = 30$, $x_2 + s_3 = 60$, $x_3 + s_4 = 10$.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \end{matrix} \\ \left[\begin{array}{cccccccc|c} 400 & 20 & 2000 & 1 & 0 & 0 & 0 & 0 & 8000 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 10 \\ \hline -4000 & -3000 & -10{,}000 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Exercises 4.2 (page 157)

For exercises . . .	1–40
Refer to example . . .	1

1. Maximum is 30 when $x_1 = 10$, $x_2 = 0$, $x_3 = 0$, $s_1 = 6$, and $s_2 = 0$. **3.** Maximum is 8 when $x_1 = 4$, $x_2 = 0$, $s_1 = 8$, $s_2 = 2$, and $s_3 = 0$. **5.** Maximum is 264 when $x_1 = 16$, $x_2 = 4$, $x_3 = 0$, $s_1 = 0$, $s_2 = 16$, and $s_3 = 0$. **7.** Maximum is 25 when $x_1 = 0$, $x_2 = 5$, $s_1 = 20$, and $s_2 = 0$. **9.** Maximum is 120 when $x_1 = 0$, $x_2 = 10$, $s_1 = 0$, $s_2 = 40$, and $s_3 = 4$. **11.** Maximum is 944 when $x_1 = 118$, $x_2 = 0$, $x_3 = 0$, $s_1 = 0$, and $s_2 = 102$. **13.** Maximum is 3300 when $x_1 = 240$, $x_2 = 60$, $x_3 = 0$, $x_4 = 0$, $s_1 = 0$, and $s_2 = 0$. **15.** No maximum **17.** Maximum is 70,818.18 when $x_1 = 181.82$, $x_2 = 0$, $x_3 = 454.55$, $x_4 = 0$, $x_5 = 1363.64$, $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, and $s_4 = 0$. **23.** 6 churches and 2 labor unions, for a maximum of \$1000 per month **25. a.** Assemble 1000 Royal Flush poker sets, 3000 Deluxe Diamond poker sets, and 0 Full House poker sets, for a maximum profit of \$104,000. **b.** $s_4 = 1000$; there are 1000 unused dealer buttons. **27. a.** No racing or touring bicycles and 2700 mountain bicycles **b.** Maximum profit is \$59,400 **c.** No; there are 1500 units of aluminum left; $s_2 = 1500$. **29. a.** 17 newspaper ads, 60 Internet banner ads, and no TV ads, for a maximum exposure of 248,000 **31. a.** 3 **b.** 4 **c.** 3 **33.** \$200, \$66.67, \$300, \$100 **35.** Rachel should run 3 hours, bike 4 hours, and walk 8 hours, for a maximum calorie expenditure of 6313 calories. **37. a.** None of species A, 114 of species B, and 291 of species C, for a maximum combined weight of 1119.72 kg **b.** No; there are 346 units of Food II available. **c.** Many answers are possible. **d.** Many answers are possible. **39.** 12 minutes to the senator, 9 minutes to the congresswoman and 6 minutes to the governor, for a maximum of 1,050,000 viewers.

Exercises 4.3 (page 168)

For exercises . . .	1–4	5–8	9–16	19–29
Refer to example . . .	2	3	4	5

1. $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 10 \\ 3 & 1 & 0 \end{bmatrix}$ **3.** $\begin{bmatrix} 4 & 7 & 5 \\ 5 & 14 & 0 \\ -3 & 20 & -2 \\ 15 & -8 & 23 \end{bmatrix}$ **5.** Minimize $w = 5y_1 + 4y_2 + 15y_3$ subject to $y_1 + y_2 + 2y_3 \geq 4$, $y_1 + y_2 + y_3 \geq 3$, $y_1 + 3y_3 \geq 2$, with $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$. **7.** Maximize $z = 150x_1 + 275x_2$ subject to $x_1 + 2x_2 \leq 3$, $x_1 + 2x_2 \leq 6$, $x_1 + 3x_2 \leq 4$, $x_1 + 4x_2 \leq 1$, with $x_1 \geq 0$ and $x_2 \geq 0$. **9.** Minimum is 14 when $y_1 = 0$ and $y_2 = 7$. **11.** Minimum is 40 when $y_1 = 10$ and $y_2 = 0$. **13.** Minimum is 30 when $y_1 = 5$ and $y_2 = 0$ or when $y_1 = 0$ and $y_2 = 3$, or any point on the line segment between (5, 0) and (0,3). **15.** Minimum is 100 when $y_1 = 0$, $y_2 = 100$, and $y_3 = 0$. **17.** a **19. a.** 25 units of regular beer and 20 units of light beer, for a minimum cost of \$1,800,000 **b.** The shadow cost is \$0.10; total production cost is \$1,850,000. **21. a.** Minimize $w = 100y_1 + 20{,}000y_2$ subject to $y_1 + 400y_2 \geq 120$, $y_1 + 160y_2 \geq 40$, $y_1 + 280y_2 \geq 60$, with $y_1 \geq 0$, $y_2 \geq 0$. **b.** 52.5 acres of potatoes and no corn or cabbage, for a profit of \$6300. **c.** 47.5 acres of potatoes and no corn or cabbage, for a profit of \$5700. **23.** 8 political interviews and no market interviews are done, for a minimum of 360 minutes. **25. a.** 1 bag of feed 1 and 2 bags of feed 2 **b.** 1.4 (or 7/5) bags of feed 1 and 1.2 (or 6/5) bags of feed 2 should be used, for a minimum cost of \$6.60 **27.** She should spend 30 minutes walking, 197.25 minutes cycling, and 75.75 minutes swimming, for a minimum time of 303 minutes per week. **29.** 8/3 units of ingredient I and 4 units of ingredient III, for a minimum cost of \$30.67.

Exercises 4.4 (page 176)

For exercises . . .	1–16	17–20,24,27,31,32	23,25,26,28,29,30,33
Refer to example . . .	1	3	2

1. $2x_1 + 3x_2 + s_1 = 8$; $x_1 + 4x_2 - s_2 = 7$ **3.** $2x_1 + x_2 + 2x_3 + s_1 = 50$; $x_1 + 3x_2 + x_3 - s_2 = 35$; $x_1 + 2x_2 - s_3 = 15$ **5.** Change the objective function to maximize $z = -3y_1 - 4y_2 - 5y_3$. The constraints are not changed. **7.** Change the objective function to maximize $z = -y_1 - 2y_2 - y_3 - 5y_4$. The constraints are not changed. **9.** Maximum is 480 when $x_1 = 40$ and $x_2 = 0$. **11.** Maximum is 750 when $x_1 = 0$, $x_2 = 150$, and $x_3 = 0$. **13.** Maximum is 135 when $x_1 = 30$ and $x_2 = 5$. **15.** Minimum is 108 when $y_1 = 0$, $y_2 = 9$, and $y_3 = 0$. **17.** Maximum is $400/3$ when $x_1 = 100/3$ and $x_2 = 50/3$. **19.** Minimum is 512 when $y_1 = 6$, $y_2 = 8$, and $y_3 = 0$. **23. a.** Ship 200 barrels of oil from supplier S_1 to distributor D_1; ship 2800 barrels of oil from supplier S_2 to distributor D_1; ship 2800 barrels of oil from supplier S_1 to distributor D_2; ship 2200 barrels of oil from supplier S_2 to distributor D_2. Minimum cost is \$180,400. **b.** $s_3 = 2000$; S_1 could furnish 2000 more barrels of oil. **25.** Make \$3,000,000 in commercial loans and \$22,000,000 in home loans, for a maximum return of \$2,940,000. **27.** Use 1500 lb of bluegrass, 2700 lb of rye, and 1800 lb of Bermuda, for a minimum cost of \$834. **29. a.** Ship 2 computers from W_1 to D_1, 20 computers from W_1 to D_2, 30 computers from W_2 to D_1, and 0 computers from W_2 to D_2, for a minimum cost of \$628. **b.** $s_3 = 3$; warehouse W_1 has 3 more computers that it could ship. **31.** Use 59.21 kg of chemical I, 394.74 kg of chemical II, and 296.05 kg of chemical III, for a minimum cost of \$600.39 **33.** 5/3 oz of I, 20/3 oz of II, and 5/3 oz of III, for a minimum cost of \$1.55 per gal; 10 oz of the additive should be used per gal of gasoline.

Chapter 4 Review Exercises (page 180)

For exercises . . .	1,15,17,18,21–24	2,3,5–11,13,37–39,41–43,47	4,12	14,16,19,20,29–32,40,44	25–28,33–36,45,46
Refer to section . . .	1	2	3	4	3, 4

1. True **2.** False **3.** True **4.** False **5.** False **6.** True **7.** True **8.** False **9.** False **10.** True **11.** False **12.** True **13.** False **14.** True **15.** When the problem has more than two variables **17. a.** $4x_1 + 6x_2 + s_1 = 60$; $3x_1 + x_2 + s_2 = 18$;

$2x_1 + 5x_2 + s_3 = 20$; $x_1 + x_2 + s_4 = 15$ **b.**

x_1	x_2	s_1	s_2	s_3	s_4	z	
4	6	1	0	0	0	0	60
3	1	0	1	0	0	0	18
2	5	0	0	1	0	0	20
1	1	0	0	0	1	0	15
−2	−7	0	0	0	0	1	0

19. a. $x_1 + x_2 + x_3 + s_1 = 90$; $2x_1 + 5x_2 + x_3 + s_2 = 120$; $x_1 + 3x_2 - s_3 = 80$ **b.**

x_1	x_2	x_3	s_1	s_2	s_3	z	
1	1	1	1	0	0	0	90
2	5	1	0	1	0	0	120
1	3	0	0	0	−1	0	80
−5	−8	−6	0	0	0	1	0

21. Maximum is 33 when $x_1 = 3$, $x_2 = 0$, $x_3 = 3$, $s_1 = 0$, and $s_2 = 0$. **23.** Maximum is 76.67 when $x_1 = 6.67$, $x_2 = 0$, $x_3 = 21.67$, $s_1 = 0$, $s_2 = 0$, and $s_3 = 35$. **25. Dual Method** Solve the dual problem: Maximize $17x_1 + 42x_2$ subject to $x_1 + 5x_2 \le 10$, $x_1 + 8x_2 \le 15$. **Method of Section 4.4** Change the objective function to maximize $z = -10y_1 - 15y_2$. The constraints are not changed. Minimum is 170 when $y_1 = 17$ and $y_2 = 0$. **27. Dual Method** Solve the dual problem: Maximize $48x_1 + 12x_2 + 10x_3 + 30x_4$ subject to $x_1 + x_2 + 3x_4 \le 7$, $x_1 + x_2 \le 2$, $2x_1 + x_3 + x_4 \le 3$. **Method of Section 4.4** Change the objective function to maximize $z = -7y_1 - 2y_2 - 3y_3$. The constraints are not changed. Minimum is 98 when $y_1 = 4$, $y_2 = 8$, and $y_3 = 18$. **29.** Maximum of 480 when $x_1 = 24$ and $x_2 = 0$ **31.** Maximum of 102 when $x_1 = 0$ and $x_2 = 8.5$ **33.** Problems with constraints involving "≤" can be solved using slack variables, while those involving "≥" or "=" can be solved using surplus and artificial variables, respectively. **35. a.** Maximize $z = 6x_1 + 7x_2 + 5x_3$, subject to $4x_1 + 2x_2 + 3x_3 \le 9$, $5x_1 + 4x_2 + x_3 \le 10$, with $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$. **b.** The first constraint would be $4x_1 + 2x_2 + 3x_3 \ge 9$. **c.** $x_1 = 0$, $x_2 = 2.1$, $x_3 = 1.6$, and $z = 22.7$ **d.** Minimize $w = 9y_1 + 10y_2$, subject to $4y_1 + 5y_2 \ge 6$, $2y_1 + 4y_2 \ge 7$, $3y_1 + y_2 \ge 5$, with $y_1 \ge 0$, $y_2 \ge 0$. **e.** $y_1 = 1.3$, $y_2 = 1.1$, and $w = 22.7$ **37. a.** Let $x_1 =$ number of cake plates, $x_2 =$ number of bread plates, and $x_3 =$ number of dinner plates. **b.** $z = 15x_1 + 12x_2 + 5x_3$ **c.** $15x_1 + 10x_2 + 8x_3 \le 1500$; $5x_1 + 4x_2 + 4x_3 \le 2700$; $6x_1 + 5x_2 + 5x_3 \le 1200$ **39. a.** Let $x_1 =$ number of gallons of Fruity wine and $x_2 =$ number of gallons of Crystal wine to be made. **b.** $z = 12x_1 + 15x_2$ **c.** $2x_1 + x_2 \le 110$; $2x_1 + 3x_2 \le 125$; $2x_1 + x_2 \le 90$ **41.** Produce no cake plates, 150 bread plates, and no dinner plates, for a maximum profit of $1800. **43.** 36.25 gal of Fruity and 17.5 gal of Crystal, for a maximum profit of $697.50 **45. a and b** Produce 660 cases of corn, 0 cases of beans, and 340 cases of carrots, for a minimum cost of $15,100. **c.** $16,100 **47.** Ginger should do $5\frac{1}{3}$ hours of tai chi, $2\frac{2}{3}$ hours of riding a unicycle, and 2 hours of fencing, for a maximum calorie expenditure of $2753\frac{1}{3}$ calories.

Chapter 5 Mathematics of Finance

Exercises 5.1 (page 197)

For exercises . . .	5–10,49,50	11,12	13,14,51–53	19–24,54a,b,55–65	25–28,54c,d,66,77	29–32,67–69	33–38,70,71	41–44	45,46	47,48,63e
Refer to example . . .	1	2	3	4,5	6	7,8	9,10	11	12	13

1. The interest rate and number of compounding periods **5.** $562.50 **7.** $59.79 **9.** $50.79 **11.** $3176.95; $51.95 **13.** 7.5% **17.** t is the number of years, while n is the number of compounding periods. **19.** $1593.85; $593.85 **21.** $890.82; $420.82 **23.** $12,630.55; $4130.55 **25.** 4.75% **27.** 5.75% **29.** 4.06% **31.** 7.38% **33.** $9677.13 **35.** $1246.33 **37.** $6864.08 **41.** 15 years **43.** 24 years, 11 months **45. a.** 21.35 years **b.** 21.21 years **47. a.** $7269.94 **b.** 3.15% **c.** 19.29 years **49.** $7534.80; $334.80 **51.** 6.8% **53.** 11.4% **55.** $41,325.95 **57.** $1000 now **59.** $136,110.16 **61.** $\$9.31 \times 10^{31}$ **63. a.** $16,288.95 **b.** $16,436.19 **c.** $16,470.09 **d.** $16,486.65 **e.** $16,487.21 **65.** 5/4 **67.** 2.48, 5.01, 4.18, 4.43, 5.15 **69.** 5.64%, 5.63%, Centennial Bank **71.** $22,829.89 **73.** About 18 years **75.** About 12 years **77. a.** 20.7% **b.** $39.6 billion

Exercises 5.2 (page 207)

For exercises . . .	1–8	9–14	17–26,47–55,65,68	29,30,63,64	31–38,56–58,66,67	39–46,59–62
Refer to example . . .	1	2	3	4	5	6

1. 48 **3.** −648 **5.** 81 **7.** 1 **9.** 15 **11.** 156/25 **13.** −208 **17.** $437.46 **19.** $2,154,099.15 **21.** $180,307.41; $128,800; $51,507.41 **23.** $28,438.21; $19,200; $9238.21 **25.** $1,145,619.96; $768,000; $377,619.96 **29.** 4.19% **31.** $628.25 **33.** $952.62 **35.** $7382.54 **37.** $5970.23 **39.** $6294.79 **41.** $136,785.74 **43.** $26,874.97; $18,000; $8874.97 **45.** $15,662.40; $12,000; $3662.40 **47. a.** $149,850.69 **b.** $137,895.79 **c.** $11,954.90 **49.** $197,750.47 **51.** $239,315.17 **53.** $312,232.31; $212,232.31 **55.** $432,548.65; $332,548.65 **57.** $1349.48 **59.** $3777.89 **61.** $67,940.98 **63.** 6.5% **65. a.** 7 yr **b.** 9 yr **67. a.** $120 **b.** $681.83 except the last payment, which is $681.80

Exercises 5.3 (page 215)

For exercises . . .	3–10,41	11–16a,b,27–30,35–40, 42,43,47,48a,49,50a–c, 51a,c,d,52,53	11–16c,21–24,44–46, 54,55	17–20	31–34	48b,49,50d,51b
Refer to example . . .	1	2,3	5	4	6	after Example 3

3. $8994.25 **5.** $209,302.93 **7.** $170,275.47 **9.** $111,183.87 **11. a.** $438.81 **b.** $2632.86; $132.86 **c.** $2632.88; $132.88 **13. a.** $10,734.93 **b.** $128,819.16; $38,819.16 **c.** $128,819.20; $38,819.20 **15. a.** $542.60 **b.** $9766.80; $2366.80 **c.** $9766.88; $2366.88 **17.** $73,015.71 **19.** $5368.98 **21.** $7.61 **23.** $35.24 **25.** $6699 **27.** $1407.76; $422,328; $223,328 **29.** $1590.82; $572,695.20; $319,659.20 **31. a.** 8 years **b.** $113,086.84 **c.** $15,732.36 **33. a.** 11 semiannual periods **b.** $8760.50 **c.** $1006.38 **35.** $1856.49; $114,168.20. The payments are $537.48 more than for the 30-yr loan, but the total interest paid is $140,675.40 less. **37. a.** $335.25 **b.** $2092 **39. a.** $844.95, $30,418.20 **b.** $655.92, $31,484.16 **41. a.** $623,110.52 **b.** $456,427.28 **c.** $563,757.78 **d.** $392,903.18 **43.** $381.74, $59,522

45.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	$110,000.00
1	$14,794.23	$4400.00	$10,394.23	$99,605.77
2	$14,794.23	$3984.23	$10,810.00	$88,795.77
3	$14,794.23	$3551.83	$11,242.40	$77,553.37
4	$14,794.23	$3102.13	$11,692.10	$65,861.27

55.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	$4836.00
1	$585.16	$175.31	$409.85	$4426.15
2	$585.16	$160.45	$424.71	$4001.43
3	$585.16	$145.05	$440.11	$3561.32
4	$585.16	$129.10	$456.06	$3105.26
5	$585.16	$112.57	$472.59	$2632.67
6	$585.16	$95.43	$489.73	$2142.94
7	$585.16	$77.68	$507.48	$1635.46
8	$585.16	$59.29	$525.87	$1109.59
9	$585.16	$40.22	$544.94	$564.65
10	$585.12	$20.47	$564.65	$0.00

47. a. $32.49 **b.** $195.52; $10.97 **49. a.** $1959.99; $127,798.20 **b.** $1677.54; $177,609.60 **c.** $1519.22; $230,766.00 **d.** After 157 payments **51. a.** $1121.63; $403,786.80; $253,786.80 **b.** $115,962.66; $201,893.40 **c.** $732.96; $267,265.60 **d.** $1010.16; $186,328.80 **53. a.** $17,584.58 **b.** $15,069.31 **57. a.** $25,000 **b.** $40,000

Chapter 5 Review Exercises (page 219)

For exercises . . .	1,11–28,64–69,74,75,81a	2–4,29–48,70,71,80,81c,e	5–10,49–63,72,73,76–79,81b,d
Refer to example . . .	1	2	3

1. True **2.** False **3.** True **4.** False **5.** True **6.** True **7.** True **8.** False **9.** False **10.** True **11.** $636.12 **13.** $1290.11 **15.** Compound interest **17.** $33,691.69 **19.** $77,860.80 **21.** $5244.50 **23.** $4725.22 **25.** $27,624.86 **27.** $1067.71 **29.** 2, 6, 18, 54, 162 **31.** -96 **33.** -120 **35.** 40.56808 **39.** $23,559.98; $5527.98 **41.** $12,302.78; $1118.78 **43.** $160,224.29; $5524.29 **45.** $955.61 **47.** $6156.14 **49.** $2945.34 **51.** $56,711.93 **53.** A home loan and an auto loan **55.** $302.59; $431.08 **57.** $1796.20; $5871.40 **59.** $1140.50; $410,580; $233,470 **61.** $132.99 **63.** $1535.61 **65.** $10,203.80; $383.80 **67.** 8.21% **69.** $2298.58 **71.** $107,892.82; $32,892.82 **73.** $8751.91; $13,263.37 **75.** 1.50% and 1.46%; Ascencia **77. a.** $266.67, $16,000 (or $16,000.20 using the rounded payments) **b.** $249.59, $17,970.48 **c.** $283.64, $13,614.72 **79. a.** $954.42 **b.** $817.92 **c.** Method 1: $109,563.99; Method 2: $109,565.13 **d.** $9650 **e.** Method 1: $118,786.01; Method 2: $118,784.87 **81. a.** 9.569% **b.** $896.44 **c.** $626,200.88 **d.** $1200.39 **e.** $478,134.14 **f.** Sue is ahead by $148,066.74.

Chapter 6 Logic

Exercises 6.1 (page 231)

For exercises . . .	1,14, 65,66, 75,76, 78,79	15,16, 67,77, 80	17–20	23–28, 68–72, 83–88	35–46, 49–56, 73,74, 89,90	57–64
Refer to example . . .	1	2	3	4	6	7

1. Statement, not compound **3.** Not a statement **5.** Statement, compound **7.** Not a statement **9.** Statement, compound **11.** Statement, not compound **13.** Statement, compound **15.** My favorite flavor is not chocolate. **17.** $y \leq 12$ **19.** $q < 5$ **23.** I'm not getting better. **25.** I'm not getting better or my parrot is dead. **27.** It is not the case that both I'm getting better and my parrot is not dead. **29.** False **31.** True **33.** Both components are false. **35.** True **37.** True **39.** False **41.** True **43.** True **45.** True **47.** Disjunction **49.** True **51.** False **53.** True **55.** True **57.** False **59.** True **61.** True **63.** True **65.** b, c, d

67. An individual has to be your biological child to be a "qualifying" child. **69.** $a \wedge j$ **71.** $\sim a \vee j$ **73.** 69, 71, and 72
75. b, c, d, e **77.** You may not find that exercise helps you cope with stress. **79.** c, d **83.** $n \wedge \sim m$ **85.** $\sim n \vee m$
87. $\sim n \wedge \sim m$ or $\sim(n \vee m)$ **89.** 85

Exercises 6.2 (page 238)

1. 4 **3.** 16 **5.** 128 **7.** 6

For exercises . . .	9–18	19–24,42,43,48	25–34,40,41, 44,46,47,49
Refer to example . . .	1,2	3,4	6

9.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

11.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

13.

p	q	$\sim p$	$\sim q$	$q \vee \sim p$	$(q \vee \sim p) \vee \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	T
F	F	T	T	T	T

In Exercises 15–23, we are using the alternate method to save space.

15.

p	q	$\sim q$	$\wedge$	$(\sim p \vee q)$
T	T	F	F	F T T
T	F	T	F	F F F
F	T	F	F	T T T
F	F	T	T	T T F
		①	④	② ③ ②

17.

p	q	$(p \vee \sim q)$	$\wedge$	$(p \wedge q)$
T	T	T T F	T	T T T
T	F	T T T	F	T F F
F	T	F F F	F	F F T
F	F	F T T	F	F F F
		① ② ①	⑤	③ ④ ③

19.

p	q	r	$(\sim p \wedge q)$	$\wedge$	r
T	T	T	F F T	F	T
T	T	F	F F T	F	F
T	F	T	F F F	F	T
T	F	F	F F F	F	F
F	T	T	T T T	T	T
F	T	F	T T T	F	F
F	F	T	T F F	F	T
F	F	F	T F F	F	F
			① ② ①	④	③

21.

p	q	r	$(\sim p \wedge \sim q)$	$\vee$	$(\sim r \vee \sim p)$
T	T	T	F F F	F	F F F
T	T	F	F F F	T	T T F
T	F	T	F F T	F	F F F
T	F	F	F F T	T	T T F
F	T	T	T F F	T	F T T
F	T	F	T F F	T	T T T
F	F	T	T T T	T	F T T
F	F	F	T T T	T	T T T
			① ② ①	⑤	③ ④ ③

23.

p	q	r	s	$\sim(\sim p \wedge \sim q)$	$\vee$	$(\sim r \vee \sim s)$
T	T	T	T	T F F F	T	F F F
T	T	T	F	T F F F	T	F T T
T	T	F	T	T F F F	T	T T F
T	T	F	F	T F F F	T	T T T
T	F	T	T	T F F T	T	F F F
T	F	T	F	T F F T	T	F T T
T	F	F	T	T F F T	T	T T F
T	F	F	F	T F F T	T	T T T
F	T	T	T	T T F F	T	F F F
F	T	T	F	T T F F	T	F T T
F	T	F	T	T T F F	T	T T F
F	T	F	F	T T F F	T	T T T
F	F	T	T	F T T T	F	F F F
F	F	T	F	F T T T	T	F T T
F	F	F	T	F T T T	T	T T F
F	F	F	F	F T T T	T	T T T
				③ ① ② ①	⑥	④ ⑤ ④

25. It's not a vacation, or I am not having fun.
27. The door was locked and the thief didn't break a window. **29.** I'm not ready to go, or Naomi Bahary is **31.** $12 \leq 4$ and $8 \neq 9$ **33.** Neither Larry nor Moe is out sick today **35.**

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

37. True **39. a.** False **b.** True **c.** False **d.** True
41. Service will not be performed at the location, and the store may not send the Covered Equipment to an Apple repair service location to be repaired.

43. $s \vee (r \wedge \sim q)$

s	r	q	s	$\vee$	$(r \wedge \sim q)$
T	T	T	T	T	T F F
T	T	F	T	T	T T T
T	F	T	T	T	F F F
T	F	F	T	T	F F T
F	T	T	F	F	T F F
F	T	F	F	T	T T T
F	F	T	F	F	F F F
F	F	F	F	F	F F T
			①	④	② ③ ②

The guarantee would be false if you are not completely satisfied, and they either don't refund your money or ask you questions.
45. Inclusive **47.** Liberty without learning is not always in peril, or learning without liberty is not always in vain.
49. You cannot reroll the die again for your Large Straight and you cannot set aside the 2 Twos and roll for your Twos or for 3 of a Kind.

Exercises 6.3 (page 247)

For exercises . . .	9–12	21–30, 91	33–42, 51–68	43–46, 89,90, 92,93	47–50	69–76, 87,88	77–86
Refer to example . . .	3	4	5	6	7	8,9	10

1. True **3.** True **5.** True **9.** True
11. True **13.** If she dances tonight, then I'm leaving early and he sings loudly.
15. If he doesn't sing loudly, then she dances tonight or I'm not leaving early. **17.** $d \vee (f \rightarrow g)$ **19.** $\sim f \rightarrow g$ **21.** False **23.** False **25.** True **27.** True

33.

p	q	$\sim q$	$\rightarrow$	p
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	F	F
		①	②	①

35.

p	$(p \vee \sim p)$	$\rightarrow$	$(p \wedge \sim p)$
T	T T F	F	T F F
F	F T T	F	F F T
	① ② ①	⑤	③ ④ ③

It is a contradiction.

37.

p	q	$(p \vee q)$	$\rightarrow$	$(q \vee p)$
T	T	T T T	T	T T T
T	F	T T F	T	F T T
F	T	F T T	T	T T F
F	F	F F F	T	F F F
		① ② ①	⑤	③ ④ ③

It is a tautology.

39.

p	q	r	r	$\rightarrow$	$(p \wedge \sim q)$
T	T	T	T	F	T F F
T	T	F	F	T	T F F
T	F	T	T	T	T T T
T	F	F	F	T	T T T
F	T	T	T	F	F F F
F	T	F	F	T	F F F
F	F	T	T	F	F F T
F	F	F	F	T	F F T
			①	④	② ③ ②

41.

p	q	r	s	$(\sim r \rightarrow s)$	$\vee$	$(p \rightarrow \sim q)$
T	T	T	T	F T T	T	T F F
T	T	T	F	F T F	T	T F F
T	T	F	T	T T T	T	T F F
T	T	F	F	T F F	F	T F F
T	F	T	T	F T T	T	T T T
T	F	T	F	F T F	T	T T T
T	F	F	T	T T T	T	T T T
T	F	F	F	T F F	T	T T T
F	T	T	T	F T T	T	F T F
F	T	T	F	F T F	T	F T F
F	T	F	T	T T T	T	F T F
F	T	F	F	T F F	T	F T F
F	F	T	T	F T T	T	F T T
F	F	T	F	F T F	T	F T T
F	F	F	T	T T T	T	F T T
F	F	F	F	T F F	T	F T T
				① ② ①	⑤	③ ④ ③

43. Your eyes are not bad or your whole body will be full of darkness. **45.** I don't have the money or I'd buy that car.
47. You ask me and I do not do it. **49.** You don't love me and I will be happy. **51.** Equivalent **53.** Not equivalent
55. Equivalent **57.** Not equivalent

59.

p	q	$p \wedge q$	$\sim (p \rightarrow \sim q)$
T	T	T T T	T T F F
T	F	T F F	F T T T
F	T	F F T	F F T F
F	F	F F F	F F T T
		① ② ①	⑤ ③ ④ ③

The columns labeled 2 and 5 are identical.

61.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F
		①	③

The columns labeled 2 and 4 are identical.

63.

p	q	r	$(p \vee q) \vee r$	$p \vee (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F
			①	⑤

The columns labeled 4 and 8 are identical.

65.

p	q	r	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F
			①	⑤

The columns labeled 4 and 9 are identical.

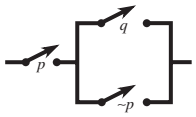
67.

p	q	$(p \wedge q) \vee p$
T	T	T
T	F	T
F	T	F
F	F	F
		①

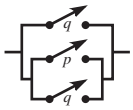
The p column and the column labeled 4 are identical.

69. $(p \wedge q) \vee (p \wedge \sim q) \equiv p$ **71.** $p \vee (\sim q \wedge r)$ **73.** $(p \vee q) \vee \sim p \equiv \mathrm{T}$ **75.** $[(p \wedge q) \vee (p \wedge p)] \vee (r \wedge \sim r) \equiv p$

77. $p \wedge (q \vee \sim p) \equiv p \wedge p$ **79.** $(p \vee q) \wedge (\sim p \wedge \sim q) \equiv F$ **81.** $\{(p \vee q) \wedge r\} \wedge \sim p \equiv r \wedge (\sim p \wedge q)$



83. $\sim q \rightarrow (\sim p \rightarrow q) \equiv p \vee q$ **85.** $[(p \wedge q) \vee p] \wedge [(p \vee q) \wedge q] \equiv p \wedge q$ **87.** False



89. **a.** You are not married at the end of the year, or you may file a joint return with your spouse. **b.** A bequest received by an executor from an estate is compensation for services, or it is tax free. **c.** A course does not improve your current job skills or does not lead to qualification for a new profession, or the course is not deductable. **91. a.** $(\nu \vee p) \rightarrow (s \wedge g)$ **b.** True **d.** The value of my portfolio exceeds \$100,000 or the price of my stock in Ford Motor Company falls below \$50 per share, and I will not sell my shares of Ford stock or I will not give the proceeds to the United Way. **93. a.** If you cannot file a civil lawsuit yourself, then your attorney can do it for you. **b.** If your driver's license does not come with restrictions, then restrictions may sometimes be added on later. **c.** If you can marry when you're not at least 18 years old, then you have the permission of your parents or guardian.

