LEARNING MADE EASY

3rd Edition

Calculus II

for dummies®
A Wiley Brand

Solve area problems using definite and indefinite integrals

Tackle *u*-substitution, integration by parts, and partial fractions

Review Pre-Calculus and Calculus I concepts

Mark Zegarelli
Math Tutor Extraordinaire

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Contents at a Glance

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Introduction

Calculus is the great Mount Everest of math. Most of the world is content to just gaze upward at it in awe. But only a few brave souls attempt the ascent.

Or maybe not.

In recent years, calculus has become a required course not only for math, engineering, and physics majors, but also for students of biology, economics, psychology, nursing, and business. Law schools and MBA programs welcome students who've taken calculus because it demonstrates discipline and clarity of mind. High schools now have multiple math tracks that include calculus, from the basic college prep track to the AP tracks that prepare students for the Advanced Placement exam.

So perhaps calculus is more like a well-traveled Vermont mountain, with lots of trails and camping spots, plus a big ski lodge on top. You may need some stamina to conquer it, but with the right guide (this book, for example!), you're not likely to find yourself swallowed up by a snowstorm half a mile from the summit.

About This Book

You *can* learn calculus. That's what this book is all about. In fact, as you read these words, you may well already be a winner, having passed a course in Calculus I. If so, then congratulations and a nice pat on the back are in order.

Having said that, I want to discuss a few rumors you may have heard about Calculus II:

- Calculus II is harder than Calculus I.
- Calculus II is harder, even, than either Calculus III or Differential Equations.
- Calculus II is more frightening than having your home invaded by zombies in the middle of the night and will result in emotional trauma requiring years of costly psychotherapy to heal.

Now, I admit that Calculus II is harder than Calculus I. Also, I may as well tell you that many — but not all — math students find it to be harder than the two semesters of math that follow. (Speaking personally, I found Calc II to be easier than Differential Equations.) But I'm holding my ground that the long-term psychological effects of a zombie attack far outweigh those awaiting you in any one-semester math course.

The two main topics of Calculus II are integration and infinite series. *Integration* is the inverse of differentiation, which you study in Calculus I. (For practical purposes, integration is a method for finding the area of unusual geometric shapes.) An *infinite series* is a sum of numbers that goes on forever, like $1 + 2 + 3 + \ldots$ or $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$. Roughly speaking, most teachers focus on integration for the first two-thirds of the semester and infinite series for the last third.

This book gives you a solid introduction to what's covered in a college course in Calculus II. You can use it either for self-study or while enrolled in a Calculus II course.

So feel free to jump around. Whenever I cover a topic that requires information from earlier in the book, I refer you to that section in case you want to refresh yourself on the basics.

Here are two pieces of advice for math students (remember them as you read the book):

- **Study a little every day.** I know that students face a great temptation to let a book sit on the shelf until the night before an assignment is due. This is a particularly poor approach for Calc II. Math, like water, tends to seep in slowly and swamp the unwary!

  So, when you receive a homework assignment, read over every problem as soon as you can and try to solve the easy ones. Go back to the harder problems every day, even if it's just to reread and think about them. You'll probably find that over time, even the most opaque problem starts to make sense.

- **Use practice problems for practice.** After you read through an example and think you understand it, copy the problem down on paper, close the book, and try to work it through. If you can get through it from beginning to end, you're ready to move on. If not, go ahead and peek, but then try solving the problem later without peeking. (Remember, on exams, no peeking is allowed!)

Conventions Used in This Book

Throughout the book, I use the following conventions:

- *Italicized* text highlights new words and defined terms.
- **Boldfaced** text indicates keywords in bulleted lists and the action parts of numbered steps.
- `Monofont` text highlights web addresses.
- Angles are measured in radians rather than degrees, unless I specifically state otherwise. (See Chapter 2 for a discussion about the advantages of using radians for measuring angles.)

What You're Not to Read

All authors believe that each word they write is pure gold, but you don't have to read every word in this book unless you really want to. You can skip over sidebars (those gray shaded boxes) where I go off on a tangent, unless you find that tangent interesting. Also feel free to pass by paragraphs labeled with the Technical Stuff icon.

If you're not taking a class where you'll be tested and graded, you can skip paragraphs labeled with the Tip icon and jump over extended step-by-step examples. However, if you're taking a class, read this material carefully and practice working through examples on your own.

Foolish Assumptions

Not surprisingly, a lot of Calculus II builds on topics introduced in Calculus I and Pre-Calculus. So here are the foolish assumptions I make about you as you begin to read this book:

- If you're a student in a Calculus II course, I assume that you passed Calculus I. (Even if you got a D-minus, your Calc I professor and I agree that you're good to go!)
- If you're studying on your own, I assume that you're at least passably familiar with some of the basics of Calculus I.

I expect that you know some things from Calculus I, Algebra, and even Pre-Algebra, but I don't throw you in the deep end of the pool and expect you to swim or drown. Chapter 2 contains a ton of useful Algebra and Pre-Algebra tidbits that you may have missed the first time around. And in Chapter 3, I give you a review of the most important topics from Calculus I that you're sure to need in Calculus II. Furthermore, throughout the book, whenever I introduce a topic that calls for previous knowledge, I point you to an earlier chapter or section so you can get a refresher.

Icons Used in This Book

Here are four useful icons to help you navigate your way through the book:

TIP

Tips are helpful hints that show you the easy way to get things done. Try them out, especially if you're taking a math course.

REMEMBER

This icon points out key ideas that you need to know. Make sure you understand these ideas before reading on.

TECHNICAL STUFF

This icon points out interesting trivia that you can read or skip over as you like.

WARNING

Warnings flag common errors that you want to avoid. Get clear where these traps are hiding so you don't fall in.

EXAMPLE

Examples walk you through a particular math exercise designed to illustrate a particular topic. Practice makes perfect!

Beyond the Book

In addition to the introduction you're reading right now, this book comes with a free, access-anywhere Cheat Sheet containing information worth remembering about Calculus II. To get this Cheat Sheet, simply go to `www.dummies.com` and type **Calculus II For Dummies Cheat Sheet** in the Search box.

Where to Go from Here

You can use this book either for self-study or to help you survive and thrive in a course in Calculus II.

If you're taking a Calculus II course, you may be under pressure to complete a homework assignment or study for an exam. In that case, feel free to skip right to the topic that you need help with. Every section is self-contained, so you can jump right in and use the book as a handy reference. And when I refer to information that I discuss earlier in the book, I give you a brief review and a pointer to the chapter or section where you can get more information if you need it.

If you're studying on your own, I recommend that you begin with Chapter 1, where I give you an overview of the entire book, and then read the chapters from beginning to end. Jump over Chapters 2 and 3 if you feel confident about your grounding in the math leading up to Calculus II. And, of course, if you're dying to read about a topic that's later in the book, go for it! You can always drop back to an easier chapter if you get lost.

1

Introduction to Integration

IN THIS PART . . .

See Calculus II as an ordered approach to finding the area of unusual shapes on the *xy*-graph

Use the definite integral to clearly define an area problem

Slice an irregularly shaped area into rectangles to approximate area

Review the math you need from Pre-Algebra, Algebra, Pre-Calculus, and Calculus I

IN THIS CHAPTER

» Measuring the area of shapes by using classical and analytic geometry

» Using integration to frame the area problem

» Approximating area using Riemann sums

» Applying integration to more complex problems

» Seeing how differential equations are related to integrals

» Looking at sequences and series

Chapter 1

An Aerial View of the Area Problem

Humans have been measuring the area of shapes for thousands of years. One practical use for this skill is measuring the area of a parcel of land. Measuring the area of a square or a rectangle is simple, so land tends to get divided into these shapes.

Discovering the area of a triangle, circle, or polygon is also relatively easy, but as shapes get more unusual, measuring them gets harder. Although the Greeks were familiar with the conic sections — parabolas, ellipses, and hyperbolas — they couldn't reliably measure shapes with edges based on these figures.

René Descartes's invention of analytic geometry — studying lines and curves as equations plotted on the *xy*-graph — brought great insight into the relationships among the conic sections. But even analytic geometry didn't answer the question of how to measure the area inside a shape that includes a curve.

This bit of mathematical history is interesting in its own right, but I tell the story in order to give you, the reader, a sense of what drove those who came up with the concepts that eventually got bundled together as part of a standard Calculus II course. I start out by showing you how *integral calculus* (*integration* for short) was developed from attempts to answer this basic question of measuring the area of weird shapes, called the *area problem.* To do this, you will discover how to approximate the area under a parabola on the *xy*-graph in ways that lead to an ordered system of measuring the exact area under any function.

First, I frame the problem using a tool from calculus called the *definite integral*. I show you how to use the definite integral to define the areas of shapes you already know how to measure, such as circles, squares, and triangles.

With this introduction to the definite integral, you're ready to look at the practicalities of measuring area. The key to approximating an area that you don't know how to measure is to slice it into shapes that you do know how to measure — for example, rectangles. This process of slicing unruly shapes into nice, crisp rectangles — called finding a *Riemann sum* — provides the basis for calculating the exact value of a definite integral.

At the end of this chapter, I give you a glimpse into the more advanced topics in a basic Calculus II course, such as finding volume of unusual solids, looking at some basic differential equations, and understanding infinite series.

Checking Out the Area

Finding the area of certain basic shapes — squares, rectangles, triangles, and circles — is easy using geometric formulas you typically learn in a geometry class. But a reliable method for finding the exact area of shapes containing more esoteric curves eluded mathematicians for centuries. In this section, I give you the basics of how this problem, called the *area problem,* is formulated in terms of a new concept, the definite integral.

The *definite integral* represents the area of a region bounded by the graph of a function, the *x*-axis, and two vertical lines located at the *bounds of integration.* Without getting too deep into the computational methods of integration, I give you the basics of how to state the area problem formally in terms of the definite integral.

Comparing classical and analytic geometry

In *classical geometry*, you discover a variety of simple formulas for finding the area of different shapes. For example, Figure 1-1 shows the formulas for the area of a rectangle, a triangle, and a circle.

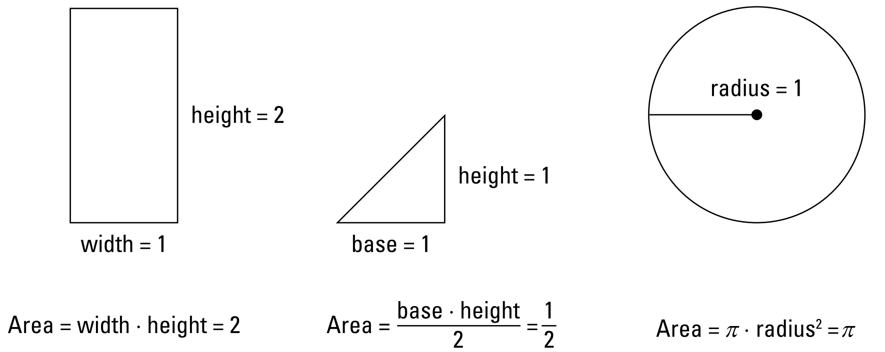

FIGURE 1-1: Formulas for the area of a rectangle, a triangle, and a circle.

On the *xy*-graph, you can generalize the problem of finding area to measure the area under any continuous function of *x*. To illustrate how this works, the shaded region in Figure 1-2 shows the area under the function $f(x)$ between the vertical lines $x = a$ and $x = b$.

The area problem is all about finding the area under a continuous function between two constant values of *x* that are called the *bounds of integration*, usually denoted by *a* and *b*. This problem is generalized as follows:

$$\text{Area} = \int_a^b f(x)\, dx$$

WISDOM OF THE ANCIENTS

Long before calculus was invented, the ancient Greek mathematician Archimedes used his *method of exhaustion* to calculate the exact area of a segment of a parabola. He was also the first mathematician to come up with an approximation for π (pi) within about a 0.2% margin of error.

Indian mathematicians also developed *quadrature* methods for some difficult shapes before Europeans began their investigations in the 17th century.

These methods anticipated some of the methods of calculus. But before calculus, no single theory could measure the area under arbitrary curves.

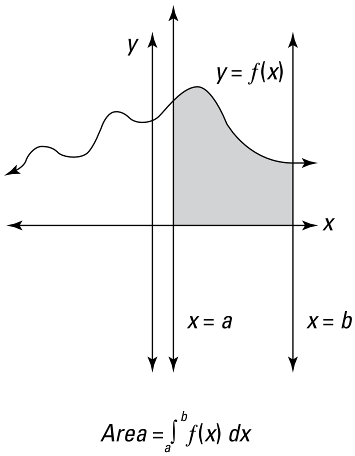

FIGURE 1-2: A typical area problem.

In a sense, this formula for the shaded area isn't much different from the geometric formulas you already know. It's just a formula, which means that if you plug in the right numbers and calculate, you get the right answer.

For example, suppose you want to measure the area under the function x^2 between $x = 1$ and $x = 5$. (You can see what this area looks like by flipping a few pages forward to Figure 1-5.) Here's how you plug these values into the area formula shown previously:

$$\text{Area} = \int_1^5 x^2\,dx$$

The catch, however, is how exactly to calculate using this new symbol. As you may have figured out, the answer is on the cover of this book: calculus. To be more specific, *integral calculus,* or *integration.*

REMEMBER

Most typical Calculus II courses taught at your friendly neighborhood college or university focus on integration — the study of how to solve the area problem. So, if what you're studying starts to get confusing (and to be honest, you probably will get confused somewhere along the way), try to relate what you're doing to this central question: "How does what I'm working on help me find the area under a function?"

Finding definite answers with the definite integral

You may be surprised to find out that you've known how to integrate some functions for years without even knowing it. (Yes, you can know something without knowing that you know it.)

For example, find the rectangular area under the function $y = 2$ between $x = 1$ and $x = 4$, as shown in Figure 1-3.

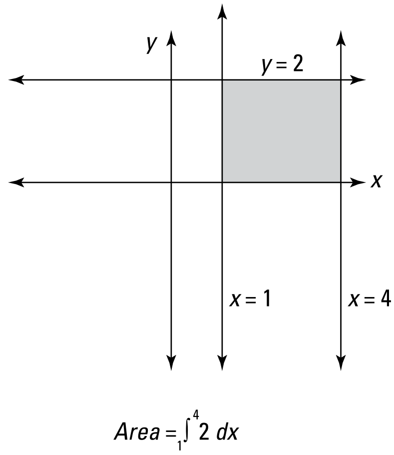

FIGURE 1-3: The rectangular area under the function $f(x) = 2$, between $a = 1$ and $b = 4$ equals 6.

This is just a rectangle with a base of 3 and a height of 2, so its area is 6. But this is also an area problem that can be stated in terms of integration as follows:

$$\text{Area} = \int_1^4 2\,dx = 6$$

As you can see, the function I'm integrating here is $f(x) = 2$. The bounds of integration are 1 and 4 (notice that the greater value goes on top). You already know that the area is 6, so you can solve this calculus problem without resorting to any scary or hairy methods. But you're still *integrating,* so please pat yourself on the back, because I can't quite reach it from here.

The following expression is called a *definite integral:*

$$\int_1^4 2\,dx$$

For now, don't spend too much time worrying about the deeper meaning behind the $\int$ symbol or the dx (which you may fondly remember from your days spent differentiating in Calculus I). Just think of $\int$ and dx as notation placed around a function — notation that means *area.*

What's so definite about a definite integral? Two things, really:

» **You definitely know the bounds of integration** (in this case, 1 and 4). Their presence distinguishes a definite integral from an indefinite integral, which you find out about in Chapter 5. Definite integrals always include the bounds of integration; indefinite integrals never include them.

- **A definite integral definitely equals a number** (assuming that its limits of integration are also numbers). This number may be simple to find or difficult enough to require a room full of math professors scribbling away with #2 pencils. But, at the end of the day, a number is just a number. And, because a definite integral is a measurement of area, you should expect the answer to be a number.

TECHNICAL STUFF

When the limits of integration *aren't* numbers, a definite integral doesn't necessarily equal a number. For example, expressions such as k and $2k$ might be used as limits of integration to stand in for constants. In such cases, the answer to a definite integral may include the letter k. Similarly, a definite integral whose limits of integration are $\sin\theta$ and $2\sin\theta$ would most likely equal a trig expression that includes θ. To sum up, because a definite integral represents an area, it always equals a number — though you may or may not be able to compute this number.

As another example, find the triangular area under the function $y = x$, between $x = 0$ and $x = 8$, as shown in Figure 1-4.

This time, the shape of the shaded area is a triangle with a base of 8 and a height of 8, so its area is 32 (because the area of a triangle is half the base times the height). But again, this is an area problem that can be stated in terms of integration as follows:

$$\text{Area} = \int_0^8 x\, dx = 32$$

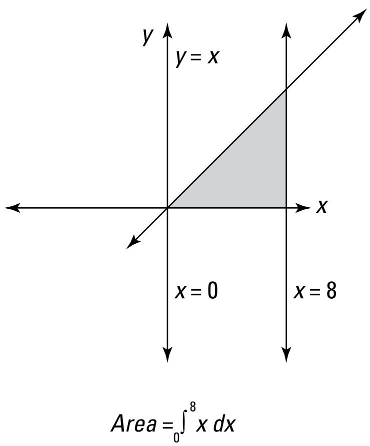

FIGURE 1-4: The triangular area under the function $y = x$, between $x = 0$ and $x = 8$ equals 32.

The function I'm integrating here is $f(x) = x$ and the bounds of integration are 0 and 8. Again, you can evaluate this integral with methods from classical and analytic geometry. And, again, the definite integral evaluates to a number, which is the area below the function and above the x-axis between $x = 0$ and $x = 8$.

Slicing Things Up

One good way of approaching a difficult task — from planning a wedding to climbing Mount Everest — is to break it down into smaller and more manageable pieces.

In this section, I show you the basics of how mathematician Bernhard Riemann used this same type of approach to calculate the definite integral using his self-named *Riemann sums*, which I introduce in the earlier section "Checking Out the Area." Throughout this section I use the example of the area under the function $y = x^2$, between $x = 1$ and $x = 5$. You can find this example in Figure 1-5.

$$\text{Area} = \int_1^5 x^2 dx$$

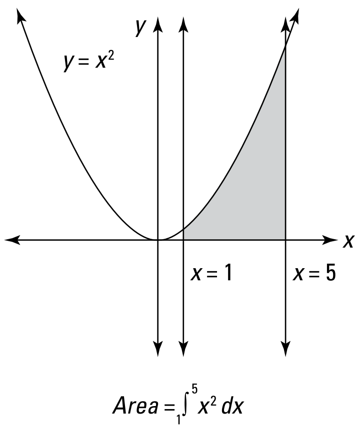

FIGURE 1-5: The area under the function $y = x^2$, between $x = 1$ and $x = 5$.

Untangling a hairy problem using rectangles

The earlier section "Checking Out the Area" tells you how to write the definite integral that represents the area of the shaded region in Figure 1-5:

$$\text{Area} = \int_1^5 x^2 dx$$

Unfortunately, this definite integral — unlike those earlier in this chapter — doesn't respond to the methods of classical and analytic geometry that I use to solve the earlier problems. (If it did, integrating would be much easier and this book would be a lot thinner!)

Even though you can't solve this definite integral directly (yet!), you can approximate it by slicing the shaded region into two pieces, as shown in Figure 1-6.

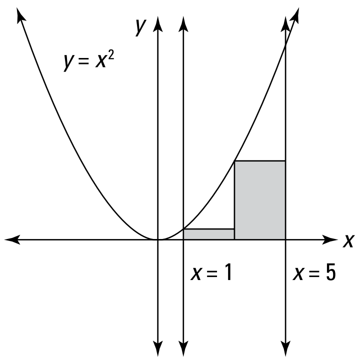

FIGURE 1-6: Area approximated by two rectangles.

Obviously, the region that's now shaded — it looks roughly like two steps going up but leading nowhere — is less than the area that you're trying to find. Fortunately, these steps do lead someplace, because calculating the area under them is fairly easy.

Each rectangle has a width of 2. The tops of the two rectangles cut across where the function x^2 meets $x = 1$ and $x = 3$, so their heights are 1 and 9, respectively. So the total area of the two rectangles is 20, because

$$2(1) + 2(9) = 2(1 + 9) = 2(10) = 20$$

With this approximation of the area of the original shaded region, here's the conclusion you can draw:

$$\int_1^5 x^2 dx \approx 20$$

Granted, this is a ballpark approximation with a really big ballpark. But even a lousy approximation is better than none at all. To get a better approximation, try cutting the figure that you're measuring into a few more slices, as shown in Figure 1-7.

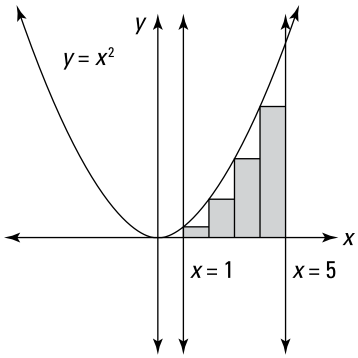

FIGURE 1-7: A closer approximation, where the area is approximated by four rectangles.

Again, this approximation is going to be less than the actual area that you're seeking. This time, each rectangle has a width of 1. And the tops of the four rectangles cut across where the function x^2 meets $x = 1$, $x = 2$, $x = 3$, and $x = 4$, so their heights are 1, 4, 9, and 16, respectively. So the total area of the four rectangles is 30, because

$$1\,(1) + 1\,(4) + 1\,(9) + 1\,(16) = 1\,(1 + 4 + 9 + 16) = 1\,(30) = 30$$

Therefore, here's a second approximation of the shaded area that you're seeking:

$$\int_1^5 x^2 dx \approx 30$$

Your intuition probably tells you that your second approximation is better than your first, because slicing the rectangles more thinly allows them to cut in closer to the function. You can verify this intuition by realizing that both 20 and 30 are *less* than the actual area, so whatever this area turns out to be, 30 must be closer to it.

You might imagine that by slicing the area into more rectangles (say 10, or 100, or 1,000,000), you'd get progressively better estimates. And, again, your intuition would be correct: As the number of slices increases, the result approaches 41.3333 . . . .

In fact, after enough calculation, you may very well decide to write:

$$\int_1^5 x^2 dx = 41.\overline{3}$$

This, in fact, is the correct answer. And it's a good start on solving the area problem. But to verify it, you'll need a more reliable overall method.

Moving left, right, or center

In the previous section, I slice the area I wish to measure into four rectangles in Figure 1-8.

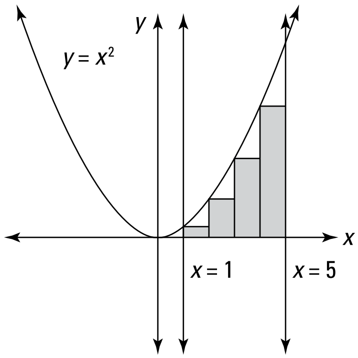

FIGURE 1-8: Approximating area with left rectangles.

As you can see, the heights of the four rectangles are determined by the value of $f(x)$ when x is equal to 1, 2, 3, and 4, respectively — that is, $f(1)$, $f(2)$, $f(3)$, and $f(4)$. Notice that the upper-left corner of each rectangle touches the function and determines the height of each rectangle. This process is called *approximating area with left rectangles*.

However, you can also *approximate area with right rectangles* by drawing the rectangles as shown in Figure 1-9.

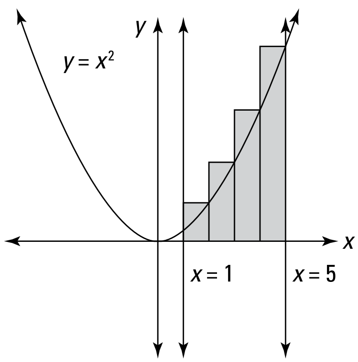

FIGURE 1-9: Approximating area with right rectangles.

In this case, the upper-right corner touches the function, so the heights of the four rectangles are $f(2)$, $f(3)$, $f(4)$, and $f(5)$.

Additionally, you can *approximate area with midpoint rectangles* by drawing the rectangles as shown in Figure 1-10.

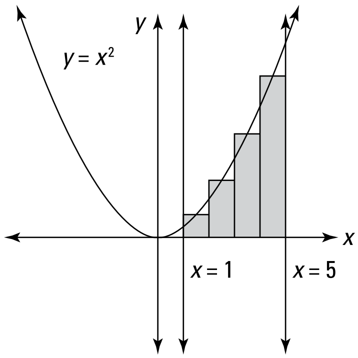

FIGURE 1-10: Approximating area with midpoint rectangles.