Exercises 6.4 (page 254)

For exercises . . .	1–10,42,43, 45,47,48	13–29,41,44,46, 55–57	32–38
Refer to example . . .	4	1,2	5

1. a. *Converse:* If I don't see it, then the exit is ahead. **b.** *Inverse:* If the exit is not ahead, then I see it. **c.** *Contrapositive:* If I see it, then the exit is not ahead. **3. a.** *Converse:* If I cleaned the house, then I knew you were coming. **b.** *Inverse:* If I didn't know you were coming, I wouldn't have cleaned the house. **c.** *Contrapositive:* If I didn't clean the house, then I didn't know you were coming. **5. a.** *Converse:* If you wear a pocket protector, then you are a mathematician. **b.** *Inverse:* If you are not a mathematician, then you do not wear a pocket protector. **c.** *Contraposition:* If you do not wear a pocket protector, then you are not a mathematician. **7. a.** *Converse:* $\sim q \rightarrow p$. **b.** *Inverse:* $\sim p \rightarrow q$. **c.** *Contrapositive:* $q \rightarrow \sim p$. **9. a.** *Converse:* $(q \vee r) \rightarrow p$. **b.** *Inverse:* $\sim p \rightarrow \sim(q \vee r)$ or $\sim p \rightarrow (\sim q \wedge \sim r)$. **c.** *Contrapositive:* $(\sim q \wedge \sim r) \rightarrow \sim p$. **13.** If you sign, then you accept the conditions. **15.** If you can take this course pass/fail, then you have prior permission. **17.** If the temperature is below 10°, then you can skate on the pond. **19.** If someone eats 10 hot dogs, then he or she will get sick. **21.** If you

travel to France, then you have a valid passport. **23.** If a number has a real square root, then it is nonnegative. **25.** If someone is a bride, then she is beautiful. **27.** If the sum of a number's digits is divisible by 3, then it is divisible by 3. **29.** d **33.** True **35.** False **37.** False

39.

p	q	$(\sim p \wedge q)$			$\leftrightarrow$	$(p \rightarrow q)$		
T	T	F	F	T	F	T	T	T
T	F	F	F	F	T	T	F	F
F	T	T	T	T	T	F	T	T
F	F	T	F	F	F	F	T	F
		①	②	①	⑤	③	④	③

41. a. If it is an employee contribution, then it must be reported on From 8889. **b.** If certain tax benefits may be claimed by married persons, then they file jointly. **c.** If he or she provides over half of his or her own support, then the child is not a qualifying child. **43.** *Converse:* If we close your account without notice, then your account is in default. *Inverse:* If your account is not in default, then we may not close your account without notice. *Contrapositive:* If we do not close your account without notice, then your account is not in default. The original statement and the contrapositive are equivalent, and the converse and inverse are equivalent. **45. a.** $p \rightarrow (q \wedge r)$ **b.** If the most persistent does not stand to gain an extra meal or it does not eat at the expense of another, then there are not triplets. **47. a.** *Converse:* If you can't get married again, then you are married. *Inverse:* If you aren't married, then you can get married again. *Contrapositive:* If you can get married again, then you are not married. **b.** *Converse:* If you are protected by the Fair Credit Billing Act, then you pay for your purchase with a credit card. *Inverse:* If you do not pay for your purchase with a credit card, then you are not protected by the Fair Credit Billing Act. *Contrapositive:* If you are not protected by the Fair Credit Billing Act, then you do not pay for your purchase with a credit card. **c.** *Converse:* If you're expected to make a reasonable effort to locate the owner, then you hit a parked car. *Inverse:* If you did not hit a parked car, then you are not expected to make a reasonable effort to locate the owner. *Contrapositive:* If you are not expected to make a reasonable effort to locate the owner, then you did not hit a parked car. The original statement and the contrapositive are equivalent, and the converse and inverse are equivalent.

49. a. $d \leftrightarrow a$,

d	a	$d \leftrightarrow a$
T	T	T
T	F	F
F	T	F
F	F	T

b. It is false **51.** If a country has democracy, then it has a high level of education. *Converse:* If a country has a high level of education, then it has democracy. *Inverse:* If a country does not have democracy, then it does not have a high level of education. *Contrapositive:* If a country does not have a high level of education, then it does not have a democracy. The contrapositive is equivalent to the original. **53.** D, 7 **55. a.** If nothing is ventured, then nothing is gained. If something is gained, then something is ventured. Something is ventured or nothing is gained. **b.** If something is one of the best things in life, then it is free. If something is not free, then it's not one of the best things in life. Something is not one of the best things in life or it is free. **c.** If something is a cloud, then it has a silver lining. If something doesn't have a silver lining, then it's not a cloud. Something is not a cloud or it doesn't have a silver lining. **57. a.** If you can score in this box, then the dice show any sequence of four numbers. You cannot score in this box, or the dice show any sequence of four numbers. **b.** If two or more words are formed in the same play, then each is scored. Two or more words are not formed in the same play, or each is scored. **c.** If words are labeled as a part of speech, then they are permitted. Words are not labeled as parts of speech, or they are permitted.

Exercises 6.5 (page 264)

1. Valid; Reasoning by Transitivity

3. Valid; Modus Ponens

5. Invalid; Fallacy of the Converse

7. Valid; Modus Tollens

For exercises . . .	1,2	3,4	5,6	7,8	9,10	11,12	13–16, 31,32, 35	17–24,29,30, 33,34, 36–44
Refer to example . . .	4	Modus Ponens	1	2	Fallacy of the Inverse	3	5,8	6,7

9. Invalid; Fallacy of the Inverse **11.** Valid; Disjunctive Syllogism **13.** Invalid; $p = \text{T}, q = \text{T}$ **15.** Invalid; $p = \text{F}, q = \text{F}$

17. Valid.

1. $\sim p \rightarrow \sim q$	Premise	
2. q	Premise	
3. p	1, 2, Modus Tollens	

19. Valid.

1. $p \rightarrow q$	Premise	
2. $\sim q$	Premise	
3. $\sim p \rightarrow r$	Premise	
4. $\sim p$	1, 2, Modus Tollens	
5. r	3, 4, Modus Ponens	

21. Valid.

1. $p \rightarrow q$	Premise	
2. $q \rightarrow r$	Premise	
3. $\sim r$	Premise	
4. $p \rightarrow r$	1, 2, Transitivity	
5. $\sim p$	3, 4, Modus Tollens	

23. Valid.

1. $p \rightarrow q$	Premise	
2. $q \rightarrow \sim r$	Premise	
3. p	Premise	
4. $r \vee s$	Premise	
5. q	1, 3, Modus Ponens	
6. $\sim r$	2, 5, Modus Ponens	
7. s	4, 6, Disjunctive Syllogism	

25.

p	q	$(p \wedge q) \rightarrow p$
T	T	T T T T T
T	F	T F F T T
F	T	F F T T F
F	F	F F F T F
		① ② ① ③ ②

27.

p	q	$(p \wedge q) \rightarrow (p \wedge q)$
T	T	T T T T T T T
T	F	T F F T T F F
F	T	F F T T F F T
F	F	F F F T F F F
		① ② ① ⑤ ③ ④ ③

29. Valid.

1.	$a \rightarrow s$	Premise
2.	$v \vee a$	Premise
3.	$\sim v$	Premise
4.	a	2, 3, Disjunctive Syllogism
5.	s	1, 4, Modus Ponens

31. Invalid; $b =$ "it is a bear market" = T, $p =$ "prices are rising" = F, $i =$ "investor will sell stocks" = T

33. Valid.

1.	$s \vee i$	Premise
2.	$s \rightarrow (l \wedge b)$	Premise
3.	$\sim l \vee \sim b$	Premise
4.	$\sim (l \wedge b)$	3, DeMorgan's Law
5.	$\sim s$	2, 4, Modus Tollens
6.	i	1, 5, Disjunctive Syllogism

35. Invalid; $M =$ "I am married to you" = T, $o =$ "we are one" = T, $r =$ "you are really a part of me" = F

37. Valid.

1.	$y \vee \sim p$	Premise
2.	$\sim p \rightarrow \sim n$	Premise
3.	n	Premise
4.	p	2, 3, Modus Tollens
5.	y	1, 4, Disjunctive Syllogism

39. a. $d \rightarrow \sim w$ **b.** $o \rightarrow w$ or $\sim w \rightarrow \sim o$ **c.** $p \rightarrow d$ **d.** $p \rightarrow \sim o$, *Conclusion:* If it is my poultry, then it is not an officer. In Lewis Carroll's words, "My poultry are not officers." **41. a.** $b \rightarrow \sim t$ or $t \rightarrow \sim b$ **b.** $w \rightarrow c$ **c.** $\sim b \rightarrow h$ **d.** $\sim w \rightarrow \sim p$, or $p \rightarrow w$ **e.** $c \rightarrow t$ **f.** $p \rightarrow h$, *Conclusion:* If one is a pawnbroker, then one is honest. In Lewis Carroll's words, "No pawnbroker is dishonest." **43. a.** $d \rightarrow p$ **b.** $\sim t \rightarrow \sim i$ **c.** $r \rightarrow \sim f$ or $f \rightarrow \sim r$ **d.** $o \rightarrow d$ or $\sim d \rightarrow \sim o$ **e.** $\sim c \rightarrow i$ **f.** $b \rightarrow s$ **g.** $p \rightarrow f$ **h.** $\sim o \rightarrow \sim c$ or $c \rightarrow o$ **i.** $s \rightarrow \sim t$ or $t \rightarrow \sim s$ **j.** $b \rightarrow \sim r$, *Conclusion:* If it is written by Brown, then I can't read it. In Lewis Carroll's words, "I cannot read any of Brown's letters."

Exercises 6.6 (page 272)

For exercises . . .	1–6,34,42	7,8,23	9,10, 21	11,12, 24	13–20 37–41 43–48	22,25–32
Refer to example . . .	1	2	4	3	6	5

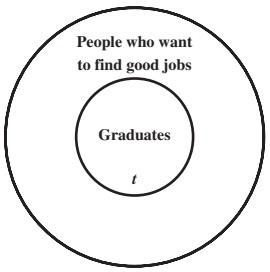
1. a. $\exists x\,[b(x) \wedge s(x)]$ **b.** $\forall x\,[b(x) \rightarrow \sim s(x)]$ **c.** No books are bestsellers. **3. a.** $\forall x\,[c(x) \rightarrow \sim s(x)]$ **b.** $\exists x\,[c(x) \wedge s(x)]$ **c.** There is a CEO who sleeps well at night. **5. a.** $\forall x\,[l(x) \rightarrow b(x)]$ **b.** $\exists x\,[l(x) \wedge \sim b(x)]$ **c.** There is a leaf that's not brown.

7. a. $\forall x[g(x) \rightarrow f(x)]$

$g(t)$

$f(t)$

b. Valid

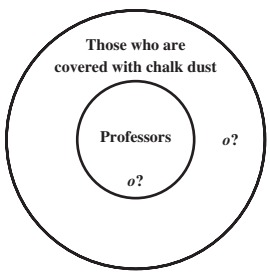


9. a. $\forall x[p(x) \rightarrow c(x)]$

$c(o)$

$p(o)$

b. Invalid

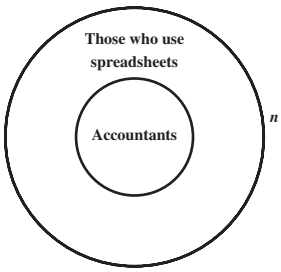


11. a. $\forall x\,[c(x) \rightarrow p(x)]$

$\sim p(n)$

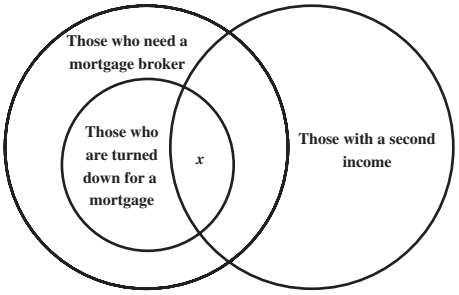
$\sim c(n)$

b. Valid



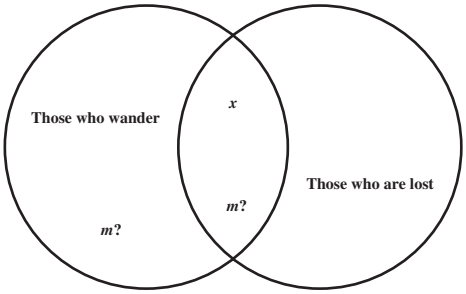
13. a. $\exists x\,[t(x) \wedge s(x)]$
$\forall x\,[t(x) \rightarrow b(x)]$
$\exists x\,[s(x) \wedge b(x)]$

b. Valid



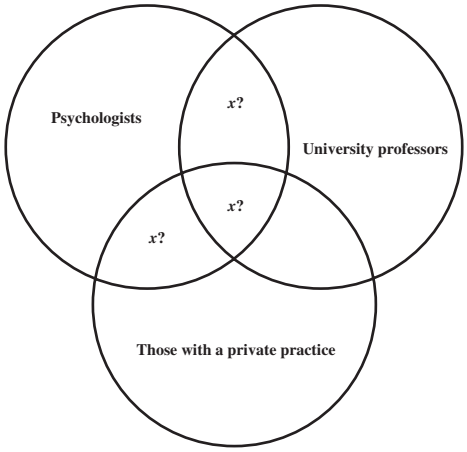
15. a. $\exists x\,[w(x) \wedge l(x)]$
$w(m)$
$l(m)$

b. Invalid



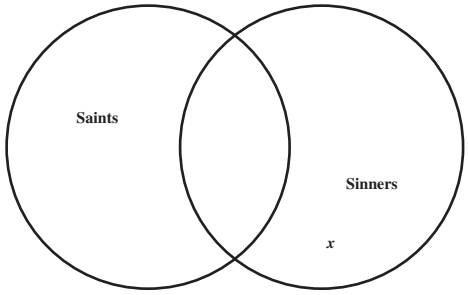
17. a. $\exists x\,[p(x) \wedge u(x)]$
$\exists x\,[p(x) \wedge r(x)]$
$\exists x\,[u(x) \wedge r(x)]$

b. Invalid



19. a. $\forall x\,[a(x) \vee i(x)]$
$\exists x\,[\sim a(x)]$
$\exists x\,[i(x)]$

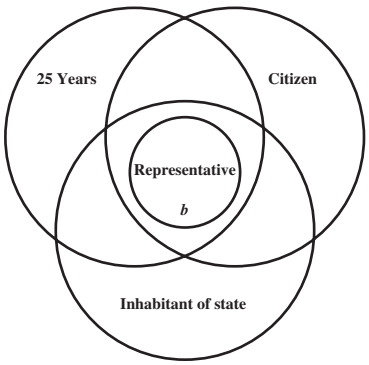
b. Valid **21.** Yes



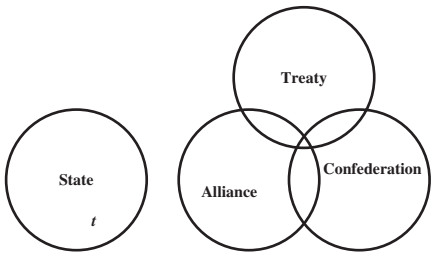
23. All major league baseball players earn at least \$300,000 a year.
Ryan Howard is a major league baseball player.
Ryan Howard earns at least \$300,000 a year.

25. Valid **27.** Invalid **29.** Invalid **31.** Invalid **37.** a, c, d **39. a.** $\forall x\{r(x) \rightarrow [a(x) \wedge c(x) \wedge i(x)]\}$ **b.** John Boehner has attained to the age of twenty-five years, and been seven years a citizen of the United States, and was, when elected, an inhabitant of that State in which he was chosen.

c.



41. a. $\forall x\,\{s(x) \rightarrow \sim[t(x) \vee a(x) \vee c(x)]\}$ **b.** Texas shall not enter into any treaty, alliance, or confederation. **c.**



43. Invalid **45.** Invalid **47.** Valid

Chapter 6 Review Exercises (Page 276)

For exercises . . .	15,19–22, 26,60–62, 69,70	14,27 39,40, 66	13,16,17,28, 35–38,41,42, 57–59,63,67	18,23–25, 29–34, 64,65	43–52	53–56,68 71–74
Refer to section . . .	1	2	3	4	5	6

1. True **3.** False **5.** False **7.** False **9.** False **11.** True **13.** She doesn't pay me and I have enough cash. **15.** $l \wedge w$ **17.** $l \rightarrow \sim w$ **19.** He doesn't lose the election and he wins the hearts of the voters. **21.** True **23.** True

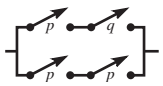
27.

p	q	p	$\wedge$	$(\sim p \vee q)$
T	T	T	T	F T T
T	F	T	F	F F F
F	T	F	F	T T T
F	F	F	F	T T F
		①	④	② ③ ②

The statement is not a tautology.

29. If someone is a mathematician, then that person is loveable. **31.** If a system has a unique solution, then it has at least as many equations as unknowns. **33. a.** If we need to change the way we do business, then the proposed regulations have been approved. **b.** If the proposed regulations have not been approved, then we do not need to change the way we do business. **c.** If we do not need to change the way we do business, then the proposed regulations have not been approved.

35. $(p \wedge p) \wedge (\sim p \vee q) \equiv p \wedge q$ **37.**



$(p \wedge q) \vee (p \wedge p) \equiv p$

39.

p	q	$p \underline{\vee} q$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T F T	T T T F F T T T
T	F	T T F	T T F T T T F F
F	T	F T T	F T T T T F F T
F	F	F F F	F F F F T F F F
		① ② ①	③ ④ ③ ⑧ ⑦ ⑤ ⑥ ⑤

The columns labeled 2 and 8 are identical.

41. a. Yes **b.** No **43.** Valid; Modus Ponens **45.** Valid; Disjunctive Syllogism **47.** Invalid; Fallacy of the Converse

49. Valid.

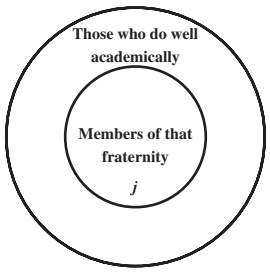
1.	$h \rightarrow t$	Premise
2.	$r \rightarrow \sim t$	Premise
3.	r	Premise
4.	$\sim t$	2, 3, Modus Ponens
5.	$\sim h$	1, 4, Modus Tollens

51. Invalid; p = F, q = F

53. a. $\forall x\,[d(x) \rightarrow l(x)]$ **b.** $\exists x\,[d(x) \wedge \sim l(x)]$ **c.** There is a dog that doesn't have a license.

55. a.
$$\begin{array}{l} \forall x\,[f(x) \rightarrow w(x)] \\ \underline{f(j)} \\ w(j) \end{array}$$

b.



Valid

57.

p	q	r	$p \rightarrow$	$(q \rightarrow r)$	$(p \rightarrow q) \rightarrow r$
T	T	T	T T	T T T	T T T T T
T	T	F	T F	T F F	T T T F F
T	F	T	T T	F T T	T F F T T
T	F	F	T T	F T F	T F F T F
F	T	T	F T	T T T	F T T T T
F	T	F	F T	T F F	F T T F F
F	F	T	F T	F T T	F T F T T
F	F	F	F T	F T F	F T F F F
			① ④	② ③ ②	⑤ ⑥ ⑤ ⑧ ⑦

No **59. a.**

p	q	$(p \wedge \sim p) \rightarrow q$
T	T	T F F T T
T	F	T F F T F
F	T	F F T T T
F	F	F F T T F
		① ② ① ④ ③

61. b, c

63. You do not use the Tax Table or you do not have to compute your tax mathematically. **65. a.** If you exercise regularly, then your heart becomes stronger and more efficient. **b.** If you are a teenager, then you need to be aware of the risks of drinking and driving. **c.** If you are visiting a country that has a high incidence of infectious diseases, then you may need extra immunizations. **d.** If you have good health, then you have food. **67.** $(w \rightarrow d) \rightarrow v$ **69.** b,c,d **71. a.** $\sim s \rightarrow g$ **b.** $l \rightarrow \sim g$ **c.** $w \rightarrow l \equiv \sim l \rightarrow \sim w$ **d.** $\sim s \rightarrow \sim w$, *Conclusion*: If the puppy does not lie still, it does not care to do worsted work. In Lewis Carroll's words, "Puppies that will not lie still never care to do worsted work." **73. a.** $f \rightarrow t \equiv \sim t \rightarrow \sim f$ **b.** $\sim a \rightarrow \sim g \equiv g \rightarrow a$ **c.** $w \rightarrow f$ **d.** $t \rightarrow \sim g \equiv g \rightarrow \sim t$ **e.** $a \rightarrow w \equiv \sim w \rightarrow \sim a$ **f.** $g \rightarrow \sim e$, *Conclusion*: If the kitten will play with a gorilla, it does not have green eyes. In Lewis Carroll's words, "No kitten with green eyes will play with a gorilla."

Chapter 7 Sets and Probability

Exercises 7.1 (page 290)

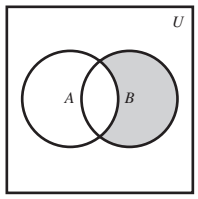
For exercises . . .	5–19	21–24,63,69	25–44	47–50,74–76,78,79	53–56,59–62,64–68,71–76,78,79	70
Refer to example . . .	2	4	5,6,7	9	8	3

1. False **3.** True **5.** True **7.** True **9.** False **11.** $\subseteq$ **13.** $\not\subseteq$ **15.** $\subseteq$ **17.** $\subseteq$ **19.** $\subset;\subset;\not\subset;\subset;\not\subset;\subset;\not\subset$ **21.** 64 **23.** 8 **25.** $\cap$ **27.** $\cup$ **29.** $\cap$ **31.** $\cup$ or $\cap$ **35.** $\{2, 4, 6\}$ **37.** $\{1, 3, 5, 7, 9\}$ **39.** $\{1, 7, 9\}$ **41.** $\{2, 3, 4, 6\}$ **43.** $\{7, 8\}$ **45.** $\{3, 6, 9\} = A$ **47.** All students in this school not taking this course **49.** All students in this school taking accounting and zoology **51.** C and D, B and E, C and E, D and E **53.** B' is the set of all stocks on the list with a closing price below \$60 or above \$70; $B' = \{$AT&T, Coca-Cola, FedEx, Disney$\}$. **55.** $(A \cap B)'$ is the set of all stocks on the list that do not have both a high price greater than \$50 and a closing price between \$60 and \$70; $A \cap B = \{$AT&T, Coca-Cola, FedEx, Disney$\}$. **57. a.** True **b.** True **c.** False **d.** False **e.** True **f.** True **g.** False **59.** {Exxon Mobil, General Electric, JPMorgan Chase & Co.} **61.** {IBM, Hewlett-Packard, Home Depot, Aflac} **63.** $2^9 = 512$ **65.** $\{i, m, h\}$ **67.** U **69.** $2^{51} \approx 2.252 \times 10^{15}$ **71.** {USA, TLC, TBS} **73.** {Discovery, TNT} **75.** {TNT, USA, TBS, Discovery}; the set of networks that features sports or that have more than 97.6 million viewers. **79. a.** The set of states who are not among those whose name contains the letter "e" or who are more than 4 million in population, and who also have an area of more than 40,000 square miles. **b.** {Alaska}

Exercises 7.2 (page 298)

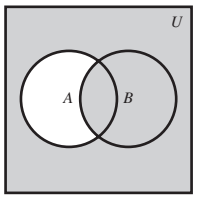
For exercises . . .	1–8,25–28	11–20,29–32	21–24	39,44,45,59	38	40,41,46–58	42,43,60,61
Refer to example . . .	1	2	6	4	5	7	8

1.



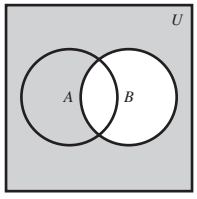
$B \cap A'$

3.



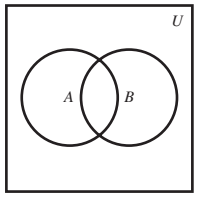
$A' \cup B$

5.



$B' \cup (A' \cap B')$

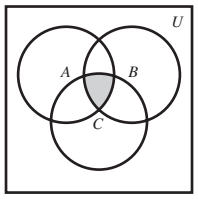
7.



$U' = \emptyset$

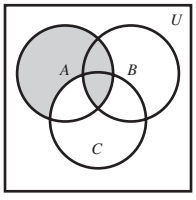
9. 8

11.



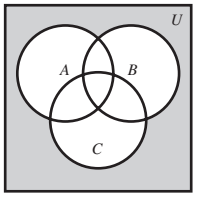
$(A \cap B) \cap C$

13.



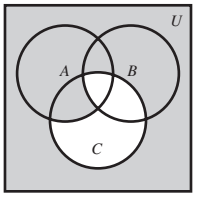
$A \cap (B \cup C')$

15.



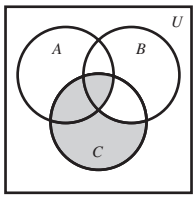
$(A' \cap B') \cap C'$

17.



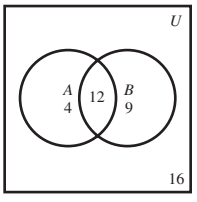
$(A \cap B') \cup C'$

19.

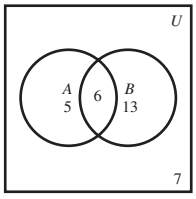


$(A \cup B') \cap C$

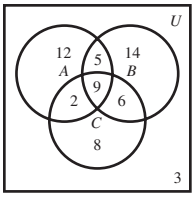
21. 13 **23.** 18 **25.**



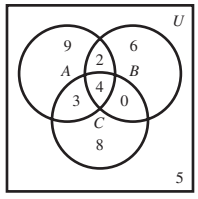
27.



29.



31.



39. a. 12 **b.** 18 **c.** 37 **d.** 97 **41. a.** 2 **b.** 60 **c.** 8 **d.** 100 **e.** 27 **f.** All those who invest in stocks or bonds and are age 18–29 **43. a.** 17 **b.** 2 **c.** 14 **45. a.** 54 **b.** 17 **c.** 10 **d.** 7 **e.** 15 **f.** 3 **g.** 12 **h.** 1 **47. a.** 34 **b.** 16 **c.** 431 **d.** 481 **49.** 110.6 million **51.** 85.4 million **53.** 71.0 million **55.** 25,568,000; Blacks or Hispanics who never married **57.** 203,803,000; Whites or divorced/separated people who are not Asian/Pacific Islanders **59. a.** 342 **b.** 192 **c.** 72 **d.** 86 **61. a.** 89 **b.** 32 **c.** 26 **d.** 30 **e.** 22 **f.** 21

Exercises 7.3 (page 307)

For exercises . . .	3–10,13–18	13–18	19–24	25–34	35–40,53,62,63	41–48,52,56,57–61	51,55,56
Refer to example . . .	1	2,3	6	7	6,7	8	4

3. {January, February, March, . . . , December} **5.** $\{0, 1, 2, 3, \ldots, 79, 80\}$ **7.** {go ahead, cancel} **9.** $\{(h, 1), (h, 2), (h, 3), (h, 4), (h, 5), (h, 6), (t, 1), (t, 2), (t, 3), (t, 4), (t, 5), (t, 6)\}$ **13.** {AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}, 10, yes **a.** {AC, BC, CD, CE} **b.** {AB, AC, AD, AE, BC, BD, BE, CD, CE} **c.** {AC} **15.** $\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$, 10, yes **a.** $\{(2, 4)\}$ **b.** $\{(1, 2), (1, 4), (2, 3), (2, 5), (3, 4), (4, 5)\}$ **c.** $\emptyset$ **17.** $\{hh, thh, hth, tthh, thth, htth, ttth, ttht, thtt, httt, tttt\}$, 11, no **a.** $\{tthh, thth, htth, ttth, ttht, thtt, httt, tttt\}$ **b.** $\{hh, thh, hth, tthh, thth, htth\}$ **c.** $\{tttt\}$ **19.** 1/6 **21.** 2/3 **23.** 1/3 **25.** 1/13 **27.** 1/26 **29.** 1/52 **31.** 2/13 **33.** 7/13 **35.** 3/20

37. 1/4 **39.** 3/5 **41.** Not empirical. **43.** Empirical **45.** Empirical **47.** Not empirical. **49.** The outcomes are not equally likely. **51. a.** Worker is male. **b.** Worker is female and has worked less than 5 years. **c.** Worker is female or does not contribute to a voluntary retirement plan. **d.** Worker has worked 5 years or more. **e.** Worker has worked less than 5 years or has contributed to a voluntary retirement plan. **f.** Worker has worked 5 years or more and does not contribute to a voluntary retirement plan. **53. a.** 8/15 **b.** 1/10 **55. a.** Person is not overweight. **b.** Person has a family history of heart disease and is overweight. **c.** Person smokes or is not overweight. **57. a.** 0.2540 **b.** 0.4851 **c.** 0.9225 **59.** 0.17 **61. a.** 0.3151 **b.** 0.3719 **c.** 0.3700 **d.** Calvary **e.** I Corps **63. a.** 25/57 **b.** 32/57 **c.** 4/19

Exercises 7.4 (page 316)

For exercises . . .	9–20,23,24,54,55,	21,22,47,52,53,57–59,61–64	27–32,48,65,70	35–40,49–51,56,66–69	60,71
Refer to example . . .	1,2,3,4	9	5	8	6,7

3. No **5.** No **7.** Yes **9. a.** 1/36 **b.** 1/12 **c.** 1/9 **d.** 5/36 **11. a.** 5/18 **b.** 5/12 **c.** 11/36 **13.** 5/18 **15. a.** 2/13 **b.** 7/13 **c.** 3/26 **d.** 3/4 **e.** 11/26 **17. a.** 5/13 **b.** 7/13 **c.** 3/13 **19. a.** 1/10 **b.** 2/5 **c.** 7/20 **21. a.** 0.51 **b.** 0.25 **c.** 0.10 **d.** 0.84 **23. a.** 5/9 **b.** 5/9 **c.** 5/9 **27.** 1 to 5 **29.** 2 to 1 **31. a.** 1 to 5 **b.** 11 to 7 **c.** 2 to 7 **d.** 7 to 2 **35.** Possible **37.** Not possible; the sum of the probabilities is less than 1. **39.** Not possible; a probability cannot be negative. **41. a.** 0.2778 **b.** 0.4167 **43. a.** 0.0463 **b.** 0.2963 **47.** 0.84 **49. a.** 0.62 **b.** 0.54 **c.** 0.43 **d.** 0.19 **51. a.** 0.061 **b.** 0.761 **c.** 0.822 **d.** 0.535 **53. a.** 0.961 **b.** 0.487 **c.** 0.513 **d.** 0.509 **e.** 0.004 **f.** 0.548 **55. a.** 1/4 **b.** 1/2 **c.** 1/4 **57.** a **59.** c **61. a.** 0.4 **b.** 0.1 **c.** 0.6 **d.** 0.9 **63.** 0 **65.** 2/5 **67. a.** 0.866 **b.** 0.478 **69. a.** 23/55 **b.** 67/220 **c.** 159/220 **71.** 0.0000000051; 0.0000012; 0.0063; 0.0166

Exercises 7.5 (page 330)

For exercises . . .	1–12	13–16	23,24,44–46, 49–53,63,71,72, 79–82,84,87–89	43,69,86	36–40,47, 54–60,62,64–67, 69,73,74,77	29,30,41,42, 75,76	48,61,68,78, 83,85,86
Refer to example . . .	3,4	8	5,6,7	2	1	9	10

1. 0 **3.** 1 **5.** 1/3 **7.** 0 **9.** 4/17 **11.** 11/51 **13.** 8/663 **15.** 25/102 **19.** Independent **21.** Dependent **23. a.** 1/4 **b.** 1/2 **25. a.** Many answers are possible **b.** Many answers are possible **29.** 1/20, 2/5 **31.** Second booth **33.** No, these events are not independent **35.** Yes **37.** The probability that a customer cashing a check will fail to make a deposit is 1/3. **39.** The probability that a customer making a deposit will not cash a check is 1/4. **41. a.** 0.3570 **43.** 0.875 **45.** 0.06 **47.** 2/3 **49.** 1/4 **51.** 1/4 **53.** 1/7 **55.** 0.039 **57.** 0.491 **59.** 0.072 **61.** Yes **63. a.** 0.3367 **b.** 0.6617 **c.** No **65.** 7/229 **67.** 191/229 **69.** e **71. a.** 0.5065 **c.** 0.2872 **d.** $p(1 - P(B)) + (1 - p)(1 - P(B))^2$ **e.** 0.2872 **f.** $2(1 - p)P(B)(1 - P(B))$ **g.** 0.2872 **73. a.** 0.2059 **b.** 0.1489 **c.** 0.0410 **d.** 0.2755 **e.** No **75. a.** 0.58 **77. a.** 0.2713 **b.** 0.3792 **c.** 0.6246 **d.** 0.3418 **e.** 0.6897 **f.** 0.5137 **g.** Not independent **79. a.** 0.052 **b.** 0.476 **81. a.** 7/10 **b.** 2/15 **83.** 10^{-12} **85.** No **87.** 0 points: 0.4; 1 point: 0.24; 2 points: 0.36. **89. c.** They are the same. **d.** The 2-points first strategy has a smaller probability of losing.