This time, the midpoint of the top edge of each rectangle touches the function, so the heights of the rectangles are $f(1.5)$, $f(2.5)$, $f(3.5)$, and $f(4.5)$.

Approximations like these are called *Riemann sums*. In Chapter 4, you work with a variety of methods for calculating Riemann sums.

Defining the Indefinite

Riemann sums allow you to approximate and even calculate areas that you can't measure using classical or analytic geometry. The downside of this method, however, is that it's quite complicated. In fact, at this point most students throw their hands up and say, "There has to be a better way!"

The better way is called the *indefinite integral.* The indefinite integral looks a lot like the definite integral. Compare for yourself:

Definite Integrals	Indefinite Integrals
$\int_1^5 x^2 dx$	$\int x^2 dx$
$\int_0^\pi \sin x\ dx$	$\int \sin x\ dx$
$\int_{-1}^1 e^x dx$	$\int e^x dx$

Like the definite integral, the indefinite integral is a tool for measuring the area under a function. A quick way to tell them apart is by noticing that the definite integral includes bounds of integration and the indefinite integral omits them.

Indefinite integrals provide an easier and faster way of finding the area under a curve. This method relies on finding a general algebraic solution to an indefinite integral, and then plugging in the bounds of integration to this solution to calculate a numerical value for the area. Chapter 5 gives you the details of how definite and indefinite integrals are related.

Indefinite integrals also provide a convenient way to calculate definite integrals. In fact, the indefinite integral is the *inverse* of the derivative, which you know from Calculus I. (Don't worry if you don't remember all about the derivative — Chapter 3 gives you a thorough review.) By inverse, I mean that the indefinite integral of a function is really the *antiderivative* of that function. This connection between integration and differentiation is more than just an odd little fact: It leads to the *Fundamental Theorem of Calculus* (FTC).

For example, you know from Calculus I that the derivative of x^2 is $2x$. So you should expect that the antiderivative — that is, the indefinite integral — of $2x$ is x^2. This is fundamentally correct with one small tweak, as I explain in Chapter 5.

Seeing integration as anti-differentiation allows you to solve tons of integrals without resorting to the Riemann sum formula for integration. But integration by finding the antiderivative can still be sticky depending on the function that you're trying to integrate. Mathematicians have developed a wide variety of algebraic techniques for evaluating integrals. Some of these methods are variable substitution (see Chapter 8), integration by parts (see Chapter 9), trig substitution (see Chapter 10), and integration by partial fractions (see Chapter 11).

Solving Problems with Integration

After you understand how to describe an area problem using the definite integral (Part 2 of this book), and how to calculate integrals (Parts 2, 3, and 4), you're ready to get into action solving a wide range of problems.

Some of these problems know their place and stay in two dimensions. Others rise up and create a revolution in three dimensions. In this section, I give you a taste of these types of problems, with an invitation to check out Part 5 of this book for a deeper look.

Three types of problems that you're almost sure to find on an exam involve finding the area between curves, the arc length of a curve, and the volume of revolution. I focus on these types of problems and many others in Chapters 12 and 13. For now, here's a preview.

We can work it out: Finding the area between curves

When you know how the definite integral represents the area under a curve, finding the area between curves isn't too difficult. Just figure out how to break the problem into several smaller versions of the basic area problem. For example, suppose that you want to find the area between the function $y = \sin x$ and $y = \cos x$, from $x = 0$ to — that is, the shaded area A in Figure 1-11.

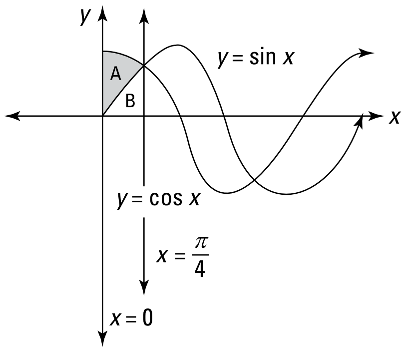

FIGURE 1-11: The area between the function $y = \sin x$ and $y = \cos x$, from $x = 0$ to $x = \frac{\pi}{4}$.

In this case, integrating $y = \cos x$ allows you to find the total area A + B. And integrating $y = \sin x$ gives you the area of B. So you can subtract A + B − B to find the area of A.

For more on how to find an area between curves, flip to Chapter 12.

Walking the long and winding road

Measuring a segment of a straight line or even the arc length of a section of a circle is relatively simple when you're using classical and analytic geometry. But how do you measure arc length along an unusual curve produced by a polynomial, exponential, or trig function?

For example, what's the distance from Point A to Point B along the curve shown in Figure 1-12?

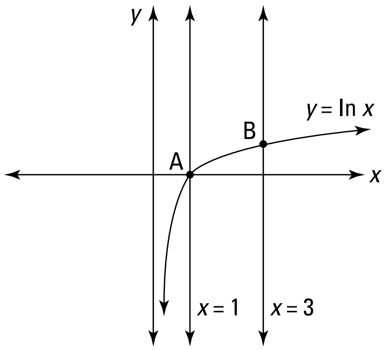

FIGURE 1-12: The distance from Point A to Point B along the function $y = \ln x$.

Once again, integration is your friend. In Chapter 12, I show you how integration provides a formula that allows you to measure arc length.

You say you want a revolution

Calculus also allows you to find the volume of unusual solids. In most cases, calculating volume involves a dimensional leap into *multivariable calculus,* the topic of Calculus III. But in a few situations, setting up an integral just right allows you to calculate volume by integrating over a single variable — that is, by using the methods you discover in Calculus II.

Among the trickiest of these problems involves the *solid of revolution* of a curve. In such problems, you're presented with a region under a curve. You imagine the solid that results when you spin this region around the axis, and then you calculate the volume of this solid as seen in Figure 1-13.

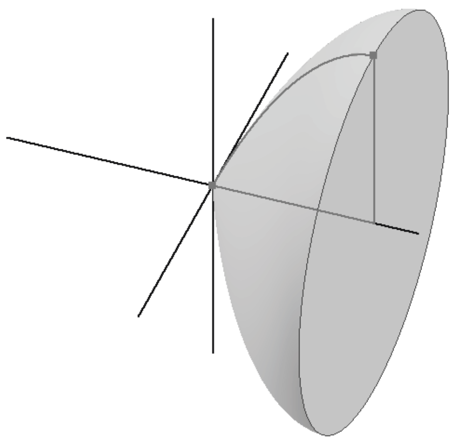

FIGURE 1-13: A solid of revolution produced by spinning the function $y = 2 \sin x$ around the x-axis.

Clearly, you need calculus to find the area of this region. Then you need more calculus and a clear plan of attack to find the volume. I give you all this and more in Chapter 13.

Differential Equations

Many Calculus II courses include a chapter on *differential equations* (*DEs* for short), which are equations that include one or more derivatives. For example:

$$y^2 \frac{dy}{dx} = \cos 2x$$

Here's what this DE is saying in words: When you square y (which is a function of x) and then multiply it by the derivative of y over x, the result is the cosine of $2x$.

Your goal is to solve this DE by finding the original function y in terms of x. To do this, treat the derivative $\frac{dy}{dx}$ as if it were a fraction. Thus, you can multiply both sides of this equation by dx, separating the x and y terms onto opposite sides of the equation:

$$y^2\ dy = \cos 2x\ dx$$

Now, you can integrate both sides to solve the problem:

$$\int y^2\ dy = \int \cos 2x\ dx$$

$$\frac{1}{3}y^3 = \frac{1}{2}\sin 2x + C$$

Don't worry if you don't understand this integration step. Just realize that now, you can use basic algebra to complete the problem:

$$y^3 = \frac{3}{2}\sin 2x + C$$

$$y = \sqrt[3]{\frac{3}{2}\sin 2x + C}$$

This solution provides you with the function y in terms of x that satisfies the original DE. All this and much more is revealed in Chapter 14.

Understanding Infinite Series

The last third of a typical Calculus II course — roughly five weeks — usually focuses on the topic of infinite series. I cover this topic in detail in Part 6. Here's an overview of some of the ideas you find out about there.

Distinguishing sequences and series

A *sequence* is a string of numbers in a determined order. For example:

$$2, 4, 6, 8, 10, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Sequences can be finite or infinite, but calculus deals well with the infinite, so it should come as no surprise that calculus concerns itself only with *infinite sequences.*

You can turn an infinite sequence into an *infinite series* by changing the commas into plus signs:

$$2+4+6+8+10+\ldots$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

Sigma notation, which I discuss further in Chapter 2, is useful for expressing infinite series more succinctly. For example, here's how to express these three series using sigma notation:

$$\sum_{n=1}^{\infty} 2n = 2+4+6+8+10+\ldots$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

Chapter 15 gets you started on sequences and series.

Evaluating series

Evaluating an infinite series is often possible. That is, you can find out what all those numbers add up to. A helpful way to get a handle on some series is to create a related *sequence of partial sums* — that is, a sequence that includes the first term, the sum of the first two terms, the sum of the first three terms, and so forth. For example, here's a sequence of partial sums for the second series shown earlier:

$$1 = 1$$
$$1 + \frac{1}{2} = 1\frac{1}{2}$$
$$1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$$
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1\frac{7}{8}$$
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$$

The resulting sequence of partial sums provides strong evidence of this conclusion:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 2$$

Chapter 15 focuses on understanding related forms of sequences and series.

Identifying convergent and divergent series

When a series evaluates to a number — as does $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ — it's called a *convergent series.* When a series isn't convergent, it's called a *divergent series.*

Identifying whether a series is convergent or divergent isn't always simple. For example, take another look at the third series I introduce earlier in this section:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = ?$$

This is called the *harmonic series,* but can you guess by looking at it whether it converges or diverges? (Before you begin adding fractions, let me warn you that the partial sum of the first 10,000 numbers is approximately 10.)

An ongoing problem as you study infinite series is deciding whether a given series is convergent or divergent. Chapter 16 gives you a slew of tests to help you find out. Then, in Chapter 17, you focus on how *power series* — convergent series that have many of the same properties as polynomials — can be "Taylored" to approximate definite integrals.

IN THIS CHAPTER

- Recalling basic operations with fractions
- Making sense of exponents of 0, negative numbers, and fractions
- Graphing common continuous functions and their transformations
- Remembering trig identities
- Playing with polar coordinates
- Working with sigma summation notation

Chapter 2

Forgotten but Not Gone: Review of Algebra and Pre-Calculus

Remember Charles Dickens's *A Christmas Carol*? You know, Scrooge and those ghosts from the past. Math can be just like that story: All the stuff you thought was dead and buried for years suddenly pays a spooky visit when you least expect it.

This quick review is designed to save you from any unnecessary sleepless nights. Before you proceed any further on your calculus quest, make sure that you're on good terms with the information in this chapter.

First, I cover a few pre-algebra topics, plus some algebra: working with fractions, factorials, exponents, and polynomials.

Next, I cover all the pre-calculus you forgot to remember: trigonometric ratios, radian measure, trig identities, graphing common functions, and basic transformations of functions.

To finish up, you get a quick review of polar coordinates and sigma notation.

If you still feel stumped after you finish this chapter, I recommend that you pick up a copy of *Pre-Calculus For Dummies* by Deborah Rumsey, PhD, for a more in-depth review.

Quick Review of Pre-Algebra and Algebra

If you're taking a Calculus II class, you've obviously done a lot of math already. Even so, sometimes it may be difficult to remember some of the easier math you learned a long time ago. In this section, I give you a quick review of a few pre-algebra topics that you'll need to use in Calculus II.

Working with fractions

If you haven't worked with fractions for a while, you might find them confusing. In this section, I give you a quick refresher on adding, subtracting, multiplying, and dividing fractions. If you're really rusty on the skills, check out my book *Basic Math and Pre-Algebra All-In-One For Dummies* (Wiley 2022) for a more detailed explanation.

Adding fractions

To add fractions with the same denominator, add the numerators and keep the denominator the same:

$$\frac{1}{5}+\frac{2}{5}=\frac{3}{5} \qquad \frac{1}{8}+\frac{5}{8}=\frac{6}{8}=\frac{3}{4} \qquad \frac{7}{10}+\frac{9}{10}=\frac{16}{10}=\frac{8}{5}$$

As you can see, you simplify your answer whenever possible. However, notice that in the last example, I simplify the fraction but I keep it as an improper fraction instead of changing it to the mixed number $1\frac{3}{5}$. Improper fractions are much easier to work with than mixed numbers, and most calculus teachers won't expect you to convert them. (If yours does, I apologize on behalf of all math teachers!)

To add a pair of fractions with different denominators, find a common denominator by increasing the terms of one or both fractions.

$$\frac{2}{3}+\frac{7}{12}=\frac{8}{12}+\frac{7}{12}=\frac{15}{12}=\frac{5}{4}$$

$$\frac{4}{5}+\frac{3}{7}=\frac{28}{35}+\frac{15}{35}=\frac{43}{35}$$

$$\frac{1}{4}+\frac{5}{6}=\frac{3}{12}+\frac{10}{12}=\frac{13}{12}$$

TIP

A common operation when using the Power Rule for Differentiation (see Chapter 3) is adding 1 to a fraction. Here's a quick way to do this in your head:

1. Add the numerator and denominator together to find the numerator of the answer.
2. Keep the denominator the same.

For example:

$$\frac{2}{3}+1=\frac{2+3}{3}=\frac{5}{3} \qquad \frac{5}{2}+1=\frac{5+2}{2}=\frac{7}{2} \qquad -\frac{7}{4}+1=\frac{-7+4}{4}=-\frac{3}{4}$$

Subtracting fractions

In a similar way to addition, to subtract fractions with the same denominator, subtract the numerators and keep the denominator the same:

$$\frac{7}{9}-\frac{2}{9}=\frac{5}{9} \qquad \frac{11}{8}-\frac{7}{8}=\frac{4}{8}=\frac{1}{2} \qquad \frac{1}{6}-\frac{5}{6}=-\frac{4}{6}=-\frac{2}{3}$$

As with addition, simplify your answer whenever possible. And when you subtract a larger fraction from a smaller one, the answer is negative.

To subtract a pair of fractions with different denominators, find a common denominator by increasing the terms of one or more fraction.

$$\frac{2}{3}-\frac{4}{15}=\frac{10}{15}-\frac{4}{15}=\frac{6}{15}=\frac{2}{5}$$

$$\frac{3}{8}-\frac{4}{5}=\frac{15}{40}-\frac{32}{40}=-\frac{17}{40}$$

$$\frac{5}{6}-\frac{3}{8}=\frac{20}{24}-\frac{9}{24}=\frac{11}{24}$$

TIP

Subtracting 1 from a fraction is also a common operation in calculus, used when applying the Power Rule for Integration (see Chapter 7). Here's a quick way to do this:

1. Subtract the numerator minus the denominator to find the numerator of the answer.
2. Keep the denominator the same.

For example:

$$\frac{9}{4}-1=\frac{9-4}{4}=\frac{5}{4} \qquad \frac{2}{5}-1=\frac{2-5}{5}=-\frac{3}{5} \qquad -\frac{3}{7}-1=\frac{-3-7}{7}=-\frac{10}{7}$$

Multiplying fractions

In contrast to addition and subtraction, you don't need to find a common denominator before multiplying fractions. Simply multiply numerators and then multiply denominators, as follows:

$$\frac{2}{5}\times\frac{4}{7}=\frac{8}{35} \qquad \frac{8}{9}\times\frac{4}{3}=\frac{32}{27} \qquad \frac{6}{25}\times\frac{5}{8}=\frac{\cancel{6}\,3}{\cancel{25}\,5}\times\frac{\cancel{5}\,1}{\cancel{8}\,4}=\frac{3}{20}$$

As you can see from the third example, in some cases you can make the multiplication easier by cross-canceling common factors before you multiply.

TIP

If you cross-cancel all the factors before you multiply, the multiplication will be relatively easy and you won't have to simplify your result at the end.

Dividing fractions

As with multiplication, you don't need to find a common denominator when dividing fractions. To divide fractions, turn the division into multiplication by applying the mnemonic **Keep-Change-Flip:**

1. **Keep** the first fraction as it is.
2. **Change** the division sign to a multiplication sign.
3. **Flip** the second fraction to its reciprocal (or inverse).

For example:

$$\frac{5}{7}\div\frac{8}{9}=\frac{5}{7}\times\frac{9}{8}=\frac{45}{56}$$

$$\frac{3}{4}\div 5=\frac{3}{4}\times\frac{1}{5}=\frac{3}{20}$$

$$\frac{5}{6}\div\frac{3}{10}=\frac{5}{6}\times\frac{10}{3}=\frac{5}{\cancel{6}\,3}\times\frac{\cancel{10}\,5}{3}=\frac{25}{9}$$

Before moving on, take a closer look at the third example. Notice that the first step when dividing is *always* **Keep-Change-Flip**. This changes the problem to multiplication.

WARNING

Never cross-cancel common factors in a division problem before you change the problem to multiplication.

TIP

After you change problem from division to multiplication, but before you multiply, cross-cancel every common factor you can. This practice makes the numbers smaller, so the calculation will be relatively easy and you won't have to simplify your result at the end.

Knowing the facts on factorials

The *factorial* of a positive integer, represented by an exclamation point (!), is that number multiplied by every positive integer less than itself. For example:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Notice that the factorial of every positive number equals that number multiplied by the next-lowest factorial. For example:

$$6! = 6(5!)$$

Generally speaking, then, the following equality is true:

$$n! = n(n-1)!$$

This equality provides the rationale for the odd-looking convention that $0! = 1$:

$$1! = 1(0!)$$
$$1 = 0!$$

When factorials show up in fractions (as they do when working with infinite series, as you see in Chapters 16 and 17), you can usually do a lot of cancellation that makes them simpler to work with. For example:

$$\frac{3!}{5!} = \frac{3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{5 \times 4} = \frac{1}{20}$$

Even when a fraction includes factorials with variables, you can usually simplify it. For example:

$$\frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1$$

Factorials turn out to be very handy when you're working with infinite series in Part 6. And they're indispensible for understanding Taylor and McLaurin series in Chapter 17.

Polishing off polynomials

A polynomial is any function of the following form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0$$

Note that every term in a polynomial is x raised to the power of a nonnegative integer, multiplied by an integer coefficient. Here are a few examples of polynomials:

$$f(x) = 3x^4 + 2x - 5 \qquad f(x) = -x^{12} + x^7 + 100x - 9 \qquad f(x) = \left(x^2 + 8\right)(x - 6)$$

Note that in the last example, expanding the right side of the equation changes the polynomial to a more recognizable form, called *standard form*:

$$f(x) = \left(x^2 + 8\right)(x - 6) = x^3 - 6x^2 + 8x - 48$$

When a polynomial is in standard form, its terms are arranged with the exponents in descending order and the constant at the end.

TIP

Integrating standard-form polynomials is relatively easy, so a good first step when working with a polynomial in any other form is to convert it to standard form.

Powering through powers (exponents)

When you find a *power* of a number, you multiply one number (the *base*) by itself the number of times indicated by another number (the *exponent*). For example:

$$4^3 = 4 \times 4 \times 4 = 64 \qquad 10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$$

$$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$$

You can use this same rule to apply a positive integer exponent to bases that are rational, irrational, or even complicated-looking algebraic expressions like the following:

$$\left(\frac{5}{6}\right)^3 = \frac{5\times5\times5}{6\times6\times6} = \frac{125}{216} \qquad \left(\frac{\sqrt{2}}{3}\right)^4 = \frac{\sqrt{2}\times\sqrt{2}\times\sqrt{2}\times\sqrt{2}}{3\times3\times3\times3} = \frac{4}{81}$$

$$\left(\frac{e^x+1}{x}\right)^2 = \frac{\left(e^x+1\right)\left(e^x+1\right)}{x\times x} = \frac{e^{2x}+2e^x+1}{x^2}$$

In this section, you extend this understanding of powers to less intuitive exponents, such as 0, negative numbers, and fractions.

Understanding zero and negative exponents

Exponents of 0 and negative numbers make sense when you observe the powers of 2, as shown in Table 2-1:

TABLE 2-1 Positive and Negative Integer Exponents of 2

2^{-5}	2^{-4}	2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3	2^4	2^5
$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32

As you can see, each power of two is twice the value of the previous one, leading to the following conclusions about exponents that aren't positive integers:

$$2^0 = 1 \qquad 2^{-1} = \frac{1}{2} \qquad 2^{-2} = \frac{1}{4}$$

This insight generalizes to other nonpositive exponents, resulting in the five rules shown in Table 2-2, . . ; which you'll use a lot in Calculus II:

TABLE 2-2 Rules for Simplifying Exponents

Rule	$x^a x^b = x^{a+b}$	$\frac{x^a}{x^b} = x^{a-b}$	$(x^a)^b = x^{ab}$	$x^{-a} = \frac{1}{x^a}$	$x^{\frac{a}{b}} = \left(\sqrt[b]{x}\right)^a$
Algebra Example	$x^3 x^5 = x^8$	$\frac{x^9}{x^4} = x^5$	$(x^6)^2 = x^{12}$	$x^{-7} = \frac{1}{x^7}$	$x^{\frac{3}{5}} = \left(\sqrt[5]{x}\right)^3$
Arithmetic Example	$2^6 \cdot 2^4 = 2^{10} = 1{,}024$	$\frac{7^8}{7^6} = 7^2 = 49$	$(10^2)^3 = 10^6 = 1{,}000{,}000$	$5^{-4} = \frac{1}{5^4} = \frac{1}{625}$	$128^{\frac{3}{7}} = \left(\sqrt[7]{128}\right)^3 = 2^3 = 8$

Understanding fractional exponents

An alternative way to express a square root is as a power of $\frac{1}{2}$:

$$x^{\frac{1}{2}} = \sqrt{x}$$

To see why this rule makes sense, square both sides of this equation and then simplify using rules you already know:

$$\left(x^{\frac{1}{2}}\right)^2 = \left(\sqrt{x}\right)^2$$

$$x = x$$

You can generalize this rule for other whole-number denominators as follows:

$$x^{\frac{1}{3}} = \sqrt[3]{x} \qquad x^{\frac{1}{4}} = \sqrt[4]{x} \qquad x^{\frac{1}{5}} = \sqrt[5]{x}$$

Finally, here's the most general form of the rule for fractional exponents, expressed in two different but equivalent ways:

$$x^{\frac{a}{b}} = \left(\sqrt[b]{x}\right)^a \qquad x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

For convenience, I include the first version of this formula in Table 2-2. In practice, I find this version the more useful of the two, especially when working with numbers rather than variables. For example:

$$8^{\frac{5}{3}} = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32 \qquad 8^{\frac{5}{3}} = \sqrt[3]{8^5} = \sqrt[3]{32{,}768} = 32$$

As you can see, the first version of the formula keeps the intermediate results small enough to calculate in your head. But you may find both versions helpful in your Calculus II course.

Expressing functions using exponents

In Calculus II, expressing functions as exponents becomes essential. Here are some examples of how to apply the rules enumerated in the previous section to some common functions.

First, remember that you can rewrite an exponent in the denominator of a rational expression as its reciprocal by negating the exponent. For example:

$$f(x) = \frac{1}{x} = x^{-1} \qquad f(x) = \frac{1}{x^2} = x^{-2} \qquad f(x) = \frac{1}{x^3} = x^{-3}$$

You can also rewrite any radical function using a fractional exponent. For example:

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \qquad f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

Combining these two rules allows you to rewrite more functions using negative fractional exponents:

$$f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \qquad f(x) = \frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}} \qquad f(x) = \frac{1}{\sqrt[4]{x}} = x^{-\frac{1}{4}}$$

More complex combinations of radicals and exponents can also be expressed as fractional exponents:

$$f(x) = \sqrt{x^3} = x^{\frac{3}{2}} \qquad f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}} \qquad f(x) = \sqrt[4]{x^3} = x^{\frac{3}{4}}$$

Finally, combining these rules with the rule for negative exponents allows you to rewrite even more complex-looking combinations of rational and radical functions:

$$f(x) = \frac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}} \qquad f(x) = \frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}} \qquad f(x) = \frac{1}{\sqrt[4]{x^3}} = x^{-\frac{3}{4}}$$

TIP

Keep these sorts of tricks in mind as you're starting to integrate functions like these in Chapters 6 and 7. Turning a weird-looking function into an exponent allows you to use the Power Rule for Integration, which is almost as simple to use as the Power Rule for Differentiation that you know from Calculus I.