Exercises 7.6 (page 340)

For exercises . . .	1,2,9,10–13,23–26, 29–32,36,39	3–8,14–17,19 27,28	18,20–22,33–35, 37,38
Refer to example . . .	1	2	3

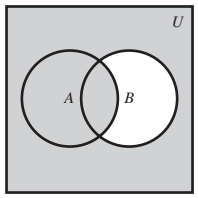
1. 1/3 **3.** 3/19 **5.** 21/38 **7.** 8/17 **9.** 85% **11.** 0.0727 **13.** 0.1765 **15.** 0.3636 **17.** 2/7 **19.** d **21.** c **23.** 0.0478 **25. a.** 0.039 **b.** 0.999 **c.** 0.001 **d.** 0.110 **27.** d **29.** b **31. a.** 0.1870 **b.** 0.9480 **33.** 0.1285 **35.** 0.1392 **37.** 0.5549 **39.** 9.9×10^{-5}

Chapter 7 Review Exercises (page 345)

For exercises . . .	1–5,13–28, 59,87,90	6,43–52, 60,61, 69,70,88, 90,103,104	7,8,53,54, 65–68, 71–73, 77,78 91,93, 95,96	9–11,55–58, 62,63,74–76, 86,90,94, 101,102, 107,108	12,79,80, 81–85, 89,97,98	39–42, 92, 99, 105,106, 109
Refer to example . . .	1	3	4	5	6	2

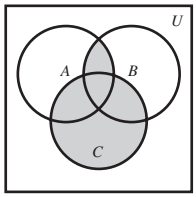
1. True **2.** True **3.** False **4.** False **5.** False **6.** True **7.** False **8.** False **9.** False **10.** True **11.** False **12.** True **13.** False **15.** False **17.** True **19.** False **21.** False **23.** 32 **25.** {a, b, g} **27.** {c, d} **29.** {a, b, e, f, g, h} **31.** U **33.** All female employees in the accounting department **35.** All employees who are in the accounting department or who have MBA degrees **37.** All male employees who are not in the sales department

39.



$A \cup B'$

41.



$(A \cap B) \cup C$

43. $\{1, 2, 3, 4, 5, 6\}$ **45.** $\{0, 0.5, 1, 1.5, 2, \ldots, 299.5, 300\}$
47. $\{(3, r), (3, g), (5, r), (5, g), (7, r), (7, g), (9, r), (9, g), (11, r), (11, g)\}$
49. $\{(3, g), (5, g), (7, g), (9, g), (11, g)\}$
51. 1/4 **53.** 11/26 **55.** 1/2 **57.** 1 **63.** No; yes **65.** 1 to 3 **67.** 2 to 11
69. 5/36 **71.** 1/6 **73.** 1/6 **75.** 2/11 **77. a.** 0.66 **b.** 0.29 **c.** 0.71
d. 0.34 **79.** 1/7 **81.** 0.8736 **83.** 0.3077 **85.** 0.87 **87. a.** $(E \cup F)'$ or $E' \cap F'$ **b.** $E \cup F$ **89. a.** 0.0297 **b.** 0.0909 **c.** 0.2626 **d.** No **91.** b

93. b **95. a.**

	N_2	T_2
N_1	N_1N_2	N_1T_2
T_1	T_1N_2	T_1T_2

b. 1/4 **c.** 1/2 **d.** 1/4 **97.** c **99. a.** 53 **b.** 7 **c.** 12 **d.** 26 **101.** 0.6279
103. 0.90 **105. a.** 4 **b.** 18 **107.** No; 2/3 **109.** d

Chapter 8 Counting Principles; Further Probability Topics

Exercises 8.1 (page 359)

For exercises . . .	1–4,37,39,54,55	5–12,19,20,32,34,40	13–16,30,31,35,43–53	19–24	23,24,33,36,38,41,42
Refer to example . . .	4,5	6	1	9,10	8

1. 720 **3.** 1.308×10^{12}
5. 156 **7.** 1.024×10^{25} **9.** 1 **11.** n **13.** 36 **15.** 20 **19.** one **21. a.** 840 **b.** 180 **c.** 420 **23. a.** 362,880 **b.** 1728
c. 1260 **d.** 24 **e.** 144 **25.** Multiply by 10 **27. a.** 2 **b.** 6 **c.** 18 **29.** Undefined **31. a.** 42 **b.** 28 **33. a.** 39,916,800
b. 172,800 **c.** 86,400 **35.** No; use at least 4 initials **37.** 720 **39.** 3.352×10^{10} **41. a.** 120 **b.** 48 **43. a.** 27,600 **b.** 35,152
c. 1104 **45. a.** 160; 8,000,000 **b.** Some numbers, such as 911, 800, and 900, are reserved for special purposes. **47. a.** 17,576,000
b. 17,576,000 **c.** 456,976,000 **49.** 100,000; 90,000 **51.** 81 **53.** 1,572,864; no **55. a.** 1.216×10^{17} **b.** 43,589,145,600

Exercises 8.2 (page 367)

For exercises . . .	3–10,39,48,58	11,12,33,35,36, 40–42,45,54–56	14,32,37,43,44, 49,53	15,16	17–24	17–24,29–31,34, 38,46,47,50–52,57
Refer to example . . .	1	6	2,3	7	4	5

3. 56 **5.** 1.761×10^{12} **7.** 1
9. n **11.** 1716 **13. a.** 10
b. 7 **15. a.** 9 **b.** 6 **c.** 3; yes, from both **17.** Combinations; **a.** 126 **b.** 462 **c.** 4620 **19.** Permutations; 479,001,600
21. Combinations; **a.** 120 **b.** 1820 **c.** 36 **23.** Combinations; **a.** 10 **b.** 0 **c.** 1 **d.** 10 **e.** 30 **f.** 15 **g.** 0 **27. a.** 30
b. $n(n-1)$ **29.** 336 **31. a.** 720 **b.** 360 **33. a.** 40 **b.** 20 **c.** 7 **35.** 4,115,439,900 **37. a.** 84 **b.** 10 **c.** 40 **d.** 74
39. No; 14,190 **41. a.** 48 **b.** 658,008 **c.** 652,080 **d.** 844,272 **e.** 79,092 **43.** 50 **45. a.** 15,504 **b.** 816 **47.** 558 **49. a** 26
b. 26 **51. a.** 3,089,157,760 **b.** 1,657,656,000 **53. a.** 838 **b.** 20,695,218,670 **55. a.** 84 **b.** 729 **d.** 680 **e.** 120
57. a. 6.402×10^{15} **b.** 3.135×10^{10}

Exercises 8.3 (page 378)

For exercises . . .	1–10,36–41,45,47,65,66,68–71,73	11–18,46–64	19–22,42–44,46,72	25–28,33	29–32
Refer to example . . .	1,2,3	4	5	6	7

1. 7/33 **3.** 14/55
5. 0.008127 **7.** 0.2980 **9.** 0.2214 **11.** 1326 **13.** 33/221 **15.** 52/221 **17.** 130/221 **19.** 8.417×10^{-8} **21.** 18,975/28,561
25. $1 - P(365, 43)/365^{43}$ **27.** 1 **29.** 13/34 **31.** 1/180 **33.** 0.6501 **35.** 3/5 and 2/5 **37.** 36/55 **39.** 21/55 **41.** 7/22
43. 1/3 **45. a.** 225/646 **b.** 15/323 **c.** 225/2584 **d.** 0 **e.** 1/2 **f.** 175/2584 **g.** 503/646 **47. a.** 0.0542 **b.** 0.0111 **c.** 0.0464
d. 0.1827 **e.** 0.3874 **f.** 0.8854 **49.** 1.385×10^{-5} **51.** 0.0039 **53.** 0.0475 **55.** 1.575×10^{-12} **57.** 0.0402 **59.** 0.4728
61. 0.0493 **63.** 0.0306 **65.** 0.0640 **67.** 4.980×10^{-7} **69.** The probability of picking 5 out of 52 is higher, 1/2,598,960 compared with 1/13,983,816. **71. a.** 0.01642 **b.** 0.01231 **73. a.** 28 **b.** 268,435,456 **c.** 40,320 **d.** 1.502×10^{-4} **e.** $n!/2^{n(n-1)/2}$
75. a. 3.291×10^{-6} **b.** 7.962×10^{-12} **c.** 5.927×10^{-11} **d.** 5.524×10^{26}

Exercises 8.4 (page 386)

For exercises . . .	1–4,9–12,15,16,24,25,29–39,33,34, 37,40,41,43,44,47,59,60	5–8,13,14,17,18,26–28,31,32, 35,36,38,39,42,45,46,48–51, 54,55,57,58,61–63	52,53,64,65
Refer to example . . .	2,3	4,5	7

1. 5/16 **3.** 1/32 **5.** 3/16
7. 13/16 **9.** 4.594×10^{-10}
11. 0.2692 **13.** 0.8748
15. 0.0002441 **17.** 0.9624 **23.** The potential callers are not likely to have birthdates that are evenly distributed throughout the twentieth century. **25.** 0.6014 **27.** 0.0876 **29.** 0.1721 **31.** 0.7868 **33.** 0.2458 **35.** 0.0170 **37.** 0.3585 **39. a.** 0.1488
b. 0.0213 **c.** 0.9787 **41. a.** 0.0222 **b.** 0.1797 **c.** 0.7766 **43.** 0.0874 **45.** 0.9126 **47.** 0.2183 **49.** 0.0025 **51. a.** 0.0478
b. 0.9767 **c.** 0.8971 **53. a.** 1 chance in 1024 **b.** About 1 chance in 1.1×10^{12} **c.** About 1 chance in 2.587×10^{6} **55.** 0.9523
57. e **59.** 0.0305 **61.** 0.8676 **63. a.** 0.2083 **b.** 0.5902 **c.** 0.1250 **d.** 0.8095 **65. a.** 0.5260 **b.** 0.9343 **67. a.** 0.125, 0.25, 0.3125, 0.3125 **b.** 0.2893, 0.3222, 0.2353, 0.1531

Exercises 8.5 (page 395)

For exercises . . .	1–4	5–8,29	9–16,30,32,36,37,42	17,18,26,45,56	19–25,29,31,38,43,44,47–54,57	33,39,40,55,56
Refer to example . . .	1	2	3	5	4,7	8

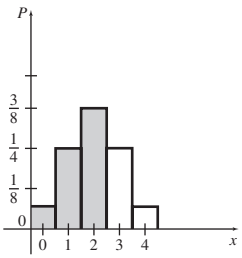
1.

Number of Heads	0	1	2	3	4
Probability	1/16	1/4	3/8	1/4	1/16

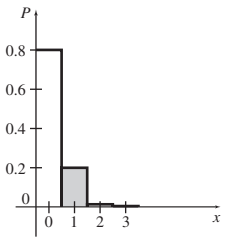
3.

Number of Aces	0	1	2	3
Probability	0.7826	0.2042	0.0130	0.0002

5.

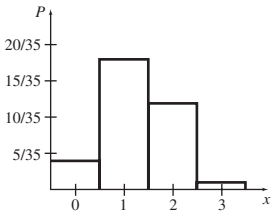


7.



9. 3.6 **11.** 14.49 **13.** 2.7 **15.** 18 **17.** 0; yes

19. a.



b. 9/7 **21. a.**

x	0	1	2	3	4
$P(x)$	625/1296	125/324	25/216	5/324	1/1296

b. 2/3

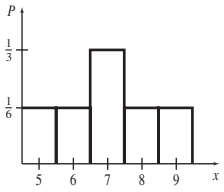
23. 3/4

25. 1/2

29. a.

Sum	5	6	7	8	9
Probability	1/6	1/6	1/3	1/6	1/6

b.



c. 1 to 2 **d.** 7

31. \$54,000 **33. a.** 30 **b.** 10 **35.** e **37. a.** \$68.51; \$72.84 **b.** Amoxicillin **39.** 185 **41. a.** 0.007230 **b.** 5.094×10^{-4} **c.** 5.5×10^{-5} **d.** 0.1143 **43. a.**

Number of Cats	0	1	2	3	4
Probability	0.2401	0.4116	0.2646	0.0756	0.0081

b. 1.2 **c.** 1.2

45. −\$0.72; no **47.** −\$0.82 **49.** −\$0.053 **51.** −\$0.50 **53.** −\$0.90; no **55. a.** 10/3 **b.** 50/9

57. a.

x	0	1	2	3	4
$P(x)$	0.1875	0.3897	0.3038	0.1053	0.0137

b. 1.37

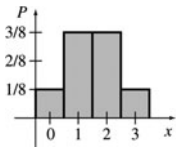
Chapter 8 Review Exercises (page 401)

For exercises . . .	1,3,13,14,19,20,53	2,15–18,21,22,54	4–6,25–30,35–40,69 76,78,79,82	7,8,31–34,49,50, 55–60,67,68,70–73	9–12,41–48,61,62, 67,70–75,77,78, 80,81
Refer to section . . .	1	2	3	4	5

1. True **2.** True **3.** True **4.** True **5.** False **6.** True **7.** True **8.** False **9.** True **10.** False **11.** True **12.** False **13.** 720 **15.** 220 **17. a.** 90 **b.** 10 **c.** 120 **d.** 220 **19. a.** 120 **b.** 24 **21. a.** 840 **b.** 2045 **25.** 2/143 **27.** 21/143 **29.** 5/13 **31.** 5/16 **33.** 11/32 **35.** 25/102 **37.** 15/34 **39.** 546/1326

41. a.

Number of Heads	0	1	2	3
Probability	0.125	0.375	0.375	0.125

b. 

c. 1.5

43. 0.6 **45.** −\$0.833; no **47. a.** 0.231 **b.** 0.75 **49.** 31/32 **51. a.** $C(n, 0)$, or 1; $C(n, 1)$, or n; $C(n, 2)$; $C(n, n)$, or 1 **b.** $C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n, n)$ **e.** The sum of the elements in row n of Pascal's triangle is 2^n. **53.** 48 **55.** 0.1122 **57.** 7.580×10^{-7} **59.** 0.6187 **61.** 2 **63.** d **65.** e

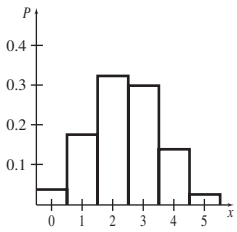
67.

x	0	1	2	3	4	5	6
Income	0	400	800	1200	800	400	0
$P(x)$	0.0041	0.0369	0.1382	0.2765	0.3110	0.1866	0.0467

a. \$780.60 **b.** \$720; \$856.32; \$868.22; 5 **69.** 0.1875

71. a.

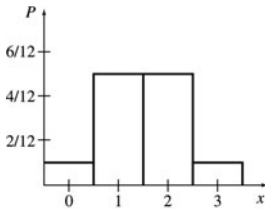
Number of Women	0	1	2	3	4	5
Probability	0.0380	0.1755	0.3240	0.2990	0.1380	0.0255

b. 

c. 2.4

73. a.

Number Who Did Not Do Homework	0	1	2	3
Probability	1/12	5/12	5/12	1/12

b. 

c. 3/2

75. −\$0.71 **77. a.** 10.95 **b.** 2.56 **79. a.** 0.4799; 0.6533 **c.** 0.3467; 0.003096

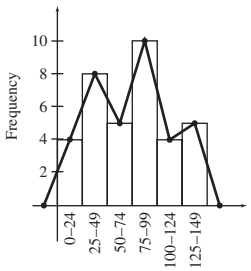
Chapter 9 Statistics

Exercises 9.1 (page 415)

For exercises . . .	1a,2a,44a	1b,2b,42a, 44b	7–12,39,40, 47	13–16,42b, 44d	17–22,48b, 50	23,24,45,46, 49	25–30,44c, 44f,48c	36–38,41,43 45,46,48
Refer to example . . .	1	2	3,7	4,5	6,7	9	8	3,6

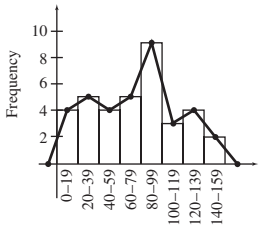
1. a.–b.

Interval	Frequency
0–24	4
25–49	8
50–74	5
75–99	10
100–124	4
125–149	5

c.–d. 

3. a.–b.

Interval	Frequency
0–19	4
20–39	5
40–59	4
60–79	5
80–99	9
100–119	3
120–139	4
140–159	2

c.–d. 

7. 16 **9.** 28,620 **11.** 7.68 **13.** 7.25 **15. a.** 73.81 **b.** 74.5 **17.** 42 **19.** 130 **21.** 29.1 **23.** 73.86, 80.5 **25.** 9 **27.** 55 and 62 **29.** 6.3 **33.** 75–99 **37.** 2095.9 million bushels, 2130 million bushels **39.** \$45,794 **41. a.** 348.4, 249.5 **c.** 1.187, 1.215 **43.** 7.38; 7; 7, 5, 4 **45. a.** 55.5°F; 50.5°F **b.** 28.9°F; 28.5°F **47. a.** \$71,349 **49. a.** \$8,253,336, \$5,500,000, \$5,500,000

Exercises 9.2 (page 425)

For exercises . . .	3–8	9,10,25,26,29	11,12	27,28	30–36
Refer to example . . .	1,3	3	4	6	5

1. The standard deviation is the square root of the variance.

3. 33; 12.6 **5.** 46; 16.1 **7.** 24; 8.1 **9.** 40.05 **11.** 39.4 **13.** 8/9 **15.** 24/25 **17.** At least 8/9 **19.** No more than 1/9

25. a. Mean = 25.5 hr; standard deviation = 7.2 hr **b.** Forever Power **c.** Forever Power **27. a.** 1/3; 2; −1/3; 0; 5/3; 7/3; 1;

4/3; 7/3; 2/3 **b.** 2.1; 2.6; 1.5; 2.6; 2.5; 0.6; 1.0; 2.1; 0.6; 1.2 **c.** 1.13 **d.** 1.68 **e.** 4.41; −2.15 **f.** 4.31; 0 **29.** Mean = 1.816 mm; standard deviation = 0.4451 mm. **31. a.** Mean = 7.3571; standard deviation = 0.1326 **b.** 100% **33. a.** 127.7 days; 30.16 days **b.** Seven **35. a.** $9,267,188 **b.** 4%

Exercises 9.3 (page 435)

For exercises . . .	5–14	15–18	22–64
Refer to example . . .	1	2	4

1. the mean **3.** z-scores are found with the formula $z = (x - \mu)/\sigma$. **5.** 45.54%
7. 48.96% **9.** 37.38% **11.** 14.78% **13.** 97.72% **15.** −1.64 or −1.65 **17.** 1.28 **19.** 0.5; 0.5 **21.** 0.8889; 0.9974 **23.** 5000
25. 6247 **27.** 7257 **29.** 0.1587 **31.** 0.0062 **33.** 84.13% **35.** 37.79% **37.** 7.03% **39. a.** 15.87% **b.** 0% **41.** 0.4325
43. About 2 **45.** 1350 units **47.** 1430 units **49.** 60.32 mph **51.** 6.68% **53.** 38.3% **55.** 87 **57.** 71 **59. b.** 55%
61. a. About 0.01; yes **b.** Essentially 0 **63. a.** 0.5596; 0.0188 **b.** Essentially 0; essentially 0 **c.** 0.9265; 0.9554
d. Essentially 0; essentially 0 **e.** Essentially 0; essentially 0

Exercises 9.4 (page 443)

For exercises . . .	3–34
Refer to example . . .	2

1. The number of trials and the probability of success on each trial **3. a.** 0.0278 **b.** 0.0279 **5. a.** 0.0106
b. 0.0122 **7.** 0.0240 **9.** 0.9463 **11.** 0.0956 **13.** 0.8643 **15.** 0.1841 **17. a.** 0.0146 **b.** 0.0537 **19. a.** Essentially 0
b. 0.0018 **21.** 0.0274 **23. a.** 0.9118 **b.** 0.9984 **25. a.** 0.0001 **b.** 0.0002 **c.** 0.0000 **27.** 0.6480 **29.** 0.9554
31. a. The numbers are too large for the calculator to handle. **b.** 0.1742 **33.** 0.0287

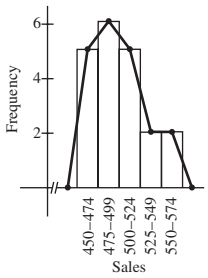
Chapter 9 Review Exercises (page 446)

For exercises . . .	4,6,10,13–25	3,5,11,26–33	7,8,12,34–38,44,52–56,58,59	1,2,9,39–41,45,46,48–51	43,47,57
Refer to section . . .	1	2	3	4	1,2

1. True **2.** False **3.** False **4.** True **5.** False **6.** False **7.** False **8.** True **9.** True **10.** True **11.** False **12.** True
15. a.

Sales	Frequency
450–474	5
475–499	6
500–524	5
525–549	2
550–574	2

b.–c.



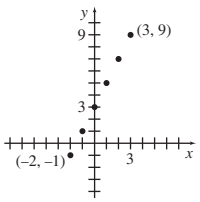
17. 29.25 **19.** 34.9 **23.** 38; 36 and 38 **25.** 30–39
29. 67; 23.9 **31.** 12.6 **33.** A skewed distribution has the largest frequency at one end. **35.** 0.7995 **37.** 0.0606
39. Because the histogram is skewed, not close to the shape of a normal distribution **41. a.** 2.1; 1.07 **b.** 2; 1
c. Answers to parts a and b should be close to each other.
43. a. Stock I: 8%, 7.9%; Stock II: 8%, 2.6% **b.** Stock II
45. a. 0.8924; 0.8962 **b.** 0.0446; 0.0477 **c.** 0.0196; 0.0150
47. Diet A: $\bar{x} = 2.7$, $s = 2.26$; Diet B: $\bar{x} = 1.3$, $s = 0.95$
a. Diet A **b.** Diet B **49.** 0.9292 **51.** 0.9534 **53.** 43.25% **55.** 56.25% **57. a.** 142.46; 140; no mode **b.** 48.67
c. 38.5% **d.** 100% **59. a.** 0.1020 **b.** 0.3707 **d.** 0.9987 **e.** 0.1336

Chapter 10 Nonlinear Functions

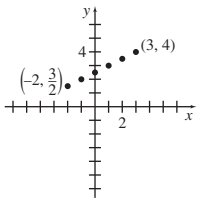
Exercises 10.1 (page 461)

For exercises . . .	1–8	9–16	17–32	33–40	41–56,76,77	57–62	63–70	71–74	75,78	79,80
Refer to example . . .	1,2	3 (b)	3(a), (c)–(e)	2 (d)	4	5	6	7	Dow Jones	8

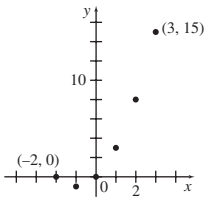
1. Not a function **3.** Function **5.** Function **7.** Not a function
9. (−2, −1), (−1, 1), (0, 3), (1, 5), (2, 7), (3, 9); range: {−1, 1, 3, 5, 7, 9}



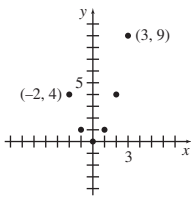
11. (−2, 3/2), (−1, 2), (0, 5/2), (1, 3), (2, 7/2), (3, 4); range: {3/2, 2, 5/2, 3, 7/2, 4}



13. (−2, 0), (−1, −1), (0, 0), (1, 3), (2, 8), (3, 15); range: {−1, 0, 3, 8, 15}



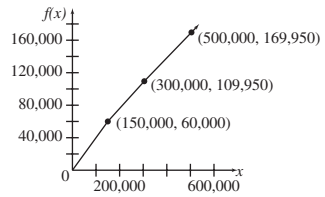
15. (−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4), (3, 9); range: {0, 1, 4, 9}



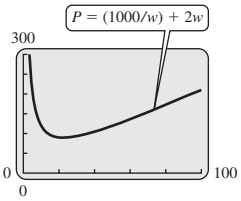
17. (−∞, ∞) **19.** (−∞, ∞) **21.** [−2, 2] **23.** [3, ∞) **25.** (−∞, −1) ∪ (−1, 1) ∪ (1, ∞)
27. (−∞, −4) ∪ (4, ∞) **29.** (−∞, −1] ∪ [5, ∞) **31.** (−∞, −1) ∪ (1/3, ∞) **33.** Domain: [−5, 4); range: [−2, 6] **35.** Domain: (−∞, ∞); range: (−∞, 12] **37.** Domain: [−2, 4]; range: [0, 4]
a. 0 **b.** 4 **c.** 3 **d.** −1.5, 1.5, 2.5 **39.** Domain: [−2, 4]; range: [−3, 2] **a.** −3 **b.** −2
c. −1 **d.** 2.5 **41. a.** 33 **b.** 15/4 **c.** $3a^2 - 4a + 1$ **d.** $12/m^2 - 8/m + 1$ or $(12 - 8m + m^2)/m^2$ **e.** 0, 4/3 **43. a.** 7 **b.** 0 **c.** $(2a + 1)/(a - 4)$ if $a \neq 4$, 7 if $a = 4$

d. $(4 + m)/(2 - 4m)$ if $m \neq 1/2$, 7 if $m = 1/2$ **e.** -5 **45.** $6t^2 + 12t + 4$ **47.** $r^2 + 2rh + h^2 - 2r - 2h + 5$
49. $9/q^2 - 6/q + 5$ or $(9 - 6q + 5q^2)/q^2$ **51. a.** $2x + 2h + 1$ **b.** $2h$ **c.** 2 **53. a.** $2x^2 + 4xh + 2h^2 - 4x - 4h - 5$
b. $4xh + 2h^2 - 4h$ **c.** $4x + 2h - 4$ **55. a.** $1/(x + h)$ **b.** $-h/[x(x + h)]$ **c.** $-1/[x(x + h)]$ **57.** Function
59. Not a Function **61.** Function **63.** Odd **65.** Even **67.** Even **69.** Odd **71. a.** \$36 **b.** \$36 **c.** \$64 **d.** \$120
e. \$120 **f.** \$148 **g.** \$148 **i.** x, the number of full and partial days **j.** S, the cost of renting a saw **73. a.** \$93,300. Attorneys can receive a maximum of \$93,300 on a jury award of \$250,000. **b.** \$124,950. Attorneys can receive a maximum of \$124,950 on a jury award of \$350,000. **c.** \$181,950. Attorneys can receive a maximum of \$181,950 on a jury award of \$550,000.

d.



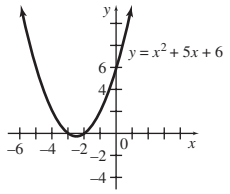
75. a. About 140 m **b.** About 250 m **77. a. i.** 3.6 kcal/km **ii.** 61 kcal/km
b. $x = g(z) = 1000z$ **c.** $y = 4.4z^{0.88}$ **79. a.** $P(w) = 1000/w + 2w$ **b.** $(0, \infty)$
c.

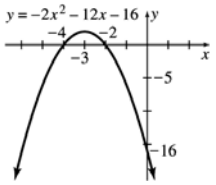


Exercises 10.2 (page 472)

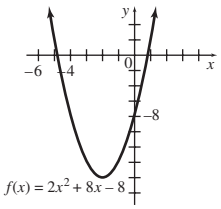
For exercises . . .	3–8	9–24,58–62,64–67	25–38,47,48	39–46	49–53	54–57,68,69
Refer to example . . .	1–3	4	8	Before Example 8	7	6

3. D **5.** A **7.** C
9. $y = 3(x + 3/2)^2 - 7/4$, $(-3/2, -7/4)$ x-intercepts are -3 and -2; y-intercept is 6.
11. $y = -2(x - 2)^2 - 1$, $(2, -1)$ **13.** Vertex is $(-5/2, -1/4)$; axis is $x = -5/2$;
15. Vertex is $(-3, 2)$; axis is $x = -3$; x-intercepts are -4 and -2; y-intercept is -16.

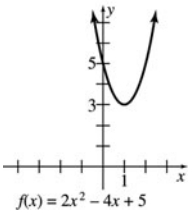




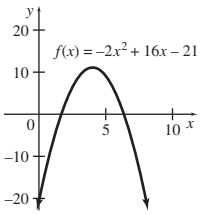
17. Vertex is $(-2, -16)$; axis is $x = -2$; x-intercepts are $-2 \pm 2\sqrt{2} \approx 0.83$ or -4.83; y-intercept is -8.



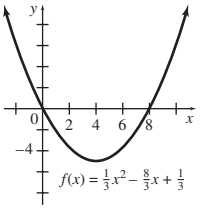
19. Vertex is $(1, 3)$; axis is $x = 1$; no x-intercepts; y-intercept is 5.



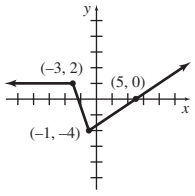
21. Vertex is $(4, 11)$; axis is $x = 4$; x-intercepts are $4 \pm \sqrt{22}/2 \approx 6.35$ or 1.65; y-intercept is -21.



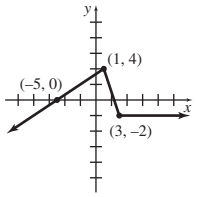
23. Vertex is $(4, -5)$; axis is $x = 4$; x-intercepts are $4 \pm \sqrt{15} \approx 7.87$ or 0.13; y-intercept is $1/3$.



25. D **27.** C **29.** E **31.**



33.