Rewriting rational functions using exponents

Recall that a rational function is a polynomial divided by another polynomial. (See the previous section for a quick review of polynomials.)

Another key algebra skill that you'll use a lot in Calculus II is splitting a single rational function into the sum of two or more rational functions.

When a rational function has a one-term polynomial in the base, you can split that function into a sum of rational functions and simplify each resulting term. For example:

$$f(x) = \frac{6x^5 + x^2 - 7}{2x^3}$$

You can split this rational function into the following three-term expression:

$$= \frac{6x^5}{2x^3} + \frac{x^2}{2x^3} - \frac{7}{2x^3}$$

Now, simplify these three terms by dividing the top exponent minus the bottom exponent:

$$= 3x^2 + \frac{1}{2}x^{-1} - \frac{7}{2}x^{-3}$$

Notice in each case how each pair of coefficients in the numerator and denominator is simplified to a single coefficient in each term. Although this result may look more complicated, it's a much more useful way to express the rational function you started with, as you'll see when you start integrating.

Simplifying rational expressions by factoring

Many students find factoring polynomials to be difficult, disagreeable, and downright depressing. I can sympathize, but unfortunately, factoring is often the only way to get an ugly rational function to behave. Here, I give you a couple of examples of rational functions that can be tamed by factoring.

To begin, remember that factoring out the greatest common factor (GCF) of both the numerator and denominator of a fraction can often be very helpful. For example:

$$\frac{9x^2 + 6x - 15}{6x^3 + 4x^2 - 10x}$$

To begin, factor out the GCF on top and on the bottom, and then cancel out a common factor:

$$= \frac{3(3x^2 + 2x - 5)}{2x(3x^2 + 2x - 5)} = \frac{3}{2x}$$

Now, rewrite this result as a negative exponent:

$$\frac{3}{2}x^{-1}$$

This form is very easy to integrate, as you'll see in Chapters 6 and 7.

Here's an example that requires a bit more advanced factoring:

$$=\frac{2x^3-6x^2-8x+24}{5x^3-5x^2-30x}$$

TIP

Whenever you decide that factoring might be helpful, always start with GCF factoring:

$$=\frac{2\left(x^3-3x^2-4x+12\right)}{5x\left(x^2-x-6\right)}$$

Now, the polynomial in the denominator is a factorable quadratic:

$$=\frac{2\left(x^3-3x^2-4x+12\right)}{5x(x+2)(x-3)}$$

The polynomial in the numerator is a factorable cubic. Here, I factor it by grouping in two steps, as you should recall from Algebra II, to remind you how to do it:

$$=\frac{2\left[x^2(x-3)-4(x-3)\right]}{5x(x+2)(x-3)}=\frac{2\left(x^2-4\right)(x-3)}{5x(x+2)(x-3)}$$

You can still factor x^2-4 in the numerator as a difference of squares:

$$=\frac{2(x+2)(x-2)(x-3)}{5x(x+2)(x-3)}$$

At last, you're ready to cancel factors:

$$=\frac{2(x-2)}{5x}$$

In an Algebra class, you'd be done. But for Calculus II, you still need to take a few more steps to break this result into a difference of exponential functions. I outline these steps earlier in this chapter:

$$=\frac{2x-4}{5x}=\frac{2x}{5x}-\frac{4}{5x}=\frac{2}{5}-\frac{4}{5}x^{-1}$$

This version of the function you started with is very easy to integrate using the Power Rule for Integration, as you'll discover in Chapters 6 and 7.

Review of Pre-Calculus

Pre-calculus covers a wide range of relatively advanced math topics dealing with the most commonly used functions, such as polynomials, trig functions, exponentials, and logarithmic functions. Calculus uses these functions frequently, so you really need to know them in order to do well in calculus.

What's also true is that a typical pre-calculus course includes a lot of material that you really don't need in order to do calculus.

In this section, I focus on the pre-calculus topics that you really, really need to understand as you proceed from Calculus I into Calculus II.

Trigonometry

If you love trigonometry more than the first snow day of the school year, you're in for a treat: Calculus II is just as chock-full of trigonometry as was Calculus I.

If you're not a fan, well, I'm not going to sugar-coat it: Trig is simply unavoidable in calculus of every variety.

In this section, I fill you in on some of the most important parts of trig that you'll need in order to do well in your Calculus II course.

Of course, I can't cover everything you need to know about trig here. For more detailed information on this topic, see *Trigonometry For Dummies* by Mary Jane Sterling (Wiley).

Noting trig notation

Trig notation can sometimes be a little confusing, so here are some basics.

For starters, when you see the notation

$$2 \cos x$$

remember that this means 2 (cos x). So to evaluate this function for $x = \pi$, evaluate the inner function cos x first, and then multiply the result by 2:

$$2 \cos \pi = 2 \times (-1) = -2$$

On the other hand, the notation

$$\cos 2x$$

means cos (2*x*). For example, to evaluate this function for $x = 0$, evaluate the inner function 2*x* first, and then take the cosine of the result:

$$\cos\ (2 \cdot 0) = \cos\ 0 = 1$$

Finally (and make sure you understand this one!), the notation

$$\cos^2 x$$

means

$$(\cos x)^2 = (\cos x) \cdot (\cos x)$$

In other words, to evaluate this function for $x = \pi$, evaluate the inner function cos *x* first, and then take the square of the result:

$$\cos^2 \pi = (\cos \pi)^2 = (-1)^2 = 1$$

WARNING

Be careful when working with notation such as $\sin^{-1} x$ and $\cos^{-1} x$. This notation denotes the inverse sine and cosine operations $\arcsin x$ and $\arccos x$. It does NOT stand in for the reciprocal functions csc *x* and sec *x*. In my humble opinion, inverse trig functions are confusing enough without throwing in ambiguous notation. So, throughout this book, I use the notation $\arcsin x$ and $\arccos x$ exclusively. But your textbook and your teacher may have other ideas, so stay alert!

Getting clear on how to evaluate trig functions really pays off when you're applying the Chain rule (which I discuss in Chapter 3) and when integrating trig functions (which I focus on in Chapter 10).

Figuring the angles with radians

When you first discovered trigonometry, you probably used degrees because they were familiar from geometry. Along the way, you were introduced to radians and forced to do a bunch of conversions between degrees and radians, and then in the next chapter you went back to using degrees.

Degrees are great for certain trig applications, such as land surveying. But for math, radians are the right tool for the job. In contrast, degrees are awkward to work with.

For example, consider the expression $\sin\ 1{,}260°$. You probably can't tell just from looking at this expression that it evaluates to 0, because $1{,}260°$ is a multiple of $180°$.

You can tell immediately that the equivalent expression $\sin 7\pi$ is a multiple of π. And, as an added bonus, when you work with radians, the numbers tend to be smaller, and you don't have to add the degree symbol (°).

I find that students tend to understand radians more and hate them less when they can see how they arise naturally from stuff they already know from geometry. To begin, recall the formula for the circumference of a circle:

$$\text{Circumference} = 2\pi r$$

Now, consider that a *unit circle* has a radius of 1, so this equation simplifies further:

$$\text{Circumference} = 2\pi$$

You probably also know that a circle has 360° all the way around its circumference, so you can substitute this value into the equation:

$$360° = 2\pi$$

This equivalence provides the basis for converting degrees to radians. For example, dividing both sides by 2 gives you the value of 180° in radians:

$$180° = \pi$$

From here, you can use the conversion factor $\frac{\pi}{180°}$ to convert from degrees to radians. For example, here's how to convert 10° to radians:

$$10° \cdot \frac{\pi}{180°} = \frac{10\pi}{180} = \frac{\pi}{18}$$

In a similar way, you can use $\frac{180°}{\pi}$ to convert from radians to degrees. For example, convert $\frac{3\pi}{10}$ radians to degrees as follows:

$$\frac{3\pi}{10} \cdot \frac{180°}{\pi} = \frac{540°}{10} = 54°$$

Figure 2-1 shows you some common angles in both degrees and radians.

TIP

Radians are the basis of polar coordinates, which I discuss later in this chapter.

Identifying some important trig identities

I know that committing trig identities to memory registers on the Fun Meter someplace between alphabetizing your spice rack and vacuuming the lint filter on your dryer. But knowing a few important trig identities can be a lifesaver when you're lost out on the misty calculus trails, so I recommend that you take a few along with you.

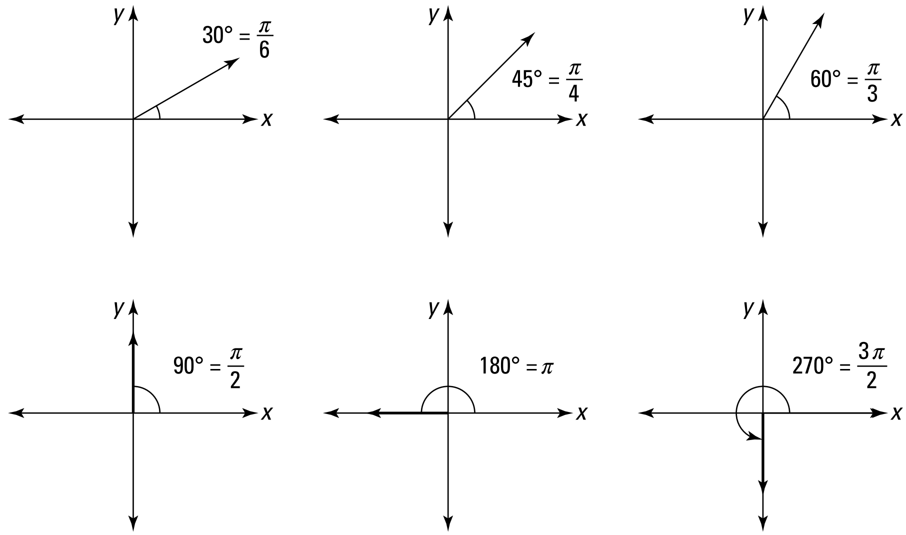

FIGURE 2-1: Some common angles in degrees and radians.

For starters, here are the three *reciprocal identities,* which you probably know already:

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

You also need these two important identities:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

I call these the Basic Five trig identities. By using them, you can express *any* trig expression in terms of sines and cosines. Less obviously, you can also express any trig expression in terms of tangents and secants, or in terms of cotangents and cosecants. Both of these facts are useful in Chapter 6, when I discuss trig integration.

Equally indispensable are the three *Pythagorean Identities.* Most students remember the first and forget about the other two, but you need to know them all:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Before moving on, take a moment to notice how these three identities break the six trig functions into three pairings:

Sines and cosines

Tangents and secants

Cotangents and cosecants

You'll find that these three pairings arise frequently when working with trig functions in Calculus II, so keep an eye on them.

For trig substitution (which I discuss in Chapter 10), you also need the two *half-angle identities* for sines and cosines:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

HOW TO AVOID AN IDENTITY CRISIS

Most students remember the first Pythagorean Identity without trouble:

$\sin^2 x + \cos^2 x = 1$

If you're worried you may forget the other two Pythagorean Identities just when you need them most, don't despair. An easy way to remember them is to divide every term in the first square identity by $\cos^2 x$ to produce one new equation and by $\sin^2 x$ to produce another.

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

Now simplify these equations using the Basic Five trig identities:

$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Finally, the *double-angle identity for sines* is also useful in Calculus II:

$$\sin 2x = 2\sin x \cos x$$

Beyond these, if you have a little spare time, you can include these *double-angle identities for cosines and tangents* in your ever-growing list of formulas to memorize:

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Asymptotes

An *asymptote* is any straight line on a graph that a function approaches but doesn't touch.

For example, in the graph of the exponential function in Figure 2-4, the *x*-axis is a horizontal asymptote, because the function approaches but doesn't touch this line as *x* decreases.

Similarly, all four graphs in Figure 2-7 have vertical asymptotes represented by dashed lines.

Another way to think of an asymptote is to see that it's a limit as a value approaches either ∞ or $-\infty$. For example, you can describe the asymptote in Figure 2-4 as follows:

$$\lim_{x \to -\infty} e^x = 0$$

I discuss asymptotes further when I focus on graphing parent functions in the next section.

Graphing common parent functions

A *parent function* is the simplest version of a function, providing a template for other functions that have similar features, collectively called a *family of functions*.

In pre-calculus, you work in-depth with a relatively short list of parent functions and their related families. This background is important in Calculus II, so you should be familiar with how certain common functions look and behave when drawn on a graph.

In this section, I show you the most common graphs of parent functions. These functions are continuous, so they're integrable at all real values of *x*.

Linear and polynomial functions

Figure 2-2 shows three simple linear functions: $y = n$, $y = x$, and $y = |x|$.

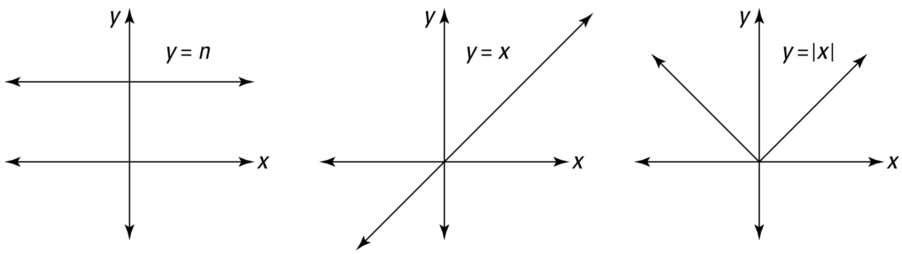

FIGURE 2-2: Graphs of two linear functions $y = n$ and $y = x$ and the absolute value function $y = |x|$.

Figure 2-3 includes a few basic polynomial functions: $y = x^2$, $y = x^3$, and $y = x^4$.

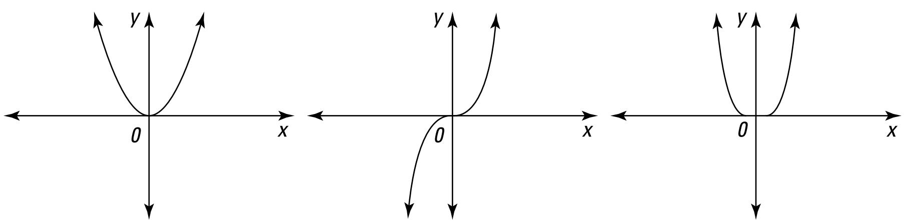

FIGURE 2-3: Graphs of three polynomial functions, $y = x^2$, $y = x^3$, and $y = x^4$.

Exponential and logarithmic functions

Here are some *exponential functions* with whole number bases:

$y = 2^x$

$y = 3^x$

$y = 10^x$

For every positive base, the exponential function

- Crosses the *y*-axis at $y = 1$
- Explodes to infinity as *x* increases (that is, it has an unbounded *y* value)
- Approaches $y = 0$ as *x* decreases (that is, in the negative direction, the *x*-axis is an *asymptote*)

The most important exponential function is e^x. See Figure 2-4 for a graph of this function.

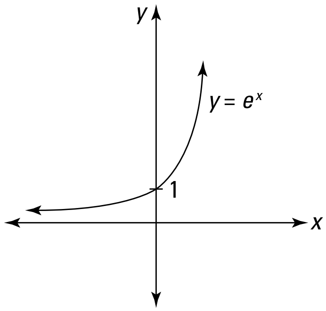

FIGURE 2-4: Graph of the exponential function $y = e^x$.

The unique feature of this exponential function is that at every value of x, its slope is e^x. That is, this function is its own derivative (see Chapter 3 for more on derivatives).

Another important function is the *logarithmic function* (also called the *natural log function*). Figure 2-5 is a graph of the logarithmic function $y = \ln x$.

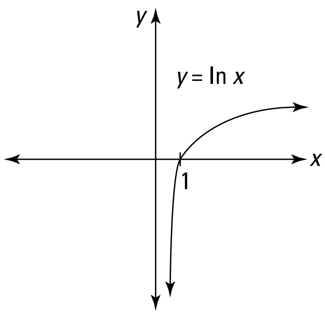

FIGURE 2-5: Graph of the logarithmic function $y = \ln x$.

Notice that this function is the reflection of e^x along the diagonal line $y = x$. So the log function does the following:

- Crosses the x-axis at $x = 1$
- Explodes to infinity as x increases (that is, it has an unbounded y value), though more slowly than any exponential function
- Produces a y value that approaches $-\infty$ as x approaches 0 from the right (that is, from the positive direction, the y-axis is an *asymptote* in which $\lim_{x \to 0^+} \ln x = -\infty$)

Furthermore, the domain of the log functions includes only positive values. That is, inputting a nonpositive value to the log function is a big no-no, on par with placing 0 in the denominator of a fraction or a negative value inside a square root.

For this reason, functions placed inside the log function often get "pretreated" with the absolute value operator. For example:

$$y = \ln |x|$$

Trigonometric functions

The two most important graphs of trig functions are the sine and cosine. See Figure 2-6 for graphs of these functions.

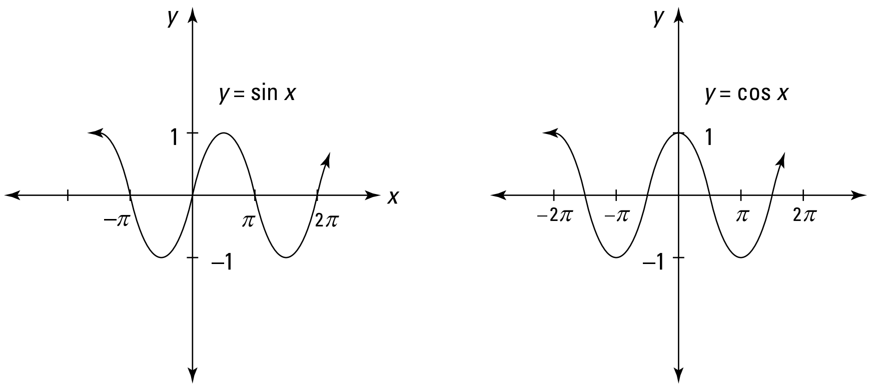

FIGURE 2-6: Graphs of the trig functions $y = \sin x$ and $y = \cos x$.

Note that the *x* values of these two graphs are typically marked off in multiples of π. Each of these functions has a *period* of 2π — in other words, it repeats its values after 2π units. And each has a maximum value of 1 and a minimum value of −1.

Remember that the sine function

- Crosses the origin
- Rises to a value of 1 at $\frac{\pi}{2}$
- Crosses the *x*-axis at all multiples of π

Remember that the cosine function

- Has a value of 1 at $x = 0$
- Drops to a value of 0 at $\frac{\pi}{2}$
- Crosses the x-axis, at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, and so on

The graphs of other trig functions are also worth knowing. Figure 2-7 shows graphs of the trig functions $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$.

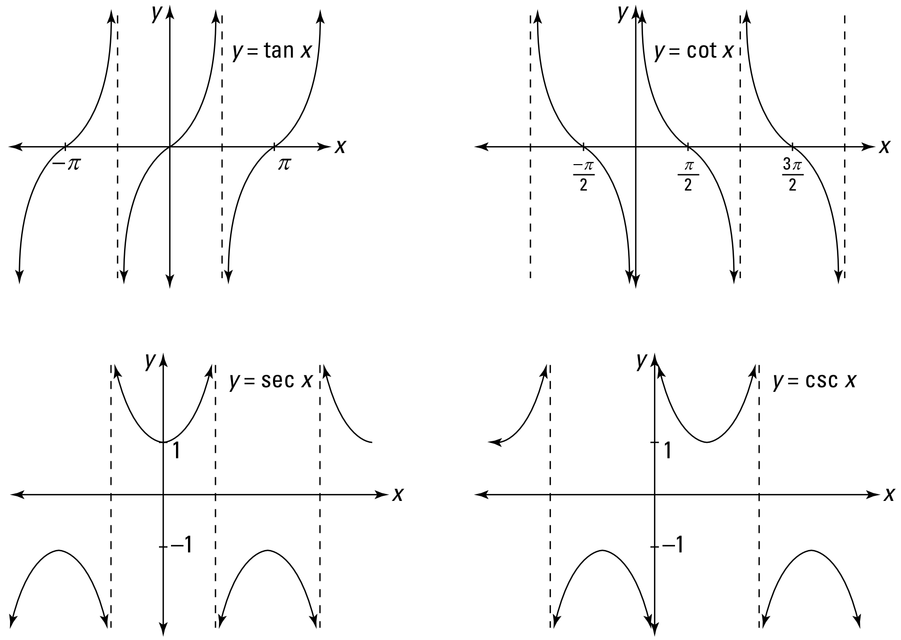

FIGURE 2-7: Graphs of the trig functions $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$.

Transforming continuous functions

When you know how to graph the most common functions, you can transform them by using a few simple tricks, as I show you in Table 2-3.

The vertical transformations are intuitive — that is, they take the function in the direction that you'd probably expect. For example, adding a constant shifts the function up and subtracting a constant shifts it down. Similarly, multiplying by a constant greater than 1 results in a vertical stretch, and dividing by a constant greater than 1 results in a vertical compression.

TABLE 2-3 Five Vertical and Five Horizontal Transformations of Functions

Axis	Direction	Transformation	Example
y-axis (vertical)	Shift up	$y = f(x) + n$	$y = e^x + 1$
	Shift down	$y = f(x) - n$	$y = x^3 - 2$
	Stretch	$y = nf(x)$	$y = 5 \sec x$
	Compress	$y = \frac{f(x)}{n}$	$y = \frac{\sin x}{10}$
	Reflect	$y = -f(x)$	$y = -(\ln x)$
x-axis (horizontal)	Shift right	$y = f(x - n)$	$y = e^{x-2}$
	Shift left	$y = f(x + n)$	$y = (x + 4)^3$
	Stretch	$y = f\left(\frac{x}{n}\right)$	$y = \sec \frac{x}{3}$
	Compress	$y = f(nx)$	$y = \sin(\pi x)$
	Reflect	$y = f(-x)$	$y = e^{-x}$

In contrast, the horizontal transformations are counterintuitive — that is, they take the function in the direction that you probably wouldn't expect. For example, adding a constant shifts the function left and subtracting a constant shifts it right. Similarly, multiplying by a constant greater than 1 results in a horizontal compression, and dividing by a constant greater than 1 results in a horizontal stretch.

Polar coordinates

Polar coordinates are an alternative to the Cartesian coordinate system. As with Cartesian coordinates, polar coordinates assign an ordered pair of values to every point on the plane. Unlike Cartesian coordinates, however, these values aren't (x, y), but rather (r, θ).

- The value r is the distance to the origin.
- The value θ is the angular distance from the polar axis, which corresponds to the positive x-axis in Cartesian coordinates. (Angular distance is always measured counterclockwise.)

Figure 2-8 shows how to plot points in polar coordinates. For example:

- To plot the point $\left(3, \frac{\pi}{4}\right)$, travel 3 units from the origin on the polar axis, and then travel in a circular arc $\frac{\pi}{4}$ units (equivalent to $45°$) counterclockwise.
- To plot $\left(4, \frac{5\pi}{6}\right)$, travel 4 units from the origin on the polar axis, and then travel in a circular arc $\frac{5\pi}{6}$ units (equivalent to $150°$) counterclockwise.
- To plot the point $\left(2, \frac{3\pi}{2}\right)$, travel 2 units from the origin on the polar axis, and then travel in a circular arc $\frac{3\pi}{2}$ units (equivalent to $270°$) counterclockwise.

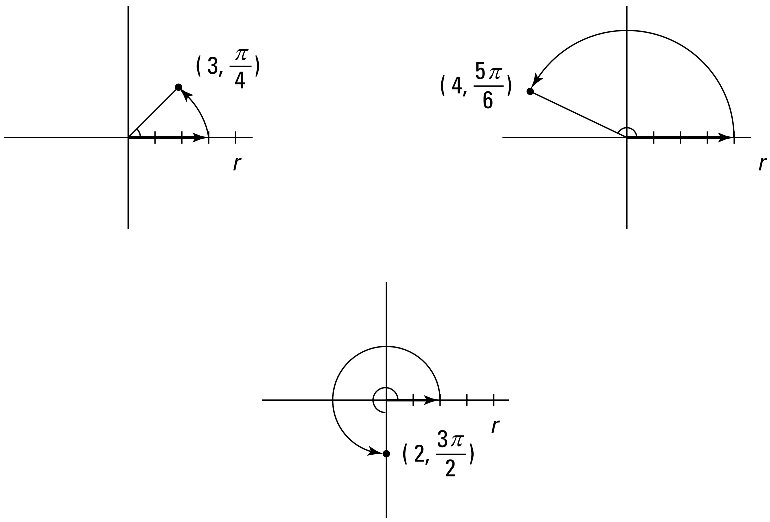

FIGURE 2-8: Plotting points in polar coordinates.

Polar coordinates allow you to plot certain shapes on the graph more simply than Cartesian coordinates. For example, here's the equation for a 3-unit circle centered at the origin in both Cartesian and polar coordinates:

$$x^2 + y^2 = 9 \qquad\qquad r = 3$$

Some problems that would be difficult to solve when expressed in terms of Cartesian variables (x and y) become much simpler when expressed in terms of polar variables (r and θ). To convert Cartesian variables to polar, use the following formulas:

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

To convert polar variables to Cartesian, use these formulas:

$$r = \pm\sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

You work more with polar coordinates when doing trig substitution in Chapter 10.

Summing up sigma notation

Mathematicians love sigma notation ($\sum$) for two reasons. First, it provides a convenient way to express a long or even infinite series. But even more important, it looks really cool and scary, which frightens people into revering mathematicians and paying them more money.

However, when you get right down to it, $\sum$ is just fancy notation for addition, and even your little brother isn't afraid of adding, so why should you be?

For example, suppose that you want to add the even numbers from 2 to 10. Of course, you can write this expression and its solution this way:

$$2 + 4 + 6 + 8 + 10 = 30$$

Or you can write the same expression by using sigma notation:

$$\sum_{n=1}^{5} 2n$$

Here, n is the *index of summation* — that is, the variable that you plug values into and then add. Below the $\sum$, you're given the starting value of n (1) and above it, the ending value (5). So here's how to expand the notation:

$$\sum_{n=1}^{5} 2n = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) = 30$$

You can also use sigma notation to stand for the sum of an infinite number of values — that is, an *infinite series.* For example, here's how to add up all the positive square numbers:

$$\sum_{n=1}^{\infty} n^2$$

This compact expression can be expanded as follows:

$$= 1^2 + 2^2 + 3^2 + 4^2 + \ldots$$

$$= 1 + 4 + 9 + 16 + \ldots$$

This sum is, of course, infinite. But not all infinite series behave in this way. In some cases, an infinite series equals a number. For example:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$$

This series expands and evaluates as follows:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

When a series evaluates to a number, the series is *convergent.* When a series isn't convergent, it's *divergent.* As you can see, expanding sigma notation is often an important first step toward understanding a series and evaluating whether it converges to a value or diverges infinitely.

You find out all about convergent and divergent series in Chapter 16.

IN THIS CHAPTER

» Understanding and evaluating limits

» Seeing the relationship between a derivative and a slope

» Differentiating with the Product and Quotient rules

» Feeling confident with the Chain rule

» Evaluating indeterminate forms of limits with L'Hôpital's rule

Chapter 3

Recent Memories: Review of Calculus I

If you're currently enrolled in a Calculus II course, I'm guessing you recently completed Calculus I. Even so, you may feel, as many students do, that Calculus I is kind of *one big blur*.

If so, this chapter is probably for you. Here, I condense down a semester's worth of information into a handful of bite-sized chunks that you can consume now or as needed as Calculus II unfolds before you.

To begin, you review limits. I discuss how limits and functions are different and give you a brief refresher on evaluating limits.

Next, I give you a review of derivatives, which is the central focus of Calculus I. Recall that the derivative of a function allows you to calculate the slope of that function on any point, provided that this point is *differentiable* — that is, provided it has a definable slope.

I also give you a refresher on the key methods for differentiating (finding the derivative of) a variety of functions, such as the Power rule, the Sum rule, the Constant Multiple rule, the Product rule, the Quotient rule, and the Chain rule.

At the end of the chapter, I discuss L'Hôpital's rule (sometimes spelled L'Hospital's rule), which is an important and useful way to evaluate limits. Depending on where you are studying calculus, you may have already learned about L'Hôpital's in Calculus I, or you may be facing it as part of your Calculus II class. Either way, I cover the topic here.

For a more thorough review of these topics, check out *Calculus For Dummies* by Mark Ryan (Wiley).

Knowing Your Limits

An important thread that runs through Calculus I is the concept of a *limit.* Limits are also important in Calculus II. In this section, I give you a review of everything you need to remember but may have forgotten about limits.

Telling functions and limits apart

A function provides a link between two variables: the independent variable (usually x) and the dependent variable (usually y). A function tells you the value y when x takes on a specific value.

EXAMPLE

For example, here's a function:

$$y = x^2$$

In this case, when x takes a value of 2, the value of y is 4.

In contrast, a limit tells you what happens to y as x approaches a certain number without actually reaching it. For example, suppose that you're working with the function $y = x^2$ and want to know the limit of this function as x approaches 2. The notation to express this idea is as follows:

$$\lim_{x \to 2} x^2$$

You can get a sense of what this limit equals by plugging successively closer approximations of 2 into the function. (See Table 3-1.)

TABLE 3-1 **Approximating $\lim_{x \to 2} x^2$**

x	1.7	1.8	1.9	1.99	1.999	1.9999
y	2.89	3.24	3.61	3.9601	3.996001	3.99960001

This table provides strong evidence that the limit evaluates to 4. That is:

$$\lim_{x \to 2} x^2 = 4$$

Remember that this limit tells you nothing about what the function actually equals when $x = 2$, or even whether the function exists at that point. It tells you only that as x approaches 2, the value of the function gets closer and closer to 4. In this case, because the function and the limit are equal, the function is *continuous* at this point.

Evaluating limits

REMEMBER

Evaluating a limit means either finding the value of the limit or showing that the limit does not exist (abbreviated DNE).

You can evaluate many limits by replacing the limit variable with the number that it approaches. For example:

EXAMPLE

$$\lim_{x \to 4} \frac{x^2}{2x} = \frac{4^2}{2 \cdot 4} = \frac{16}{8} = 2$$

Sometimes this replacement shows you that a limit doesn't exist. For example:

$$\lim_{x \to \infty} x = \infty \text{ does not exist}$$

REMEMBER

When you find that a limit explodes to either ∞ or $-\infty$, the limit *does not exist*.

TECHNICAL STUFF

Technically speaking, using an equals sign to indicate that a limit "equals" ∞ or $-\infty$ is mathematically sketchy, because infinity is not a value that can be placed in an equation in this way. Most math teachers, however, will allow this type of description of what's happening with a limit.

Some replacements lead to apparently untenable situations, such as division by zero. For example:

EXAMPLE

Evaluate $\lim_{x \to 0} \frac{e^x}{x}$.

To begin, plug in 0 for x:

$$= \frac{e^0}{0} = \frac{1}{0}$$

This looks like a dead end, because division by zero is undefined. But, in fact, you can actually get an answer to this problem. Remember that this limit tells you nothing about what happens when x actually equals 0, only what happens as x *approaches* 0: The denominator shrinks toward 0, while the numerator never falls below 1, so the value fraction becomes indefinitely large. Therefore, the limit does not exist, so here's the answer to the problem:

$$\lim_{x \to 0} \frac{e^x}{x} \text{ DNE}$$

Here's another example:

EXAMPLE

$$\lim_{x \to \infty} \frac{1{,}000{,}000}{x} = \frac{1{,}000{,}000}{\infty}$$

This is another apparent dead end, because ∞ isn't really a number, so how can it be the denominator of a fraction? Again, the limit saves the day. It doesn't tell you what happens when x actually equals ∞ (if such a thing were possible), only what happens as x *approaches* ∞. In this case, the denominator becomes indefinitely large while the numerator remains constant. So

$$\lim_{x \to \infty} \frac{1{,}000{,}000}{x} = 0$$

For quick reference, here are four limits that sometimes confuse the unwary:

$$\lim_{x \to 0} \frac{x}{k} = 0 \qquad \lim_{x \to 0} \frac{k}{x} \text{ DNE} \qquad \lim_{x \to \infty} \frac{x}{k} \text{ DNE} \qquad \lim_{x \to \infty} \frac{k}{x} = 0$$

In each case, k stands for a non-zero constant. Make sure you understand why each limit is evaluated as it is (or shown not to exist):

REMEMBER

Some limits are more difficult to evaluate because they're one of several *indeterminate forms.* A good way to solve them is to use L'Hôpital's rule, which I discuss in detail at the end of this chapter.

If you need more background in limits, refer to *Calculus For Dummies* by Mark Ryan (Wiley).

Hitting the Slopes with Derivatives

The *derivative* at a given point on a function is the slope of the tangent line to that function at that point. The derivative of a function provides a "slope map" of that function.

A good way to compare a function with its derivative is by lining them up vertically. (See Figure 3-1 for an example.)

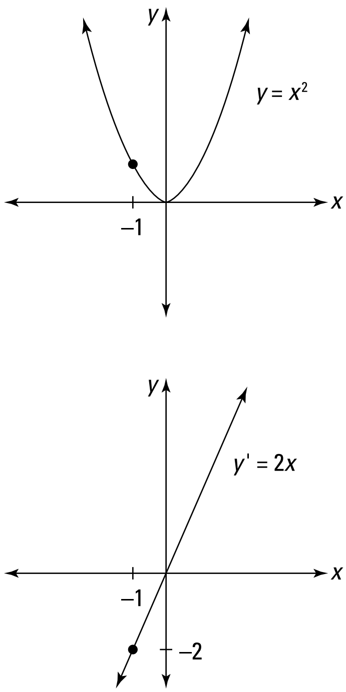

FIGURE 3-1: Comparing a graph of the function $y = x^2$ with its derivative function $y' = 2x$.

Looking at the top graph, you can see that when $x = 0$, the slope of the function $y = x^2$ is 0. The bottom graph verifies this because at $x = 0$, the derivative function $y = 2x$ is also 0.

You probably can't tell exactly what the slope of the top graph is at $x = -1$, although you can see that it's negative. To find out, look at the bottom graph and notice that at $x = -1$, the derivative function equals -2, so -2 is also the slope of the top graph at this point. Similarly, the derivative function tells you the slope at every point on the original function.

Referring to the limit formula for derivatives

In Calculus I, you develop two formulas for the derivative of a function. These formulas are both based on limits, and they're both equally valid:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

REMEMBER

You probably won't need to refer to these formulas much as you study Calculus II. Still, please keep in mind that the official definition of a function's derivative is always cast in terms of a limit.

For a more detailed look at how these formulas are developed, see *Calculus For Dummies* by Mark Ryan (Wiley).

Knowing two notations for derivatives

Students often find the notation for derivatives confusing. To make things simple, think of this notation as a *unary operator* that works in a similar way to a minus sign.

A minus sign attaches to the front of an expression, changing the value of that expression to its negative. For example:

$$-(x^2 + 4x - 5) = -x^2 - 4x + 5$$

Similarly, the notation $\frac{d}{dx}$ attaches to the front of an expression, changing the value of that expression to its *derivative.* Evaluating the effect of this notation on the expression is called *differentiation,* which also produces a new but equivalent expression.

For example:

EXAMPLE

$$\frac{d}{dx}\left(x^2 + 4x - 5\right) = 2x + 4$$

This basic notation remains the same even when an expression is recast as a function.

EXAMPLE

For example, differentate the function $y = f(x) = x^2 + 4x - 5$ as follows:

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = 2x + 4$$

The notation $\frac{dy}{dx}$, which means "the change in *y* as *x* changes," was first used by Gottfried Leibniz, one of the two inventors of calculus (the other inventor was Isaac Newton). An advantage of Leibniz notation is that it explicitly tells you the variable over which you're differentiating — in this case, *x*. When this information is easily understood in context, a shorter notation is also available:

$$y' = f'(x) = 2x + 4$$

You should be comfortable with both of these forms of notation as you move forward into Calculus II. I use them interchangeably throughout this book.

Understanding Differentiation

Differentiation — the calculation of derivatives — is the central topic of Calculus I and makes an encore appearance in Calculus II.

In this section, I give you a refresher on some of the key topics of differentiation. I also include the 17 need-to-know derivatives in this chapter. And if you're shaky on the Chain rule, I offer a clear explanation that gets you up to speed.

Memorizing key derivatives

The derivative of any constant is always 0:

$$\frac{d}{dx}a = 0$$

The derivative of the variable by which you're differentiating (in most cases, *x*) is 1:

$$\frac{d}{dx}x = 1$$

Here are three more derivatives that are important to remember:

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x} \qquad \frac{d}{dx}a^x = a^x \ln a$$

You need to know each of these derivatives as you move on in your study of calculus.

Derivatives of the trig functions

The derivatives of the six trig functions are as follows:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

You need to know all of these by heart.

Derivatives of the inverse trig functions

Two notations are commonly used for inverse trig functions. One is the addition of the power of -1 to the function: $\sin^{-1}$, $\cos^{-1}$, and so forth. The second is the addition of *arc* to the function: arcsin, arccos, and so forth. They both mean the same thing, but I prefer the *arc* notation, because it's less likely to be mistaken for an exponent.

I know that asking you to memorize these functions seems like a cruel joke. But you really need them when you get to trig substitution in Chapter 10, so at least have a look-see:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{x\sqrt{x^2-1}}$$

TIP

Notice that derivatives of the three "co" functions — arccosine, arccotangent, and arccosecant — are just negations of the three other functions, so when you get down to memorizing these derivatives, your work is cut in half: just focus on the derivatives of the arcsin, arctan, and and arcsec functions.

The Power rule

The Power rule tells you that to find the derivative of x raised to any power, you bring down the exponent as the coefficient of x, and then subtract 1 from the exponent and use this as your *new* exponent. Here's the general form:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Here are a few examples:

$$\frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}x^3 = 3x^2 \qquad \frac{d}{dx}x^{10} = 10x^9$$

When the function that you're differentiating already has a coefficient, multiply the exponent by this coefficient. For example:

$$\frac{d}{dx}2x^4 = 8x^3 \qquad \frac{d}{dx}7x^6 = 42x^5 \qquad \frac{d}{dx}4x^{100} = 400x^{99}$$

The Power rule also extends to negative exponents, which allows you to differentiate many fractions. For example, to differentiate $\frac{1}{x}$, rewrite it in its equivalent form x^{-1} and then use the Power rule:

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

As another example, here's how you differentiate $\frac{1}{x^5}$:

$$\frac{d}{dx}\frac{1}{x^5} = \frac{d}{dx}x^{-5} = -5x^{-6} = -\frac{5}{x^6}$$

The Power rule also extends to fractional exponents, which allows you to differentiate square roots and other roots:

$$\frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}}$$

For most teachers, this is a good final answer. However, an optional step that some teachers may insist on is to change the exponential notation back to root notation:

$$= \frac{1}{3\sqrt[3]{x^2}}$$

The Sum rule

In textbooks, the Sum rule is often phrased like this: The derivative of the sum of functions equals the sum of the derivatives of those functions. Here's the mathematical translation:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Simply put, the Sum rule tells you that differentiating expressions term by term is okay.

EXAMPLE

For example, suppose you want to evaluate the following:

$$\frac{d}{dx}\left(\sin x + x^4 - \ln x\right)$$

The expression that you're differentiating has three terms, so by using the Sum rule, you can break the expression into three separate derivatives and solve them separately:

$$= \frac{d}{dx}\sin x + \frac{d}{dx}x^4 - \frac{d}{dx}\ln x = \cos x + 4x^3 - \frac{1}{x}$$

As you can see, the Sum rule also applies to expressions of more than two terms. It also applies regardless of whether the term is positive or negative. (Some books call this variation the Difference rule, but you get the idea.)

The Constant Multiple rule

A typical textbook gives you this sort of definition for the Constant Multiple rule: The derivative of a constant multiplied by a function equals the product of that constant and the derivative of that function. Check out the mathematical translation:

$$\frac{d}{dx}nf(x) = n\frac{d}{dx}f(x)$$

In plain English, this rule tells you that moving a constant multiple — that is, a coefficient — outside of a derivative before you differentiate is okay.

EXAMPLE

For example:

$$\frac{d}{dx}5\tan x$$

To solve this, move the 5 outside the derivative, and then differentiate:

$$= 5\frac{d}{dx}\tan x = 5\sec^2 x$$

The Product rule

The derivative of the product of two functions $f(x)$ and $g(x)$ is equal to the derivative of $f(x)$ multiplied by $g(x)$ plus the derivative of $g(x)$ multiplied by $f(x)$. That is:

$$\frac{d}{dx}[f(x)\cdot g(x)] = \frac{d}{dx}f(x)\cdot g(x) + \frac{d}{dx}g(x)\cdot f(x)$$

TIP

Practice saying the Product rule like this: "The derivative of the first times the second *plus* the derivative of the second times the first." This encapsulates the Product rule and sets you up to remember the Quotient rule. (See the next section.)

EXAMPLE

For example, suppose that you want to differentiate $e^x \sin x$. Start by breaking the problem out as follows:

$$\frac{d}{dx} e^x \sin x = \left(\frac{d}{dx} e^x \right) \sin x + \left(\frac{d}{dx} \sin x \right) e^x$$

Now you can evaluate both derivatives without much confusion:

$$= e^x \cdot \sin x + \cos x \cdot e^x$$

You can clean this up a bit as follows:

$$= e^x (\sin x + \cos x)$$

The Quotient rule

Here's what the Quotient rule looks like:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{\frac{d}{dx} f(x) \cdot g(x) - \frac{d}{dx} g(x) \cdot f(x)}{g(x)^2} \qquad \text{provided } g(x) \neq 0$$

TIP

Practice saying the Quotient rule like this: "The derivative of the top times the bottom *minus* the derivative of the bottom times the top, over the bottom squared." This is similar enough to the Product rule that you can remember it.

EXAMPLE

For example, suppose that you want to differentiate the following:

$$\frac{d}{dx} \left(\frac{x^4}{\tan x} \right)$$

As you do with the Product rule example in the preceding section, start by breaking the problem out as follows:

$$\frac{\left(\frac{d}{dx} x^4 \right) \cdot \tan x - \left(\frac{d}{dx} \tan x \right) \cdot x^4}{\tan^2 x}$$

Now evaluate the two derivatives:

$$= \frac{4x^3 \cdot \tan x - \sec^2 x \cdot x^4}{\tan^2 x}$$

This answer is fine, but you can clean it up by using some algebra plus the Basic Five trig identities from earlier in this chapter. (Don't worry too much about these steps unless your professor is particularly unforgiving.)

$$\begin{aligned}
&= \frac{4x^3 \tan x}{\tan^2 x} - \frac{\sec^2 x \cdot x^4}{\tan^2 x} \\
&= \frac{4x^3}{\tan x} - x^4 \sec^2 x \cot^2 x \\
&= 4x^3 \cot x - x^4 \left(\frac{1}{\cos^2 x}\right)\left(\frac{\cos^2 x}{\sin^2 x}\right) \\
&= 4x^3 \cot x - x^4 \csc^2 x \\
&= x^3 \left(4 \cot x - x \csc^2 x\right)
\end{aligned}$$

The Chain rule

I'm aware that the Chain rule is considered a major sticking point in Calculus I, so I want to take a little time to review it.

The Chain rule allows you to differentiate *nested functions* — that is, functions within functions. It places no limit on how deeply nested these functions are. In this section, I show you an easy way to think about nested functions, and then I show you how to apply the Chain rule simply.

Evaluating functions from the inside out

When you're evaluating a nested function, you begin with the *inner* function and move *outward.* For example:

EXAMPLE

$f(x) = e^{2x}$

In this case, $2x$ is the inner function. To see why, suppose that you want to evaluate $f(x)$ for a given value of x. To keep things simple, say that $x = 0$. After plugging in 0 for x, your first step is to evaluate the inner function, which I underline:

Step 1: $e^{2(0)} = e^0$

Your next step is to evaluate the outer function:

Step 2: $e^0 = 1$

The terms *inner function* and *outer function* are determined by the order in which the functions get evaluated. This is true no matter how deeply nested these functions are. For example:

EXAMPLE

$$g(x) = \left(\ln\sqrt{e^{3x-6}}\right)^3$$

Suppose that you want to evaluate $g(x)$. To keep the numbers simple, this time let $x = 2$. After plugging in 2 for x, here's the order of evaluation from the inner function to the outer function:

Step 1: $\left(\ln\sqrt{e^{3(2)-6}}\right)^3 = \left(\ln\sqrt{e^0}\right)^3$

Step 2: $\left(\ln\sqrt{e^0}\right)^3 = (\ln\sqrt{1})^3$

Step 3: $(\ln\sqrt{1})^3 = (\ln 1)^3$

Step 4: $(\ln 1)^3 = 0^3$

Step 5: $0^3 = 0$

The process of evaluation clearly lays out the five nested functions of $g(x)$ from inner to outer.

Differentiating functions from the outside in

In contrast to evaluation, differentiating a function using the Chain rule forces you to begin with the *outer* function and move *inward.*

Here's the basic Chain rule the way that you find it in textbooks:

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}f(g(x))\frac{d}{dx}g(x)$$

REMEMBER

To differentiate nested functions using the Chain rule, write down the derivative of the outer function, copying everything inside it, and multiply this result by the derivative of the next function inward.

TIP

Memorize these words: "The derivative of the outside with respect to the inside *times* the derivative of the inside."

This explanation of the Chain rule may seem a bit confusing, but it's a lot easier to understand when you know how to find the outer function, which I explain in the previous section, "Evaluating functions from the inside out." A couple of examples should help.

EXAMPLE

Suppose that you want to evaluate $\frac{d}{dx}\sin 2x$. The outer function is the sine portion, so this is where you start:

$$\frac{d}{dx}\sin 2x = \cos 2x \cdot \frac{d}{dx}2x$$

As you can see, I write down the derivative of the outside function (cosine), copying everything inside it, then multiply by the derivative of the inside function (2*x*).

To finish, you still need to find $\frac{d}{dx}2x$:

$$= \cos 2x \cdot 2$$

Rearranging this solution to make it more presentable gives you your final answer:

$$= 2 \cos 2x$$

When you differentiate more than two nested functions, the Chain rule really lives up to its name: As you break down the problem step by step, you string out a *chain* of multiplied expressions.

EXAMPLE

For example, suppose that you want to differentiate $\sin^3 e^x$. Keep in mind that the notation $\sin^3 e^x$ really means $(\sin e^x)^3$. This rearrangement makes clear that the outer function is the power of 3, so begin differentiating with this function using the Power rule:

$$\frac{d}{dx}\left(\sin e^x\right)^3 = 3\left(\sin e^x\right)^2 \cdot \frac{d}{dx}\left(\sin e^x\right)$$

Now you have a smaller derivative, $\frac{d}{dx}\left(\sin e^x\right)$, to evaluate. This time, the outer function is the sine:

$$= 3\left(\sin e^x\right)^2 \cdot \cos e^x \cdot \frac{d}{dx}e^x$$

You're almost there, with only one more derivative to go, $\frac{d}{dx}e^x$:

$$= 3\left(\sin e^x\right)^2 \cdot \cos e^x \cdot e^x$$

Again, rearranging your answer is customary:

$$= 3e^x \cos(e^x)\sin^2(e^x)$$

Notice that for this final answer, I rewrote "$\sin^2$" using the standard (though confusing) notation for powers of trig functions.

Finding Limits Using L'Hôpital's Rule

L'Hôpital's rule is all about limits and derivatives, so it fits better with Calculus I than Calculus II. But some colleges save this topic for Calculus II.

So even though I'm addressing this as a review topic, fear not: Here, I give you the full story of L'Hôpital's rule, starting with how to pronounce L'Hôpital (low-pee-tahl). L'Hôpital's rule provides a method for evaluating certain *indeterminate forms* of limits. First, I show you what an indeterminate form of a limit looks like, with a list of all common indeterminate forms. Next, I show you how to use L'Hôpital's rule to evaluate some of these forms. And finally, I show you how to work with the other indeterminate forms so that you can evaluate them.