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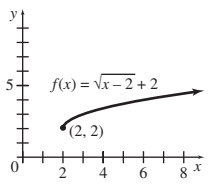
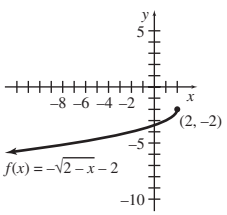
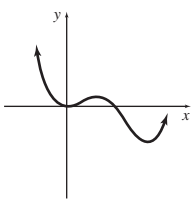


37.



39.



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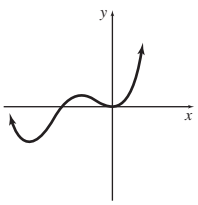

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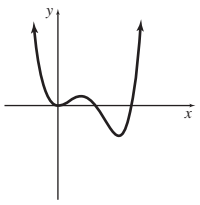
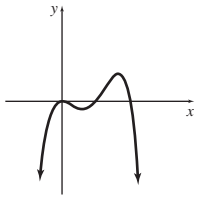
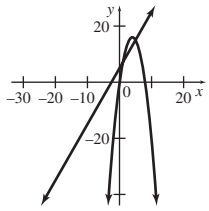


45.

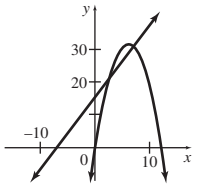


47. a. r **b.** $-r$ **c.** $-r$ **49. a.**



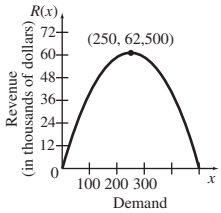
b. 1 batch
c. \$16,000
d. \$4000

51. a.



b. 2.5 batches **c.** \$31,250 **d.** \$5000 **53.** Maximum revenue is \$9225; 35 seats are unsold.

55. a. $R(x) = x(500 - x) = 500x - x^2$ **b.**

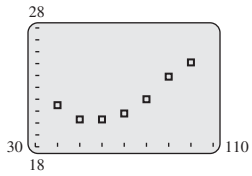


c. \$250 **d.** \$62,500

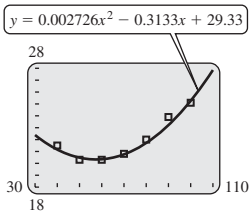
57. a. $800 + 25x$ **b.** $80 - x$ **c.** $R(x) = 64{,}000 + 1200x - 25x^2$ **d.** 24 **e.** \$78,400 **59. a.** 87 yr **b.** 98 yr

61. a. 28.5 weeks **b.** 0.81 **c.** 0 weeks or 57 weeks of gestation; no

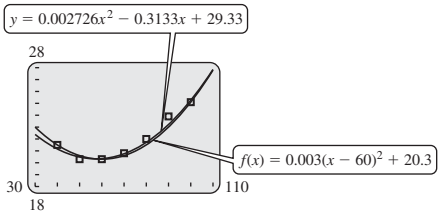
63. a.



b. Quadratic **c.** $y = 0.002726x^2 - 0.3113x + 29.33$; **d.** $f(x) = 0.003(x - 60)^2 + 20.3$

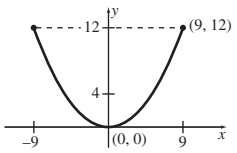


e.



65. 49 yr; 3.98 **67. a.** 61.70 ft **b.** 43.08 mph **69.** 9025 ft^2

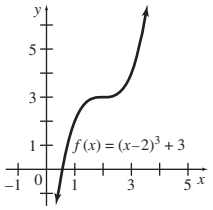
71. $y = (4/27)x^2$; $6\sqrt{3}$ ft ≈ 10.39 ft



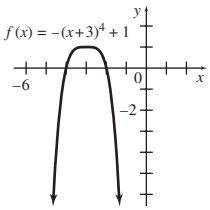
Exercises 10.3 (page 481)

For exercises . . .	3–6	7–15,21–26	16–20,27–42,57,58	46,47,50–52	54–56,59
Refer to example . . .	1	4	6	7	2,3

3.

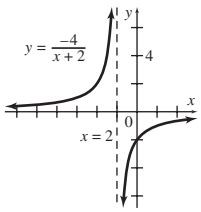


5.

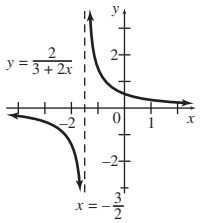


7. D **9.** E **11.** I **13.** G **15.** A **17.** D **19.** E **21.** 4, 6, etc. (true degree $= 4$); + **23.** 5, 7, etc. (true degree $= 5$); + **25.** 7, 9, etc. (true degree $= 7$); −

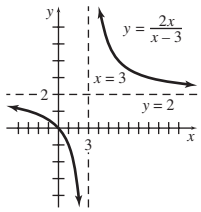
27. Horizontal asymptote: $y = 0$; vertical asymptote: $x = -2$; no x-intercept; y-intercept $= -2$



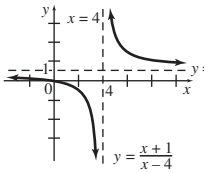
29. Horizontal asymptote: $y = 0$; vertical asymptote: $x = -3/2$; no x-intercept; y-intercept $= 2/3$



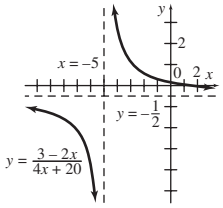
31. Horizontal asymptote: $y = 2$; vertical asymptote: $x = 3$; x-intercept $= 0$; y-intercept $= 0$



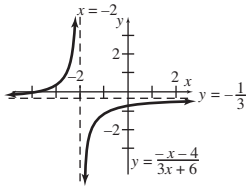
33. Horizontal asymptote: $y = 1$; vertical asymptote: $x = 4$; x-intercept $= -1$; y-intercept $= -1/4$



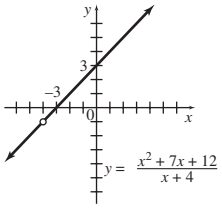
35. Horizontal asymptote: $y = -1/2$; vertical asymptote: $x = -5$; x-intercept $= 3/2$; y-intercept $= 3/20$



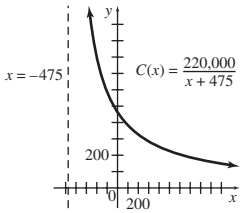
37. Horizontal asymptote: $y = -1/3$; vertical asymptote: $x = -2$; x-intercept $= -4$; y-intercept $= -2/3$



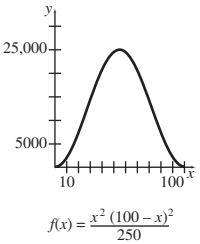
39. No asymptotes; hole at $x = -4$; x-intercept $= -3$; y-intercept $= 3$



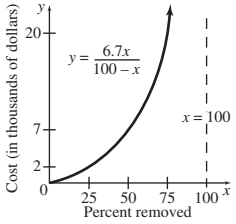
41. One possible answer is $y = 2x/(x - 1)$. **43. a.** 0 **b.** $2, -3$ **d.** $(x + 1)(x - 1)(x + 2)$ **e.** $3(x + 1)(x - 1)(x + 2)$ **f.** $(x - a)$ **45. a.** Two; one at $x = -1.4$ and one at $x = 1.4$ **b.** Three; one at $x = -1.414$, one at $x = 1.414$, and one at $x = 1.442$ **47. a.** \$440; \$419; \$383; \$326; \$284; \$251 **b.** Vertical asymptote at $x = -475$; horizontal asymptote at $y = 0$ **c.** $y = 463.2$ **d.**



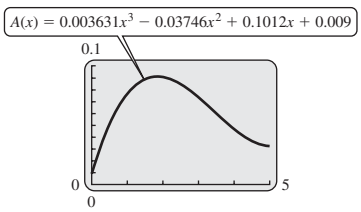
49. $f_1(x) = x(100 - x)/25$, $f_2(x) = x(100 - x)/10$, $f(x) = x^2(100 - x)^2/250$



51. a. \$6700; \$15,600; \$26,800; \$60,300; \$127,300; \$328,300; \$663,300 **b.** No **c.**

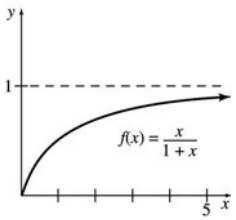


53. a. $a = 337/d$ **b.** 8.32 (using $k = 337$) **55. a.**

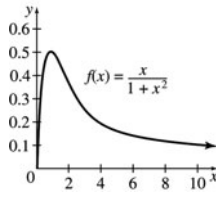


b. Close to 2 hours **c.** About 1.1 to 2.7 hours

57. a. $[0, \infty)$ **b.**

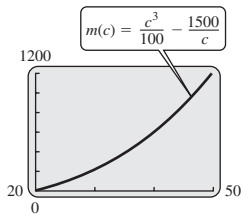


c.



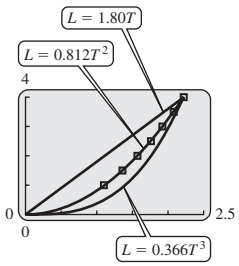
d. The population of the next generation, $f(x)$, gets smaller when the current generation, x, is larger.

59. a. 220 g; 602.5 g; 1220 g **b.** $c < 19.68$ **c.**



d. 41.9 cm

61. a. 1.80, 0.812, 0.366 **b.**



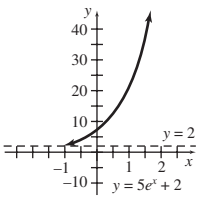
c. 2.48 sec **d.** The period increases by a factor of $\sqrt{2}$.
e. $L = 0.822T^2$, which is very close to the function found in part b.

Exercises 10.4 (page 494)

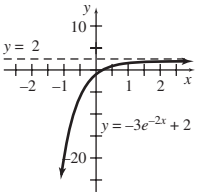
For exercises . . .	3–11,29–32	13–28	37–45,46	47–49	50–55
Refer to example . . .	1,2	3	4,5	6	7

1. 2, 4, 8, 16, 32, . . . , 1024; $1.125899907 \times 10^{15}$ **3.** E **5.** C **7.** F
9. A **11.** C **13.** 5 **15.** -4 **17.** -3 **19.** $21/4$ **21.** $-12/5$ **23.** $2, -2$ **25.** $0, -1$ **27.** $0, 1/2$

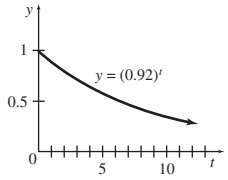
29.



31.

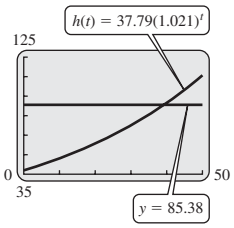


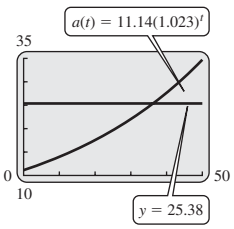
37. a. \$2166.53 **b.** \$2189.94 **c.** \$2201.90 **d.** \$2209.97 **e.** \$2214.03
39. He should choose the 5.9% investment, which would yield \$23.74 additional interest. **41. a.** \$10.94 **b.** \$11.27 **c.** \$11.62 **43.** 6.30%
45. a. 1, 0.92, 0.85, 0.78, 0.72, 0.66, 0.61, 0.56, 0.51, 0.47, 0.43
b.

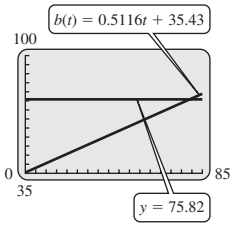


c. About \$384,000 **d.** About \$98
47. a. The function gives a population of about 3660 million, which is close to the actual population.
b. 6022 million **c.** 7725 million

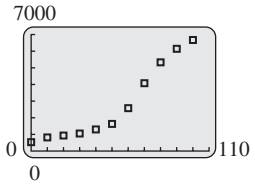
49. a. 41.93 million **b.** 12.48 million **c.** 2.1%, 2.3%; Asian **d.** 37.99 million
e. Hispanic: 2039; Asian: 2036; Black: 2079





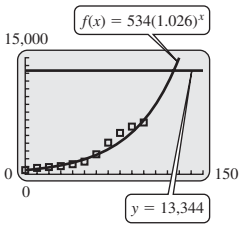


51. a. exponentially

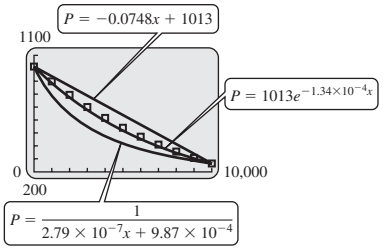


b. $f(x) = 534(1.026)^x$ **c.** 2.6%

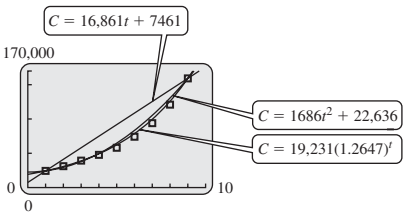
d.



2026 **53. a.** $P = 1013e^{-1.34\times10^{-4}x}$; $P = -0.0748x + 1013$; $P = 1/(2.79 \times 10^{-7}x + 9.87 \times 10^{-4})$
b. $P = 1013e^{-1.34\times10^{-4}x}$ is the best fit.
c. 829 millibars, 232 millibars
d. $P = 1038(0.99998661)^x$. This is slightly different from the function found in part b, which can be rewritten as $P = 1013(0.99998660)^x$.



55. a. $C = 16{,}861t + 7461$, $C = 1686t^2 + 22{,}636$, $C = 19{,}231(1.2647)^t$ **b.**



$C = 19{,}231(1.2647)^t$ is the best fit.
c. $C = 19{,}250(1.2579)^t$; this is close to the function found in part b.
d. 176,100, 191,200, 201,300, 190,900

Exercises 10.5 (page 506)

For exercises . . .	1–12	13–24	27–36	37–40	41–56	57–64	65–68	69–70	75–77,79,87	75d,76c,78	81–83
Refer to example . . .	1	2	3	4	5	6	7	first CAUTION	8	9	10

1. $\log_5 125 = 3$ **3.** $\log_3 81 = 4$ **5.** $\log_3(1/9) = -2$ **7.** $2^5 = 32$ **9.** $e^{-1} = 1/e$ **11.** $10^5 = 100{,}000$ **13.** 2 **15.** 3
17. -4 **19.** $-2/3$ **21.** 1 **23.** $5/3$ **25.** $\log_3 4$ **27.** $\log_5 3 + \log_5 k$ **29.** $1 + \log_3 p - \log_3 5 - \log_3 k$
31. $\ln 3 + (1/2)\ln 5 - (1/3)\ln 6$ **33.** $5a$ **35.** $2c + 3a + 1$ **37.** 2.113 **39.** -0.281 **41.** $x = 1/6$ **43.** $z = 4/3$
45. $r = 25$ **47.** $x = 1$ **49.** No solution **51.** $x = 3$ **53.** $x = 5$ **55.** $x = 1/\sqrt{3e} \approx 0.3502$ **57.** $x = (\ln 6)/(\ln 2) \approx 2.5850$
59. $k = 1 + \ln 6 \approx 2.7918$ **61.** $x = (\ln 3)/(\ln (5/3)) \approx 2.1507$ **63.** $x = (\ln 1.25)/(\ln 1.2) \approx 1.224$ **65.** $e^{(\ln 10)(x+1)}$
67. 20.09^x **69.** $x < 5$ **75. a.** 23.4 yr **b.** 11.9 yr **c.** 9.0 yr **d.** 23.3 yr; 12 yr; 9 yr **77.** 6.25% **79.** 2035
81. a. About 0.693 **b.** $\ln 2$ **c.** Yes **83. a.** About 1.099 **b.** About 1.386 **85.** About every 7 hr, $T = (3\ln 5)/\ln 2$
87. a. 2039 **b.** 2036 **89.** $s/n = 2^{C/B} - 1$ **91.** No; $1/10$ **93. a.** 1000 times greater **b.** 1,000,000 times greater

Exercises 10.6 (page 515)

For exercises . . .	6,28b,29b	7–10,15–17,21c	11–14,18–20	19d,20d	23,24	25–28,30,31,42	21d,e,22,29,41	32–40
Refer to example . . .	3	4	6	7	8	1	5	2

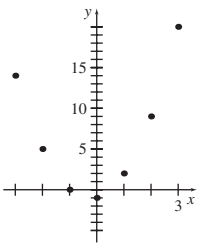
7. 4.06% **9.** 8.33% **11.** \$6209.93 **13.** \$6283.17 **15.** 9.20% **17.** 6.17% **19. a.** \$257,107.67 **b.** \$49,892.33
c. \$68,189.54 **d.** 14.28% **21. a.** The 8% investment compounded quarterly. **b.** \$759.26 **c.** 8.24% and 8.06% **d.** 3.71 yr
e. 3.75 yr **23. a.** 200 **b.** About 1/2 year **c.** No **d.** Yes; 1000 **25. a.** $0.003285e^{0.007232t}$ **b.** 3300 **c.** No; it is too small.
Exponential growth does not accurately describe population growth for the world over a long period of time.
27. a. $y = 25{,}000e^{0.047t}$ **b.** $y = 25{,}000(1.048)^t$ **c.** About 18.6 hours **29. a.** 17.9 days **b.** January 17 **31. a.** 2, 5, 24, and
125 **b.** 0.061 **c.** 0.24 **d.** No, the values of k are different. **e.** Between 3 and 4 **33.** About 13 years
35. a. 0.0193 gram **b.** 69 years **37. a.** $y = 500e^{-0.0863t}$ **b.** $y = 500(386/500)^{t/3}$ **c.** 8.0 days **39. a.** 19.5 watts
b. 173 days **41. a.** 67% **b.** 37% **c.** 23 days **d.** 46 days **43.** 18.02° **45.** 1 hour

Chapter 10 Review Exercises (page 521)

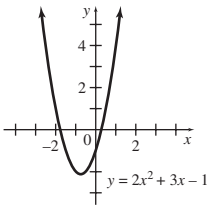
1. True **2.** False **3.** True
4. True **5.** False **6.** False
7. True **8.** False **9.** False
10. False **11.** False **12.** False
13. False **14.** True **15.** False
16. False **17.** True **18.** True

For exercises . . .	1,2,6,20,21, 35–42,87,110, 115,122	3,4,31–34, 105,109	5,9,43–46, 51–54,83, 88–93,101,108, 111,112,114, 116–118,120	7,10–17,22, 29,30,47–50, 55–82,84, 94–95,121	8,23–28,86	18,96–100. 102–104
Refer to section . . .	3	2	4	5	1	6

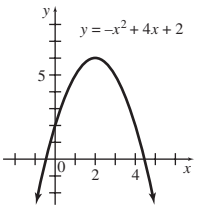
23. $(-3, 14), (-2, 5), (-1, 0), (0, -1), (1, 2), (2, 9), (3, 20)$; range: $\{-1, 0, 2, 5, 9, 14, 20\}$



25. a. 17 **b.** 4 **c.** $5k^2 - 3$ **d.** $-9m^2 + 12m + 1$
e. $5x^2 + 10xh + 5h^2 - 3$ **f.** $-x^2 - 2xh - h^2 + 4x + 4h + 1$
g. $10x + 5h$ **h.** $-2x - h + 4$ **27.** $(-\infty, 0) \cup (0, \infty)$
29. $(-7, \infty)$ **31.**



33.



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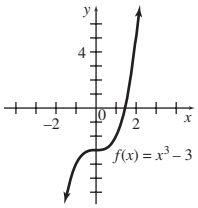
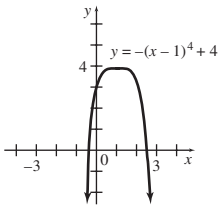


37.



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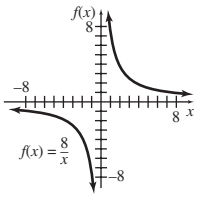

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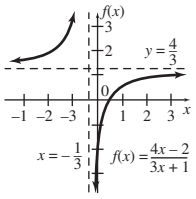
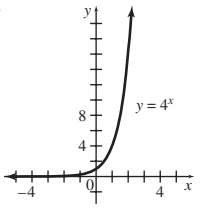
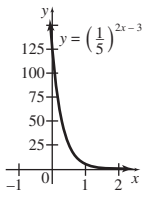


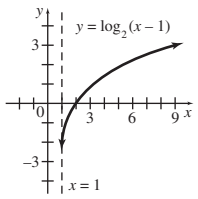
43.



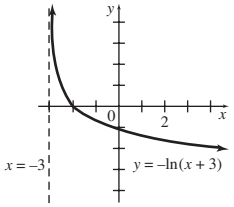
45.



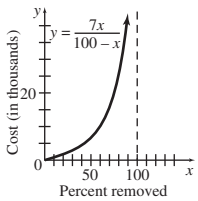
47.



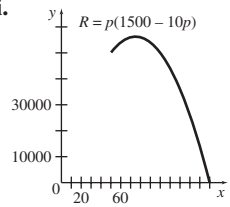
49.



51. -5 **53.** -6 **55.** $\log_3 243 = 5$ **57.** $\ln 2.22554 = 0.8$ **59.** $2^5 = 32$ **61.** $e^{4.41763} = 82.9$ **63.** 4 **65.** 3/2
67. $\log_5(21k^4)$ **69.** $\log_3(y^4/x^2)$ **71.** $p = 1.581$ **73.** $m = -1.807$ **75.** $x = -3.305$ **77.** $m = 2.156$ **79.** $k = 2$
81. $p = 3/4$ **83. a.** $(-\infty, \infty)$ **b.** $(0, \infty)$ **c.** 1 **d.** $y = 0$ **e.** Greater than 1 **f.** Between 0 and 1 **87. a.** \$28,000
b. \$7000 **c.** \$63,000 **d.**

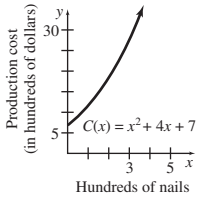


e. No. **89.** \$921.95 **91.** \$15,510.79 **93.** \$17,901.90 **95.** 70 quarters or 17.5 years; 111 quarters or 27.75 years **97.** 6.17% **99.** \$1494.52 **101.** \$17,339.86
103. About 9.59% **105. a.** $n = 1500 - 10p$ **b.** $R = p(1500 - 10p)$
c. $50 \le p \le 150$ **d.** $R = (1500n - n^2)/10$ **e.** $0 \le n \le 1000$ **f.** \$75
g. 750 **h.** \$56,250 **i.**



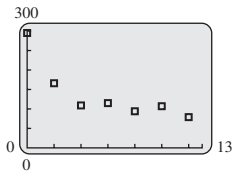
j. The revenue starts at \$50,000 when the price is \$50, rises to a maximum of \$56,250 when the price is \$75, and falls to 0 when the price is \$150.

107. a.

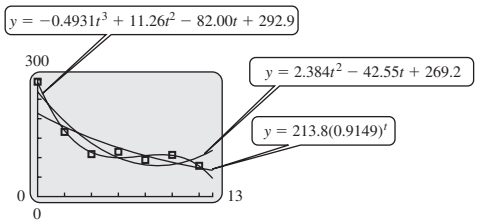


b. $2x + 5$ **c.** $A(x) = x + 4 + 7/x$
d. $1 - 7/[x(x + 1)]$ **109.** The third day; 104.2°F

111. a.

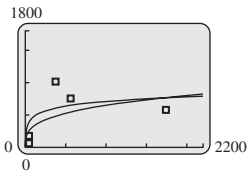


b. $y = 2.384t^2 - 42.55t + 269.2$, $y = -0.4931t^3 + 11.26t^2 - 82.00t + 292.9$, $y = 213.8(0.9149)^t$
c.



d. 372, −788, 36 **113.** 187.9 cm; 345 kg
115. a. 10.915 billion; this is about 4.006 billion more than the estimate of 6.909 billion. **b.** 26.56 billion; 96.32 billion
117. 0.25; 0.69 minutes

119. a. $S = -3.404 + 103.2 \ln A$ **b.** $S = 81.26\, A^{0.3011}$ **c.**



d. 234, 163
121. a. 0 yr **b.** 1.85×10^9 yr
c. As r increases, t increases, but at a slower and slower rate. As r decreases, t decreases at a faster and faster rate.

Chapter 11 The Derivative

Exercises 11.1 (page 543)

For exercises . . .	1,7b, 8b,9a, 82–84	2,7a, 10a, 53–56	3,5a, 6a, 17–20	4	5a,6b, 8a,9b, 10b,15,16	11, 12	21–30	31–38, 41,42	39,40	43–52, 75–81, 84–93	57–63, 65, 71–74
Refer to example . . .	5	3	2	4	1	11	6	8	9	12	10

1. c **3.** b **5. a.** 3 **b.** 1 **7. a.** 0 **b.** Does not exist **9. a. i.** -1; **ii.** $-1/2$ **iii.** Does not exist; **iv.** Does not exist
b. i. $-1/2$ **ii.** $-1/2$ **iii.** $-1/2$ **iv.** $-1/2$ **11.** 3 **15.** 4 **17.** 10 **19.** Does not exist **21.** -18 **23.** 1/3 **25.** 3
27. 512 **29.** 2/3 **31.** 6 **33.** 3/2 **35.** -5 **37.** $-1/9$ **39.** 1/10 **41.** $2x$ **43.** 3/7 **45.** 3/2 **47.** 0 **49.** ∞ (does not exist) **51.** $-\infty$ (does not exist) **53.** 1 **55. a.** 2 **b.** Does not exist **57.** 6 **59.** 1.5 **61. a.** Does not exist **b.** $x = -2$
c. If $x = a$ is a vertical asymptote for the graph of $f(x)$, then $\lim_{x \to a} f(x)$ does not exist. **65. a.** 0 **b.** $y = 0$ **67. a.** $-\infty$ (does not exist) **b.** $x = 0$ **71.** 5 **73.** 0.3333 or 1/3 **75. a.** 1.5 **77. a.** -2 **79. a.** 8 **83. a.** 3 cents **b.** 7.25 cents
c. 8.25 cents **d.** Does not exist **e.** 8.25 cents **85.** \$6; the average cost approaches \$6 as the number of DVDs produced becomes very large. **87.** 63 items; the number of items a new employee produces gets closer and closer to 63 as the number of

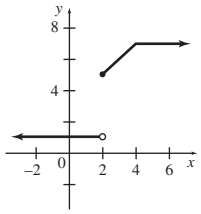
days of training increases. **89.** $R/(i-g)$ **91. a.** 36.2 cm; the depth of the sediment layer deposited below the bottom of the lake in 1970 is 36.2 cm. **b.** 155 cm; the depth of the sediment approaches 155 cm going back in time. **93. a.** 0.572 **b.** 0.526 **c.** 0.503 **d.** 0.5; the numbers in a, b, and c give the probability that the legislator will vote yes on the second, fourth, and eighth votes. In d, as the number of roll calls increases, the probability of a yes vote approaches 0.5 but is never less than 0.5.

Exercises 11.2 (page 554)

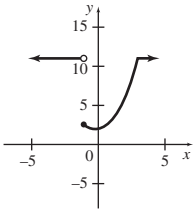
For exercises . . .	1–6,34,35	7–18,31–33	19–28	36–41
Refer to example . . .	1	2	3	4

1. $a=-1$: **a.** $f(-1)$ does not exist. **b.** $1/2$ **c.** $1/2$ **d.** $1/2$ **e.** $f(-1)$ does not exist. **3.** $a=1$: **a.** 2 **b.** -2 **c.** -2 **d.** -2 **e.** $f(1)$ does not equal the limit. **5.** $a=-5$: **a.** $f(-5)$ does not exist **b.** ∞ (does not exist) **c.** $-\infty$ (does not exist) **d.** Limit does not exist. **e.** $f(-5)$ does not exist and the limit does not exist; $a=0$: **a.** $f(0)$ does not exist. **b.** 0 **c.** 0 **d.** 0 **e.** $f(0)$ does not exist. **7.** $a=0$, limit does not exist; $a=2$, limit does not exist. **9.** $a=2$, limit is 4 **11.** Nowhere **13.** $a=-2$, limit does not exist. **15.** $a<1$, limit does not exist. **17.** $a=0, -\infty$ (limit does not exist); $a=1, \infty$ (limit does not exist).

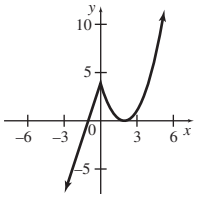
19. a.