Introducing L'Hôpital's rule

REMEMBER

Suppose that you're attempting to evaluate the limit of a function of the form $\frac{f(x)}{g(x)}$. When replacing the limit variable with the number that it approaches results in either $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, L'Hôpital's rule tells you that the following equation holds true:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Note that c can be any real number as well as ∞ or $-\infty$.

EXAMPLE

As an example, suppose that you want to evaluate the following limit:

$$\lim_{x \to 0} \frac{x^3}{\sin x}$$

Replacing x with 0 in the function leads to the following result:

$$\frac{0^3}{\sin 0} = \frac{0}{0}$$

This is one of the two indeterminate forms that L'Hôpital's rule applies to, so you can draw the following conclusion:

$$\lim_{x \to 0} \frac{x^3}{\sin x} = \lim_{x \to 0} \frac{\left(x^3\right)'}{(\sin x)'}$$

Next, evaluate the two derivatives:

$$= \lim_{x \to 0} \frac{3x^2}{\cos x}$$

Now use this new function to try another replacement of x with 0 and see what happens:

$$\frac{3\left(0^2\right)}{\cos 0} = \frac{0}{1}$$

This time, the result is a determinate form, so you can evaluate the original limit as follows:

$$\lim_{x\to 0}\frac{x^3}{\sin x}=0$$

In some cases, you may need to apply L'Hôpital's rule more than once to get an answer. For example:

EXAMPLE

$$\lim_{x\to\infty}\frac{e^x}{x^5}$$

Replacement of x with ∞ results in the indeterminate form $\frac{\infty}{\infty}$, so you can use L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{e^x}{x^5}=\lim_{x\to\infty}\frac{e^x}{5x^4}$$

In this case, the new function gives you the same indeterminate form, so use L'Hôpital's rule again:

$$=\lim_{x\to\infty}\frac{e^x}{20x^3}$$

Again, replacing x with ∞ results in the indeterminate form $\frac{\infty}{\infty}$, but again you can use L'Hôpital's rule. You can probably see where this example is going, so I fast-forward to the end:

$$=\lim_{x\to\infty}\frac{e^x}{60x^2}=\lim_{x\to\infty}\frac{e^x}{120x}=\lim_{x\to\infty}\frac{e^x}{120}$$

WARNING

When you apply L'Hôpital's rule repeatedly to a problem, make sure that every step along the way results in one of the two indeterminate forms that the rule applies to.

At last! The process finally yields a function with a determinate form:

$$\frac{e^\infty}{120}=\frac{\infty}{120}=\infty$$

Therefore, the original limit doesn't exist:

$$\lim_{x\to\infty}\frac{e^x}{x^5}\quad\text{DNE}$$

Alternative indeterminate forms

L'Hôpital's rule applies only to the two indeterminate forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$.

But limits can result in a variety of other indeterminate forms for which L'Hôpital's rule doesn't hold. Table 3-2 is a list of the indeterminate forms that you're most likely to see.

TABLE 3-2 **Cases of Indeterminate Forms Where You Can't Apply L'Hôpital's rule Directly**

Case	$f(x)$	$g(x)$	Function	Form
#1	0	∞	$f(x) \cdot g(x)$	Indeterminate
#2	∞	∞	$f(x) - g(x)$	Indeterminate
#3	0	0	$f(x)^{g(x)}$	Indeterminate
	∞	0		
	1	∞		

WARNING

Because L'Hôpital's rule doesn't hold for these indeterminate forms, applying the rule directly gives you the wrong answer.

These indeterminate forms require special attention. In this section, I show you how to rewrite these functions so you can then apply L'Hôpital's rule.

Case #1: 0 · ∞

When $f(x) = 0$ and $g(x) = \infty$, the limit of $f(x) \cdot g(x)$ is the indeterminate form $0 \cdot \infty$, which doesn't allow you to use L'Hôpital's rule. To evaluate this limit, place the reciprocal of $g(x)$ in the denominator of the expression:

$$f(x) \cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}}$$

The limit of this equivalent function is the indeterminate form $\frac{0}{0}$, which allows you to use L'Hôpital's rule. For example, suppose that you want to evaluate the following limit:

$$\lim_{x \to 0^+} x \cot x$$

Replacing x with 0 gives you the indeterminate form $0 \cdot \infty$, so rewrite the limit as follows:

$$= \lim_{x \to 0^+} \frac{x}{\left(\frac{1}{\cot x}\right)}$$

This can be simplified a little by using the inverse trig identity for cot *x*:

$$= \lim_{x \to 0^+} \frac{x}{\tan x}$$

Now, replacing *x* with 0 gives you the indeterminate for $\frac{0}{0}$, so you can apply L'Hôpital's rule:

$$= \lim_{x \to 0^+} \frac{(x)'}{(\tan x)'} = \lim_{x \to 0^+} \frac{1}{\sec^2 x}$$

At this point, you can evaluate the limit directly by replacing *x* with 0:

$$= \frac{1}{1} = 1$$

Therefore, the limit evaluates to 1.

Case #2: ∞ – ∞

When $f(x) = \infty$ and $g(x) = \infty$, the limit of $f(x) - g(x)$ is the indeterminate form $\infty - \infty$, which doesn't allow you to use L'Hôpital's rule. To evaluate this limit, try to find a common denominator that turns the subtraction into a fraction. For example:

$$\lim_{x \to 0^+} (\cot x - \csc x)$$

In this case, replacing *x* with 0 gives you the indeterminate form $\infty - \infty$. A little tweaking with the Basic Five trig identities (see Chapter 2) does the trick:

$$= \lim_{x \to 0^+} \left(\frac{\cos x}{\sin x} - \frac{1}{\sin x} \right)$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x}$$

Now, replacing *x* with 0 gives you the indeterminate form $\frac{0}{0}$, so you can use L'Hôpital's rule:

$$= \lim_{x \to 0^+} \frac{(\cos x - 1)'}{(\sin x)'} = \lim_{x \to 0^+} \frac{-\sin x}{\cos x}$$

At last, you can evaluate the limit by directly replacing *x* with 0.

$$= \frac{0}{1} = 0$$

Therefore, the limit evaluates to 0.

Case #3: Indeterminate powers

Case #3 comprises three distinct ways that the limit of $f(x)^{g(x)}$ can result in a form of indeterminate power that doesn't allow you to use L'Hôpital's rule:

- When $f(x) = 0$ and $g(x) = 0$
- When $f(x) = \infty$ and $g(x) = 0$
- When $f(x) = 1$ and $g(x) = \infty$

WARNING

This indeterminate form 1^∞ is easy to forget because it seems weird. After all, $1^x = 1$ for every real number, so why should 1^∞ be any different? In this case, infinity plays one of its many tricks on mathematics. You can find out more about some of these tricks in Chapter 10.

For example, suppose that you want to evaluate the following limit:

$$\lim_{x \to 0} x^x$$

As it stands, this limit is of the indeterminate form 0^0.

Fortunately, I can show you a trick to handle indeterminate powers. As with so many things mathematical, mere mortals such as you and me (you and I?) probably wouldn't discover this trick, short of being washed up on a desert island with nothing to do but solve math problems and eat coconuts. However, somebody did the hard work already. Remembering this recipe is a small price to pay:

1. **Set the limit equal to *y*.**

   $$y = \lim_{x \to 0} x^x$$

2. **Take the natural log of both sides, and then do some *log rolling:***

   $$\ln y = \ln \lim_{x \to 0} x^x$$

   Here are the two log-rolling steps:

   - First, roll the log inside the limit:

     $$= \lim_{x \to 0} \ln x^x$$

     This step is valid because the limit of a log equals the log of a limit (I know, those words veritably *roll* off the tongue).

- Next, *roll* the exponent over the log:

  $$= \lim_{x \to 0} x \ln x$$

  This step is also valid, by the laws of logarithms.

3. **Evaluate this limit as I show you in "Case #1: 0 · ∞."**

   Begin by changing the limit to a determinate form:

   $$= \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}}$$

   At last, you can apply L'Hôpital's rule:

   $$= \lim_{x \to 0} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

   Now evaluating the limit isn't too bad:

   $$= \lim_{x \to 0} -\frac{x^2}{x} = \lim_{x \to 0} -x = 0$$

   Wait! Remember that way back in Step 2, you set this limit equal to ln *y*. So you have one more step!

4. **Solve for *y*.**

   $$\ln y = 0$$
   $$y = 1$$

   Yes, this is your final answer, so $\lim_{x \to 0} x^x = 1$.

This recipe works with all three indeterminate forms that I talk about at the beginning of this section. Just make sure that you keep tweaking the limit until you have one of the two forms that are compatible with L'Hôpital's rule.

2

From Definite to Indefinite Integrals

IN THIS PART . . .

Approximate area using Riemann sums

See how the Fundamental Theorem of Calculus (FTC) frames integration as the inverse of differentiation

Calculate definite integrals using anti-differentiation

IN THIS CHAPTER

» Calculating Riemann sums using left and right rectangles

» Using midpoint rectangles to improve Riemann sum approximation

» Approximating area using trapezoids

» Understanding and applying Simpson's rule for finding Riemann sums

Chapter 4

Approximating Area with Riemann Sums

In Chapter 1, I show you how to use a definite integral to state an area problem on the *xy*-graph in mathematically precise terms. You find that, in some cases, you can calculate the exact value of a definite integral simply by applying a bit of geometry.

In this chapter, I show you how to approximate the value of a definite integral by slicing an area into a finite number of rectangles, and how to calculate the resulting Riemann sum.

If you've already taken a college-level Calculus 1 course, this material may be review; otherwise, you may be discovering it for the first time. Either way, here you review the three methods for finding a Riemann sum that I covered briefly in Chapter 1. Next, I show you two more methods — the Trapezoid rule and Simpson's rule — which both provide successively better estimates.

Riemann sums — by whatever method you choose — are the basis for Bernhard Riemann's formula for calculating the definite integral. I end the chapter with an explanation of how this formula arises by applying a limit to a Riemann sum.

Three Ways to Approximate Area with Rectangles

Slicing an irregular shape into rectangular partitions is the most common approach to approximating its area. (See Chapter 1 for more details on this approach.) In this section, I show you three different techniques for approximating area with rectangles.

Using left rectangles

In Chapter 1, I give you a first look at how to use left rectangles to approximate the solution to an area problem.

For example, suppose that you want to approximate the shaded area in the left side of Figure 4-1. Begin by defining it as follows:

$$\int_0^8 \left(x^2+1\right)dx \approx w(h_1+h_2+h_3+h_4)$$

In Figure 4-1, I also show you how to make this approximation by using four left rectangles. To draw these four rectangles, start by dropping a vertical line from the function to the x-axis at the *left-hand* bound of integration — that is, $x = 0$. Then, slice the area into four regions of equal width by dropping three more vertical lines from the function to the x-axis at $x = 2$, 4, and 6.

Next, at the four points where these lines cross the function, draw horizontal lines *from left to right* to make the top edges of the four rectangles. The left and top edges define the size and shape of each left rectangle.

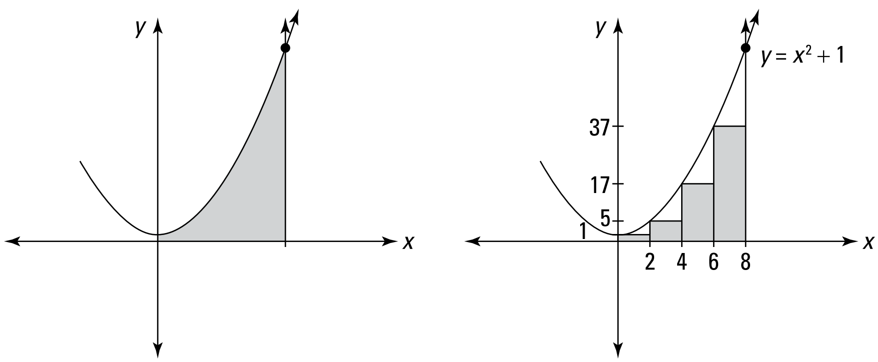

FIGURE 4-1: Approximating $\int_0^8 \left(x^2+1\right)dx$ by using four left rectangles.

To measure the areas of these four rectangles, you need the width and height of each. The width of each rectangle is obviously 2. The height and area of each is determined by the value of the function at its left edge.

To approximate the shaded area, set up a Riemann sum to add up the areas of these four rectangles:

$$\approx 2(1+5+17+37)$$

Notice that to set up this Riemann sum, I place the common width of the rectangles (2) outside the parentheses and then add up each of the heights $(1+5+17+37)$ separately inside them. This evaluation is very simple:

$$= 2(60) = 120$$

The result of this Riemann sum using four left rectangles provides an approximation of 120 for the value of the definite integral.

Using right rectangles

Using right rectangles to approximate the solution to an area problem is very similar to using left rectangles. For example, suppose that you want to use six right rectangles to approximate the shaded area in Figure 4-2.

To draw these rectangles, start by dropping a vertical line from the function to the x-axis at the *right-hand* limit of integration — that is, $x = 3$. Next, drop five more vertical lines from the function to the x-axis at $x = 0.5$, 1, 1.5, 2, and 2.5. Then, at the six points where these lines cross the function, draw horizontal lines *from right to left* to make the top edges of the six rectangles. The right and top edges define the size and shape of each right rectangle.

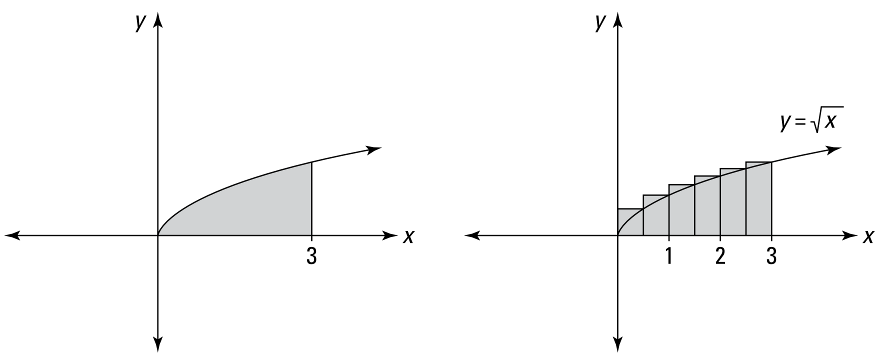

FIGURE 4-2: Approximating $\int_0^3 \sqrt{x}\,dx\; by$ by using six right rectangles.

To approximate the shaded area, set up the Riemann sum as follows:

$$\int_0^3 \sqrt{x}dx \approx w(h_1 + h_2 + h_3 + h_4 + h_5 + h_6)$$

To measure the areas of these six rectangles, you need the width and height of each. Each rectangle's width is 0.5. Its height and area are determined by the value of the function at its *right* edge.

$$\approx 0.5(0.707 + 1 + 1.225 + 1.414 + 1.581 + 1.732)$$

Now, crunch the numbers:

$$= 0.5(7.659) = 3.8295$$

Thus, using a Riemann sum with six right rectangles produces an approximation of 3.831 for the shaded area.

Finding a middle ground: The Midpoint rule

Both left and right rectangles give you a decent approximation of area. So it stands to reason that slicing an area vertically and measuring the height of each rectangle from the *midpoint* of each slice might give you a slightly better approximation of area.

For example, suppose that you want to use midpoint rectangles to approximate the shaded area in Figure 4-3.

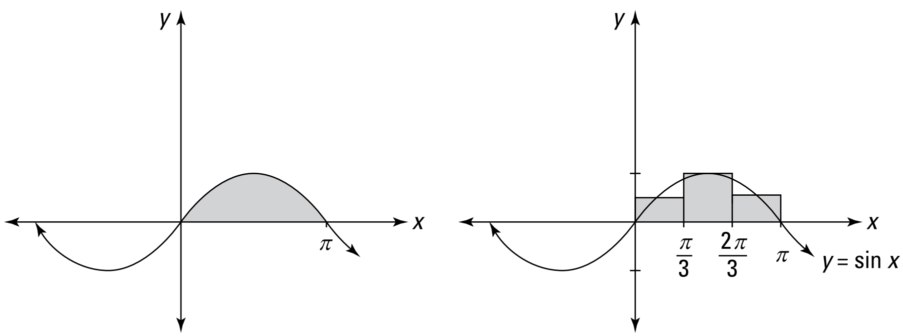

FIGURE 4-3: Approximating $\int_0^{\pi} \sin x \, dx$ by using three midpoint rectangles.

Begin by defining the definite integral as a Riemann sum with three rectangles:

$$\int_0^{\pi} \sin x \, dx \approx w(h_1 + h_2 + h_3)$$

To draw these three rectangles, start by drawing vertical lines that intersect both the function and the x-axis at $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, and π. Next, find where the midpoints of these three regions — that is, $\frac{\pi}{6}, \frac{\pi}{2}$, and $\frac{5\pi}{6}$ — intersect the function. Now draw horizontal lines through these three points to make the tops of the three rectangles.

To measure these three rectangles, you need the width and height of each to compute the area. The width of each rectangle is $\frac{\pi}{3}$, and the heights are $\frac{1}{2}$, 1, and $\frac{1}{2}$.

To approximate the shaded area, add up the areas of the three rectangles:

$$\approx \frac{\pi}{3}\left(\frac{1}{2}+1+\frac{1}{2}\right)=\frac{2\pi}{3}\approx 2.0944$$

Two More Ways to Approximate Area

Although slicing a region into rectangles is the simplest way to approximate its area, rectangles aren't the only shape that you can use. For finding many areas, other shapes can yield a better approximation in fewer slices.

In this section, I show you two common alternatives to rectangular slicing: the Trapezoid rule (which, not surprisingly, uses trapezoids) and Simpson's rule (which uses rectangles topped with parabolas).

Feeling trapped? The Trapezoid rule

In case you feel restricted — dare I say *boxed in?* — by estimating areas with only rectangles, you can get an even closer approximation by drawing partitions as trapezoids instead of rectangles.

For example, suppose that you want to use six trapezoids to estimate this area:

$$\int_{-3}^{3} 9 - x^2 dx$$

You can probably tell just by looking at the graph in Figure 4-4 that using trapezoids gives you a closer approximation than rectangles. In fact, the area of a trapezoid drawn on any slice of a function will be the average of the areas of the left and right rectangles drawn on that slice.

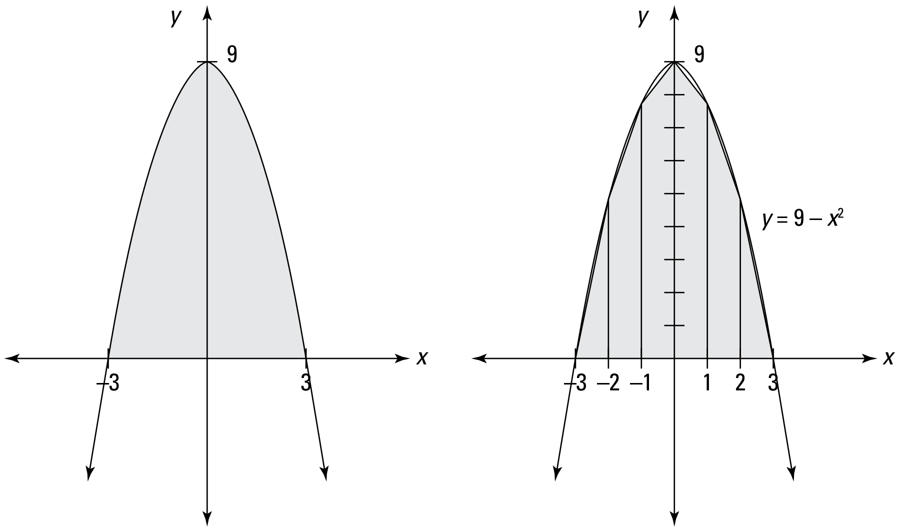

FIGURE 4-4: Approximating $\int_{-3}^{3} 9 - x^2 dx$ by using six trapezoids.

To draw these six trapezoids, first plot points along the function at $x = -3, -2, -1, 0, 1, 2,$ and 3. Next, connect adjacent points to make the top edges of the trapezoids. Finally, draw vertical lines through these points.

WARNING

Two of the six "trapezoids" are actually triangles. This fact doesn't affect the calculation; just think of each triangle as a trapezoid with one height equal to zero.

To find the area of these six trapezoids, use the formula for the area of a trapezoid that you know from geometry: $\frac{w(b_1 + b_1)}{2}$. In this case, however, the two bases — that is, the parallel sides of the trapezoid — are the heights on the left and right sides. As always, the width is easy to calculate — in this case, it's 1. Table 4-1 shows the rest of the information for calculating the area of each trapezoid.

To approximate the shaded area, find the sum of the six areas of the trapezoids:

$$\int_{-3}^{3} 9 - x^2 dx \approx 2.5 + 6.5 + 8.5 + 8.5 + 6.5 + 2.5 = 35$$

Don't have a cow! Simpson's rule

You may recall from geometry that you can draw exactly one circle through any three nonlinear points. You may not recall, however, that the same is true of parabolas: Just three nonlinear points determine a parabola.

TABLE 4-1 **Approximating Area by Using Trapezoids**

Trapezoid	Width	Left Height	Right Height	Area
#1	1	$9-(-3)^2=0$	$9-(-2)^2=5$	$\frac{1(0+5)}{2}=2.5$
#2	1	$9-(-2)^2=5$	$9-(-1)^2=8$	$\frac{1(5+8)}{2}=6.5$
#3	1	$9-(-1)^2=8$	$9-(0)^2=9$	$\frac{1(8+9)}{2}=8.5$
#4	1	$9-(0)^2=9$	$9-(1)^2=8$	$\frac{1(9+8)}{2}=8.5$
#5	1	$9-(1)^2=8$	$9-(2)^2=5$	$\frac{1(8+5)}{2}=6.5$
#6	1	$9-(2)^2=5$	$9-(3)^2=0$	$\frac{1(5+0)}{2}=2.5$

Simpson's rule relies on this geometric theorem. When using Simpson's rule, you use left and right endpoints as well as midpoints as these three points for each slice.

1. **Begin slicing the area that you want to approximate into strips that intersect the function.**
2. **Mark the left endpoint, midpoint, and right endpoint of each strip.**
3. **Top each strip with the section of the parabola that passes through these three points.**
4. **Add up the areas of these parabola-topped strips.**

At first glance, Simpson's rule seems a bit circular: You're trying to approximate the area under a curve, but this method forces you to measure the area inside a region that includes a curve. Fortunately, Thomas Simpson, who invented this rule, is way ahead on this one. His method allows you to measure these strangely shaped regions without too much difficulty.

Without further ado, here's Simpson's rule:

Given that n is an even number,

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[\begin{array}{c} f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_{n-3})+2f(x_{n-2}) \\ +4f(x_{n-1})+f(x_n) \end{array}\right]$$

What does it all mean? As with every approximation method you've encountered, the key to Simpson's rule is measuring the width and height of each of these regions (with some adjustments):

- **The width is represented by** $\frac{b-a}{n}$ — but Simpson's rule adjusts this value to $\frac{b-a}{3n}$.
- **The heights are represented by *f*(*x*) taken at various values of *x*** — but Simpson's rule multiplies some of these by a coefficient of either 4 or 2. (By the way, these choices of coefficients are based on the known result of the area under a parabola — not just picked out of the air!)

The best way to show you how this rule works is with an example. Suppose that you want to use Simpson's rule to approximate the following:

$$\int_1^5 \frac{1}{x}dx$$

First, divide the area that you want to approximate into an *even* number of regions — say, eight — by drawing nine vertical lines from $x = 1$ to $x = 5$. Now top these regions off with parabolas as I show you in Figure 4-5.

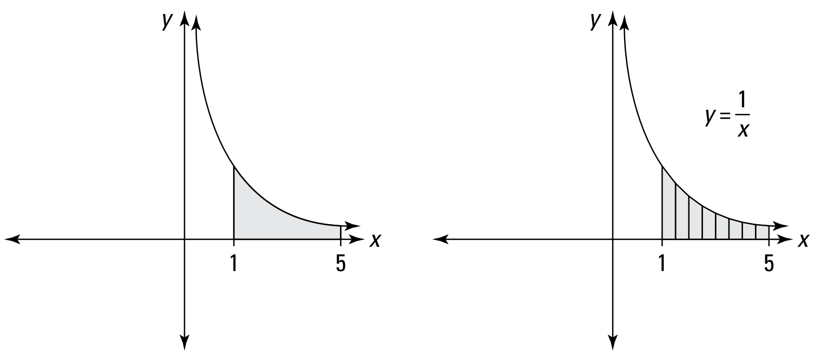

FIGURE 4-5: Approximating $\int_1^5 \frac{1}{x}dx$ by using Simpson's rule.