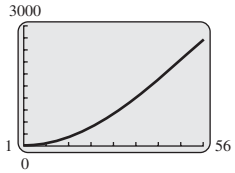
b. 2 **c.** 1, 5 **21. a.**



b. -1 **c.** 11, 3 **23. a.**



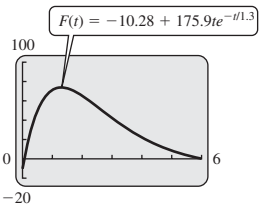
b. None **25.** 2/3 **27.** 4 **31. a.** Discontinuous at $x=1.2$ **33.** a **35. a.** \$500 **b.** \$1500 **c.** \$1000 **d.** Does not exist **e.** Discontinuous at $x=10$; a change in shifts **f.** 15 **37. a.** \$100 **b.** \$150 **c.** \$125 **d.** At $x=100$ **39. a.** \$2.92 **b.** \$3.76 **c.** Does not exist **d.** \$2.92 **e.** \$10.56 **f.** \$10.56 **g.** \$10.56 **h.** \$10.56 **i.** 1, 2, 3, 4, 5, 6, 7, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60 **41.** About 687 g **b.** No **c.**



Exercises 11.3 (page 566)

For exercises . . .	1–8	9–18,33, 36,38,44	19–22, 28,29,37	25–27	30–32, 34,35, 39–41	42,43
Refer to example . . .	1	3	6	4 and 5	2	7

1. 6 **3.** -15 **5.** 1/3 **7.** 0.4323 **9.** 17 **11.** 18 **13.** 5 **15.** 2 **17.** 2 **19.** 6.773 **21.** 1.121 **25. a.** \$700 per item **b.** \$500 per item **c.** \$300 per item **d.** \$1100 per item **27. a.** -25 boxes per dollar **b.** -20 boxes per dollar **c.** -30 boxes per dollar **d.** Demand is decreasing. Yes, a higher price usually reduces demand. **29. a.** \$56.81 per year **b.** \$72.94 per year **c.** \$64.20 per year **31. a.** 17.7 cents per gallon per month **b.** –48.0 cents per gallon per month **c.** –11.2 cents per gallon per month **33. a.** 6% per day **b.** 7% per day **35. a.** 2; from 1 min to 2 min, the population of bacteria increases, on the average, 2 million per min. **b.** -0.8; from 2 min to 3 min, the population of bacteria decreases, on the average, 0.8 million or 800,000 per min. **c.** -2.2; from 3 min to 4 min, the population of bacteria decreases, on the average, 2.2 million per min. **d.** -1; from 4 min to 5 min, the population decreases, on the average, 1 million per min. **e.** 2 min **f.** 3 min **37. a.**



b. 81.51 kilojoules per hour per hour **c.** 18.81 kilojoules per hour per hour **d.** 1.3 hours **39. a.** $-15{,}760$ immigrants per year **b.** 17,680 immigrants per year **c.** 960 immigrants per year **d.** They are equal; no **e.** 1,126,000 immigrants **41. a.** 4° per 1000 ft **b.** 1.75° per 1000 ft **c.** $-4/3$° per 1000 ft **d.** 0° per 1000 ft **e.** 3000 ft; 1000 ft; if 7000 ft is changed to 10,000 ft, the lowest temperature would be at 10,000 ft. **f.** 9000 ft **43. a.** 50 mph **b.** 44 mph

Exercises 11.4 (page 584)

For exercises . . .	5–10, 54,55, 58–61	11–14	15, 16	17, 18	19, 20	21–26	27–34, 42–45, 53	35–38	49–52, 56–57
Refer to example . . .	3	4	6	7	5	1 and 9	2	10	8

1. a. 0 **b.** 1 **c.** -1 **d.** Does not exist **e.** m **3.** At $x=-2$ **5.** 2 **7.** 1/4 **9.** 0 **11.** 3; 3; 3; 3 **13.** $-8x+9$; 25; 9; -15 **15.** $-12/x^2$; -3; does not exist; $-4/3$ **17.** $1/(2\sqrt{x})$; does not exist; does not exist; $1/(2\sqrt{3})$ **19.** $6x^2$; 24; 0; 54 **21. a.** $y=10x-15$ **b.** $y=8x-9$ **23. a.** $y=-(1/2)x+7/2$ **b.** $y=-(5/4)x+5$ **25. a.** $y=(4/7)x+48/7$ **b.** $y=(2/3)x+6$ **27.** -5; -117; 35 **29.** 7.389; 8,886,112; 0.0498 **31.** 1/2; 1/128; 2/9 **33.** $1/(2\sqrt{2})$; 1/8; does not exist

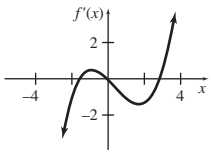
35. 0 **37.** $-3; -1; 0; 2; 3; 5$ **39. a.** $(a, 0)$ and (b, c) **b.** $(0, b)$ **c.** $x = 0$ and $x = b$ **41. a.** Distance **b.** Velocity **43.** 56.66 **45.** -0.0158 **49. a.** $-4p - 4$ **b.** -44; demand is decreasing at a rate of about 44 items for each increase in price of \$1. **51. a.** \$16 per table **b.** \$15.996 (or \$16) **c.** \$15.998 (or \$16) **d.** The marginal revenue gives a good approximation of the actual revenue from the sale of the 1001st table. **53.** Answers are in trillion of dollars. **a.** 2.54; 2.03; 1.02 **b.** 0.061; -0.019; -0.079; -0.120; -0.133 **55.** 1000; the population is increasing at a rate of 1000 shellfish per time unit. 570; the population is increasing more slowly at 570 shellfish per time unit. 250; the population is increasing at a much slower rate of 250 shellfish per time unit. **57. a.** 1690 m per sec **b.** 4.84 days per m per sec; an increase in velocity from 1700 m per sec to 1701 m per sec indicates an approximate increase in the age of the cheese of 4.84 days. **59. a.** About 270; the temperature was increasing at a rate of about 270° per hour at 9:00 a.m. **b.** About -150; the temperature was decreasing at a rate of about 150° per hour at 11:30 a.m. **c.** About 0; the temperature was staying constant at 12:30 p.m. **d.** About 11:15 a.m. **61. a.** 0; about 0.5 mph/oz

Exercises 11.5 (page 592)

For exercises . . .	2–6	7–24
Refer to example . . .	4	1,2, and 3

3. $f{:}Y_2$; $f'{:}Y_1$ **5.** $f{:}Y_1$; $f'{:}Y_2$

7.



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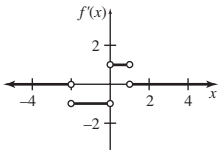

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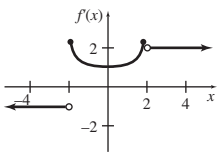

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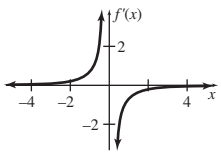

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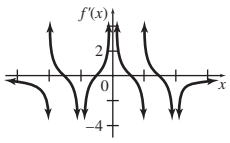

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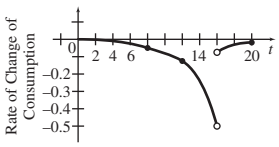

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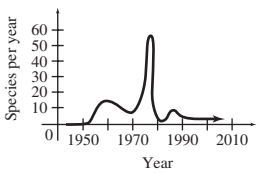

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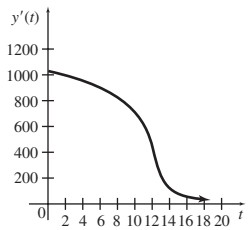
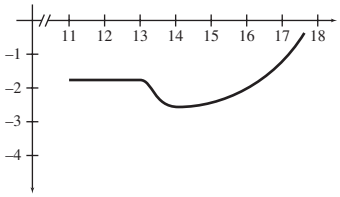


23.

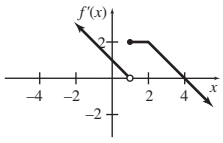


About 9 cm; about 2.6 cm less per year

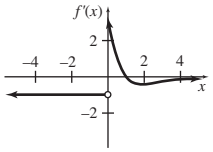
Chapter 11 Review Exercises (page 596)

For exercises . . .	1–3, 17–34, 45–46, 61,66	4–6,15, 35–44, 63,74	7–9, 47–50, 63–65	10–12,16, 51–58,62, 68–69,73	59–60,67, 70–72,74
Refer to section . . .	1	2	3	4	5

1. True **2.** True **3.** True **4.** False **5.** True **6.** False **7.** False **8.** True **9.** True **10.** True **11.** False **12.** False **17. a.** 4 **b.** 4 **c.** 4 **d.** 4 **19. a.** ∞ **b.** $-\infty$ **c.** Does not exist **d.** Does not exist **21.** ∞ **23.** 19/9 **25.** 8 **27.** -13 **29.** 1/6 **31.** 2/5 **33.** 3/8 **35.** Discontinuous at x_2 and x_4 **37.** 0, does not exist, does not exist; $-1/3$, does not exist, does not exist **39.** -5, does not exist, does not exist **41.** Continuous everywhere **43. a.**

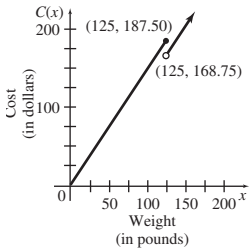


b. 1 **c.** 0, 2 **45.** 2 **47.** 126; 18 **49.** 9/77; 18/49 **51. a.** $y = 13x - 17$ **b.** $y = 7x - 5$ **53. a.** $y = -x + 9$ **b.** $y = -3x + 15$ **55.** $8x + 3$ **57.** 1.332 **59.**



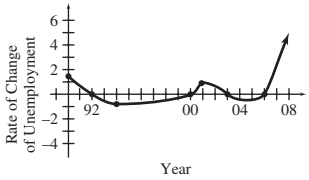
61. e **63. a.** \$150 **b.** \$187.50 **c.** \$189

d.



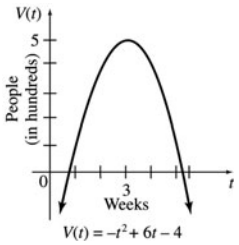
e. Discontinuous at $x =$ \$125 **f.** \$1.50 **g.** \$1.50 **h.** \$1.35 **i.** 1.5; when 100 lb are purchased, an additional pound will cost about \$1.50 more. **j.** 1.35; when 140 lb are purchased, an additional pound will cost about \$1.35 more. **65. b.** $x = 7.5$ **c.** The marginal cost equals the average cost at the point where the average cost is smallest.

67.



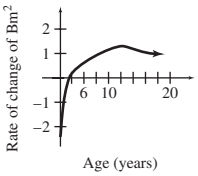
; about 5.8; about 2

69. a.

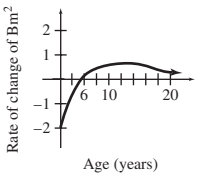


b. $[0.8, 5.2]$ **c.** 3 weeks; 500 cases **d.** $V'(t) = -2t + 6$ **e.** 0 **f.** +; −

71. a.

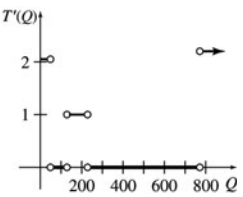


b.



73. a. Nowhere **b.** 50, 130, 230, 770

c.



Chapter 12 Calculating the Derivative

Exercises 12.1 (page 615)

For exercises . . .	1–22,27–30	47,55,56	51,53	52,54	57,58,60–62, 64,68–75	59,63, 65,67	31–43
Refer to example . . .	1,2,3,5	6	7	8	9	4	1 (in 4th section of previous chapter)

1. $dy/dx = 36x^2 - 16x + 7$ **3.** $dy/dx = 12x^3 - 18x^2 + (1/4)x$ **5.** $f'(x) = 21x^{2.5} - 5x^{-0.5}$ or $21x^{2.5} - 5/x^{0.5}$
7. $dy/dx = 4x^{-1/2} + (9/2)x^{-1/4}$ or $4/x^{1/2} + 9/(2x^{1/4})$ **9.** $dy/dx = -30x^{-4} - 20x^{-5} - 8$ or $-30/x^4 - 20/x^5 - 8$
11. $f'(t) = -7t^{-2} + 15t^{-4}$ or $-7/t^2 + 15/t^4$ **13.** $dy/dx = -24x^{-5} + 21x^{-4} - 3x^{-2}$ or $-24/x^5 + 21/x^4 - 3/x^2$
15. $p'(x) = 5x^{-3/2} - 12x^{-5/2}$ or $5/x^{3/2} - 12/x^{5/2}$ **17.** $dy/dx = (-3/2)x^{-5/4}$ or $-3/(2x^{5/4})$
19. $f'(x) = 2x - 5x^{-2}$ or $2x - 5/x^2$ **21.** $g'(x) = 256x^3 - 192x^2 + 32x$ **23.** b **27.** $-(9/2)x^{-3/2} - 3x^{-5/2}$ or $-9/(2x^{3/2}) - 3/x^{5/2}$ **29.** $-25/3$ **31.** -28; $y = -28x + 34$ **33.** $25/6$ **35.** $(4/9, 20/9)$ **37.** $-5, 2$ **39.** $(4 \pm \sqrt{37})/3$
41. $(-1/2, -19/2)$ **43.** $(-2, -24)$ **45.** 7 **51. a.** 30 **b.** 0 **c.** -10 **53.** \$990 **55. a.** 100 **b.** 1 **57. a.** 18¢; 36¢
b. 0.755¢ per year; 1.09¢ per year **c.** $C(t) = -0.0001549t^3 + 0.02699t^2 - 0.6484t + 3.212$; 0.889¢ per year; 0.853¢ per year
59. a. 0.4824 **b.** 2.216 **61. a.** 1232.62 cm^3 **b.** 948.08 cm^3/yr **63.** $5.00l^{0.86}$ **65. a.** 3 minutes, 58.1 seconds **b.** 0.118 sec/m; at 100 meters, the fastest possible time increases by 0.118 seconds for each additional meter. c. Yes **67. a.** 27.5
b. 23 pounds **c.** $-175{,}750/h^3$ **d.** -0.64; for a 125-lb female with a height of 65 in. (5′5″), the BMI decreases by 0.64 for each additional inch of height. **69. a.** $v(t) = 36t - 13$ **b.** -13; 167; 347 **71. a.** $v(t) = -9t^2 + 8t - 10$ **b.** -10; -195; -830
73. 0 ft/sec; -32 ft/sec **b.** 2 seconds **c.** 64 ft **75. a.** 35, 36 **b.** When $x = 5$, $dy_1/dx = 4.13$ and $dy_2/dx \approx 4.32$. These values are fairly close and represent the rate of change of four years for a dog for one year of a human, for a dog that is actually 5 years old. **c.** $y = 4x + 16$

Exercises 12.2 (page 624)

For exercises . . .	1–10,29,34,39, 42,44–47,50	11–26,31–33,35,40, 43,45,48,49,51–54	27,28,30	41,42
Refer to example . . .	1,2	3	4	5

1. $dy/dx = 18x^2 - 6x + 4$ **3.** $dy/dx = 8x - 20$
5. $k'(t) = 4t^3 - 4t$
7. $dy/dx = (3/2)x^{1/2} + (1/2)x^{-1/2} + 2$ or $3x^{1/2}/2 + 1/(2x^{1/2}) + 2$ **9.** $p'(y) = -8y^{-5} + 15y^{-6} + 30y^{-7}$
11. $f'(x) = 57/(3x + 10)^2$ **13.** $dy/dt = -17/(4 + t)^2$ **15.** $dy/dx = (x^2 - 2x - 1)/(x - 1)^2$
17. $f'(t) = 2t/(t^2 + 3)^2$ **19.** $g'(x) = (4x^2 + 2x - 12)/(x^2 + 3)^2$ **21.** $p'(t) = [-\sqrt{t}/2 - 1/(2\sqrt{t})]/(t - 1)^2$ or $(-t - 1)/[2\sqrt{t}(t - 1)^2]$ **23.** $dy/dx = (5/2)x^{-1/2} - 3x^{-3/2}$ or $(5x - 6)/(2x\sqrt{x})$
25. $h'(z) = (-z^{4.4} + 11z^{1.2})/(z^{3.2} + 5)^2$ **27.** $f'(x) = (60x^3 + 57x^2 - 24x + 13)/(5x + 4)^2$ **29.** 77
31. In the first step, the numerator should be $(x^2 - 1)2 - (2x + 5)(2x)$. **33.** $y = -2x + 9$

35. a. $f'(x) = (7x^3 - 4)/x^{5/3}$ **b.** $f'(x) = 7x^{4/3} - 4x^{-5/3}$ **39.** 0, −1.307, and 1.307 **41. a.** \$22.86 per unit **b.** \$12.92 per unit **c.** $(3x + 2)/(x^2 + 4x)$ per unit **d.** $\overline{C}'(x) = (-3x^2 - 4x - 8)/(x^2 + 4x)^2$ **43. a.** $M'(d) = 2000d/(3d^2 + 10)^2$ **b.** 8.3; the new employee can assemble about 8.3 additional bicycles per day after 2 days of training. 1.4; the new employee can assemble about 1.4 additional bicycles per day after 5 days of training. **47.** Increasing at a rate of \$0.03552 per gallon per month **49. a.** $AK/(A + x)^2$ **b.** $K/(4A)$ **51. a.** 8.57 min **b.** 16.36 min **c.** 6.12 min^2/kcal; 2.48 min^2/kcal **53. a.** −100 facts/hr **b.** −0.01 facts/hr

Exercises 12.3 (page 633)

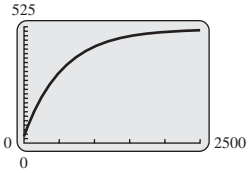
For exercises . . .	1–6	7–14,53,59	15–20	21–28,54,57, 62,63,66,67	29–34,64	35–40,56	58,65	55	60,61	45–50
Refer to example . . .	1	2	3	5,6	7	8	9	10	4	1 (in 3rd section of previous chapter)

1. 1767 **3.** 131 **5.** $320k^2 + 224k + 39$ **7.** $(6x + 55)/8$; $(3x + 164)/4$ **9.** $1/x^2$; $1/x^2$ **11.** $\sqrt{8x^2 - 4}$; $8x + 10$ **13.** $\sqrt{(x - 1)/x}$; $-1/\sqrt{x + 1}$ **15.** If $f(x) = x^{3/5}$ and $g(x) = 5 - x^2$, then $y = f[g(x)]$. **17.** If $f(x) = -\sqrt{x}$ and $g(x) = 13 + 7x$, then $y = f[g(x)]$. **19.** If $f(x) = x^{1/3} - 2x^{2/3} + 7$ and $g(x) = x^2 + 5x$, then $y = f[g(x)]$. **21.** $dy/dx = 4(8x^4 - 5x^2 + 1)^3(32x^3 - 10x)$ **23.** $k'(x) = 288x(12x^2 + 5)^{-7}$ **25.** $s'(t) = (1215/2)t^2(3t^3 - 8)^{1/2}$ **27.** $g'(t) = -63t^2/(2\sqrt{7t^3 - 1})$ **29.** $m'(t) = -6(5t^4 - 1)^3(85t^4 - 1)$ **31.** $dy/dx = 3x^2(3x^4 + 1)^3(19x^4 + 64x + 1)$ **33.** $q'(y) = 2y(y^2 + 1)^{1/4}(9y^2 + 4)$ **35.** $dy/dx = 60x^2/(2x^3 + 1)^3$ **37.** $r'(t) = 2(5t - 6)^3(15t^2 + 18t + 40)/(3t^2 + 4)^2$ **39.** $dy/dx = (-18x^2 + 2x + 1)/(2x - 1)^6$ **43. a.** −2 **b.** −24/7 **45.** $y = (3/5)x + 16/5$ **47.** $y = x$ **49.** 1, 3 **53.** $D(c) = (-c^2 + 10c + 12{,}475)/25$ **55. a.** \$101.22 **b.** \$111.86 **c.** \$117.59 **57. a.** −\$10,500 **b.** −\$4570.64 **59.** $P[f(a)] = 18a^2 + 24a + 9$ **61. a.** $A[r(t)] = A(t) = 4\pi t^2$; this function gives the area of the pollution in terms of the time since the pollutants were first emitted. **b.** 32π; at 12 P.M. the area of pollution is changing at the rate of 32π mi^2 per hour **63. a.** −0.5 **b.** $-1/54 \approx -0.02$ **c.** $-1/128 \approx -0.008$ **d.** Always decreasing; the derivative is negative for all $t \geq 0$. **65. a.** 34 minutes **b.** $-(108/17)\pi$ mm^3 per minute, $-(72/17)\pi$ mm^2 per minute **67. a.** $12x^{11}$ **b.** x^{12}; $12x^{11}$

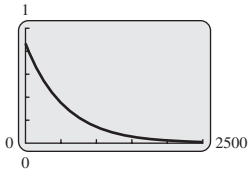
Exercises 12.4 (page 640)

For exercises . . .	1–12,25–30,33,34, 38,41,44–45,48,50, 51–54,56,59–63	13–16,23,24, 39,40,42, 49	17–22,31,32,51	43,46,47,55,57	58
Refer to example . . .	1	2	3	5	4

1. $dy/dx = 4e^{4x}$ **3.** $dy/dx = -24e^{3x}$ **5.** $dy/dx = -32e^{2x+1}$ **7.** $dy/dx = 2xe^{x^2}$ **9.** $dy/dx = 12xe^{2x^2}$ **11.** $dy/dx = 16xe^{2x^2-4}$ **13.** $dy/dx = xe^x + e^x = e^x(x + 1)$ **15.** $dy/dx = 2(x + 3)(2x + 7)e^{4x}$ **17.** $dy/dx = (2xe^x - x^2e^x)/e^{2x} = x(2 - x)/e^x$ **19.** $dy/dx = [x(e^x - e^{-x}) - (e^x + e^{-x})]/x^2$ **21.** $dp/dt = 8000e^{-0.2t}/(9 + 4e^{-0.2t})^2$ **23.** $f'(z) = 4(2z + e^{-z^2})(1 - ze^{-z^2})$ **25.** $dy/dx = 3(\ln 7)7^{3x+1}$ **27.** $dy/dx = 6x(\ln 4)4^{x^2+2}$ **29.** $ds/dt = (\ln 3)3^{\sqrt{t}}/\sqrt{t}$ **31.** $dy/dt = [(1 - t)e^{3t} - 4e^{2t} + (1 + t)e^t]/(e^{2t} + 1)^2$ **33.** $f'(x) = (9x + 4)e^{x\sqrt{3x+2}}/[2\sqrt{3x + 2}]$ **39. a.** \$3.81 **b.** \$0.20 **c.** $C'(x)$ approaches zero. **41. a.** 100% **b.** 94% **c.** 89% **d.** 83% **e.** −3.045 **f.** −2.865 **g.** The percent of these cars on the road is decreasing but at a slower rate as they age. **43. a.** $G(t) = 250/(1 + 124e^{-0.45t})$ **b.** 17.8 million; 7.4 million/yr **c.** 105.2 million; 27.4 million/yr **d.** 246.2 million; 1.7 million/yr **e.** It increases for a while and then gradually decreases to 0. **45. a.** 51,600,000 **b.** 1,070,000 people/year **47. a.** $5200/(1 + 12e^{-0.52t})$ **b.** 639, 292 **c.** 2081, 649 **d.** 4877, 157 **e.** It increases for a while and then gradually decreases to 0. **49. a.** 3.857 cm^3 **b.** 0.973 cm **c.** 18 years **d.** 1100 cm^3 **e.** 0.282; at 240 months old, the tumor is increasing in volume at the instantaneous rate of 0.282 cm^3/month. **51. a.** 0.180 **b.** 2024 **c.** The marginal increase in the proportion per year in 2010 is approximately 0.023. **53. a.** 509.7 kg, 498.4 kg **b.** 1239 days, 1095 days **c.** 0.22 kg/day, 0.22 kg/day **d.** The growth patterns of the two functions are very similar.



e. The graphs of the rates of change of the two functions are also very similar.



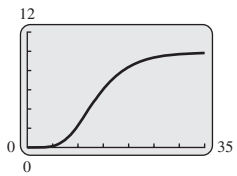
55. a. $G(t) = 1/(1 + 270e^{-3.5t})$ **b.** 0.109, 0.341 per century **c.** 0.802, 0.555 per century **d.** 0.993, 0.0256 per century **e.** It increases for a while and then gradually decreases to 0. **57. a.** $G(t) = 6.8/(1 + 3.242e^{-0.2992t})$ **b.** 2.444 million students; 0.468 million students/yr **c.** 3.435 million students; 0.509 million students/yr **d.** 5.247 million students; 0.359 million students/yr **e.** The rate increases at first and then decreases toward 0. **59. a.** $(V/R)e^{-t/RC}$ **b.** 1.35×10^{-7} amps **61. a.** 218.5 seconds **b.** The record is decreasing by 0.034 seconds per year at the end of 2010. **c.** 218 seconds. If the estimate is correct, then this is the least amount of time that it will ever take for a human to run a mile. **63. a.** 625.14 ft **b.** 0; yes **c.** –1.476 ft per ft from center

Exercises 12.5 (page 648)

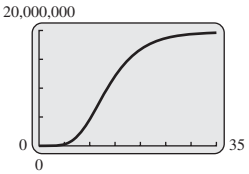
For exercises . . .	1,2,59,62,66	3,4,7–10,31,32,56,67	5,6,11–30,33–44, 57,58,60,61,64,65	68
Refer to example . . .	1	2	3	4

1. $dy/dx = 1/x$ **3.** $dy/dx = -3/(8 - 3x)$ or $3/(3x - 8)$ **5.** $dy/dx = (8x - 9)/(4x^2 - 9x)$
7. $dy/dx = 1/[2(x + 5)]$ **9.** $dy/dx = 3(2x^2 + 5)/[x(x^2 + 5)]$ **11.** $dy/dx = -15x/(3x + 2) - 5\ln(3x + 2)$
13. $ds/dt = t + 2t\ln|t|$ **15.** $dy/dx = [2x - 4(x + 3)\ln(x + 3)]/[x^3(x + 3)]$
17. $dy/dx = (4x + 7 - 4x\ln x)/[x(4x + 7)^2]$ **19.** $dy/dx = (6x\ln x - 3x)/(\ln x)^2$ **21.** $dy/dx = 4(\ln|x + 1|)^3/(x + 1)$
23. $dy/dx = 1/(x\ln x)$ **25.** $dy/dx = e^{x^2}/x + 2xe^{x^2}\ln x$ **27.** $dy/dx = (xe^x\ln x - e^x)/[x(\ln x)^2]$
29. $g'(z) = 3(e^{2z} + \ln z)^2(2ze^{2z} + 1)/z$ **31.** $dy/dx = 1/(x\ln 10)$ **33.** $dy/dx = -1/[(\ln 10)(1 - x)]$ or $1/[(\ln 10)(x - 1)]$
35. $dy/dx = 5/[(2\ln 5)(5x + 2)]$ **37.** $dy/dx = 3(x + 1)/[(\ln 3)(x^2 + 2x)]$ **39.** $dw/dp = (\ln 2)2^p/[(\ln 8)(2^p - 1)]$
41. $f'(x) = e^{\sqrt{x}}\{1/[2\sqrt{x}(\sqrt{x} + 5)] + \ln(\sqrt{x} + 5)/[2\sqrt{x}]\}$
43. $f'(t) = [(t^2 + 1)\ln(t^2 + 1) - t^2 + 2t + 1]/\{(t^2 + 1)[\ln(t^2 + 1) + 1]^2\}$
55. $h(x) = (x^2 + 1)^{5x}\left[\dfrac{10x^2}{x^2 + 1} + 5\ln(x^2 + 1)\right]$ **57. a.** $dR/dq = 100 + 50(\ln q - 1)/(\ln q)^2$ **b.** \$112.48
c. To decide whether it is reasonable to sell additional items. **59. a.** −0.19396 **b.** −0.06099 **61. a.** $N(t) = 1000e^{9.8901e^{-e^{2.54197 - 0.2167t}}}$
b. 1,307,416 bacteria per hour; the number of bacteria is increasing at a rate of 1,307,416 per hour, 20 hours after the experiment began.

c.



d.



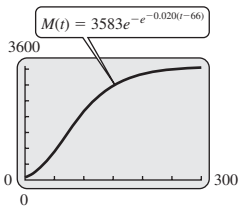
e. 9.8901; $1000e^{9.8901} \approx 19{,}734{,}033$
63. b. i. 3.343 **ii.** 1.466 **c. i.** −0.172 **ii.** −0.0511 **65.** 26.9; 13.1
67. a. 1.567×10^{11} kWh **b.** 63.4 months **c.** 4.14×10^{-6}
d. *dM/dE* decreases and approac.hes zero.

Chapter 12 Review Exercises (page 652)

For exercises . . .	1,2,4,11–16, 53,54,73,79–81	3,17–20,55,56, 74,75,88	5,21–30,51,52, 57,58,67–70,76,91	6–7,31–36,43–46,59,60, 71,77,82–87,89,90	8–10,37–42,47–50, 61,62,72,78
Refer to section . . .	1	2	3	4	5

1. False **2.** True **3.** False **4.** True **5.** False **6.** False **7.** False **8.** True **9.** True **10.** False
11. $dy/dx = 15x^2 - 14x - 9$ **13.** $dy/dx = 24x^{5/3}$ **15.** $f'(x) = -12x^{-5} + 3x^{-1/2}$ or $-12/x^5 + 3/x^{1/2}$
17. $k'(x) = 21/(4x + 7)^2$ **19.** $dy/dx = (x^2 - 2x)/(x - 1)^2$ **21.** $f'(x) = 24x(3x^2 - 2)^3$ **23.** $dy/dt = 7t^6/(2t^7 - 5)^{1/2}$
25. $dy/dx = 3(2x + 1)^2(8x + 1)$ **27.** $r'(t) = (-15t^2 + 52t - 7)/(3t + 1)^4$ **29.** $p'(t) = t(t^2 + 1)^{3/2}(7t^2 + 2)$
31. $dy/dx = -12e^{2x}$ **33.** $dy/dx = -6x^2e^{-2x^3}$ **35.** $dy/dx = 10xe^{2x} + 5e^{2x} = 5e^{2x}(2x + 1)$ **37.** $dy/dx = 2x/(2 + x^2)$
39. $dy/dx = (x - 3 - x\ln|3x|)/[x(x - 3)^2]$ **41.** $dy/dx = [e^x(x + 1)(x^2 - 1)\ln(x^2 - 1) - 2x^2e^x]/[(x^2 - 1)[\ln(x^2 - 1)]^2]$
43. $ds/dt = 2(t^2 + e^t)(2t + e^t)$ **45.** $dy/dx = -6x(\ln 10)\cdot 10^{-x^2}$ **47.** $g'(z) = (3z^2 + 1)/[(\ln 2)(z^3 + z + 1)]$
49. $f'(x) = (x + 1)e^{3x}/(xe^x + 1) + 2e^{2x}\ln(xe^x + 1)$ **51. a.** −3/2 **b.** −24/11 **53.** -2; $y = -2x - 4$
55. $-3/4$; $y = -(3/4)x - 9/4$ **57.** 3/4; $y = (3/4)x + 7/4$ **59.** 1; $y = x + 1$ **61.** 1; $y = x - 1$ **63.** No points if $k > 0$; exactly one point if $k = 0$ or if $k < -1/2$; exactly two points if $-1/2 \le k < 0$. **65.** 5%; 5.06%
67. $\overline{C}'(x) = (-x - 2)/[2x^2(x + 1)^{1/2}]$ **69.** $\overline{C}'(x) = (x^2 + 3)^2(5x^2 - 3)/x^2$ **71.** $\overline{C}'(x) = [e^{-x}(x + 1) - 10]/x^2$
73. a. 22; sales will increase by \$22 million when \$1000 more is spent on research. **b.** 19.5; sales will increase by \$19.5 million when \$1000 more is spent on research. **c.** 18; sales will increase by \$18 million when \$1000 more is spent on research.
d. As more is spent on research, the increase in sales is decreasing. **75. a.** −2.201; costs will decrease by \$2201 for the next \$100 spent on training. **b.** −0.564; costs will decrease by \$564 for the next \$100 spent on training. **c.** Decreasing
77. \$218.65. The balance increases by roughly \$218.65 for every 1% increase in the interest rate when the rate is 5%.

79. a. 4.839 billion/yr; the volume of mail is increasing by about 4,839,000,000 pieces per year **b.** –2.582 billion/yr; the volume of mail is decreasing by about 2,582,000,000 pieces per year **81. a.** $y = 1.799 \times 10^{-5}t^3 + 3.177 \times 10^{-4}t^2 - 0.06866t + 2.504$, $y = -1.112 \times 10^{-6}t^4 + 2.920 \times 10^{-4}t^3 - 0.02238t^2 + 0.6483t - 4.278$ **b.** \$0.59/yr, \$0.46/yr
83. a. $G(t) = 30{,}000/(1 + 14e^{-0.15t})$ **b.** 4483; 572 **85. a.** 3493.76 grams **b.** 3583 grams **c.** 84 days **d.** 1.76 g/day
e.



Growth is initially rapid, then tapers off. **f.**

Day	Weight	Rate
50	904	24.90
100	2159	21.87
150	2974	11.08
200	3346	4.59
250	3494	1.76
300	3550	0.66

87. a. 14,612 megawatts/yr **b.** 47,276 megawatts/yr **c.** 152,960 megawatts/yr **89.** 0.242; the production of corn is increasing at a rate of 0.242 billion bushels a year in 2000. **91. a.** −0.4677 fatalities per 1000 licensed drivers per 100 million miles per year; at the age of 20, each extra year results in a decrease of 0.4677 fatalities per 1000 licensed drivers per 100 million miles.
b. 0.003672 fatalities per 1000 licensed drivers per 100 million miles per year; at the age of 60, each extra year results in an increase of 0.003672 fatalities per 1000 licensed drivers per 100 million miles.

Chapter 13 Graphs and the Derivative

Exercises 13.1 (page 668)

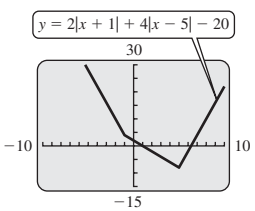
For exercises . . .	1–8, 43,44 50,51, 60,61	13–22,25, 29–34,38, 39,42,49, 52,53,56–59	21–24	26–28, 35,36	45,54,55	46–48
Refer to example . . .	1	2	4	3	6	5

1. a. $(1, \infty)$ **b.** $(-\infty, 1)$ **3. a.** $(-\infty, -2)$ **b.** $(-2, \infty)$ **5. a.** $(-\infty, -4), (-2, \infty)$ **b.** $(-4, -2)$ **7. a.** $(-7, -4), (-2, \infty)$ **b.** $(-\infty, -7), (-4, -2)$
9. a. $(-\infty, -1), (3, \infty)$ **b.** $(-1, 3)$ **11. a.** $(-\infty, -8), (-6, -2.5), (-1.5, \infty)$ **b.** $(-8, -6), (-2.5, -1.5)$ **13. a.** 17/12 **b.** $(-\infty, 17/12)$ **c.** $(17/12, \infty)$ **15. a.** $-3, 4$ **b.** $(-\infty, -3), (4, \infty)$ **c.** $(-3, 4)$ **17. a.** $-3/2, 4$ **b.** $(-\infty, -3/2), (4, \infty)$ **c.** $(-3/2, 4)$ **19. a.** $-2, -1, 0$ **b.** $(-2, -1), (0, \infty)$ **c.** $(-\infty, -2), (-1, 0)$ **21. a.** None **b.** None **c.** $(-\infty, \infty)$ **23. a.** None **b.** None **c.** $(-\infty, -1), (-1, \infty)$ **25. a.** 0 **b.** $(0, \infty)$ **c.** $(-\infty, 0)$ **27. a.** 0 **b.** $(0, \infty)$ **c.** $(-\infty, 0)$ **29. a.** 7 **b.** $(7, \infty)$ **c.** $(3, 7)$ **31. a.** 1/3 **b.** $(-\infty, 1/3)$ **c.** $(1/3, \infty)$ **33. a.** $0, 2/\ln 2$ **b.** $(0, 2/\ln 2)$ **c.** $(-\infty, 0), (2/\ln 2, \infty)$ **35. a.** $0, 2/5$ **b.** $(0, 2/5)$ **c.** $(-\infty, 0), (2/5, \infty)$ **39.** Vertex: $(-b/(2a), (4ac - b^2)/(4a))$; increasing on $(-\infty, -b/(2a))$, decreasing on $(-b/(2a), \infty)$ **41.** On $(0, \infty)$; nowhere; nowhere
43. a. About $(567, \infty)$ **b.** About $(0, 567)$ **45. a.** Nowhere **b.** $(0, \infty)$ **47.** (0, 2200) **49. a.** Increasing on (0, 7.4) or from 2000 to about the middle of 2007 **b.** Decreasing on (7.4, 50) or from about the middle of 2007 to 2050 **51. a.** Yes **b.** April to July; July to November; January to April and November to December **c.** January to April and November to December
53. a. (0, 1.85) **b.** (1.85, 5) **55. a.** (0, 1) **b.** $(1, \infty)$ **57. a.** $F'(t) = 175.9e^{-t/1.3}(1 - 0.769t)$ **b.** (0, 1.3); $(1.3, \infty)$
59. $(-\infty, 0); (0, \infty)$ **61. a.** (2500, 5750) **b.** (5750, 6000) **c.** (2800, 4800) **d.** (2500, 2800) and (4800, 6000)

Exercises 13.2 (page 679)

For exercises . . .	1–8, 37–39, 50	13–20, 35,36, 51	21–24	25–34, 52–54, 56	41–44, 46–49	45,55,57
Refer to example . . .	2	3a	3b	3c	4	1

1. Relative minimum of −4 at 1 **3.** Relative maximum of 3 at −2 **5.** Relative maximum of 3 at −4; relative minimum of 1 at −2 **7.** Relative maximum of 3 at −4; relative minimum of −2 at −7 and −2 **9.** Relative maximum at −1; relative minimum at 3 **11.** Relative maxima at −8 and −2.5; relative minimum at −6 and −1.5 **13.** Relative minimum of 8 at 5 **15.** Relative maximum of −8 at −3; relative minimum of −12 at −1 **17.** Relative maximum of 827/96 at −1/4; relative minimum of −377/6 at −5 **19.** Relative maximum of −4 at 0; relative minimum of −85 at 3 and −3 **21.** Relative maximum of 3 at −8/3 **23.** Relative maximum of 1 at −1; relative minimum of 0 at 0 **25.** No relative extrema **27.** Relative maximum of 0 at 1; relative minimum of 8 at 5 **29.** Relative maximum of −2.46 at −2; relative minimum of −3 at 0 **31.** No relative extrema **33.** Relative minimum of $e \ln 2$ at $1/\ln 2$
35. (3, 13) **37.** Relative maximum of 6.211 at 0.085; relative minimum of −57.607 at 2.161
39. Relative minimum at $x = 5$



41. a. 13 **b.** \$44 **c.** \$258 **43. a.** 100 **b.** \$14.72 **c.** \$635.76 **45.** Relative maximum of 20,470 megawatts at midnight ($t = 0$); relative minimum of 20,070 megawatts at 1:40 A.M.; relative maximum of 30,060 megawatts at 4:44 P.M.; relative minimum of 20,840 megawatts at midnight ($t = 24$). **47.** $q = 10$; $p \approx \$73.58$ **49.** 120 units **51.** 5:04 P.M.; 6:56 A.M. **53.** 4.96 years; 458.22 kg **55.** 10 **57. a.** 28 ft **b.** 2.57 sec

Exercises 13.3 (page 691)

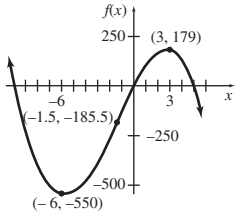
For exercises . . .	1–16	17–26, 76,77	27–48,54, 65–68, 70,80, 85–88	57–64,79, 81,82, 90,92,93	72–75 78, 83,84	91,92, 94,95
Refer to example . . .	2	3	5	6	7	4

1. $f''(x) = 30x - 14$; -14; 46
3. $f''(x) = 48x^2 - 18x - 4$; -4; 152
5. $f''(x) = 6$; 6; 6
7. $f''(x) = 2/(1 + x)^3$; 2; $2/27$
9. $f''(x) = 4/(x^2 + 4)^{3/2}$; $1/2$; $1/(4\sqrt{2})$ **11.** $f''(x) = -6x^{-5/4}$ or $-6/x^{5/4}$; $f''(0)$ does not exist; $-3/2^{1/4}$
13. $f''(x) = 20x^2e^{-x^2} - 10e^{-x^2}$; -10; $70e^{-4} \approx 1.282$ **15.** $f''(x) = (-3 + 2\ln x)/(4x^3)$; does not exist; -0.050
17. $f'''(x) = 168x + 36$; $f^{(4)}(x) = 168$ **19.** $f'''(x) = 300x^2 - 72x + 12$; $f^{(4)}(x) = 600x - 72$ **21.** $f'''(x) = 18(x + 2)^{-4}$ or $18/(x + 2)^4$; $f^{(4)}(x) = -72(x + 2)^{-5}$ or $-72/(x + 2)^5$ **23.** $f'''(x) = -36(x - 2)^{-4}$ or $-36/(x - 2)^4$; $f^{(4)}(x) = 144(x - 2)^{-5}$ or $144/(x - 2)^5$ **25. a.** $f'(x) = 1/x$; $f''(x) = -1/x^2$; $f'''(x) = 2/x^3$; $f^{(4)}(x) = -6/x^4$; $f^{(5)}(x) = 24/x^5$ **b.** $f^{(n)}(x) = (-1)^{n-1}[1 \cdot 2 \cdot 3 \cdots (n - 1)]/x^n$ or, using factorial notation, $f^{(n)}(x) = (-1)^{n-1}(n - 1)!/x^n$
27. Concave upward on $(2, \infty)$; concave downward on $(-\infty, 2)$; inflection point at $(2, 3)$ **29.** Concave upward on $(-\infty, -1)$ and $(8, \infty)$; concave downward on $(-1, 8)$; inflection points at $(-1, 7)$ and $(8, 6)$ **31.** Concave upward on $(2, \infty)$; concave downward on $(-\infty, 2)$; no inflection points **33.** Always concave upward; no inflection points **35.** Concave upward on $(-\infty, 3/2)$; concave downward on $(3/2, \infty)$; inflection point at $(3/2, 525/2)$ **37.** Concave upward on $(5, \infty)$; concave downward on $(-\infty, 5)$; no inflection points **39.** Concave upward on $(-10/3, \infty)$; concave downward on $(-\infty, -10/3)$; inflection point at $(-10/3, -250/27)$ **41.** Never concave upward; always concave downward; no inflection points
43. Concave upward on $(-\infty, 0)$ and $(1, \infty)$; concave downward on $(0, 1)$; inflection points at $(0, 0)$ and $(1, -3)$
45. Concave upward on $(-1, 1)$; concave downward on $(-\infty, -1)$ and $(1, \infty)$; inflection points at $(-1, \ln 2)$ and $(1, \ln 2)$
47. Concave upward on $(-\infty, -e^{-3/2})$ and $(e^{-3/2}, \infty)$; concave downward on $(-e^{-3/2}, 0)$ and $(0, e^{-3/2})$; inflection points at $(-e^{-3/2}, -3e^{-3}/(2\ln 10))$ and $(e^{-3/2}, -3e^{-3}/(2\ln 10))$ **49.** Concave upward on $(-\infty, 0)$ and $(4, \infty)$; concave downward on $(0, 4)$; inflection points at 0 and 4 **51.** Concave upward on $(-7, 3)$ and $(12, \infty)$; concave downward on $(-\infty, -7)$ and $(3, 12)$; inflection points at -7, 3, and 12 **53.** Choose $f(x) = x^k$ where $1 < k < 2$. For example, $f(x) = x^{4/3}$ has a relative minimum at $x = 0$, and $f(x) = x^{5/3}$ has an inflection point at $x = 0$. **55. a.** Close to 0 **b.** Close to 1 **57.** Relative maximum at -5.
59. Relative maximum at 0; relative minimum at $2/3$. **61.** Relative minimum at -3 **63.** Relative maximum at $-4/7$; relative minimum at 0 **65. a.** Minimum at about -0.4 and 4.0; maximum at about 2.4 **b.** Increasing on about $(-0.4, 2.4)$ and about $(4.0, \infty)$; decreasing on about $(-\infty, -0.4)$ and $(2.4, 4.0)$ **c.** About 0.7 and 3.3 **d.** Concave upward on about $(-\infty, 0.7)$ and $(3.3, \infty)$; concave downward on about $(0.7, 3.3)$ **67. a.** Maximum at 1; minimum at -1 **b.** Increasing on $(-1, 1)$; decreasing on $(-\infty, -1)$ and $(1, \infty)$ **c.** About -1.7, 0, and about 1.7 **d.** Concave upward on about $(-1.7, 0)$ and $(1.7, \infty)$; concave downward on about $(-\infty, -1.7)$ and $(0, 1.7)$ **71.** 45.6; mid 2045 **73.** (22, 6517.9) **75.** (2.06, 20.8) **79. a.** 4 hours **b.** 1160 million **81. a.** After 2 hours **b.** $3/4\%$ **83.** (38.92, 5000) **85.** Inflection point at $t = (\ln c)/k \approx 2.96$ years; this signifies the time when the rate of growth begins to slow down, since L changes from concave up to concave down at this inflection point.
87. Always concave down **89.** $f(t)$ is decreasing and concave up; $f'(t) < 0$, $f''(t) > 0$. **91. a.** -96 ft/sec **b.** -160 ft/sec **c.** -256 ft/sec **d.** -32 ft/sec^2 **93.** $v(t) = 256 - 32t$; $a(t) = -32$; 1024 ft; 16 seconds after being thrown **95.** $t = 6$

Exercises 13.4 (page 702)

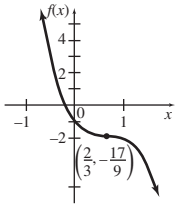
For exercises . . .	3–10,29–30	11,12	13–20	21–28
Refer to example . . .	1	2	3	4

1. 0 **3.**



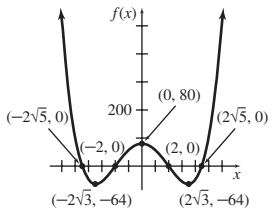
$f(x) = -2x^3 - 9x^2 + 108x - 10$

5.



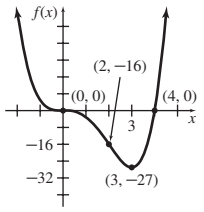
$f(x) = -3x^3 + 6x^2 - 4x - 1$

7.



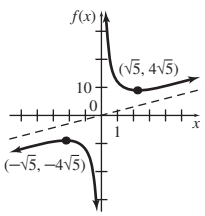
$f(x) = x^4 - 24x^2 + 80$

9.



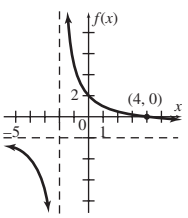
$f(x) = x^4 - 4x^3$

11.



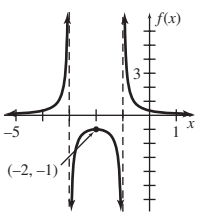
$f(x) = 2x + \frac{10}{x}$

13.



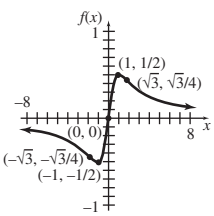
$f(x) = \frac{-x + 4}{x + 2}$

15.



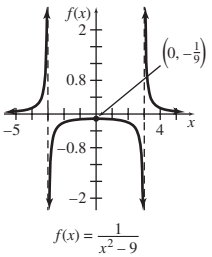
$f(x) = \frac{1}{x^2 + 4x + 3}$

17.



$f(x) = \frac{x}{x^2 + 1}$

19.



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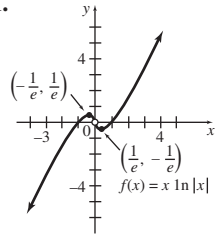

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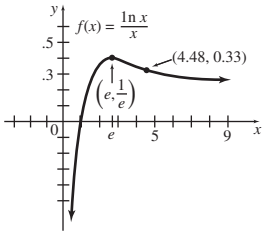

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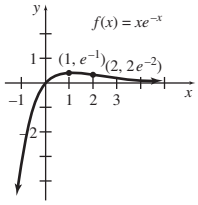

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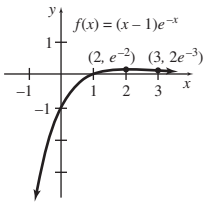
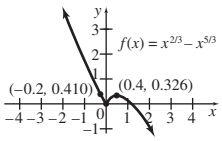
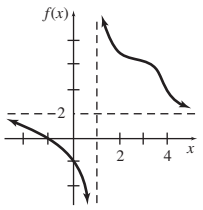


29.

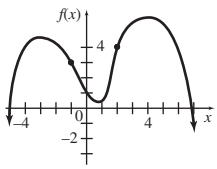


31. 3, 7, 9, 11, 15 **33.** 17, 19, 23, 25, 27

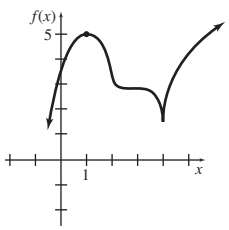
In Exercises 35–39, other answers are possible. **35.**



37.



39.



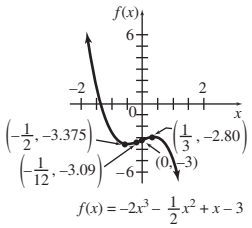
Chapter 13 Review Exercises (page 705)

1. True **2.** False **3.** False **4.** True **5.** False
6. True **7.** False **8.** False **9.** False
10. False **11.** True **12.** False

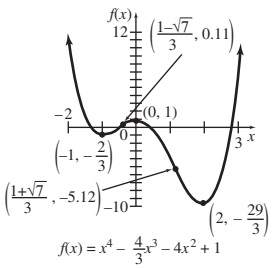
For exercises . . .	1,2,12,13, 17–24,70	3,4,14,15, 25–32,63,64, 72,73a,74b,c	5–9,16,33–38, 61,62,65,66,71, 73b,74a	10,11, 39–60,63e, 67–69
Refer to section . . .	1	2	3	4

17. Increasing on $(-9/2, \infty)$; decreasing on $(-\infty, -9/2)$ **19.** Increasing on $(-5/3, 3)$; decreasing on $(-\infty, -5/3)$ and $(3, \infty)$
21. Never decreasing; increasing on $(-\infty, 3)$ and $(3, \infty)$ **23.** Decreasing on $(-\infty, -1)$ and $(0, 1)$; increasing on $(-1, 0)$ and $(1, \infty)$ **25.** Relative maximum of -4 at 2 **27.** Relative minimum of -7 at 2 **29.** Relative maximum of 101 at -3; relative minimum of -24 at 2 **31.** Relative maximum at $(-0.618, 0.206)$; relative minimum at $(1.618, 13.203)$
33. $f''(x) = 36x^2 - 10$; 26; 314 **35.** $f''(x) = 180(3x - 6)^{-3}$ or $180/(3x - 6)^3$; $-20/3$; $-4/75$
37. $f''(t) = (t^2 + 1)^{-3/2}$ or $1/(t^2 + 1)^{3/2}$; $1/2^{3/2} \approx 0.354$; $1/10^{3/2} \approx 0.032$

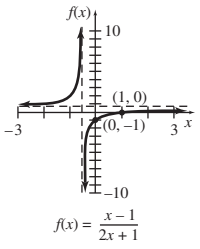
39.



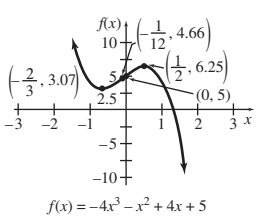
41.



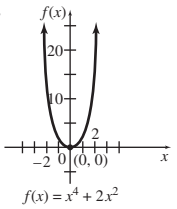
43.



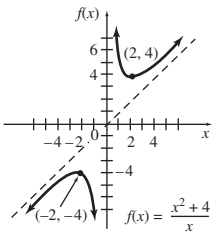
45.



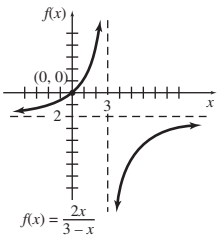
47.



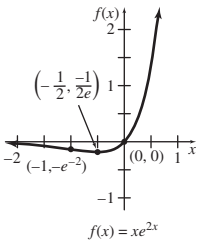
49.



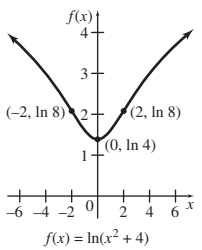
51.



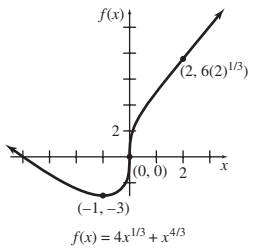
53.



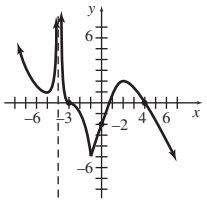
55.



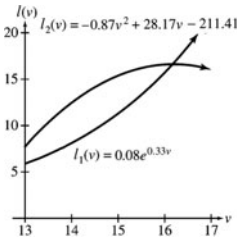
57.



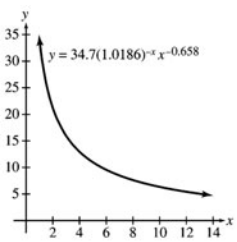
In Exercise 59, other answers are possible. **59.**



61. a. Both are negative. **63. a.** $P(q) = -q^3 + 7q^2 + 49q$ **b.** 7 brushes **c.** \$229 **d.** \$343 **e.** $q = 7/3$; between 2 and 3 brushes **65. a.** The first derivative has many critical numbers **b.** The curve is always decreasing except at frequent inflection points. **67.**



69.



71. 7.405 yr; the age at which the rate of learning to pass the test begins to slow down **73. a.** At 1965, 1973, 1976, 1983, 1986, and 1988 **b.** Concave upward; this means that the stockpile was increasing at an increasingly rapid rate.

Chapter 14 Applications of the Derivative

Exercises 14.1 (page 717)

For exercises . . .	1–6,31–38	7,8,11–30,39	41–46,51–60	47–50
Refer to example . . .	2	1	3	On Graphical Optimization

1. Absolute maximum at x_3; no absolute minimum **3.** No absolute extrema **5.** Absolute minimum at x_1; no absolute maximum **7.** Absolute maximum at x_1; absolute minimum at x_2 **11.** Absolute maximum of 12 at $x = 5$; absolute minimum of -8 at $x = 0$ and $x = 3$ **13.** Absolute maximum of 19.67 at $x = -4$; absolute minimum of -1.17 at $x = 1$ **15.** Absolute maximum of 1 at $x = 0$; absolute minimum of -80 at $x = -3$ and $x = 3$ **17.** Absolute maximum of 1/3 at $x = 0$; absolute minimum of $-1/3$ at $x = 3$ **19.** Absolute maximum of 0.21 at $x = 1 + \sqrt{2} \approx 2.4$; absolute minimum of 0 at $x = 1$ **21.** Absolute maximum of 1.710 at $x = 3$; absolute minimum of -1.587 at $x = 0$ **23.** Absolute maximum of 7 at $x = 1$; absolute minimum of 0 at $x = 0$ **25.** Absolute maximum of 4.910 at $x = 4$; absolute minimum of -1.545 at $x = 2$ **27.** Absolute maximum of 19.09 at $x = -1$; absolute minimum of 0.6995 at $x = (\ln 3)/3$ **29.** Absolute maximum of 1.356 at $x = 0.6085$; absolute minimum of 0.5 at $x = -1$ **31.** Absolute minimum of 7 at $x = 2$; no absolute maximum **33.** Absolute maximum of 137 at $x = 3$; no absolute minimum **35.** Absolute maximum of 0.1 at $x = 4$; absolute minimum of -0.5 at $x = -2$ **37.** Absolute maximum of 0.1226 at $x = e^{1/3}$; no absolute minimum **39. a.** Absolute minimum of -5 at $x = -1$; absolute maximum of 0 at $x = 0$ **b.** Absolute maximum of about -0.76 at $x = 2$; absolute minimum of -1 at $x = 1$ **41. a.** Relative maxima of 8496 in 2001, 7556 in 2004, 6985 in 2006, and 6700 in 2008; relative minima of 7127 in 2000, 7465 in 2003, 6748 in 2005, 5933 in 2007, and 5943 in 2009. **b.** Absolute maximum of 8496 in 2001 and absolute minimum of 5933 in 2007. **43.** The maximum profit is \$700,000 when 1,000,000 tires are sold. **45. a.** 112 **b.** 162 **47.** About 11.5 units **49.** 100 units **51.** 6 mo; 6% **53.** About 7.2 mm **55.** Maximum of 25 mpg at 45 mph; minimum of 16.1 mpg at 65 mph **57.** The piece formed into a circle should have length $12\pi/(4 + \pi)$ ft, or about 5.28 ft.

Exercises 14.2 (page 726)

For exercises . . .	1–8,15–18, 33,34,37,44	9–14, 19–24,47	25–30	31,32,41, 42,45,46	35,36, 38–40,43
Refer to example . . .	1	3	4	2	5

1. a. $y = 180 - x$ **b.** $P = x(180 - x)$ **c.** $[0, 180]$ **d.** $dP/dx = 180 - 2x$; $x = 90$ **e.** $P(0) = 0$; $P(180) = 0$; $P(90) = 8100$ **f.** 8100; 90 and 90 **3. a.** $y = 90 - x$ **b.** $P = x^2(90 - x)$ **c.** $[0, 90]$ **d.** $dP/dx = 180x - 3x^2$; $x = 0$, $x = 60$ **e.** $P(0) = 0$, $P(60) = 108{,}000$, $P(90) = 0$ **f.** 108,000; 60 and 30 **5.** $\overline{C}(x) = x^2/2 + 2x - 3 + 35/x$; $x = 2.722$ **7. a.** $R(x) = 160{,}000x - 100x^2$ **b.** 800 **c.** \$640,000 **9. a.** $1400 - 2x$ **b.** $A(x) = 1400x - 2x^2$ **c.** 350 m **d.** 245,000 m^2 **11.** 405,000 m^2 **13.** \$960 **15. a.** 125 passengers **b.** \$156,250 **17.** In 10 days; \$960 **19.** 4 in. by 4 in. by 2 in. **21.** 2/3 ft (or 8 in.) **23.** 20 cm by 20 cm by 40 cm; \$7200 **27.** Radius = 5.206 cm, height = 11.75 cm **29.** Radius = 5.242 cm; height = 11.58 cm **31.** 1 mile from point A **33. a.** 15 days **b.** 16.875% **35.** 12.98 thousand **37. a.** 12 days **b.** 50 per ml **c.** 1 day **d.** 81.365 per ml

39. 49.37 **41.** Point P is $3\sqrt{7}/7 \approx 1.134$ mi from Point A **43. a.** Replace a with e^r and b with r/P.
b. Shepherd: $f'(S) = a[1 + (1 - c)(S/b)^c]/[1 + (S/b)^c]^2$; Ricker: $f'(S) = ae^{-bS}(1 - bS)$; Beverton-Holt: $f'(S) = a/[1 + (S/b)]^2$ **c.** Shepherd: a; Ricker: a; Beverton-Holt: a; the constant a represents the slope of the graph of $f(S)$ at $S = 0$. **d.** 194,000 tons **e.** 256,000 tons **45.** $(56 - 2\sqrt{21})/7 \approx 6.7$ mi **47.** 36 in. by 18 in. by 18 in.

Exercises 14.3 (page 737)

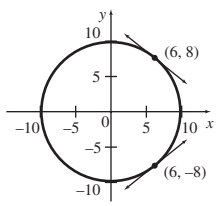
For exercises . . .	1,2,9–12, 15–18	3,13,14	4,27,32	5,7,8,25,26, 28–30	19–24,31
Refer to example . . .	1	2	3	4	5

3. c **5.** It is negative. **9.** 10,000 **11.** 10
13. 4899 **19. a.** $E = p/(200 - p)$ **b.** 25
21. a. $E = 2p^2/(7500 - p^2)$ **b.** 25,000 **23. a.** $E = 5/q$ **b.** 5 **25. a.** $E = 0.5$; inelastic; total revenue increases as price increases. **b.** $E = 8$; elastic; total revenue decreases as price increases. **27.** 0.06; the demand is inelastic. **29.** 2.826; the demand is elastic. **31. a.** 0.071 **b.** Inelastic **c.** \$1255

Exercises 14.4 (page 742)

For exercises . . .	1–16,38–40	17–37	41–50
Refer to example . . .	1,2	3	4

1. $dy/dx = -6x/(5y)$ **3.** $dy/dx = (8x - 5y)/(5x - 3y)$
5. $dy/dx = 15x^2/(6y + 4)$ **7.** $dy/dx = -3x(2 + y)^2/2$
9. $dy/dx = \sqrt{y}/[\sqrt{x}(5\sqrt{y} - 2)]$ **11.** $dy/dx = (4x^3y^3 + 6x^{1/2})/(9y^{1/2} - 3x^4y^2)$ **13.** $dy/dx = (5 - 2xye^{x^2y})/(x^2e^{x^2y} - 4)$
15. $dy/dx = y(2xy^3 - 1)/(1 - 3x^2y^3)$ **17.** $y = (3/4)x + 25/4$ **19.** $y = x + 2$ **21.** $y = x/64 + 7/4$
23. $y = (11/12)x - 5/6$ **25.** $y = -(37/11)x + 59/11$ **27.** $y = (5/2)x - 1/2$ **29.** $y = 1$ **31.** $y = -2x + 7$
33. $y = -x + 2$ **35.** $y = (8/9)x + 10/9$ **37. a.** $y = -(3/4)x + 25/2$; $y = (3/4)x - 25/2$
b.