The width of each region is 0.5, so adjust this by dividing by 3:

$$\frac{b-a}{3n} = \frac{1}{3}\cdot\frac{5-1}{8} = \frac{1}{3}\cdot 0.5 \approx 0.167$$

Moving on to the heights, find $f(x)$ when $x = 1, 1.5, 2, \ldots, 4.5$, and 5. Adjust all these values except the first and the last by multiplying by 4 or 2, alternately.

Now apply Simpson's rule as follows:

$$\int_1^5 \frac{1}{x}dx$$
$$\approx 0.167(1+2.668+1+1.6+0.666+1.16+0.5+0.888+0.2)$$
$$=0.167(9.682)\approx 1.617$$

So Simpson's rule approximates the area of the shaded region in Figure 4-5 as 1.617. (By the way, the actual area to three decimal places is about 1.609 — so Simpson's rule provides a pretty good estimate.)

TECHNICAL STUFF

In fact, Simpson's rule often provides an even better estimate than this example leads you to believe, because a lot of inaccuracy arises from rounding off decimals. In this case, when you perform the calculations with enough precision, Simpson's rule provides the correct area to three decimal places!

Building the Riemann Sum Formula

Riemann sums are not only a clever numerical way to *estimate* the area under a function. They also provide the basis for the Riemann sum formula, which is the official definition of the definite integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}f\left(x_i^*\right)$$

This formula provides precise algebraic method for calculating the *exact* area under a function as the sum of infinitely many smaller estimated areas. In this section, I show you how this eye-glazing formula arises from one of the simplest geometric formulas you know: $\text{Area} = \text{width} \times \text{height}$.

Approximating the definite integral with the area formula for a rectangle

I start with the simple formula introduced earlier in this section:

$$\text{Area} = w(h_1 + h_2 + h_3 + \ldots + h_n)$$

This is the Riemann sum that lies at the heart of the Riemann sum formula. From here, the first step is simply to introduce the definite integral for area into the mix:

$$\int_a^b f(x)dx \approx w(h_1 + h_2 + \ldots + h_n)$$

As you can see, I changed the equals sign (=) to the symbol for "approximately equals" (≈). In other words, the equation has been demoted to an approximation. This change is appropriate because the definite integral is the *precise* area inside the specified bounds, which the area of the partitions merely approximates.

Widening your understanding of width

The next step is to replace the variable *w*, which stands for the width of each rectangle, with an expression that's more useful.

Remember that the limits of integration tell you the width of the area that you're trying to measure, with *a* as the smaller value and *b* as the greater value. So you can write the width of the entire area quite simply as follows:

$$b-a$$

And when you divide this area into *n* rectangles, each rectangle has the following width:

$$w=\frac{b-a}{n}$$

Substituting this expression into the approximation results in the following:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}(h_1+h_2+\ldots+h_n)$$

As you can see, all I'm doing here is expressing the variable *w* in terms of *a*, *b*, and *n*. That's not only way more useful, but it also removes the unknown *w* from the equation.

Limiting the margin of error

Recall that when approximating area, as *n* increases — that is, the more rectangles you draw — your approximation gets better and better. In other words, as *n* approaches infinity, the area of the rectangles approaches the area you're trying to find.

So you may not be surprised to find that when you express this approximation in terms of a limit, you remove the margin of error and restore the approximation to the status of an equation:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\frac{b-a}{n}(h_1+h_2+\ldots+h_n)$$

This limit simply states mathematically what makes intuitive sense: As n approaches infinity, the area of all the rectangles approaches the *exact* area that the definite integral represents.

That's why I've put the equals sign back in its rightful place in the equation.

Summing things up with sigma notation

Sigma notation — the Greek symbol Σ used in equations — allows you to streamline equations that have long strings of numbers added together. Chapter 2 gives you a review of sigma notation, so check it out if you need a refresher.

The expression $h_1 + h_2 + \ldots + h_n$ is a great candidate for sigma notation:

$$\sum_{i=1}^{n} h_i = h_1 + h_2 + \ldots + h_n$$

So in the equation that you're working with, you can make a simple substitution as follows:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} h_i$$

Heightening the functionality of height

Remember that the variable h_i represents the height of a single rectangle that you're measuring. (The sigma notation takes care of adding up these heights.) The last step is to replace h_i with something more functional. And *functional* is the operative word, because the *function* determines the height of each rectangle, so I provisionally replace h_i with $f(x)$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f(x_i)$$

Finishing with the slack factor

I'm almost done, but there's one final adjustment I need to make:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(x_i^*\right)$$

TECHNICAL STUFF

The notation x_i^* represents what I call a *slack factor* that's present when calculating Riemann sums: When estimating an area by dividing it into partitions, you don't have to use a single slicing method. Instead, you can use any combination you like of Left Rectangles, Right Rectangles, and so forth. In fact, you don't even have to divide the area into partitions that all have the same width. Partitions of essentially random widths are acceptable and don't affect the calculation.

This slack factor comes about because the limit does its job of *limiting* the inaccuracies that arise from using a variety of shapes to define a Riemann sum. As n approaches infinity — that is, as the number of shapes increases, and the area of each individual shape decreases — all of these inconsistencies are smoothed over.

This point is a little arcane because, in practice, when mathematicians use the Riemann sum formula for a calculation, they generally stick to a single method for drawing the partitions (Right Rectangles is a reliable favorite).

IN THIS CHAPTER

» Seeing how the indefinite integral is the inverse of the derivative

» Clarifying the differences between definite and indefinite integrals

Chapter 5

There Must Be a Better Way — Introducing the Indefinite Integral

In Chapter 4, I discuss the Riemann sum formula, which provides the formal definition for the definite integral. Although this formula can be used to calculate the definite integral, it usually results in lengthy and difficult calculations.

There must be a better way! And, indeed, there is.

In this chapter, I introduce you to the Fundamental Theorem of Calculus (FTC), which provides the link between the slope of a curve (the derivative) and the area under it (the integral). This connection provides a way to calculate definite integrals without resorting to the Riemann sum formula. Instead, you use the FTC to evaluate integrals as antiderivatives — that is, by understanding integration as the inverse of differentiation.

This insight leads an important new concept: the *indefinite integral*. The indefinite integral looks similar to the definite integral but provides the power to calculate the values of infinitely many related definite integrals using anti-differentiation.

Spoiler alert: Using indefinite integrals is the most common way you'll evaluate definite integrals — and, thus, solve area problems — throughout the remainder of Calculus II. This chapter provides the bridge from what you've studied in Parts 1 and 2 of this book to what you'll focus on in Parts 3, 4, and 5.

FTC2: The Saga Begins

The importance of the Fundamental Theorem of Calculus (FTC for short) can hardly be overstated. It connects the two main branches of calculus — differentiation and integration — showing that they're inverse operations.

This insight seems to have originated with Isaac Barrow, a mathematician at Cambridge University at the time that, by luck, Isaac Newton happened to be a student there. (You kind of have to ask: What are the odds that a budding theoretical scientist rivaled only by Albert Einstein would attend classes taught by probably the only person on the planet who fully understood what calculus could become, and that they'd both have the same first name?)

Newton took Barrow's insight and formulated a set of mathematical tools that he called *The Calculus*, which he then employed in calculations related to his formulation of physics. As with Facebook (originally *The Facebook*), eventually calculus also dropped the *The*.

The FTC is actually two theorems — the Fundamental Theorems of Calculus 1 and 2, which are shortened to FTC1 and FTC2. Both are important and you need to understand and be able to work with them. But in my humble opinion, FTC2 is the better place to begin this understanding.

In this section, I introduce you to FTC2, which you'll use throughout the rest of Calculus II. It means that you'll almost never need to rely on the Riemann sum formula to calculate another integral, except once or perhaps twice on your final exam. And for that, you should be grateful!

Later in the chapter, I show you the FTC1. I also walk you through the types of related problems that you'll typically be expected to solve using this theorem.

Introducing FTC2

Without further ado, here's the Fundamental Theorem of Calculus 2 (FTC2):

REMEMBER

Given that $f'(x)$ is continuous on the closed interval $[a, b]$,

$$\int_a^b f'(x)dx = f(b) - f(a)$$

The mainspring of this equation is the connection between f and its derivative function f'. To solve an integral, you need to be able to *undo* differentiation and find the original function f.

TECHNICAL STUFF

Many math books use the following notation for the FTC:

$$\int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

Both notations are equally valid, but I find this version a bit less intuitive than the version that I just gave you, which made it clear that integration and differentiation are inverse operations.

Evaluating definite integrals using FTC2

FTC2 makes evaluating definite integrals a whole lot easier. For example, suppose that you want to evaluate the following:

EXAMPLE

$$\int_0^\pi \sin x \; dx$$

This is the shaded area shown in Figure 5-1. Note that you also worked with this integral in Chapter 4, where, using the Midpoint rule, you approximated it to be 2.0944.

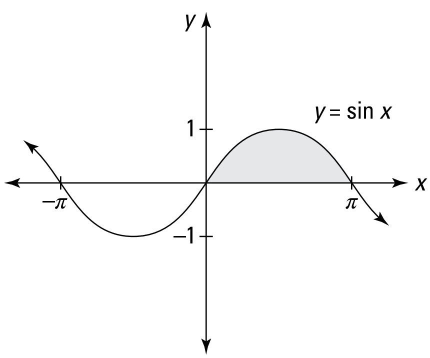

FIGURE 5-1: The area $\int_0^\pi \sin x \; dx$

The FTC allows you to evaluate this definite integral by thinking about it in a new way. First, notice the following connection between the two functions sin *x* and −cos *x*:

$f(x)$	$f'(x)$
$-\cos x$	$\sin x$

Because sin *x* is the derivative of −cos *x*, the FTC allows you to draw this conclusion:

$$\int_0^{\pi} \sin x \, dx = -\cos \pi - (-\cos 0)$$

Now you can solve this problem using information you know from working with the unit circle in trig:

$$= -(-1) - (-1) = 1 + 1 = 2$$

So the *exact* (not approximate) shaded area in Figure 5-1 is 2. You found this value quickly, without drawing a bunch of rectangles!

EXAMPLE

As another example, here's an integral you've seen before:

$$\int_0^4 x^2 \, dx$$

Begin by noticing that the following statement is true:

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

Now use the FTC to write this equation:

$$\int_0^4 x^2 dx = \frac{1}{3}\left(4^3\right) - \frac{1}{3}\left(0^3\right)$$

At this point, the solution becomes a matter of arithmetic:

$$\frac{64}{3} - 0 = \frac{64}{3}$$

In just three simple steps, the definite integral is solved without resorting to Riemann sums!

Your New Best Friend: The Indefinite Integral

The Fundamental Theorem of Calculus provides insight into the connection between a function's slope and the area underneath it — that is, between differentiation and integration.

On a practical level, the FTC also gives you an easier way to integrate, without resorting to the Riemann sum formula. This easier way is called *anti-differentiation* — in other words, undoing differentiation. Anti-differentiation is the method that you'll use to integrate throughout the remainder of Calculus II. It leads quickly to a new key concept: the *indefinite integral.*

In this section, I show you step by step how to use the indefinite integral to solve definite integrals, and I introduce the important concept of signed area. To finish the chapter, I make sure that you understand the important distinctions between definite and indefinite integrals.

Introducing anti-differentiation

Integration without resorting to Riemann sums depends on undoing differentiation (anti-differentiation). Earlier in this chapter, I calculate a few areas informally by reversing a few differentiation formulas that you know from Calculus I. But anti-differentiation is so important that it deserves its own notation: the indefinite integral.

An *indefinite integral* is simply the notation representing the inverse of the derivative function. Thus, the following statement is true of any integrable function:

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

This statement simply tells you that if you integrate a function and then differentiate the result, the process brings you back to the function you started with.

WARNING

Be careful not to confuse the indefinite integral with the definite integral. For the moment, notice that the indefinite integral has *no limits of integration.* Later in this section, I outline the differences between these two types of integrals.

Here are a few examples that clarify this important implication of the FTC:

$$\frac{d}{dx}\int \sec^3 x\, dx = \sec^3 x \qquad \frac{d}{dx}\int 3x^5 e^x dx = 3x^5 e^x$$

$$\frac{d}{dx}\int \ln|\cos 7x^2|\, dx = \ln|\cos 7x^2|$$

The takeaway here is that when you *first* integrate and *then* differentiate any integrable function (that is, a function that can be integrated — see Chapter 6 for a look at this important concept), the result is the function you started with.

Watch what happens, however, when you reverse this process, *differentiating* first and then *integrating*:

$$\int\left(\frac{d}{dx}\sec^3 x\right) dx = \sec^3 x + C \qquad \int\left(\frac{d}{dx}3x^5 e^x\right) dx = 3x^5 e^x + C$$

$$\int \frac{d}{dx}\left(\ln|\cos 7x^2|\right) dx = \ln|\cos 7x^2| + C$$

To understand why this happens, I'll take a simpler example that you know how to differentiate easily:

$$\frac{d}{dx}\sin x + 1 = \cos x \qquad \frac{d}{dx}\sin x - 100 = \cos x \qquad \frac{d}{dx}\sin x + 1{,}000{,}000 = \cos x$$

As you already know from Calculus I, any constant differentiates to 0. So whenever you integrate, your final answer needs to account for the possible presence of a constant that may have disappeared from the expression on the previous differentiation step.

REMEMBER

The formal solution of every indefinite integral is an antiderivative up to the addition of a constant C, which is called the *constant of integration.* So just mechanically attach $+ C$ whenever you evaluate an indefinite integral.

Solving area problems without the Riemann sum formula

After you know how to solve an indefinite integral by using anti-differentiation (as I show you in the previous section), you have at your disposal a useful method for evaluating definite integrals. This announcement should come as a great relief, especially after reading Chapter 4, where you see that the hairy Riemann sum formula is the official definition of the definite integral.

Here's how you solve an area problem using indefinite integrals:

1. Formulate the area problem as a definite integral (as I show you in Chapter 1).
2. Solve the definite integral as an indefinite integral evaluated between the given limits of integration.
3. Plug the limits of integration into this expression and simplify to find the area.

This method is, in fact, the one that you'll use for solving area problems for the rest of Calculus II.

EXAMPLE

For example, suppose that you want to find the shaded area in Figure 5-2.

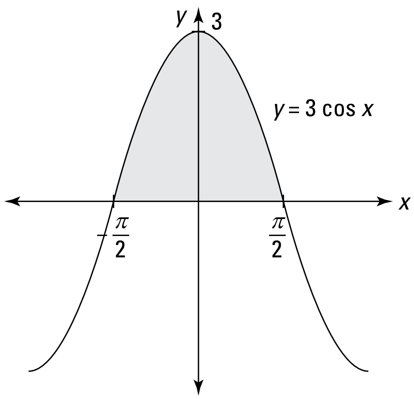

FIGURE 5-2: The shaded area $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cos x\ dx$

Here's how you do it:

1. **Formulate the area problem as a definite integral:**

   You're getting good at this, right?

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cos x\ dx$$

2. **Solve this definite integral as an indefinite integral:**

$$= 3\sin x \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}}$$

   In this step, I replace the integral with the expression 3 sin x, because $\frac{d}{dx}3\sin x = 3\cos x$.

I also introduce the notation $\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}}$. You can read it (from bottom to top) as "evaluated from x equals $-\frac{\pi}{2}$ to x equals $\frac{\pi}{2}$." This notation is commonly used so you can show your teacher that you know how to integrate and postpone worrying about the limits of integration until the next step.

3. **Plug these limits of integration into the expression and simplify:**

$$= 3\sin\frac{\pi}{2} - 3\sin\left(-\frac{\pi}{2}\right)$$

As you can see, this step comes straight from the FTC, subtracting $f(b) - f(a)$. Now I just simplify this expression to find the area:

= 3 – (–3) = 6

So the area of the shaded region in Figure 5-2 equals 6.

NO C? NO PROBLEM!

You may wonder why the constant of integration C — which is so important when you're evaluating an indefinite integral — gets dropped when you're evaluating a definite integral. This one is easy to explain.

Remember that every definite integral is expressed as the difference between a function evaluated at one point and the same function evaluated at another point. If this function includes a constant C, one C cancels out the other.

For example, take the definite integral $\int_0^{\frac{\pi}{6}} \cos x \; dx$. Technically speaking, this integral is evaluated as follows:

$$\sin x + C\Big|_{x=0}^{x=\frac{\pi}{6}}$$

$$= \left(\sin\frac{\pi}{6} + C\right) - (\sin 0 + C)$$

$$= \frac{1}{2} + C - 0 - C = \frac{1}{2}$$

As you can clearly see, the two *C*'s cancel each other out, so there's no harm in dropping them at the beginning of the evaluation rather than at the end.

Understanding signed area

In the real world, area is always positive, so the value representing area is always greater than 0. On the *xy*-graph, however, the value of an integral can be positive, negative, or 0.

REMEMBER

The value of an integral is negative when a function dips below the *x*-axis.

The definite integral takes this important distinction into account. It provides not just the area, but the *signed area* of a region on the graph. For example, suppose that you want to measure the shaded area in Figure 5-3.

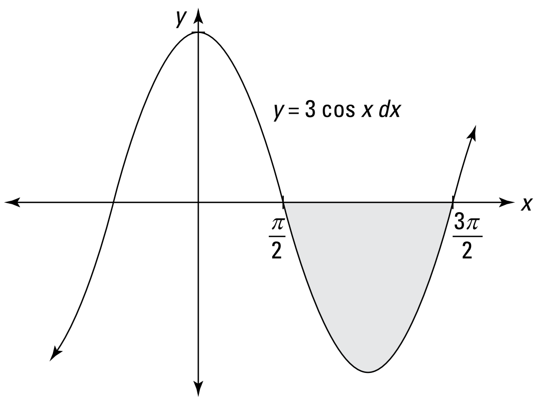

FIGURE 5-3: Measuring signed area on the graph $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3\cos x \, dx$.

Here's how you do it using the steps that I outline in the previous section:

1. **Formulate the area problem as a definite integral — as above, so below:**

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3\cos x \, dx$$

2. **Solve this definite integral as an indefinite integral — once more, with feeling:**

$$= 3\sin x \Big|_{x=\frac{\pi}{2}}^{x=\frac{3\pi}{2}}$$

3. **Plug these limits of integration into the expression and simplify:**

$$= 3\sin\frac{3\pi}{2} - 3\sin\frac{\pi}{2}$$
$$= -3 - 3 = -6$$

In this case, the signed area of the shaded region in Figure 5-3 equals −6. As you can see, the computational method for evaluating the definite integral gives the signed area automatically.

EXAMPLE

As another example, suppose that you want to find the total area of the two shaded regions in Figure 5-2 and Figure 5-3. Here's how you do it using the steps that I outline in the previous section:

1. **Formulate the area problem as a definite integral:**

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx$$

2. **Solve this definite integral as an indefinite integral:**

$$= 3\sin x \Big|_{x=-\frac{\pi}{2}}^{x=\frac{3\pi}{2}}$$

3. **Plug these limits of integration into the expression and simplify:**

$$= -3\sin\frac{3\pi}{2} - 3\sin\left(-\frac{\pi}{2}\right)$$

$$= 3 - 3 = 0$$

This time, the signed area of the shaded region is 0. This answer makes sense, because the unsigned area above the x-axis equals the unsigned area below it, so these two areas cancel each other out.

Distinguishing definite and indefinite integrals

Don't confuse definite and indefinite integrals. Here are the key differences between them:

A definite integral

- Includes limits of integration (*a* and *b*)
- Represents the exact area of a specific set of points on a graph
- Evaluates to a number

An indefinite integral

- Doesn't include limits of integration
- Can be used to evaluate an infinite number of related definite integrals
- Evaluates to a function

For example, here's a *definite* integral:

$$\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$$

As you can see, it includes limits of integration (0 and $\frac{\pi}{4}$), so you can draw a graph of the area that it represents. You can then use a variety of methods to evaluate this integral as a *number.* This number equals the signed area between the function and the x-axis inside the limits of integration, as I discuss earlier in this section.

In contrast, here's an *indefinite* integral:

$$\int \sec^2 x\, dx$$

This time, the integral doesn't include limits of integration, so it doesn't represent a specific area. Thus, it doesn't evaluate to a number, but to a function:

$$= \tan x + C$$

You can use this function to evaluate any related definite integral. For example, here's how to use it to evaluate the definite integral I just gave you:

$$\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$$

$$= \tan x \Big|_{x=0}^{x=\frac{\pi}{4}}$$

$$= \tan \frac{\pi}{4} - \tan 0$$

$$= 1 - 0 = 1$$

So the value of this definite integral is *definitely* 1.

As you can see, the indefinite integral encapsulates infinitely many related definite integrals. It also provides a practical means for evaluating definite integrals. Small wonder that much of Calculus II focuses on evaluating indefinite integrals.

In Part 3, you discover an ordered approach to evaluating indefinite integrals.

FTC1: The Journey Continues

Earlier in this chapter, I give you this piece of the Fundamental Theorem of Calculus, lovingly labeled FTC2:

$$\int_a^b f'(x) = f(b) - f(a)$$

By now, you've had some practice using this theorem to solve area problems by using the indefinite integral (anti-differentiation) rather than the hairy Riemann sum formula. In this section, I roll out the other part of the Fundamental Theorem of Calculus, FTC1.

To do this, I first introduce you to a new concept, the *area function* (or *accumulation function*), which is a function defined by a definite integral. Then, as you get comfortable working with area functions, I slip you the final piece of the FTC puzzle.

Understanding area functions

To begin, I'm going to show you the bare bones of a new type of function, called an *area function*, defined as follows:

$$A(x) = \int_s^x f(t)\, dt$$

This function $A(x)$ is an area function defined in terms of a function t. It measures the area under $f(t)$ from an arbitrary point s to any value of x you'd like to know about.

Take a look at Figure 5-4 to begin getting familiar with how this type of function, well, functions.

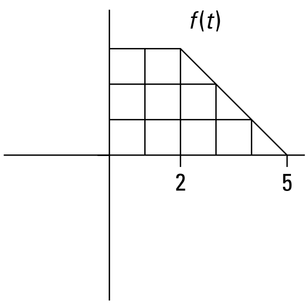

FIGURE 5-4: Understanding the area function $A(x) = \int_0^x f(t)dt$

Questions about area functions are often framed as visual problems, asking you to find $A(x)$ for a specific value of x.

EXAMPLE

For the function $f(x)$ shown in Figure 5-4, what is $A(2)$?

To begin, note in this specific example that the lower bound of the definite integral is $s = 0$. I choose s here to represent the *starting point* for this area function.

To solve this problem, plug in 2 for x, as follows:

$$A(2) = \int_0^2 f(t)\,dt$$

This area corresponds to the shaded area in Figure 5-5. You don't need calculus to solve it. As I show you in Chapter 1, simply use geometry, calculating the area of a rectangle with a width of 2 and a height of 3 (or, even easier, by counting the shaded boxes):

$$w \times h = 2 \times 3 = 6$$

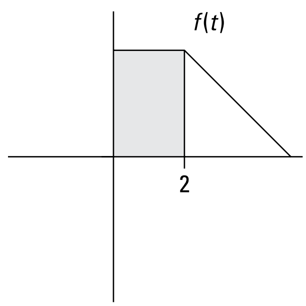

FIGURE 5-5: The shaded region representing $A(2) = \int_0^2 f(t)\,dt$

EXAMPLE

To find $A(5)$ for the same function, plug in 5 for x:

$$A(5) = \int_0^5 f(t)\,dt$$

This time, you're measuring the shaded area in Figure 5-6. Calculate it by measuring the area of a trapezoid with bases of 2 and 5 and with a height of 3:

$$= \frac{b_1 + b_2}{2} \cdot h = \frac{2+5}{2} \cdot 3 = 10.5$$

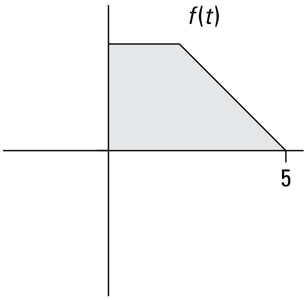

FIGURE 5-6: The shaded region representing $A(5)=\int_0^5 f(t)\,dt$

Making sense of FTC1

When you're relatively comfortable working with area functions, the Fundamental Theorem of Calculus 1 becomes much simpler to understand and work with. Here is FTC1:

REMEMBER

Given that $f'(x)$ is continuous on the closed interval $[a, b]$, for all x such that $a \le x \le b$:

If $A(x)=\int_s^x f(t)\,dt$, then $\frac{d}{dx}A(x)=f(x)$.