39. a. $du/dv = -2u^{1/2}/(2v + 1)^{1/2}$ **b.** $dv/du = -(2v + 1)^{1/2}/(2u^{1/2})$ **c.** They are reciprocals.
41. $dy/dx = -x/y$; there is no function $y = f(x)$ that satisfies $x^2 + y^2 + 1 = 0$.
43. a. \$0.94; the approximate increase in cost of an additional unit
b. \$0; the approximate change in revenue for a unit increase in sales **45 a.** 0.44; inelastic
b. 0.44 **47.** $1/(3\sqrt{3})$ **49.** $ds/dt = (4s - 6t^2 + 5)/(3s^2 - 4t)$

Exercises 14.5 (page 749)

For exercises . . .	1–8, 16–22	9–14	15	23,24, 29,31,32	25	26–28,30
Refer to example . . .	1	5	6	3	2	4

1. -64 **3.** $-9/7$ **5.** $1/5$ **7.** $-3/2$ **9.** \$384 per month
11. a. Revenue is increasing at a rate of \$180 per day.
b. Cost is increasing at a rate of \$50 per day. **c.** Profit is increasing at a rate of \$130 per day. **13.** Demand is decreasing at a rate of approximately 98 units per unit time.
15. 0.067 mm per min **17.** About 1.9849 g per day **19. a.** $105.15\ \text{m}^{-0.25}\ dm/dt$ **b.** About 52.89 kcal per day^2
21. 25.6 crimes per month **23.** 24/5 ft/min **25.** 16π ft^2/min **27.** 2/27 cm/min **29.** 62.5 ft per min **31.** $\sqrt{2} \approx 1.41$ ft per sec

Exercises 14.6 (page 756)

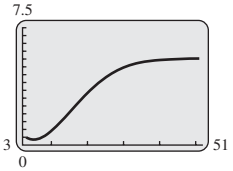
For exercises . . .	1–8	9–16	17–20	21–35	36–41
Refer to example . . .	1	2	3	4	5

1. 1.9 **3.** 0.1 **5.** 0.060 **7.** -0.023 **9.** 12.0417; 12.0416; 0.0001
11. 0.995; 0.9950; 0 **13.** 1.01; 1.0101; 0.0001 **15.** 0.05; 0.0488; 0.0012 **17. a.** -4.4 thousand lb **b.** -52.2 thousand lb
19. \$60 **21.** About 9600 in^3 **23. a.** 0.007435 **b.** -0.005105 **25. a.** 0.347 million **b.** -0.022 million **27.** 1568π mm^3
29. 80π mm^2 **31. a.** About 9.3 kg **b.** About 9.5 kg **33.** -7.2π cm^3 **35.** 0.472 cm^3 **37.** 0.00125 cm **39.** ± 1.273 in^3
41. ± 0.116 in^3

Chapter 14 Review Exercises (page 758)

For exercises . . .	1–3, 11–18	4,5, 49–54	6, 19–30, 43,44,56	7,8, 31–38, 55,59–61	9,10, 39–42, 62,63	45–48, 57,58, 64–68
Refer to section . . .	1	3	4	5	6	2

1. False **2.** True **3.** False **4.** True **5.** True
6. True **7.** True **8.** True **9.** True **10.** True
11. Absolute maximum of 33 at 4; absolute minimum of 1 at 0 and 6 **13.** Absolute maximum of 39 at -3; absolute minimum of $-319/27$ at $5/3$ **17. a.** Maximum = 0.37; minimum = 0
b. Maximum = 0.35; minimum = 0.13 **21.** $dy/dx = (2x - 9x^2y^4)/(8y + 12x^3y^3)$ **23.** $dy/dx = 6\sqrt{y - 1}/[x^{1/3}(1 - \sqrt{y - 1})]$
25. $dy/dx = -(30 + 50x)/3$ **27.** $dy/dx = (2xy^4 + 2y^3 - y)/(x - 6x^2y^3 - 6xy^2)$ **29.** $y = (-16/23)x + 94/23$ **33.** 272
35. -2 **37.** $-8e^3$ **41.** 0.00204 **43. a.** $(2, -5)$ and $(2, 4)$ **b.** $(2, -5)$ is a relative minimum; $(2, 4)$ is a relative maximum.
c. No **45. a.** 600 boxes **b.** \$720 **47.** 3 in. **49.** 1789 **51.** 80 **53.** 0.47; inelastic **55.** 56π ft^2 per min
57. a.



b. About the 15th day **59.** 8/3 ft per min **61.** 21/16 = 1.3125 ft per min **63.** $\pm$ 0.736 in^2
65. $1.25 + 2 \ln 1.5$ **67.** 10 ft; 18.67 sec

Chapter 15 Integration

Exercises 15.1 (page 774)

For exercises . . .	5–24, 27,28,	25,26, 29–32, 35,36, 41,42, 60,63, 66	33,34, 37–40, 61	43,44	45–52, 57–59, 65	53–56	67–74
Refer to example . . .	4,5	7,8	6	12	9	10	11

1. They differ only by a constant.
5. $6k + C$ **7.** $z^2 + 3z + C$
9. $2t^3 - 4t^2 + 7t + C$
11. $z^4 + z^3 + z^2 - 6z + C$
13. $10z^{3/2}/3 + \sqrt{2}z + C$
15. $5x^4/4 - 20x^2 + C$
17. $8v^{3/2}/3 - 6v^{5/2}/5 + C$
19. $4u^{5/2} - 4u^{7/2} + C$ **21.** $-7/z + C$ **23.** $-\pi^3/(2y^2) - 2\sqrt{\pi y} + C$ **25.** $6t^{-1.5} - 2\ln|t| + C$ **27.** $-1/(3x) + C$
29. $-15e^{-0.2x} + C$ **31.** $-3\ln|x| - 10e^{-0.4x} + e^{0.1}x + C$ **33.** $(1/4)\ln|t| + t^3/6 + C$ **35.** $e^{2u}/2 + 2u^2 + C$
37. $x^3/3 + x^2 + x + C$ **39.** $6x^{7/6}/7 + 3x^{2/3}/2 + C$ **41.** $10^x/(\ln 10) + C$ **43.** $f(x) = 3x^{5/3}/5$
45. $C(x) = 2x^2 - 5x + 8$ **47.** $C(x) = 3e^{0.01x} + 5$ **49.** $C(x) = 3x^{5/3}/5 + 2x + 114/5$
51. $C(x) = 5x^2/2 - \ln|x| - 153.50$ **53.** $p = 175 - 0.01x - 0.01x^2$ **55.** $p = 500 - 0.1\sqrt{x}$
57. a. $f(t) = 3.75t^2 - 16.8t + 0.05$ **b.** Approximately 152.6 billion monthly text messages **59. a.** $P(x) = 25x^4/2 + 10x^3 - 40$
b. \$240 **61.** $a\ln x - bx + C$ **63. a.** $N(t) = 155.3e^{0.3219t} + 144.7$ **b.** 7537
65. a. $B(t) = 0.02016t^3 - 0.6460t^2 + 15.86t + 839.7$ **b.** About 2,082,000 **67.** $v(t) = 5t^3/3 + 4t + 6$
69. $s(t) = -16t^2 + 6400$; 20 sec **71.** $s(t) = 2t^{5/2} + 3e^{-t} + 1$ **73.** 160 ft/sec, 12 ft

Exercises 15.2 (page 782)

For exercises . . .	3,4,29, 33,34	5–8, 21,22	9,10,27, 28,30	11–16, 31,35, 36	17–20, 32	23–26	39–44
Refer to example . . .	1	3	2	5	4	6	7

3. $2(2x + 3)^5/5 + C$
5. $-(2m + 1)^{-2}/2 + C$
7. $-(x^2 + 2x - 4)^{-3}/3 + C$
9. $(4z^2 - 5)^{3/2}/12 + C$ **11.** $e^{2x^3}/2 + C$ **13.** $e^{2t - t^2}/2 + C$ **15.** $-e^{1/z} + C$ **17.** $[\ln(t^2 + 2)]/2 + C$
19. $[\ln(x^4 + 4x^2 + 7)]/4 + C$ **21.** $-1/[2(x^2 + x)^2] + C$ **23.** $(p + 1)^7/7 - (p + 1)^6/6 + C$
25. $2(u - 1)^{3/2}/3 + 2(u - 1)^{1/2} + C$ **27.** $(x^2 + 12x)^{3/2}/3 + C$ **29.** $(1 + 3\ln x)^3/9 + C$ **31.** $(1/2)\ln(e^{2x} + 5) + C$
33. $(\ln 10)(\log x)^2/2 + C$ **35.** $8^{3x^2 + 1}/(6\ln 8) + C$ **39. a.** $R(x) = 6(x^2 + 27{,}000)^{1/3} - 180$ **b.** 150 players
41. a. $C(x) = 6\ln(5x^2 + e) + 4$ **b.** Yes **43. a.** $f(t) = 4.0674 \times 10^{-4}[(t - 1970)^{2.4}/2.4 + 1970(t - 1970)^{1.4}/1.4] + 61.298$ **b.** About 181,000

Exercises 15.3 (page 791)

For exercises . . .	5–14,17–22	15,16,24–30, 33–35,39	31,32,36–38
Refer to example . . .	1,2	3	4

3. a. 88 **b.** $\int_0^8 (2x + 5)\,dx$
5. a. 21 **b.** 23 **c.** 22 **d.** 22 **7. a.** 10 **b.** 10 **c.** 10 **d.** 11 **9. a.** 8.22 **b.** 15.48 **c.** 11.85 **d.** 10.96 **11. a.** 6.70 **b.** 3.15 **c.** 4.93 **d.** 4.17 **13. a.** 4 **b.** 4 **15. a.** 4 **b.** 5 **17.** 4π **19.** 24 **21. b.** 0.385 **c.** 0.33835 **d.** 0.334334 **e.** 0.333333 **25.** Left: 1410 trillion BTUs; right: 3399 trillion BTUs; average: 2404.5 trillion BTUs **27. a.** Left: about 582,000 cases; right: about 580,000 cases; average: about 581,000 cases **b.** Left: about 146,000 cases; right: about 144,000 cases; average: about 145,000 cases **29.** About 1300 ft; yes **31.** 2751 ft, 3153 ft, 2952 ft **33. a.** About 660 BTU/ft^2 **b.** About 320 BTU/ft^2 **35. a.** 9 ft **b.** 2 sec **c.** 4.6 ft **d.** Between 3 and 3.5 sec **37.** 22.5 and 18 ft **39. a.** About 75,600 **b.** About 77,300

Exercises 15.4 (page 803)

For exercises . . .	1–6,9–12, 15–20, 23,24	7,8,13,14, 21,22, 27–30,54	31–44,47	55–61, 63–72
Refer to example . . .	1,2,3	4	5, 6	7

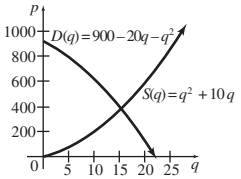
1. -18 **3.** $-3/2$ **5.** $28/3$ **7.** 13
9. $-16/3$ **11.** 76 **13.** $4/5$ **15.** $108/25$
17. $20e^{0.3} - 20e^{0.2} + 3\ln 2 - 3\ln 3 \approx 1.353$
19. $e^8/4 - e^4/4 - 1/6 \approx 731.4$ **21.** $91/3$
23. $447/7 \approx 63.86$ **25.** $(\ln 2)^2/2 \approx 0.2402$ **27.** 49 **29.** $1/8 - 1/[2(3 + e^2)] \approx 0.07687$ **31.** 10 **33.** 76 **35.** $41/2$
37. $e^2 - 3 + 1/e \approx 4.757$ **39.** $e - 2 + 1/e \approx 1.086$ **41.** $23/3$ **43.** $e^2 - 2e + 1 \approx 2.952$
45. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ **47.** -8 **51.** -12 **53. a.** $x^5/5 - 1/5$ **c.** $f'(1) \approx 2.746$, and $g(1) = e \approx 2.718$
55. a. $(9000/8)(17^{4/3} - 2^{4/3}) \approx \$46{,}341$ **b.** $(9000/8)(26^{4/3} - 17^{4/3}) \approx \$37{,}477$ **c.** It is slowly increasing without bound.
57. No **59. a.** 0.8778 ft **b.** 0.6972 ft **61. a.** 18.12 **b.** 8.847 **63. b.** $\int_0^{60} n(x)\,dx$ **c.** $2(51^{3/2} - 26^{3/2})/15 \approx 30.89$ million

65. a. $Q(R) = \pi k R^4/2$ **b.** $0.04k$ mm per min **67. b.** About 505,000 kJ/W$^{0.67}$ **69. a.** About 286 million; the total population aged 0 to 90 **b.** About 64 million **71. a.** $c'(t) = 1.2e^{0.04t}$ **b.** $\int_0^{10} 1.2e^{0.04t}\,dt$ **c.** $30e^{0.4} - 30 \approx 14.75$ billion **d.** About 12.8 yr **e.** About 14.4 yr

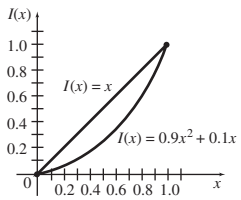
Exercises 15.5 (page 813)

For exercises . . .	1,2	3–6, 11–18,20, 23,24	7–10, 19,21,22, 25,26,41,42	27–30, 39,40	31–37
Refer to example . . .	1	3	2	4	5

1. 21 **3.** 20 **5.** 23/3 **7.** 366.2
9. 4/3 **11.** $2\ln 2 - \ln 6 + 3/2 \approx 1.095$
13. $6\ln(3/2) - 6 + 2e^{-1} + 2e \approx 2.605$
15. $(e^{-2} + e^4)/2 - 2 \approx 25.37$ **17.** 1/2 **19.** 1/20 **21.** $3(2^{4/3})/2 - 3(2^{7/3})/7 \approx 1.620$ **23.** $(e^9 + e^6 + 1)/3 \approx 2836$
25. −1.9241, −0.4164, 0.6650 **27. a.** 8 yr **b.** About \$148 **c.** About \$771 **29. a.** 39 days **b.** \$3369.18 **c.** \$484.02 **d.** \$2885.16 **31.** 12,931.66 **33.** 54
35. a.



b. (15, 375) **c.** \$4500 **d.** \$3375 **37. a.** 12 **b.** \$5616, \$1116 **c.** \$1872, \$1503 **d.** \$387 **39. a.** About 71.25 gal **b.** About 25 hr **c.** About 105 gal **d.** About 47.91 hr
41. a. 0.019; the lower 10% of the income producers earn 1.9% of the total income of the population. **b.** 0.184; the lower 40% of the income producers earn 18.4% of the total income of the population. **c.**

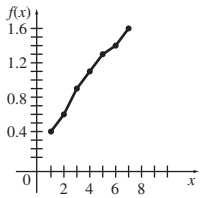


d. 0.15 **e.** Income is distributed less equally in 2008 than in 1968.

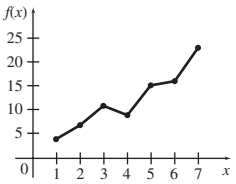
Exercises 15.6 (page 821)

For exercises . . .	1a–12a,15, 23a–24a,29	1b–12b,17,19,20, 23b–24b,29,30, 33–35	21,22,25–28, 31,32
Refer to example . . .	1	2	3

1. a. 12.25 **b.** 12 **c.** 12 **3. a.** 3.35 **b.** 3.3
c. $3\ln 3 \approx 3.296$ **5. a.** 11.34 **b.** 10.5 **c.** 10.5
7. a. 0.9436 **b.** 0.8374 **c.** $4/5 = 0.8$ **9. a.** 1.236
b. 1.265 **c.** $2 - 2e^{-1} \approx 1.264$ **11. a.** 5.991 **b.** 6.167
c. 6.283; Simpson's rule **13.** b is true. **15. a.** 0.2 **b.** 0.220703, 0.205200, 0.201302, 0.200325, 0.020703, 0.005200, 0.001302, 0.000325 **c.** $p = 2$ **17. a.** 0.2 **b.** 0.2005208, 0.2000326, 0.2000020, 0.2000001, 0.0005208, 0.0000326, 0.0000020, 0.0000001
c. $p = 4$ **19.** $M = 0.7355$; $S = 0.8048$
21. a.



b. 6.3 **c.** 6.27 **23. a.** 1.831 **b.** 1.758 **25.** About 30 mcg(h)/ml; this represents the total amount of drug available to the patient for each ml of blood. **27.** About 9 mcg(h)/ml; this represents the total effective amount of the drug available to the patient for each ml of blood.
29. a. $y = b_0(t/7)^{b_1}e^{-b_2 t/7}$ **b.** About 1212 kg; about 1231 kg **c.** About 1224 kg; about 1250 kg
31. a.



b. 71.5 **c.** 69.0 **33.** 3979 **35. a.** 0.6827 **b.** 0.9545 **c.** 0.9973

Chapter 15 Review Exercises (page 825)

For exercises . . .	1–4, 19–30, 79,80, 84	5,16,17, 31–40, 87,94	6–8,15, 41–43, 81,92,96	9,10, 45–62, 82,83, 89–91, 95	11,12, 63–66, 85,86	13,14, 18, 44, 67–77, 93
Refer to section . . .	1	2	3	4	5	6

1. True **2.** False **3.** False **4.** True
5. True **6.** False **7.** False **8.** True
9. True **10.** False **11.** True **12.** False
13. False **14.** True **19.** $x^2 + 3x + C$
21. $x^3/3 - 3x^2/2 + 2x + C$ **23.** $2x^{3/2} + C$ **25.** $2x^{3/2}/3 + 9x^{1/3} + C$ **27.** $2x^{-2} + C$ **29.** $-3e^{2x}/2 + C$ **31.** $e^{3x^2}/6 + C$
33. $(3\ln|x^2 - 1|)/2 + C$ **35.** $-(x^3 + 5)^{-3}/9 + C$ **37.** $-e^{-3x^4}/12 + C$ **39.** $(3\ln x + 2)^5/15 + C$ **41.** 20 **43.** 24
45. a. $s(T) - s(0)$ **b.** $\int_0^T v(t)\,dt = s(T) - s(0)$ is equivalent to the Fundamental Theorem with $a = 0$ and $b = T$ because $s(t)$ is an antiderivative of $v(t)$. **47.** 12 **49.** $3\ln 5 + 12/25 \approx 5.308$ **51.** 19/15 **53.** $3(1 - e^{-4})/2 \approx 1.473$ **55.** $\pi/32$
57. $25\pi/4$ **59.** 13/3 **61.** $(e^4 - 1)/2 \approx 26.80$ **63.** 64/3 **65.** 149/3 **67.** 0.5833; 0.6035 **69.** 4.187; 4.155 **71.** 0.6011
73. 4.156 **75. a.** 4/3 **b.** 1.146 **c.** 1.252 **77. a.** 0 **b.** 0 **79.** $C(x) = (2x - 1)^{3/2} + 145$ **81.** \$96,000 **83.** \$38,000
85. a. \$916.67 **b.** \$666.67 **87. a.** 17.718 billion barrels **b.** 17.718 billion barrels **d.** $y = -0.03545x + 2.1347$; 17.777 billion barrels **89.** 782 **91. a.** 0.2784 **b.** 0.2784 **93. a.** About 8208 kg **b.** About 8430 kg **c.** About 8558 kg
95. $s(t) = t^3/3 - t^2 + 8$.

Chapter 16 Further Techniques and Applications of Integration

Exercises 16.1 (page 840)

For exercises . . .	1–4,36,40,41,43,44	5,6,35,39,42	7–10,20–22	13–19	23–28
Refer to example . . .	1,3	2	4	1–3	5

1. $xe^x - e^x + C$ **3.** $(-x/2 + 23/16)\,e^{-8x} + C$ **5.** $(x^2 \ln x)/2 - x^2/4 + C$ **7.** $-5e^{-1} + 3 \approx 1.161$
9. $26 \ln 3 - 8 \approx 20.56$ **11.** $e^4 + e^2 \approx 61.99$ **13.** $x^2e^{2x}/2 - xe^{2x}/2 + e^{2x}/4 + C$
15. $(2/7)(x+4)^{7/2} - (16/5)(x+4)^{5/2} + (32/3)(x+4)^{3/2} + C$ or $(2/3)x^2(x+4)^{3/2} - (8/15)x(x+4)^{5/2} + (16/105)(x+4)^{7/2} + C$
17. $(4x^2 + 10x)\ln 5x - 2x^2 - 10x + C$ **19.** $(-e^2/4)(3e^2 + 1) \approx -42.80$ **21.** $2\sqrt{3} - 10/3 \approx 0.1308$
23. $16 \ln |x + \sqrt{x^2 + 16}| + C$ **25.** $-(3/11) \ln |(11 + \sqrt{121 - x^2})/x| + C$ **27.** $-1/(4x+6) - (1/6) \ln |x/(4x+6)| + C$
31. -18 **33.** 15 **37. a.** $(2/3)x(x+1)^{3/2} - (4/15)(x+1)^{5/2} + C$ **b.** $(2/5)(x+1)^{5/2} - (2/3)(x+1)^{3/2} + C$
39. $(169/2) \ln 13 - 42 \approx \$174.74$ **41.** $15e^6 + 3 \approx 6054$ **43.** About 219 kJ

Exercises 16.2 (page 847)

For exercises . . .	1–23,32,33,40,43	24–31,34–37,39,41,42	38
Refer to example . . .	1–3	4	Derivation of volume formula

1. 9π **3.** $364\pi/3$ **5.** $386\pi/27$ **7.** $15\pi/2$ **9.** 18π **11.** $\pi(e^4 - 1)/2 \approx 84.19$ **13.** $4\pi \ln 3 \approx 13.81$ **15.** $3124\pi/5$
17. $16\pi/15$ **19.** $4\pi/3$ **21.** $4\pi r^3/3$ **23.** $\pi r^2 h$ **25.** $13/3 \approx 4.333$ **27.** $38/15 \approx 2.533$ **29.** $e - 1 \approx 1.718$
31. $(5e^4 - 1)/8 \approx 34.00$ **33.** 3.758 **35.** \$42.49 **37.** 200 cases **39. a.** $110e^{-0.1} - 120e^{-0.2} \approx 1.284$
b. $210e^{-1.1} - 220e^{-1.2} \approx 3.640$ **c.** $330e^{-2.3} - 340e^{-2.4} \approx 2.241$ **41. a.** $9(6 \ln 6 - 5) \approx 51.76$ **b.** $5(10 \ln 10 - 9) \approx 70.13$
c. $3(31 \ln 31 - 30)/2 \approx 114.7$ **43.** $1.083 \times 10^{21}\ \text{m}^3$

Exercises 16.3 (page 855)

For exercises . . .	1(a)–8(a),16	1(b)–8(b),15,17,18	9–14,19,20
Refer to example . . .	2	3,4	5

1. a. \$6883.39 **b.** \$15,319.26 **3. a.** \$3441.69 **b.** \$7659.63 **5. a.** \$3147.75 **b.** \$7005.46 **7. a.** \$32,968.35
b. \$73,372.42 **9. a.** \$746.91 **b.** \$1662.27 **11. a.** \$688.64 **b.** \$1532.59 **13. a.** \$11,351.78 **b.** \$25,263.84
15. \$63,748.43 **17.** \$28,513.76, \$54,075.81 **19.** \$4560.94

Exercises 16.4 (page 860)

For exercises . . .	1–26,31–34,37	27–30,35,36	42–47,50	48,49,51,52
Refer to example . . .	1	2	4	3

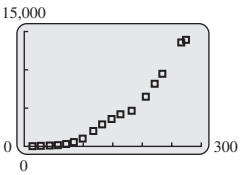
1. 1/3 **3.** Divergent **5.** -1 **7.** 10,000 **9.** 1/10
11. 3/5 **13.** 1 **15.** 1000 **17.** Divergent **19.** 1 **21.** Divergent
23. Divergent **25.** Divergent **27.** 0 **29.** Divergent **31.** Divergent **33.** 1 **35.** 0 **39. a.** 2.808, 3.724, 4.417, 6.720, 9.022
b. Divergent **c.** 0.8770, 0.9070, 0.9170, 0.9260, 0.9269 **d.** Convergent **41. a.** 9.9995, 49.9875, 99.9500, 995.0166
b. Divergent **c.** 100,000 **43.** \$20,000,000 **45.** \$30,000 **47.** \$30,000 **49.** $Na/[b(b+k)]$ **51.** About 833.3

Exercises 16.5 (page 871)

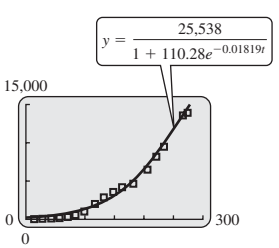
For exercises . . .	1,2	3,4,17–20	5–16,21–32, 38,39,53	35	36,37,41,42, 49–51,54	40,47,48,52	43–46,56–59
Refer to example . . .	1	3	4,5,6	2	5	7	6

1. $y = -2x^2 + 2x^3 + C$ **3.** $y = x^4/2 + C$ **5.** $y^2 = 2x^3/3 + C$ **7.** $y = ke^{x^2}$ **9.** $y = ke^{x^3 - x^2}$ **11.** $y = Cx$
13. $\ln(y^2 + 6) = x + C$ **15.** $y = -1/(e^{2x}/2 + C)$ **17.** $y = x^2 - x^3 + 5$ **19.** $y = -2xe^{-x} - 2e^{-x} + 44$ **21.** $y^2 = x^4/2 + 25$
23. $y = e^{x^2 + 3x}$ **25.** $y^2/2 - 3y = x^2 + x - 4$ **27.** $y = -3/(3 \ln |x| - 4)$ **29.** $y = (e^{x-1} - 3)/(e^{x-1} - 2)$ **35. a.** \$1011.75
b. \$1024.52 **c.** No **37.** About 13.9 yr **39.** $q = C/p^2$ **41.** d **43. a.** $I = 2.4 - 1.4e^{-0.088W}$ **b.** I approaches 2.4.
45. a. $dw/dt = k(C - 17.5w)$; the calorie intake per day is constant. **b.** lb/calorie **c.** $dw/dt = (C - 17.5w)/3500$
d. $w = C/17.5 - e^{-0.005M}e^{-0.005t}/17.5$ **e.** $w = C/17.5 + (w_0 - C/17.5)e^{-0.005t}$

47. a.

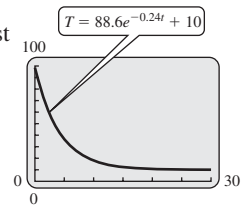


b. $y = \dfrac{25{,}538}{1 + 110.28e^{-0.01819t}}$ **c.**



d. 25,538

49. $y = 35.6e^{0.02117t}$ **51. a.** $k \approx 0.8$ **b.** 11 **c.** 55 **d.** About 3000 **53.** About 10 **55.** 7:22:55 A.M. **57.** The temperature approaches T_M, the temperature of the surrounding medium. **59. a.** $T = 88.6e^{-0.24t} + 10$ **b.**



c. Just after death—the graph shows that the most rapid decrease occurs in the first few hours. **d.** About 43.9°F **e.** About 4.5 hours

Chapter 16 Review Exercises (page 876)

For exercises . . .	1–4,18–29	5–7,70–39,91	8,9,48,65–75,81	10,40–47,76,82	11–14,49–64,71–80,83–90,92,93
Refer to section . . .	1	2	3	4	5

1. False **2.** True **3.** False **4.** True **5.** False **6.** False **7.** True **8.** True **9.** True **10.** False **11.** True **12.** False **13.** True **14.** True **19.** $6x(x-2)^{1/2} - 4(x-2)^{3/2} + C$ **21.** $-(x+2)e^{-3x} - (1/3)e^{-3x} + C$ **23.** $(x^2/2 - x)\ln|x| - x^2/4 + x + C$ **25.** $(1/8)\sqrt{16 + 8x^2} + C$ **27.** $10e^{1/2} - 16 \approx 0.4872$ **29.** $234/7 \approx 33.43$ **31.** $81\pi/2 \approx 127.2$ **33.** $\pi \ln 3 \approx 3.451$ **35.** $64\pi/5 \approx 40.21$ **39.** 2,391,484/3 **41.** 1/5 **43.** $6/e \approx 2.207$ **45.** Divergent **47.** 3 **51.** $y = x^3 + 3x^2 + C$ **53.** $y = 2e^{2x} + C$ **55.** $y^2 = 3x^2 + 2x + C$ **57.** $y = (Cx^2 - 1)/2$ **59.** $y = x^3/3 - 3x^2 + 3$ **61.** $y = -\ln[5 - (x+2)^4/4]$ **63.** $y^2 + 6y = 2x - 2x^2 + 352$ **65.** $16{,}250/3 \approx \$5416.67$ **67.** \$174,701.45 **69.** \$15.58 **71.** \$5354.97 **73.** \$30,035.17 **75.** \$176,919.15 **77. a.** \$10,099 **b.** \$71,196 **79. a.** $dA/dt = 0.05A - 20{,}000$ **b.** \$235,127.87 **81.** 0.4798 **83. a.** About 40 **b.** About 1.44×10^{10} hours **87. a.** $N = 329$, $b = 7.23$; $k = 0.247$ **b.** $y \approx 268$ million, which is less than the table value of 308.7 million. **c.** About 289 million for 2030, about 303 million for 2050 **89. a.** $x = 1/k + Ce^{-kt}$ **b.** $1/k$ **91. a.** 158.3° **b.** 125° **c.** 133.3° **93.** 93.3 hr

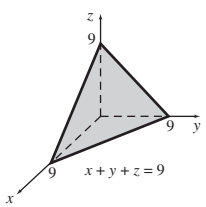
Chapter 17 Multivariable Calculus

Exercises 17.1 (page 891)

For exercises . . .	1–4,27,28,32, 33,38–45,48	5–12	13–16	21–26	23–28	31,34–36
Refer to example . . .	1–3	4–6	7,8	material after Example 8	5	8

1. a. 12 **b.** −6 **c.** 10 **d.** −19 **3. a.** $\sqrt{43}$ **b.** 6 **c.** $\sqrt{19}$ **d.** $\sqrt{11}$

5.



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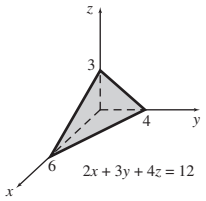

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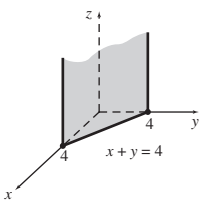
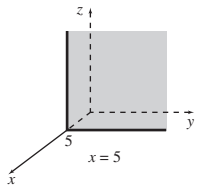
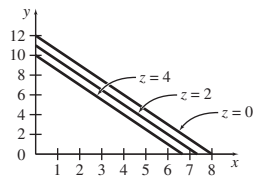


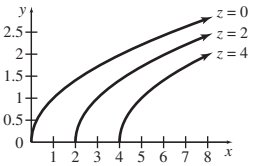
11.



13.

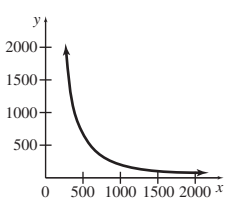


15.



21. c **23.** e **25.** b **27. a.** $8x + 4h$ **b.** $-4y - 2h$ **c.** $8x$ **d.** $-4y$ **29. a.** $3e^2$; slope of tangent line in the direction of x at $(1, 1)$ **b.** $3e^2$; slope of tangent line in the direction of y at $(1, 1)$ **31. a.** 1987 (rounded) **b.** 595 (rounded) **c.** 359,768 (rounded) **33.** 1.416; the IRA account grows faster.

35. $y = 500^{5/2}/x^{3/2} \approx 5{,}590{,}170/x^{3/2}$



37. $C(x, y, z) = 250x + 150y + 75z$ **39. a.** 1.89 m^2 **b.** 1.62 m^2 **c.** 1.78 m^2 **41. a.** 8.7% **b.** 48% **c.** Multiple solutions: $W = 19.75$, $R = 0$, $A = 0$ or $W = 10$, $R = 10$, $A = 4.59$ **d.** Wetland percentage **43. a.** 397 accidents **45. a.** $T = 242.257\,C^{0.18}/F^3$ **b.** 58.82; a tethered sow spends nearly 59% of the time doing repetitive behavior when she is fed 2 kg of food a day and neighboring sows spend 40% of the time doing repetitive behavior. **47.** $g(L, W, H) = 2LW + 2WH + 2LH$ ft^2

Exercises 17.2 (page 902)

For exercises . . .	1,2,33–36,43,44	3–20	21–32	37–42	45–47,53–67	48–52
Refer to example . . .	3	1–3	6,7	8	4	5

1. a. $12x - 4y$ **b.** $-4x + 18y$ **c.** 12 **d.** -40 **3.** $f_x(x, y) = -4y$; $f_y(x, y) = -4x + 18y^2$; 4; 178 **5.** $f_x(x, y) = 10xy^3$; $f_y(x, y) = 15x^2y^2$; -20; 2160 **7.** $f_x(x, y) = e^{x+y}$; $f_y(x, y) = e^{x+y}$; e^1 or e; e^{-1} or $1/e$ **9.** $f_x(x, y) = -24e^{4x-3y}$; $f_y(x, y) = 18e^{4x-3y}$; $-24e^{11}$; $18e^{-25}$ **11.** $f_x(x, y) = (-x^4 - 2xy^2 - 3x^2y^3)/(x^3 - y^2)^2$; $f_y(x, y) = (3x^3y^2 - y^4 + 2x^2y)/(x^3 - y^2)^2$; $-8/49$; $-1713/5329$ **13.** $f_x(x, y) = 15x^2y^2/(1 + 5x^3y^2)$; $f_y(x, y) = 10x^3y/(1 + 5x^3y^2)$; 60/41; 1920/2879 **15.** $f_x(x, y) = e^{x^2y}(2x^2y + 1)$; $f_y(x, y) = x^3e^{x^2y}$; $-7e^{-4}$; $-64e^{48}$ **17.** $f_x(x, y) = (1/2)(4x^3 + 3y)/(x^4 + 3xy + y^4 + 10)^{1/2}$; $f_y(x, y) = (1/2)(3x + 4y^3)/(x^4 + 3xy + y^4 + 10)^{1/2}$; $29/(2\sqrt{21})$; $48/\sqrt{311}$ **19.** $f_x(x, y) = [6xy(e^{xy} + 2) - 3x^2y^2e^{xy}]/(e^{xy} + 2)^2$; $f_y(x, y) = [3x^2(e^{xy} + 2) - 3x^3ye^{xy}]/(e^{xy} + 2)^2$; $-24(e^{-2} + 1)/(e^{-2} + 2)^2$; $(624e^{-12} + 96)/(e^{-12} + 2)^2$ **21.** $f_{xx}(x, y) = 8y^2 - 32$; $f_{yy}(x, y) = 8x^2$; $f_{xy}(x, y) = f_{yx}(x, y) = 16xy$ **23.** $R_{xx}(x, y) = 8 + 24y^2$; $R_{yy}(x, y) = -30xy + 24x^2$; $R_{xy}(x, y) = R_{yx}(x, y) = -15y^2 + 48xy$ **25.** $r_{xx}(x, y) = 12y/(x + y)^3$; $r_{yy}(x, y) = -12x/(x + y)^3$; $r_{xy}(x, y) = r_{yx}(x, y) = (6y - 6x)/(x + y)^3$ **27.** $z_{xx} = 9ye^x$; $z_{yy} = 0$; $z_{xy} = z_{yx} = 9e^x$ **29.** $r_{xx} = -1/(x + y)^2$; $r_{yy} = -1/(x + y)^2$; $r_{xy} = r_{yx} = -1/(x + y)^2$ **31.** $z_{xx} = 1/x$; $z_{yy} = -x/y^2$; $z_{xy} = z_{yx} = 1/y$ **33.** $x = -4$, $y = 2$ **35.** $x = 0$, $y = 0$; or $x = 3$, $y = 3$ **37.** $f_x(x, y, z) = 4x^3$; $f_y(x, y, z) = 2z^2$; $f_z(x, y, z) = 4yz + 4z^3$; $f_{yz}(x, y, z) = 4z$ **39.** $f_x(x, y, z) = 6/(4z + 5)$; $f_y(x, y, z) = -5/(4z + 5)$; $f_z(x, y, z) = -4(6x - 5y)/(4z + 5)^2$; $f_{yz}(x, y, z) = 20/(4z + 5)^2$ **41.** $f_x(x, y, z) = (2x - 5z^2)/(x^2 - 5xz^2 + y^4)$; $f_y(x, y, z) = 4y^3/(x^2 - 5xz^2 + y^4)$; $f_z(x, y, z) = -10xz/(x^2 - 5xz^2 + y^4)$; $f_{yz}(x, y, z) = 40xy^3z/(x^2 - 5xz^2 + y^4)^2$ **43. a.** 6.773 **b.** 3.386 **45. a.** 80 **b.** 150 **c.** 80 **d.** 440 **47. a.** \$902,100 **b.** $f_p(p, i) = 99 - 0.5i - 0.005p$; $f_i(p, i) = -0.5p$; the rate at which weekly sales are changing per unit of change in price when the interest rate remains constant ($f_p(p, i)$) or per unit change in interest rate when the price remains constant ($f_i(p, i)$) **c.** A weekly sales decrease of \$9700 **49. a.** 50.57 hundred units **b.** $f_x(16, 81) = 1.053$ hundred units and is the rate at which production is changing when labor changes by 1 unit (from 16 to 17) and capital remains constant; $f_y(16, 81) = 0.4162$ hundred units and is the rate at which production is changing when capital changes by 1 unit (from 81 to 82) and labor remains constant. **c.** Production would increase by approximately 105 units. **51.** $0.4x^{-0.6}y^{0.6}$; $0.6x^{0.4}y^{-0.4}$ **53. a.** 1279 kcal per hr **b.** 2.906 kcal per hr per g; the instantaneous rate of change of energy usage for a 300-kg animal traveling at 10 km per hr is about 2.9 kcal per hr per g. **55. a.** 0.0142 m^2 **b.** 0.00442 m^2 **57. a.** 4.125 lb **b.** $\partial f/\partial n = n/4$; the rate of change of weight loss per unit change in workouts **c.** An additional loss of 3/4 lb **59. a.** $(2ax - 3x^2)t^2e^{-t}$ **b.** $x^2(a - x)(2t - t^2)e^{-t}$ **c.** $(2a - 6x)t^2e^{-t}$ **d.** $(2ax - 3x^2)(2t - t^2)e^{-t}$ **e.** $\partial R/\partial x$ gives the rate of change of the reaction per unit of change in the amount of drug administered. $\partial R/\partial t$ gives the rate of change of the reaction for a 1-hour change in the time after the drug is administered. **61. a.** -24.9°F **b.** 15 mph **c.** $W_V(20, 10) = -1.114$; while holding the temperature fixed at 10°F, the wind chill decreases approximately 1.1°F when the wind velocity increases by 1 mph; $W_T(20, 10) = 1.429$; while holding the wind velocity fixed at 20 mph, the wind chill increases approximately 1.429°F if the actual temperature increases from 10°F to 11°F.

d. Sample table

T\V	5	10	15	20
30	27	16	9	4
20	16	3	−5	−11
10	6	−9	−18	−25
0	−5	−21	−32	−39

63. -10 ml per year, 100 ml per in. **65. a.** $F_m = gR^2/r^2$; the rate of change in force per unit change in mass while the distance is held constant; $F_r = -2mgR^2/r^3$; the rate of change in force per unit change in distance while the mass is held constant **67. a.** 1055 **b.** $T_s(3, 0.5) = 127.4$ msec per ft. If the distance to move an object increases from 3 ft to 4 ft, while keeping w fixed at 0.5, the approximate increase in movement time is 127.4 msec. $T_w(3, 0.5) = -764.6$ msec per ft. If the width of the target area increases by 1 ft, while keeping s fixed at 3 ft, the approximate decrease in movement time is 764.6 msec.

Exercises 17.3 (page 912)

For exercises . . .	1–18,21–28	34–40,42
Refer to example . . .	1–3	4

1. Saddle point at $(-1, 2)$ **3.** Relative minimum at $(-3, -3)$ **5.** Relative minimum at $(-2, -2)$ **7.** Relative minimum at $(15, -8)$ **9.** Relative maximum at $(2/3, 4/3)$ **11.** Saddle point at $(2, -2)$ **13.** Saddle point at $(0, 0)$; relative minimum at $(27, 9)$ **15.** Saddle point at $(0, 0)$; relative minimum at $(9/2, 3/2)$ **17.** Saddle point at $(0, -1)$ **21.** Relative maximum of 9/8 at $(-1, 1)$; saddle point at $(0, 0)$; a **23.** Relative minima of $-33/16$ at $(0, 1)$ and at $(0, -1)$; saddle point at $(0, 0)$; b **25.** Relative maxima of 17/16 at $(1, 0)$ and $(-1, 0)$; relative minima of $-15/16$ at $(0, 1)$ and $(0, -1)$; saddle points at $(0, 0)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$, and $(-1, -1)$; e **31. a.** all values of k **b.** $k \geq 0$ **35.** Minimum cost of \$59 when $x = 4$, $y = 5$ **37.** Sell 12 spas and 7 solar heaters for a maximum revenue of \$166,600. **39.** \$2000 on quality control and \$1000 on consulting, for a minimum time of 8 hours **41. a.** $r = -1.722$, $s = 0.3652$, $y = 0.1787(1.441)^t$ **b.** Same as a **c.** Same as a

Exercises 17.4 (page 922)

For exercises . . .	1–8,11,12,15,21	9,10,13,14,37–42	23–26	27–36
Refer to example . . .	1	3	material after Example 3	2

1. $f(8,8) = 256$ **3.** $f(5,5) = 125$ **5.** $f(5,3) = 28$ **7.** $f(20,2) = 360$ **9.** $f(3/2, 3/2, 3) = 81/4 = 20.25$
11. $x = 8, y = 16$ **13.** 30, 30, 30 **15.** Minimum value of 128 at $(2,5)$, maximum value of 160 at $(-2,7)$
21. a. Minimum value of -5 at $(3,-2)$. **d.** $(3,-2)$ is a saddle point. **23.** Purchase 20 units of x and 20 units of y for a maximum utility of 8000. **25.** Purchase 20 units of x and 5 units of y for a maximum utility of 4,000,000.
27. 60 feet by 60 feet **29.** 10 large kits and no small kits **31.** 167 units of labor and 178 units of capital **33.** 125 m by 125 m **35.** Radius = 5 in.; height = 10 in. **37.** 12.91 m by 12.91 m by 6.455 m **39.** 5 m by 5 m by 5 m
41. b. 2 yd by 1 yd by 1/2 yd

Exercises 17.5 (page 926)

For exercises . . .	1–6,18–20,23–28	7–14	15–17,21,22,29–32	33,34
Refer to example . . .	1	2	3	4

1. 0.12 **3.** 0.0311 **5.** -0.335
7. 10.022; 10.0221; 0.0001 **9.** 2.0067; 2.0080; 0.0013 **11.** 1.07; 1.0720; 0.0020 **13.** -0.02; -0.0200; 0 **15.** 20.73 cm^3
17. 86.4 in^3 **19.** 0.07694 unit **21.** 6.65 cm^3 **23.** 2.98 liters **25. a.** 0.2649 **b.** Actual 0.2817; approximation 0.2816
27. a. 87% **b.** 75% **d.** 89%; 87% **29.** 26.945 cm^2 **31.** 3% **33.** 8

Exercises 17.6 (page 937)

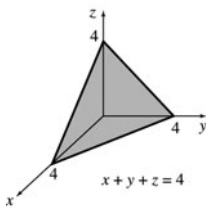
For exercises . . .	1–10	11–20	21–28,61–64,66–71	29–38,65	39–46	47–56	57,58
Refer to example . . .	1	2,3	4	5,6	7	8	9

1. $630y$ **3.** $(2x/9)[(x^2 + 15)^{3/2} - (x^2 + 12)^{3/2}]$ **5.** $6 + 10y$ **7.** $(1/2)(e^{12+3y} - e^{4+3y})$ **9.** $(1/2)(e^{4x+9} - e^{4x})$ **11.** 945
13. $(2/45)(39^{5/2} - 12^{5/2} - 7533)$ **15.** 21 **17.** $(\ln 3)^2$ **19.** $8 \ln 2 + 4$ **21.** 171 **23.** $(4/15)(33 - 2^{5/2} - 3^{5/2})$
25. $-3 \ln(3/4)$ or $3 \ln(4/3)$ **27.** $(1/2)(e^7 - e^6 - e^3 + e^2)$ **29.** 96 **31.** 40/3 **33.** $(2/15)(2^{5/2} - 2)$
35. $(1/4) \ln(17/8)$ **37.** $e^2 - 3$ **39.** 97,632/105 **41.** 128/9 **43.** $\ln 16$ or $4 \ln 2$ **45.** 64/3 **47.** 34 **49.** 10/3
51. $7(e - 1)/3$ **53.** 16/3 **55.** $4 \ln 2 - 2$ **57.** 1 **61.** 49 **63.** $(e^6 + e^{-10} - e^{-4} - 1)/60$ **65.** 9 in^3 **67.** 14,753 units
69. \$34,833 **71.** \$32,000

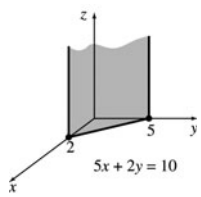
Chapter 17 Review Exercises (page 942)

For exercises . . .	1–5,17–26,87	6,13,14,27–44, 89,98–101	7,8,45–53,90, 103,104	9,54–57, 91,92	10–12,67–86, 100	15,59–66,88, 93–96,98
Refer to section . . .	1	2	3	4	6	5

1. True **2.** True **3.** True **4.** True **5.** False **6.** False **7.** False **8.** True **9.** False **10.** False **11.** True **12.** False
17. -19; -255 **19.** $-5/9$; $-4/3$ **21.**



23.



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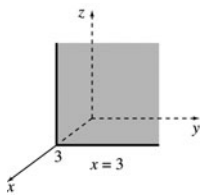


27. a. $9x^2 + 8xy$ **b.** -12 **c.** 16 **29.** $f_x(x,y) = 12xy^3$; $f_y(x,y) = 18x^2y^2 - 4$ **31.** $f_x(x,y) = 4x/(4x^2 + y^2)^{1/2}$; $f_y(x,y) = y/(4x^2 + y^2)^{1/2}$ **33.** $f_x(x,y) = 3x^2e^{3y}$; $f_y(x,y) = 3x^3e^{3y}$ **35.** $f_x(x,y) = 4x/(2x^2 + y^2)$; $f_y(x,y) = 2y/(2x^2 + y^2)$
37. $f_{xx}(x,y) = 30xy$; $f_{xy}(x,y) = 15x^2 - 12y$ **39.** $f_{xx}(x,y) = 12y/(2x - y)^3$; $f_{xy}(x,y) = (-6x - 3y)/(2x - y)^3$
41. $f_{xx}(x,y) = 8e^{2y}$; $f_{xy}(x,y) = 16xe^{2y}$ **43.** $f_{xx}(x,y) = (-2x^2y^2 - 4y)/(2 - x^2y)^2$; $f_{xy}(x,y) = -4x/(2 - x^2y)^2$
45. Saddle point at (0, 2) **47.** Relative minimum at (2, 1) **49.** Saddle point at (3, 1) **51.** Saddle point at $(-1/3, 11/6)$; relative minimum at $(1, 1/2)$ **55.** Minimum of 18 at $(-3, 3)$ **57.** $x = 25, y = 50$ **59.** 1.22 **61.** 13.0846; 13.0848; 0.0002
63. $8y - 6$ **65.** $(3/2)[(100 + 2y^2)^{1/2} - (2y^2)^{1/2}]$ **67.** 1232/9 **69.** $(2/135)[(42)^{5/2} - (24)^{5/2} - (39)^{5/2} + (21)^{5/2}]$
71. $2 \ln 2$ or $\ln 4$ **73.** 110 **75.** $(4/15)(782 - 8^{5/2})$ **77.** 105/2 **79.** 1/2 **81.** 1/48 **83.** $\ln 2$ **85.** 3
87. a. $\$(325 + \sqrt{10}) \approx \$328.16$ **b.** $\$(800 + \sqrt{15}) \approx \$803.87$ **c.** $\$(2000 + \sqrt{20}) \approx \$2004.47$ **89. a.** $0.7y^{0.3}/x^{0.3}$
b. $0.3x^{0.7}/y^{0.7}$ **91.** Purchase 10 units of x and 15 units of y for a maximum utility of 33,750. **93.** Decrease by \$243.82
95. 4.19 ft^3 **97. a.** \$200 spent on fertilizer and \$80 spent on seed will produce a maximum profit of \$266 per acre.
b. Same as a **c.** Same as a **99. a.** 49.68 liters **b.** -0.09, the approximate change in total body water if age is increased by 1 yr and mass and height are held constant is -0.09 liters; 0.34, the approximate change in total body water if mass is increased by 1 kg and age and height are held constant is 0.34 liters; 0.25, the approximate change in total body water if height is increased by 1 cm and age and mass are held constant is 0.25 liters. **101. a.** 50; in 1900, 50% of those born 60 years earlier are still alive.
b. 75; in 2000, 75% of those born 70 years earlier are still alive. **c.** -1.25; in 1900, the percent of those born 60 years earlier who are still alive was dropping at a rate of 1.25 percent per additional year of life. **d.** -2; in 2000, the percent of those born 70 years earlier who are still alive was dropping at a rate of 2 percent per additional year of life. **103.** 5 in. by 5 in. by 5 in.

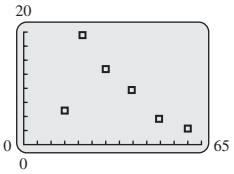
Chapter 18 Probability and Calculus

Exercises 18.1 (page 957)

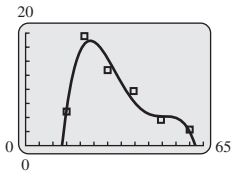
For exercises . . .	1–10	11–18,29–34	19–24,35c,d,36c,d, 48d,e,49d,e	35a,b,36a,b,39–42,44–47, 48a,b,c,49a,b,c,50	43,51
Refer to example . . .	1	2	5	3	4

1. Yes **3.** Yes **5.** No; $\int_0^3 4x^3\,dx \neq 1$ **7.** No; $\int_{-2}^{2} x^2/16\,dx \neq 1$ **9.** No; $f(x) < 0$ for some x values in $[-1, 1]$. **11.** $k = 3/14$ **13.** $k = 3/125$ **15.** $k = 2/9$ **17.** $k = 1/12$ **19.** $F(x) = (x^2 - x - 2)/18, 2 \le x \le 5$ **21.** $F(x) = (x^3 - 1)/63, 1 \le x \le 4$ **23.** $F(x) = (x^{3/2} - 1)/7, 1 \le x \le 4$ **25.** 1 **29. a.** 0.4226 **b.** 0.2071 **c.** 0.4082 **31. a.** 0.3935 **b.** 0.3834 **c.** 0.3679 **33. a.** 1/3 **b.** 2/3 **c.** 295/432 **35. a.** 0.9975 **b.** 0.0024 **c.** $F(t) = 1 - e^{-t/2}, t \ge 0$ **d.** 0.9502 **37.** c **39. a.** 0.2679 **b.** 0.4142 **c.** 0.3178 **41. a.** 0.8131 **b.** 0.4901

43. a.



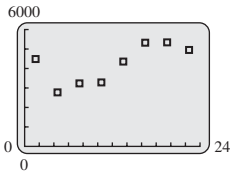
polynomial function **b.** $N(t) = -0.00007445t^4 + 0.01243t^3 - 0.7419t^2 + 18.18t - 137.5$



Yes

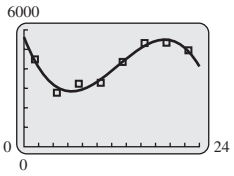
c. $S(t) = \dfrac{1}{466.26}(-0.00007445t^4 + 0.01243t^3 - 0.7419t^2 + 18.18t - 137.5)$ **d.** Estimates: 0.1688, 0.5896, 0.1610; actual: 0.1730, 0.5865, 0.1325 **45. a.** 0.2 **b.** 0.6 **c.** 0.6 **47. a.** 0.1640 **b.** 0.1353 **49. a.** 0.2829 **b.** 0.4853 **c.** 0.2409 **d.** $F(t) = 1.8838(0.5982 - e^{-0.03211t}), 16 \le t \le 84$ **e.** 0.1671

51. a.



A polynomial function

b. $T(t) = -2.564t^3 + 99.11t^2 - 964.6t + 5631$



c. $S(t) = \dfrac{1}{101{,}370}(-2.564t^3 + 99.11t^2 - 964.6t + 5631)$ **d.** 0.09457; 0.07732

Exercises 18.2 (page 967)

For exercises . . .	1–6,11–14	7,8	11d,e–14d,e,24–26,31–33, 34,35,38–42	15a–20a,24d,26d,31d, 33d,36,40d,41d	37,43
Refer to example . . .	1	2	3	5	4

1. $\mu = 5$; $\text{Var}(X) \approx 1.33$; $\sigma \approx 1.15$ **3.** $\mu = 14/3 \approx 4.67$; $\text{Var}(X) \approx 0.89$; $\sigma \approx 0.94$ **5.** $\mu = 2.83$; $\text{Var}(X) \approx 0.57$; $\sigma \approx 0.76$ **7.** $\mu = 4/3 \approx 1.33$; $\text{Var}(X) = 2/9 \approx 0.22$; $\sigma \approx 0.47$ **11. a.** 5.40 **b.** 5.55 **c.** 2.36 **d.** 0.5352 **e.** 0.6043 **13. a.** 1.6 **b.** 0.11 **c.** 0.33 **d.** 0.5904 **e.** 0.6967 **15. a.** 5 **b.** 0 **17. a.** 4.828 **b.** 0.0553 **19. a.** $\sqrt[4]{2} \approx 1.189$ **b.** 0.1836 **21.** 16/5; does not exist; does not exist **23.** d **25. a.** 6.409 yr **b.** 1.447 yr **c.** 0.4910 **27.** c **29.** c **31. a.** 6.342 seconds **b.** 5.135 sec **c.** 0.7518 **d.** 4.472 sec **33. a.** 2.333 cm **b.** 0.8692 cm **c.** 0 **d.** 2.25 cm **35.** 111 **37.** 31.75 years; 11.55 years **39. a.** 1.806 **b.** 1.265 **c.** 0.1886 **41. a.** 38.51 years **b.** 17.56 years **c.** 0.1656 **d.** 34.26 years **43.** About 1 pm

Exercises 18.3 (page 979)

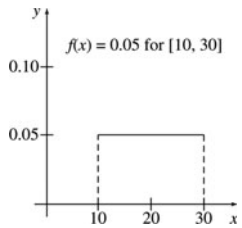
For exercises . . .	1,2,29,35,39,48	3–6,30,31,36–38,40, 41,43–45,47,49–51	7–14,32–34,42,52	46
Refer to example . . .	1	2	3	4

1. a. 3.7 cm **b.** 0.4041 cm **c.** 0.2886 **3. a.** 0.25 years **b.** 0.25 years **c.** 0.2325 **5. a.** 3 days **b.** 3 days **c.** 0.2325 **7.** 49.98% **9.** 8.01% **11.** −1.28 **13.** 0.92 **19.** $m = (-\ln 0.5)/a$ or $(\ln 2)/a$ **23. a.** 1.00000 **b.** 1.99999 **c.** 8.00000 **25. a.** $\mu \approx 0$ **b.** $\sigma = 0.9999999251 \approx 1$ **27.** $F(x) = (x - a)/(b - a), a \le x \le b$ **29. a.** \$47,500 **b.** 0.4667 **31. a.** $f(x) = 0.235e^{-0.235x}$ on $[0, \infty)$ **b.** 0.0954 **33. a.** 0.1587 **b.** 0.7698 **35.** c **37.** d **39. a.** 28 days **b.** 0.375 **41. a.** 1 hour **b.** 0.3935 **43. a.** 58 minutes **b.** 0.0907 **45. a.** 0.1967 **b.** 0.2468 **47. a.** 4.37 millennia; 4.37 millennia **b.** 0.6325 **49. a.** 0.2865 **b.** 0.2212 **51. a.** 0.5457 **b.** 0.0039

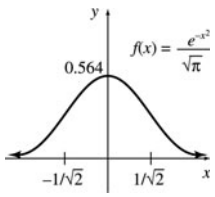
Chapter 18 Review Exercises (page 983)

For exercises . . .	1–4,11–22,41, 46a–47a,60a-d	5,6,23–31,42, 46b,c–47b,c,52, 55,60e,f,61	7–10,33–40,43, 48–51,53,54,56, 57–59,62
Refer to section . . .	1	2	3

1. True **2.** True **3.** True **4.** False **5.** False **6.** True **7.** True **8.** True **9.** False **10.** False **11.** probabilities **13.** 1. $f(x) \geq 0$ for all x in $[a, b]$; 2. $\int_a^b f(x)dx = 1$ **15.** Not a probability density function **17.** Probability density function **19.** $k = 1/21$ **21. a.** $1/5 = 0.2$ **b.** $9/20 = 0.45$ **c.** 0.54 **25. a.** 4 **b.** 0.5 **c.** 0.7071 **d.** 4.121 **e.** $F(x) = (x-2)^2/9, 2 \leq x \leq 5$ **27. a.** 5/4 **b.** $5/48 \approx 0.1042$ **c.** 0.3227 **d.** 1.149 **e.** $F(x) = 1 - 1/x^5, x \geq 1$ **29. a.** 0.5833 **b.** 0.2444 **c.** 0.4821 **d.** 0.6114 **31. a.** 100 **b.** 100 **c.** 0.8647 **33.** 33.36% **35.** 34.31% **37.** 11.51% **39.** −0.05 **41. a.** Uniform **b.** Domain: [10, 30], range: {0.05} **c.**



d. $\mu = 20; \sigma \approx 5.77$ **e.** 0.577 **43. a.** Normal **b.** Domain: $(-\infty, \infty)$, range: $(0, 1/\sqrt{\pi}]$ **c.**



d. $\mu = 0; \sigma = 1/\sqrt{2}$ **e.** 0.6826 **45. b.** 0.6819 **c.** 0.9716 **d.** 1; yes **47. a.** 0.9107 **b.** 13.57 years **c.** 6.68 years **49. a.** $f(x) = e^{-x/8}/8; [0, \infty)$ **b.** 8 **c.** 8 **d.** 0.2488 **51.** d **53.** 0.6321 **55. a.** 40.07°C **b.** 0.4928 **57.** 0.2266 **59. a.** 0.2921 **b.** 0.1826 **61.** 3650.1 days; 3650.1 days

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This is a sample of the comprehensive source list available for the tenth edition of Calculus with Applications. The complete list is available at the Downloadable Student Resources site, www.pearsonhighered.com/mathstatsresources, as well as to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center, www.pearsonhighered.com/irc.

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KEY DEFINITIONS, THEOREMS, AND FORMULAS

2.2 Row Operations

For any augmented matrix of a system of equations, the following operations produce the augmented matrix of an equivalent system:

1. interchanging any two rows;
2. multiplying the elements of a row by any nonzero real number;
3. adding a nonzero multiple of the elements of one row to the corresponding elements of a nonzero multiple of some other row.

2.5 Finding a Multiplicative Inverse Matrix

To obtain A^{-1} for any $n \times n$ matrix A for which A^{-1} exists, follow these steps.

1. Form the augmented matrix $[A|I]$, where I is the $n \times n$ identity matrix.
2. Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$ if this is possible.
3. Matrix B is A^{-1}.

4.2 Simplex Method for Standard Maximization Problems

1. Determine the objective function.
2. Write all necessary constraints.
3. Convert each constraint into an equation by adding a slack variable in each.
4. Set up the initial simplex tableau.
5. Locate the most negative indicator. If there are two such indicators, choose the one farther to the left.
6. Form the necessary quotients to find the pivot. Disregard any quotients with 0 or a negative number in the denominator. The smallest nonnegative quotient gives the location of the pivot. If all quotients must be disregarded, no maximum solution exists. If two quotients are both equal and smallest, choose the pivot in the row nearest the top of the matrix.
7. Use row operations to change all other numbers in the pivot column to zero by adding a suitable multiple of the pivot row to a positive multiple of each row.
8. If the indicators are all positive or 0, this is the final tableau. If not, go back to Step 5 and repeat the process until a tableau with no negative indicators is obtained.
9. Read the solution from this final tableau.

5.1 Compound Amount

$$A = P(1 + i)^n$$

where $i = \dfrac{r}{m}$ and $n = mt$,
A is the future (maturity) value;
P is the principal;
r is the annual interest rate;
m is the number of compounding periods per year;
t is the number of years;
n is the number of compounding periods;
i is the interest rate per period.

5.2 Future Value of an Ordinary Annuity

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] \qquad \text{or} \qquad S = Rs_{\overline{n}|i}$$

where
S is the future value;
R is the payment;
i is the interest rate per period;
n is the number of periods.

6.1, 6.3 Truth Tables

The following truth table defines the logical operators in this chapter.

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

7.3 Basic Probability Principle

Let S be a sample space of equally likely outcomes, and let event E be a subset of S. Then the probability that event E occurs is

$$P(E) = \frac{n(E)}{n(S)}.$$

7.4 Union Rule

For any two events E and F from a sample space S,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

7.5 Product Rule

If E and F are events, then $P(E \cap F)$ may be found by either of these formulas.

$$P(E \cap F) = P(F) \cdot P(E|F) \qquad \text{or} \qquad P(E \cap F) = P(E) \cdot P(F|E)$$

7.6 Bayes' Theorem

$$P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + \cdots + P(F_n) \cdot P(E|F_n)}$$

8.2 Permutations and Combinations

Permutations Different orderings or arrangements of the r objects are different permutations.

$$P(n, r) = \frac{n!}{(n-r)!}$$

Clue words: arrangement, schedule, order
Order matters!

Combinations Each choice or subset of r objects gives one combination. Order within the group of r objects does not matter.

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

Clue words: group, committee, set, sample
Order does not matter!

8.4 Binomial Probability

If p is the probability of success in a single trial of a binomial experiment, the probability of x successes and $n - x$ failures in n independent repeated trials of the experiment, known as binomial probability, is

$$P(x \text{ successes in } n \text{ trials}) = C(n, x) \cdot p^x \cdot (1-p)^{n-x}.$$

8.5 Expected Value

Suppose the random variable x can take on the n values $x_1, x_2, x_3, \ldots, x_n$. Also, suppose the probabilities that these values occur are, respectively, $p_1, p_2, p_3, \ldots, p_n$. Then the expected value of the random variable is

$$E(x) = x_1p_1 + x_2p_2 + x_3p_3 + \cdots + x_np_n.$$

9.2 Variance and Standard Deviation

The variance of a sample of n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\overline{x}$, is

$$s^2 = \frac{\Sigma x^2 - n\overline{x}^2}{n-1}.$$

The standard deviation of a sample of n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\overline{x}$, is

$$s = \sqrt{\frac{\Sigma x^2 - n\overline{x}^2}{n-1}}.$$

11.4 Derivative

The derivative of the function f at x, written $f'(x)$, is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{provided this limit exists.}$$

Rules for Derivatives

The following rules for derivatives are valid when all the indicated derivatives exist.

12.1 *Constant Rule* If $f(x) = k$ where k is any real number, then $f'(x) = 0$.

12.1 *Power Rule* If $f(x) = x^n$ for any real number n, then $f'(x) = nx^{n-1}$.

12.1 *Constant Times a Function* Let k be a real number. Then the derivative of $f'(x) = k \cdot g(x)$ is

$$f'(x) = k \cdot g'(x).$$

12.1 *Sum or Difference Rule* If $f(x) = u(x) \pm v(x)$ then

$$f'(x) = u'(x) \pm v'(x).$$

12.2 *Product Rule* If $f(x) = u(x) \cdot v(x)$ then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x).$$

12.2 *Quotient Rule* If $f(x) = u(x)/v(x)$, and $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

12.3 *Chain Rule* If y is a function of u, say $y = f(u)$, and if u is a function of x, say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

12.3 *Chain Rule* (Alternate Form) If $y = f[g(x)]$, then $dy/dx = f'[g(x)], g'(x)$.

12.4 *Exponential Function*

$$\frac{d}{dx}[a^{g(x)}] = (\ln a)a^{g(x)}g'(x)$$

$$\frac{d}{dx}[e^{g(x)}] = e^{g(x)}g'(x)$$

12.5 *Logarithmic Function*

$$\frac{d}{dx}[\log_a|g(x)|] = \frac{1}{\ln a} \cdot \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}[\ln|g(x)|] = \frac{g'(x)}{g(x)}$$

15.2 Substitution Each of the following forms can be integrated using the substitution $u = f(x)$.

Form of the Integral	**Result**
1. $\int [f(x)]^n f'(x)\,dx, \quad n \neq -1$	$\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{[f(x)]^{n+1}}{n+1} + C$
2. $\int \frac{f'(x)}{f(x)}\,dx$	$\int \frac{1}{u}\,du = \ln\|u\| + C = \ln\|f(x)\| + C$
3. $\int e^{f(x)}f'(x)\,dx$	$\int e^u\,du = e^u + C = e^{f(x)} + C$

15.4 Fundametal Theorem of Calculus Let f be continuous on the interval $[a, b]$, and let F be *any* antiderivative of f. Then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b.$$

17.3 Test for Relative Extrema For a function $z = f(x, y)$ let $f_{xx}f_{yy}$, and f_{xy} all exist in a circular region contained in the xy-plane with center (a, b) Further, let

$$f_x(a, b) = 0 \quad \text{and} \quad f_x(a, b) = 0.$$

Define the number D by

$$D = f_{xx}(a, b) + f_{xx}(a, b) - [f_{xx}(a, b)]^2.$$

Then

(a) $f(a, b)$ is a relative maximum if $D > 0$ and $f_{xx}(a, b) < 0$;

(b) $f(a, b)$ is a relative minimum if $D > 0$ and $f_{xx}(a, b) > 0$;

(c) $f(a, b)$ is a saddle point (neither a maximum nor a minimum) if $D < 0$;

(d) if $D = 0$, the test gives no information.