In plain English, this theorem tells you that differentiation undoes integration: When you define an area function in terms of a definite integral with an upper bound of x and then differentiate it, the result is the function you started with, now framed in terms of x.

An example or two should help you see how relatively straightforward this process can be. Here's a new area function to work with:

EXAMPLE

$$A(x)=\int_{97}^x \sin t\,dt$$

I choose a lower bound of 97 here specifically to show you how little you have to worry about it. Watch what happens when you differentiate this function by x:

$$\frac{d}{dx}A(x)=\sin x$$

Differentiating by x returns the function inside the integral, with all the t's changed to x's. And the lower bound of 97 completely drops out of the resulting equation.

Here's another example that's just about as ornery as I know how to make it:

EXAMPLE

$$A(x) = \int_{-273.15}^{x} 85t^9 - \frac{\sec t}{\ln t} - e^{\sin 2t^3}\, dt$$

Differentiating this area function produces the following result:

$$\frac{d}{dx} A(x) = 85x^9 - \frac{\sec x}{\ln x} - e^{\sin 2x^3}$$

Again, the upper bound x replaces t in the resulting function, and the lower bound drops out entirely.

3 Evaluating Indefinite Integrals

IN THIS PART . . .

Evaluate 17 indefinite integrals as antiderivatives

Apply the Sum rule, Constant Multiple rule, and Power rule for Integrals

Use algebra and trigonometry to prepare functions to be integrated

Integrate functions using variable substitution

IN THIS CHAPTER

» Calculating simple integrals as antiderivatives

» Using 17 integral formulas and 3 integration rules

» Integrating more difficult functions using more than one integration tool

» Clarifying the difference between integrable and nonintegrable functions

Chapter 6

Instant Integration: Just Add Water (And *C*)

In Chapter 5, I show you how to use indefinite integrals to solve area problems framed by the definite integral.

Indefinite integrals are essentially antiderivatives, so the good news here is that you can already evaluate a variety of definite integrals by reversing the differentiation process — a major advance over using the difficult Riemann sum formula.

Throughout the remainder of Calculus II, anti-differentiation becomes your go-to method of evaluating definite integrals. Thus, the problem evaluating indefinite integrals as functions takes center stage.

Now some bad news: In practice, finding indefinite integrals is often a lot trickier than differentiation. In some cases, you'll need to revisit a variety of algebra and trig tricks from your past. In others, integration requires a variety of new techniques.

In this chapter — and also in Chapters 7 through 11 — I focus exclusively on one question: How do you integrate every single function on the planet? Okay, I'm exaggerating, but not by much. I give you a manageable set of integration techniques that you can do with a pencil and paper, and if you know when and how to apply them, you'll be able to integrate everything but the kitchen sink.

First, you start integrating by thinking about integration as anti-differentiation — that is, as the inverse of differentiation. I give you a not-too-long list of basic integrals, which mirrors the list of basic derivatives from Chapter 3. I also give you a few rules for breaking down functions into manageable chunks that are easier to integrate.

I also explain the concept of integrable versus differentiable functions. You see why, even though integration is more difficult to do in practice, more functions are actually integrable than differentiable.

By the end of this chapter, you'll have the tools to integrate dozens of functions quickly and easily.

Evaluating Basic Integrals

In Calculus I (which I review in Chapter 3), you find that a few algorithms — such as the Product rule, Quotient rule, and Chain rule — give you the tools to differentiate just about every function your professor could possibly throw at you. In Calculus II, students often greet the news that "there's no Chain rule for integration" with celebratory cheers. By the middle of the semester, they usually revise this opinion.

Using the 17 basic antiderivatives for integrating

In Chapter 3, I give you a list of 17 derivatives to know, cherish, and above all *memorize* (yes, I said *memorize*). Reading that list may lead you to believe that I'm one of those harsh über-math dudes who takes pleasure in cruel and unusual curricular activities.

But math is kind of like the Ghost of Christmas Past — the stuff you thought was long ago dead and buried comes back to haunt you. And so it is with derivatives. If you already know them, you'll find this section easy.

The Fundamental Theorem of Calculus shows that integration is the inverse of differentiation up to a constant C. This key theorem gives you a way to begin integrating a short list of functions. In Table 6-1, I show you how to integrate these functions by identifying them as the derivatives of functions you already know.

TABLE 6-1 The 17 Basic Integrals (Antiderivatives)

Derivative	Integral (Antiderivative)
$\frac{d}{dx}a = 0$	$\int 0\,dx = C$
$\frac{d}{dx}x = 1$	$\int 1\,dx = x + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln x + C$
$\frac{d}{dx}n^x = n^x \ln n$	$\int n^x dx = \frac{n^x}{\ln n} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}\text{arccot}\, x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2}\,dx = \text{arccot}\, x + C$
$\frac{d}{dx}\text{arcsec}\, x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \text{arcsec}\, x + C$
$\frac{d}{dx}\text{arccsc}\, x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}} = \text{arccsc}\, x + C$

As I discuss in Chapter 5, you need to add the constant of integration C whenever you integrate, because constants differentiate to 0. For example:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\sin x + 1 = \cos x \qquad \frac{d}{dx}\sin x - 100 = \cos x$$

So when you integrate using anti-differentiation, you need to account for the potential presence of this constant:

$$\int \cos x\, dx = \sin x + C$$

Three important integration rules

After you know how to integrate using the 17 basic antiderivatives in Table 6-1, you can expand your repertoire of functions with three additional integration rules: the Sum rule, the Constant Multiple rule, and the Power rule. These three rules mirror those that you know from differentiation.

The Sum rule for integration

The Sum rule for integration tells you that integrating long expressions term by term is okay. Here it is formally:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

EXAMPLE

For example:

$$\int \left(\cos x + e^x - \frac{1}{x}\right)dx = \int \cos x\, dx + \int e^x dx - \int \frac{1}{x}dx$$

Note that the Sum rule also applies to expressions of more than two terms. It also applies regardless of whether the term is positive or negative. Splitting this integral into three parts allows you to integrate each part separately by using a different anti-differentiation rule:

$$= \sin x + e^x - \ln x + C$$

Notice that I add only one C at the end. Technically speaking, you should add one variable of integration (say, C_1, C_2, and C_3) for each integral that you evaluate. But, at the end, you can still declare the variable $C = C_1 + C_2 + C_3$ to consolidate all these variables. In most cases when you use the Sum rule, you can skip this step and just tack a C onto the end of the answer.

TECHNICAL STUFF

Some math books present the Sum rule as the Sum and Difference rule, to clarify that adding negative values is allowed. Others present the Sum rule and the Difference rule as two distinct rules. For simplicity, I simply present the Sum rule as a single rule for integrating the sums as well as differences of functions.

The Constant Multiple rule for integration

The Constant Multiple rule tells you that you can move the coefficient of a function outside of the integral before you integrate. Here it is expressed in symbols:

$$\int nf(x)dx = n\int f(x)dx$$

EXAMPLE

For example:

$$\int 3\tan x\sec x\; dx = 3\int \tan x\sec x\; dx$$

As you can see, this rule mirrors the Constant Multiple rule for differentiation, allowing you to move the coefficient outside the integral to simplify the integration process. With the coefficient out of the way, integrating is now easy using an anti-differentiation rule:

$$= 3\sec x + C$$

The Power rule for integration

The Power rule for integration allows you to integrate any real power of x (except −1). Here's the Power rule expressed formally:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

EXAMPLE

For example:

$$\int x\; dx = \frac{1}{2}x^2 + C \qquad \int x^2 dx = \frac{1}{3}x^3 + C \qquad \int x^{100} dx = \frac{1}{101}x^{101} + C$$

TIP

The Power rule works fine for negative powers of x, which are powers of x in the denominator. For example, to integrate the function $\frac{1}{x^2}$, rewrite it as a negative power:

$$\int \frac{1}{x^2}dx = \int x^{-2}dx$$

Now, you can integrate easily using the Power rule, and then simplify:

$$= -x^{-1} + C = -\frac{1}{x} + C$$

Notice before moving on that I introduce the $+ C$ on the step where the integration actually takes place, and then carry it along.

TIP

The Power rule also works for rational powers of x, which are roots of x.

EXAMPLE

For example, to integrate the function $\sqrt{x^3}$, rewrite it as a fractional power:

$$\int \sqrt{x^3}\,dx = \int x^{\frac{3}{2}}\,dx$$

Now you can apply the Power rule and simplify:

$$= \frac{2}{5}x^{\frac{5}{2}} + C = \frac{2}{5}\sqrt{x^5} + C$$

Here, I complete the problem by putting the result back into radical form, but if your teacher doesn't require this step, happy joy!

TIP

The *only* real-number exponent that the Power rule doesn't work for is –1:

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx$$

Now if you try using the Power rule with –1, here's what you get:

$$= \frac{1}{0}x^0 + C \qquad \text{WRONG!}$$

Fortunately, you have an anti-differentiation rule to handle this case:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

In Chapter 7, you further explore how to use the Power rule for integrating a variety of rational and radical functions.

What happened to the other rules?

Integration has formulas that mirror the Sum rule, the Constant Multiple rule, and the Power rule when it comes to differentiation. But it lacks formulas that look like the Product rule, Quotient rule, and Chain rule. This fact may sound like good news, but the lack of formulas makes integration a lot trickier in practice than differentiation is.

In fact, Chapters 7 through 11 focus on a bunch of methods that mathematicians have devised for getting around this difficulty. Chapter 7 extends your understanding of the methods in this chapter and gives you a few additional tools. Chapter 8 focuses on variable substitution, which is a limited form of the Chain rule. And in Part 3, I show you a set of more advanced ways to integrate a wider variety of functions.

Evaluating More Difficult Integrals

The anti-differentiation rules for integrating, which I explain earlier in this chapter, greatly limit how many integrals you can compute easily. In many cases, however, you can tweak a function to make it easier to integrate.

In this section, I show you how to integrate certain fractions and roots using the Power rule. I also show you how to use the trig identities in Chapter 2 to stretch your capacity to integrate trig functions.

Integrating polynomials

You can integrate *any* polynomial in three steps using the rules from this section:

1. Use the Sum rule to break the polynomial into its terms and integrate each of these separately.
2. Use the Constant Multiple rule to move the coefficient of each term outside its respective integral.
3. Use the Power rule to evaluate each integral. (You only need to add a single *C* to the end of the resulting expression.)

EXAMPLE

For example, suppose that you want to evaluate the following integral:

$$\int 10x^6 - 3x^3 + 2x - 5\ dx$$

1. **Break the expression into four separate integrals:**

   $$= \int 10x^6 dx - \int 3x^3 dx + \int 2x\ dx - \int 5\ dx$$

2. **Move each of the four coefficients outside its respective integral:**

   $$= 10\int x^6 dx - 3\int x^3 dx + 2\int x\ dx - 5\int dx$$

3. **Integrate each term separately using the Power rule:**

   $$= \frac{10}{7}x^7 - \frac{3}{4}x^4 + x^2 - 5x + C$$

EXAMPLE

Here's another function that seems trickier at first:

$$\int (x^4 + 7)(x^3 - 4)\ dx$$

Because no Product rule exists for integration, you may feel stuck. However, you can transform this polynomial to standard form by distributing using the FOIL method (First, Outside, Inside, and Last):

$$= \int x^7 - 4x^4 + 7x^3 - 28\ dx$$

Now, you can apply the Sum rule, Constant Multiple rule, and Power rule to integrate this function. Technically, these are three separate steps, but in practice you'll probably soon grow comfortable handling these calculations in one step, as you likely did when differentiating polynomials:

$$= \frac{1}{8}x^8 - \frac{4}{5}x^5 + \frac{7}{4}x^4 - 28x + C$$

REMEMBER

You can integrate *any* polynomial using this method. Many integration methods I introduce later in this book rely on this fact. So I recommend that you practice integrating polynomials until you feel so comfortable that you could do it in your sleep.

Integrating more complicated-looking functions

You can combine the Sum rule, Constant Multiple rule, and Power rule to integrate some truly hairy-looking functions. For example:

EXAMPLE

$$\int 27x^9 + 5\sec^2 x - \frac{3e^x}{4} + \frac{2}{3x}\ dx$$

1. **Begin by using the Sum rule to split this function into separate terms, just as when integrating polynomials in the previous section:**

$$= \int 27x^9 dx + \int 5\sec^2 x\ dx - \int \frac{3e^x}{4} dx + \int \frac{2}{3x} dx$$

2. **Next, move the coefficients in each case outside the integral:**

$$= 27\int x^9 dx + 5\int \sec^2 x\ dx - \frac{3}{4}\int e^x dx + \frac{2}{3}\int \frac{1}{x} dx$$

3. **Finally, evaluate each separate integral using the appropriate antiderivative:**

$$= \frac{27}{10}x^{10} + 5\tan x - \frac{3}{4}e^x + \frac{2}{3}\ln|x| + C$$

As when integrating polynomials, you can just append a single constant C to the result rather than add a different constant for each integral evaluated.

When you're comfortable integrating complicated functions in this way, you can combine this process into a single step. For example:

$$\int -\frac{1}{4}x^6 - 8\sin x + \frac{2}{5}e^x + \frac{4}{9\sqrt{1-x^2}}\,dx$$
$$= -\frac{1}{28}x^7 + 8\cos x + \frac{2}{5}e^x + \frac{4}{9}\arcsin x + C$$

Understanding Integrability

By now, you've probably figured out that, in practice, integration is usually harder than differentiation. The lack of any set formulas for integrating products, quotients, and compositions of functions makes integration something of an art rather than a science.

So you may think that a large number of functions are differentiable, with a smaller subset of these being integrable. It turns out that this conclusion is false. In fact, the set of integrable functions is larger, with a smaller subset of these being differentiable. To understand this fact, you need to be clear on what the words *integrable* and *differentiable* really mean.

In this section, I shine some light on two common mistakes that students make when trying to understand what integrability is all about. After that, I discuss what it means for a function to be integrable, and I show you why many functions that are integrable aren't differentiable.

Taking a look at two red herrings of integrability

In trying to understand what makes a function integrable, you first need to understand two related issues: the difficulties in *computing integrals* as well as the difficulties in *representing integrals as functions.* These issues are valid concerns, but they're red herrings — that is, they don't really affect whether a function is integrable.

Computing integrals

For many input functions, integrals are more difficult to compute than derivatives. For example, suppose that you want to differentiate and integrate the following function:

EXAMPLE

$$y = 3x^5e^{2x}$$

You can differentiate this function easily by using the Product rule (I take an additional step to simplify the answer):

$$\begin{aligned}\frac{dy}{dx} &= 3\left[\frac{d}{dx}\left(x^5\right)e^{2x} + \frac{d}{dx}\left(e^{2x}\right)x^5\right]\\ &= 3\left(5x^4e^{2x} + 2e^{2x}x^5\right)\\ &= 3x^4e^{2x}(2x+5)\end{aligned}$$

Because no such rule exists for integration, in this example you're forced to seek another method. (You find this method in Chapter 9, where I discuss integration by parts.)

Finding solutions to integrals can be tricky business. In comparison, finding derivatives is relatively simple — you learned most of what you need to know about it in Calculus I.

Representing integrals as elementary functions

Beyond difficulties in computation, the integrals of certain functions simply can't be represented by using the functions that you're used to.

More precisely, some integrals can't be represented as *elementary functions* — those familiar functions that you work with all the time in calculus. They include:

- Addition, subtraction, multiplication, and division
- Powers and roots
- Exponential functions and logarithms (usually the natural log)
- Trig and inverse trig functions
- All combinations and compositions of these functions

EXAMPLE

For example, consider the following function:

$$y = e^{x^2}$$

You can find the derivative of this function easily using the Chain rule:

$$\frac{d}{dx}e^{x^2}$$
$$= e^{x^2}\left(\frac{d}{dx}x^2\right)$$
$$= e^{x^2}(2x)$$
$$= 2xe^{x^2}$$

This result is an elementary function, so it's also differentiable. And because the derivative of every elementary function is an elementary function, every elementary function is *infinitely differentiable* — that is, you can continue this cycle of differentiation as long as you like.

However, the integral of the same function, $\int e^{x^2}dx$, can't be expressed as an elementary function — that is, it can't be expressed as a function that you're used to working with.

Instead, you can express this integral either *exactly* (as an infinite series; see Part 6 for more on infinite series) or *approximately* (as a function that approximates the integral to a given level of precision). Alternatively, you can just leave it as an integral $\int e^{x^2}dx$, which also expresses this function well for some purposes.

Getting an idea of what integrable really means

When mathematicians discuss whether a function is integrable, they aren't talking about the difficulty of computing that integral — or even whether a method has been discovered. Each year, mathematicians find new ways to integrate classes of obscure functions you'll probably never have to worry about. However, this fact doesn't mean that previously nonintegrable functions are now integrable.

Similarly, a function's integrability also doesn't hinge on whether its integral can be easily represented as another function, without resorting to infinite series.

In fact, when mathematicians say that a function is integrable, they mean only that the integral is *well defined* — that is, that the integral makes mathematical sense.

TECHNICAL STUFF

In practical terms, integrability hinges on continuity: If a function is continuous on a given interval, it's integrable on that interval. Additionally, if a function has only a finite number of discontinuities on an interval, it's also integrable on that interval: Its integral equals the sum of the finite number of continuous integrals on that interval.

You probably remember from Calculus I that many functions — such as those with discontinuities, sharp turns, and vertical slopes — are non-differentiable. Discontinuous functions are also nonintegrable. However, continuous functions with sharp turns and vertical slopes are integrable.

EXAMPLE

For example, the function $y = |x|$, shown in Figure 6-1, contains a sharp point at $x = 0$, so the function is non-differentiable at this point — that is, attempting to compute the slope at 0 makes no mathematical sense. However, the same function is integrable for all values of x, because the area under this function between any two limits of integration you care to name is a well-defined concept.

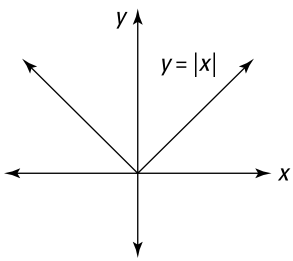

FIGURE 6-1: The function $y = |x|$ is integrable for all values of x, but non-differentiable at $x = 0$.

This function is just one of infinitely many examples of a function that's integrable but not differentiable over its entire domain.

So, surprisingly, the set of differentiable functions is actually a subset of the set of integrable functions. In practice, however, computing the integral of most functions is more difficult than computing the derivative.

IN THIS CHAPTER

» Understanding a variety of ways to integrate rational and radical functions

» Anti-differentiating inverse trig functions to integrate difficult rational functions

» Using trig identities to integrate trig functions

» Integrating compositions of functions with linear input functions

Chapter 7

Sharpening Your Integration Moves

In Chapter 6, you discover how to evaluate 17 indefinite integrals using anti-differentiation — that is, by reversing the process of finding a derivative. You also use the Sum rule, Constant Multiple rule, and Power rule to extend the range of functions that you can integrate.

In this chapter, you solidify these skills and extend them to integrate a wider variety of functions. You also incorporate tricks from algebra and trigonometry to rewrite functions in equivalent forms that are more conducive to integration.

Additionally, you discover a quick way to integrate certain compositions of functions. This insight gives you a glimpse at u-substitution, a powerful way to integrate more complicated functions, which I discuss in greater detail in Chapter 8.

Integrating Rational and Radical Functions

In Chapter 6, you discover the Power rule for Integration — the Power rule, for short. Here it is again in all its splendor:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

When you're comfortable using this rule, you can apply it to integrate all sorts of complicated-looking functions. In this section, I show you how to apply the Power rule to integrate a variety of functions that don't look like powers at first glance.

Integrating simple rational functions

EXAMPLE

Integrating a simple rational function such as $\frac{1}{x^2}$ may look tricky at first, until you realize you can express this type of function using a negative exponent (see Chapter 2 for a refresher on negative exponents):

$$\int \frac{1}{x^2} \, dx = \int x^{-2} \, dx$$

From here, you integrate using the Power rule, and simplify the result:

$$= \frac{1}{-1} x^{-1} + C \quad = -\frac{1}{x} + C$$

EXAMPLE

This next example includes numerical values in the numerator and denominator, which can both be moved outside the integral using the Constant Multiple rule before you change the exponent to a negative value:

$$\int \frac{4}{5x^9} \, dx = \frac{4}{5} \int \frac{1}{x^9} \, dx = \frac{4}{5} \int x^{-9} \, dx$$

Now you're ready to use the Power rule to integrate:

$$= \frac{4}{5}\left(-\frac{1}{8}\right) x^{-8} + C$$

To complete the problem, multiply the two fractions and simplify the result:

$$= -\frac{4}{40} x^{-8} + C = -\frac{1}{10} x^{-8} + C$$

Integrating radical functions

EXAMPLE

As with rational functions, a radical such as $\sqrt{x}$ may look difficult to integrate, but expressing it using a fractional power allows you to integrate it easily:

$$\int \sqrt{x}\ dx = \int x^{\frac{1}{2}}\ dx$$

Now, integrate using the Power rule:

$$= \frac{2}{3}x^{\frac{3}{2}} + C$$

EXAMPLE

Here's a two-term function that you can integrate by similar means. Begin by using the Sum rule to split the function into two separate integrals, each expressed as a fractional power:

$$\int \sqrt[4]{x^3} + \frac{1}{\sqrt{x}}\ dx = \int x^{\frac{3}{4}}\ dx + \int x^{-\frac{1}{2}}\ dx$$

Now, evaluate both integrals using the Power rule and finish with the optional step of putting the result back into radical notation, if this is your prof's preference:

$$= \frac{4}{7}x^{\frac{7}{4}} + 2x^{\frac{1}{2}} + C = \frac{4}{7}\sqrt[4]{x^7} + 2\sqrt{x} + C$$

EXAMPLE

Here's an unfun-looking final example that pulls together a variety of skills:

$$\int \frac{2}{7x^4} - \frac{6\sqrt[10]{x^7}}{11} - \frac{4}{5\sqrt{x^3}}\ dx$$

To begin, apply the Sum rule and the Constant Multiple rule to separate this expression into three integrals, pulling out a fractional coefficient in each case:

$$= \frac{2}{7}\int \frac{1}{x^4}dx - \frac{6}{11}\int \sqrt[10]{x^7}dx - \frac{4}{5}\int \frac{1}{\sqrt{x^3}}\ dx$$

Next, express each term as a power:

$$= \frac{2}{7}\int x^{-4}dx - \frac{6}{11}\int x^{\frac{7}{10}}dx - \frac{4}{5}\int x^{-\frac{3}{2}}\ dx$$

Now, use the Power rule to integrate:

$$= \frac{2}{7}\left(-\frac{1}{3}\right)x^{-3} - \frac{6}{11}\left(\frac{10}{17}\right)x^{\frac{17}{10}} - \frac{4}{5}(-2)x^{-\frac{1}{2}} + C$$

Most teachers will want to see this result simplified a bit:

$$= -\frac{2}{21}x^{-3} - \frac{60}{187}x^{\frac{17}{10}} + \frac{8}{5}x^{-\frac{1}{2}} + C$$

With luck, you won't have to do this next step, but if your teacher insists on it, here's how to change the negative and fractional exponents back into more standard notation:

$$= -\frac{2}{21x^3} - \frac{60\sqrt[10]{x^{17}}}{187} + \frac{8}{5\sqrt{x}} + C$$

Using Algebra to Integrate Using the Power Rule

In many cases, you can use algebra to untangle hairy expressions so that you can integrate them using the Power rule. I discuss some of these techniques in Chapter 2 as well. For example:

EXAMPLE

$$\int \frac{x^6 + x}{x^3}\, dx$$

Begin by splitting this rational expression into two separate expressions with the same denominator:

$$= \int \frac{x^6}{x^3} + \frac{x}{x^3} dx$$

This step allows you to take advantage of the Sum rule, by splitting this expression into two separate integrals:

$$= \int \frac{x^6}{x^3} dx + \int \frac{x}{x^3}\, dx$$

Next, simplify each rational expression, writing each one using a single exponent:

$$= \int x^3 dx + \int x^{-2} dx$$

Now you can integrate and, if needed, put the second term of the result back into rational form:

$$= \frac{1}{4}x^4 - x^{-1} + C = \frac{1}{4}x^4 - \frac{1}{x} + C$$

EXAMPLE

Here's a more difficult example that requires you to apply algebra before you integrate:

$$\int \frac{(x^2+5)(x-3)^2}{\sqrt{x}}dx$$

You could split the function into the product of several rational functions, but without the Product rule or Quotient rule, you're now stuck. Instead, bite the bullet and expand the numerator by distribution (FOILing), and put the denominator in exponential form:

$$= \int \frac{x^4 - 6x^3 + 14x^2 - 30x + 45}{x^{\frac{1}{2}}}dx$$

Next, split the expression into five separate rational terms:

$$= \int \frac{x^4}{x^{\frac{1}{2}}} - \frac{6x^3}{x^{\frac{1}{2}}} + \frac{14x^2}{x^{\frac{1}{2}}} - \frac{30x}{x^{\frac{1}{2}}} + \frac{45}{x^{\frac{1}{2}}}dx$$

This result looks worse than what you started with, but you can now express each term as an exponent of *x* times a coefficient:

$$= \int x^{\frac{7}{2}} - 6x^{\frac{5}{2}} + 14x^{\frac{3}{2}} - 30x^{\frac{1}{2}} + 45x^{-\frac{1}{2}}dx$$

Now, use the Sum rule to separate the integral into five separate integrals, and the Constant Multiple rule to move the coefficient outside the integral in each case:

$$= \int x^{\frac{7}{2}}dx - 6\int x^{\frac{5}{2}}dx + 14\int x^{\frac{3}{2}}dx - 30\int x^{\frac{1}{2}}dx + 45\int x^{-\frac{1}{2}}dx$$

Finally, you can integrate each term separately using the Power rule:

$$= \frac{2}{9}x^{\frac{9}{2}} - \frac{12}{7}x^{\frac{7}{2}} + \frac{28}{5}x^{\frac{5}{2}} - 20x^{\frac{3}{2}} + 90x^{\frac{1}{2}} + C$$

I'm hoping that, when you're faced with a problem this time-consuming, your teacher will be content with a final answer stated in terms of rational exponents of *x*.

EXAMPLE

To finish up, here's a very tough example of a rational expression that requires a lot of care and feeding before you're able to integrate it:

$$\int \frac{4x^2 - 4x - 80}{3x^3 - 15x^2}dx$$

To begin, factor both the numerator and the denominator:

$$= \int \frac{4(x+4)(x-5)}{3x^2(x-5)}\, dx$$

This step allows you to cancel out a factor of $x-5$:

$$= \int \frac{4(x+4)}{3x^2}\, dx$$

Now, distribute the 4 in the numerator:

$$= \int \frac{4x+16}{3x^2}\, dx$$

Finally, you can separate this rational expression into two separate expressions and then use the Sum rule and Constant Multiple rule to further simplify the resulting integrals. I do this in separate steps, but feel free to jump to the end if you're getting comfortable with this set of moves:

$$= \int \frac{4x}{3x^2} + \frac{16}{3x^2}\, dx = \int \frac{4x}{3x^2}\, dx + \int \frac{16}{3x^2}\, dx = \frac{4}{3} \int \frac{x}{x^2}\, dx + \frac{16}{3} \int \frac{1}{x^2}\, dx$$

Now, express the rational expressions that remain inside the integrals as powers of x:

$$= \frac{4}{3} \int x^{-1}\, dx + \frac{16}{3} \int x^{-2}\, dx$$

Okay, I know that was a lot, but you may also be starting to see that I'm using the same tricks over and over. And, at last, you're ready to integrate:

$$= \frac{4}{3} \ln|x| + \frac{16}{3}\left(-\frac{1}{x}\right) + C = \frac{4}{3} \ln|x| - \frac{16}{3x} + C$$

Before moving on, notice that I didn't use the Power rule to integrate x^{-1}, which is equivalent to $\frac{1}{x}$. Instead, remember that I used $\ln|x|$ when integrating this term. And you should, too.

Integrating by using inverse trig functions

As you discover in Chapter 6, a short list of functions can be anti-differentiated as the six inverse trig functions. Many students respond to this news by asking, "Do I *really* have to memorize them?"

Well, it's a good idea to at least have a passing familiarity with these functions, because they can magically turn an impossible-looking integration into an easy one.

Table 7-1 gives you another look at this list.

TABLE 7-1 **Functions That Anti-differentiate to Inverse Trig Functions**

Function	Antiderivative
$\int \frac{1}{\sqrt{1-x^2}}\,dx$	$= \arcsin x + C$
$\int -\frac{1}{\sqrt{1-x^2}}\,dx$	$= \arccos x + C$
$\int \frac{1}{1+x^2}\,dx$	$= \arctan x + C$
$\int -\frac{1}{1+x^2}\,dx$	$= \operatorname{arccot} x + C$
$\int \frac{1}{x\sqrt{x^2-1}}\,dx$	$= \operatorname{arcsec} x + C$
$\int -\frac{1}{x\sqrt{x^2-1}}$	$= \operatorname{arccsc} x + C$

Notice that only three distinct forms of these functions exist, because each pair of co-functions differs only by a minus sign. Thus, for example, you can evaluate the integral $\int -\frac{1}{\sqrt{1-x^2}}\,dx$ as either $\arccos x + C$ or $-\arcsin x + C$, which are equivalent expressions. So, even if you forget about the others, I recommend that you absolutely, positively memorize the arcsin x, arctan x, and arcsec x forms.

As you can see, these functions look a little like the rational and radical functions you've been working with in this chapter. However, you can't use the Power rule to evaluate them. For example:

EXAMPLE

$$\int \frac{2}{3x\sqrt{x^2-1}} - \frac{3}{4(1+x^2)}\,dx$$

As always, use the Sum rule and Constant Multiple rule to make the integration a bit more manageable:

$$= \frac{2}{3}\int \frac{1}{x\sqrt{x^2-1}} - \frac{3}{4}\int \frac{1}{(1+x^2)}\,dx$$

At this point, it looks like you're stuck — *unless* you recognize that these two functions are actually boilerplate derivatives of inverse trig functions, in which case the problem is actually super easy:

$$= \frac{2}{3} \operatorname{arcsec} x - \frac{3}{4} \arctan x + C$$

In some cases, you may need to do a little algebra to set up a function so that it can be integrated in this fashion. For example:

EXAMPLE

$$\int \frac{1}{\sqrt{(2+2x)(3-3x)}}\, dx$$

In this case, you can FOIL the contents of the parentheses inside the radical and then factor out a constant:

$$= \int \frac{1}{\sqrt{6-6x^2}}\, dx = \int \frac{1}{\sqrt{6(1-x^2)}}\, dx$$

Now, separate this radical into the product of two radicals, and use the Constant Multiple rule to move the resulting constant outside the integral:

$$= \int \frac{1}{\sqrt{6}\sqrt{1-x^2}}\, dx = \frac{1}{\sqrt{6}} \int \frac{1}{\sqrt{1-x^2}}\, dx$$

Voila! The result integrates easily:

$$= \frac{1}{\sqrt{6}} \arcsin x + C$$

Calculus teachers tend to be perfectly happy with radicals in the denominator, but just in case your teacher isn't, you can rationalize the denominator if necessary:

$$= \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \arcsin x + C = \frac{\sqrt{6}}{6} \arcsin x + C$$

TIP

As a final note before moving on, trig substution (which I discuss in Chapter 10) relies on patterns based on the functions that anti-differentiate to arcsin *x*, arctan *x*, and arcsec *x*. So, if you bite the bullet and memorize them now, you'll be one step ahead later in your Calculus II course.

Integrating Trig Functions

If you didn't guess from Calculus I, trigonometry is a big topic in calculus, and this goes double for Calculus II. And if you thought differentiating trig functions using the Chain rule was tricky, you'll probably find integrating them a lot harder without it.

In this section, you expand your ability to integrate trig functions.

Recalling how to anti-differentiate the six basic trig functions

In Chapter 6, I start you off on integrating trig functions with a list of six antiderivatives for the six basic trig functions. For convenience, Table 7-2 recaps this list:

TABLE 7-2 Anti-differentiating the six basic trig functions

Trig function	Antiderivative
$\int \cos x\, dx$	$= \sin x + C$
$\int \sin x\, dx$	$= -\cos x + C$
$\int \sec^2 x\, dx$	$= \tan x + C$
$\int \csc^2 x\, dx$	$= -\cot x + C$
$\int \sec x \tan x\, dx$	$= \sec x + C$
$\int \csc x \cot x\, dx$	$= -\csc x + C$

Although this list of functions is pretty limited, you can extend it a bit by applying trig identities, and by applying the Sum rule and Constant Multiple rule wherever applicable.

Using the Basic Five trig identities

At first glance, some products or quotients of trig functions may seem impossible to integrate using the formulas I give you earlier in this chapter. But you'll be surprised how much headway you can often make when you integrate an unfamiliar trig function by first tweaking it using the Basic Five trig identities that I list in Chapter 2.

The unseen power of these identities lies in the fact that they allow you to express any combination of trig functions into a combination of sines and cosines. Generally speaking, the trick is to simplify an unfamiliar trig function and turn it into something that you know how to integrate.

When you're faced with an unfamiliar product or quotient of trig functions, follow these steps:

1. **Use trig identities to turn all factors into sines and cosines.**
2. **Cancel factors wherever possible.**
3. **If necessary, use trig identities to eliminate all fractions.**

For example:

EXAMPLE

$$\int \sin^2 x \cot x \sec x \, dx$$

In its current form, you can't integrate this expression by using the rules from this chapter. So follow these steps to turn it into an expression you can integrate:

1. **Use the identities $\cot x = \frac{\cos x}{\sin x}$ and $\sec x = \frac{1}{\cos x}$:**

$$= \int \sin^2 x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} dx$$

2. **Cancel both sin *x* and cos *x* in the numerator and denominator:**

$$= \int \sin x \, dx$$

In this example, even without Step 3, you have a function that you can integrate.

$$= -\cos x + C$$

Here's another example:

EXAMPLE

$$\int \tan x \sec x \csc x \, dx$$

Again, this integral looks like a dead end before you apply the Basic Five trig identities to it:

1. **Turn all three factors into sines and cosines:**

$$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\sin x} dx$$

2. **Cancel sin *x* in the numerator and denominator:**

$$= \int \frac{1}{\cos^2 x} dx$$

3. **Use the identity $\cos x = \frac{1}{\sec x}$ to eliminate the fraction:**

$$= \int \sec^2 x \, dx$$
$$= \tan x + C$$

Again, you turn an unfamiliar function into one of the trig functions that you know how to integrate. I show you lots more tricks for integrating trig functions in Chapter 10.

Applying the Pythagorean trig identities

The Pythagorean trig identities can also be useful for turning a big old trigonometric mess into a function that you know how to integrate. I list all three of them in Chapter 2. In this section, I show you how to use them all to assist you when integrating trig functions.

Using $\sin^2 x + \cos^2 x = 1$ to integrate trig functions

Everybody's favorite Pythagorean Identity is probably $\sin^2 x + \cos^2 x = 1$. It's relatively easy to remember, and you can use it to great effect when trying to simplify complicated-looking trig functions so you can integrate them.

For example, here's a trig function that looks difficult until you remember this particular identity:

EXAMPLE

$$\int \frac{1 - \sin^2 x}{\cos x} \, dx$$

You might be tempted to break this function into $\frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$, but this step would lead you down a blind alley. Instead, apply the Pythagorean Identity in its equivalent form $1 - \sin^2 x = \cos^2 x$:

$$= \int \frac{\cos^2 x}{\cos x} \, dx$$

This step allows you to simplify and then integrate as follows:

$$= \int \cos x \, dx = \sin x + C$$

Here's a problem that allows you two chances to apply the same Pythagorean Identity:

EXAMPLE

$$\int \frac{\sin^2 x + \cos^2 x}{1 - \cos^2 x}\, dx$$

Use $\sin^2 x + \cos^2 x = 1$ to rewrite both the numerator and denominator:

$$= \int \frac{1}{\sin^2 x}\, dx$$

Now, applying the reciprocal identity for sines puts the expression in a form that you can integrate:

$$= \int \csc^2 x\, dx = -\cot x + C$$

Using $1 + \tan^2 x = \sec^2 x$ to integrate trig functions

You already know that $\int \sec^2 x\, dx = \tan x + C$. This fact makes the identity $1 + \tan^2 x = \sec^2 x$ particularly useful for integration. For example:

EXAMPLE

$$\int \frac{\tan x + \cot x}{\cot x}\, dx$$

To begin, split this into two functions (be careful — *don't* use the Sum rule to break this integral into two integrals):

$$= \int \frac{\tan x}{\cot x} + \frac{\cot x}{\cot x}\, dx$$

Next, simplify both functions:

$$= \int \tan x \tan x + 1\, dx = \int \tan^2 x + 1\, dx$$

Now, apply the Pythagorean Identity $1 + \tan^2 x = \sec^2 x$, then integrate:

$$= \int \sec^2 x\, dx = \tan x + C$$

Using $1 + \cot^2 x = \csc^2 x$ to integrate trig functions

An important integral that's easy to evaluate is $\int \csc^2 x\, dx = -\cot x + C$. This evaluation makes the Pythagorean Identity $1 + \cot^2 x = \csc^2 x$ very useful for integration. For example:

EXAMPLE

$$\int \frac{\sin x + \cos x \cot x}{\sin x}\, dx$$

Split this function into two pieces and simplify each:

$$\int \frac{\sin x}{\sin x} + \frac{\cos x \cot x}{\sin x}\,dx = \int 1 + \cot x \cot x\,dx = \int 1 + \cot^2 x\,dx$$

Now, apply the Pythagorean Identity $1 + \cot^2 x = \csc^2 x$ and integrate:

$$= \int \csc^2 x\,dx = \cot x + C$$

Integrating Compositions of Functions with Linear Inputs

In Chapter 6, I break the news that the Chain rule for differentiation doesn't carry over into integration. Lots of students greet this announcement as good news until they realize that without a Chain rule, integration becomes much more difficult in practice than differentiation.

What makes the Chain rule so useful is that it allows you to differentiate compositions of functions — even long chains of functions within functions within functions.

For example, consider the function e^{x^2}. This is the composition of functions $f(g(x))$ with:

An outer (or output) function $f(x) = e^x$

An inner (or input) function $g(x) = x^2$

Before moving on, spend a moment getting comfortable with this notation. Can you see why $f(g(x)) = e^{x^2}$, given the functions $f(x)$ and $g(x)$ as previously defined?

Now, recall the Chain rule:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

The Chain rule allows you to differentiate this composition of functions as follows:

$$\frac{d}{dx} e^{x^2} = e^{x^2} \cdot 2x = 2xe^{x^2}$$

Because no Chain rule exists for integration, integrating the function e^{x^2} isn't a simple matter. In fact, as you discover in Chapter 6, while this function is integrable, its integral cannot be expressed as an elementary function — that is, as a function that you're used to seeing and working with in virtually all the math classes you've ever taken.

However, some actual good news for Calculus II is that, in certain cases, you can integrate compositions of functions of the form $f(g(x))$ provided that:

- The outer function $f(x)$ has a formula that allows you to integrate it.
- The inner function $g(x)$ is of the form $ax + b$.

Even better, when you get comfortable with this process, you'll find that you can integrate some relatively hairy-looking compositions of functions very quickly!

If this information doesn't jump out at you and make sense immediately, don't worry. In this section, I show you how to apply this method to a variety of common functions that you're probably beginning to feel comfortable integrating. After that, I discuss why this method works and then connect it to the Chain rule.

Understanding how to integrate familiar functions that have linear inputs

If you've read through and practiced the material in Chapter 6 and this chapter, I suspect that you're beginning to grow a bit more comfortable when integrating functions such as e^x, the six simple trig functions that arise from derivatives, and functions that yield nicely to the Power rule.

In this section, I show you how to integrate compositions of functions based on those that you already understand, but with linear inputs.

Integrating the e^x function composed with a linear input

You already know that $\int e^x dx = e^x$. Building from this information, you can also integrate any function of the form e^{ax+b} using the following formula:

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b}$$

The table includes four examples to show you how this works.

$\int e^{2x}\,dx$	$= \frac{1}{2}e^{2x} + C$
$\int e^{-2x+5}\,dx$	$= -\frac{1}{2}e^{-2x+5} + C$
$\int e^{\frac{1}{3}x-1}\,dx$	$= 3e^{\frac{1}{3}x-1} + C$
$\int e^{-\frac{9}{10}x+5}\,dx$	$= -\frac{10}{9}e^{-\frac{9}{10}x+5} + C$

EXAMPLE

Here's an example:

$$\int \frac{1}{2}e^{\frac{3}{4}x+2}\,dx$$

To begin, use the Constant Multiple rule to move the coefficient of $\frac{1}{2}$ outside the integral:

$$\frac{1}{2}\int e^{\frac{3}{4}x+2}\,dx$$

Now, apply the formula:

$$= \frac{1}{2}\left(\frac{4}{3}\right)e^{\frac{3}{4}x} + C$$

Complete the problem by simplifying the fractions:

$$= \frac{2}{3}e^{\frac{3}{4}x} + C$$

This formula works for any function of this form.

Integrating the six basic trig functions with linear inputs

Table 7-2 earlier in this chapter gives you six trig functions that are easy to integrate because they're the derivatives of the six original trig functions. As with e^x, you can also integrate any of these six trig functions with an input of the form $ax + b$ instead of x. Table 7-3 shows you some examples.

TABLE 7-3 Formulas for anti-differentiating the six basic trig functions with linear inputs

Formula	Example
$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+C$	$\int \cos 5x\,dx = \frac{1}{5}\sin 5x + C$
$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+C$	$\int \sin(8x+1)\,dx = -\frac{1}{8}\cos(8x+1)+C$
$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+C$	$\int \sec^2\left(-\frac{1}{4}x-3\right)dx = -4\tan\left(-\frac{1}{4}x-3\right)+C$
$\int \csc^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+C$	$\int \csc^2(10x+20)\,dx = -\frac{1}{10}\cot(10x+20)+C$
$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+C$	$\int \sec 25x \tan 25x\,dx = \frac{1}{25}\sec 25x + C$
$\int \csc(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\csc(ax+b)+C$	$\int \csc(0.01x+1)\cot(0.01x+1)\,dx = -100\csc(0.01x+1)+C$

As you can see, when using this rule for integrating variations of either sec x tan x or csc x cot x, the linear inputs $ax+b$ must match each other exactly. Here's an example:

EXAMPLE

$$\int \cos\frac{1}{2}x - \csc\frac{3}{5}x\cot\frac{3}{5}x\,dx$$

To begin, split this integral into two separate integrals using the Sum rule:

$$= \int \cos\frac{1}{2}x\,dx - \int \csc\frac{3}{5}x\cot\frac{3}{5}x\,dx$$

Now, integrate each term separately using the appropriate formula:

$$= 2\sin\frac{1}{2}x - \left(-\frac{5}{3}\csc\frac{3}{5}x\right) + C$$

Notice that when applying the Sum rule, I add a single $+\,C$ at the end. To complete the problem, simplify the result:

$$= 2\sin\frac{1}{2}x + \frac{5}{3}\csc\frac{3}{5}x + C$$

As you can see, the work here isn't too difficult. But it does require you to keep clear throughout the process about which antiderivative you're using to integrate each term, as well as what the linear input is in each case. Welcome to Calculus II!

Integrating power functions composed with a linear input

You can also use a modified version of the Power rule to integrate power functions with a linear input of the form $ax + b$. Here's the formula:

$$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$$

And here are a few examples that show you how to integrate in this way:

$\int (5x-1)^2 \ dx$	$= \frac{1}{15}(5x-1)^3 + C$
$\int (-2x+9)^3 \ dx$	$= -\frac{1}{8}(-2x+9)^4 + C$
$\int \left(\frac{1}{5}x+2\right)^7 \ dx$	$= \frac{5}{8}\left(\frac{1}{5}x+2\right)^8 + C$
$\int \left(-\frac{1}{11}x-3\right)^{19} \ dx$	$= -\frac{11}{20}\left(-\frac{1}{11}x-3\right)^{20} + C$

As with the Power rule, this variation also allows you to integrate negative and fractional powers. For example, to integrate the following, rewrite it as a linear function raised to a negative power:

EXAMPLE

$$\int \frac{1}{(5x+6)^3} dx = \int (5x+6)^{-3} dx$$

Now, apply this formula and, if your teacher insists, rewrite the solution as a rational function:

$$= -\frac{1}{10}(5x+6)^{-2} + C = -\frac{1}{10(5x+6)^2} + C$$

Here's an example with a linear function embedded inside a root function. As usual, begin by casting the function you're trying to integrate as a fractional exponent:

EXAMPLE

$$\int \sqrt[3]{(13x-1)^4} dx = \int (13x-1)^{\frac{4}{3}} dx$$

Now, use the formula to integrate (be careful when working with those fractions!):

$$= \frac{3}{91}(13x-1)^{\frac{7}{3}} + C$$

This solution should be enough to please even the stuffiest professor, but just in case, here's how you'd rewrite it using the root symbol:

$$= \frac{3}{91}\sqrt[3]{(13x-1)^7} + C$$

EXAMPLE

Here's an example that requires you to rewrite an integral using a negative exponent:

$$\int \frac{1}{\sqrt{\left(-\frac{8}{7}x-4\right)^3}}dx = \int \left(-\frac{8}{7}x-4\right)^{-\frac{3}{2}}dx$$

Once you've got this key step done, just be very careful as you use the formula to handle the numerical values. I break this out into a couple of steps so you can see the process:

$$= -2\left(-\frac{7}{8}\right)\left(-\frac{8}{7}x-4\right)^{-\frac{1}{2}} + C = \frac{7}{4}\left(-\frac{8}{7}x-4\right)^{-\frac{1}{2}} + C$$

If you must, make your teacher happy and recast this as a square root in the denominator:

$$= \frac{7}{4\sqrt{\left(-\frac{8}{7}x-4\right)}} + C$$

EXAMPLE

Here's a final example that contains a trap that's easy to step in:

$$\int \frac{1}{3x-5}dx = \int (3x-5)^{-1}dx \quad \text{NOT HELPFUL!}$$

Did you spot the misstep? Recall that you can't use the Power rule to integrate exponents of −1. Instead, use the following formula:

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$$

So, evaluate this integral as follows:

$$\int \frac{1}{3x-5}dx = \frac{1}{3}\ln|3x-5| + C$$

Knowing the handy arctan formula

I shy away from recommending that students integrate linear inputs to rational equations that integrate to inverse trig functions. These functions can get confusing, and usually require stronger medicine, such as variable substitution (Chapter 8), trig substitution (Chapter 10), or integration with partial fractions (Chapter 11).

Even so, I want to alert you to one handy formula that you may need later, when integrating with partial fractions (Chapter 11):

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

Notice that this formula allows you to integrate the sum of squares in the denominator of a rational expression.

EXAMPLE

$$\int \frac{5}{x^2+7} - \frac{8}{x^2+6x+19}\,dx$$

To begin, apply the Sum rule and Constant Multiple rule as usual:

$$= 5\int \frac{1}{x^2+7}\,dx - 8\int \frac{1}{x^2+6x+19}\,dx$$

Now, apply the formula to the first integral. I do this in two steps: first, explicitly stating the denominator as the sum of two squares, and then applying the formula:

$$= 5\int \frac{1}{x^2+\left(\sqrt{7}\right)^2}\,dx - 8\int \frac{1}{x^2+6x+19}\,dx$$

$$= \frac{5}{\sqrt{7}}\arctan\frac{x}{\sqrt{7}} - 8\int \frac{1}{x^2+6x+19}\,dx$$

I haven't finished integrating yet, so I refrain from adding C until the end of the problem. To evaluate the second integral, you also need to write the denominator explicitly as the sum of squares. To do this, start by completing the square:

$$= \frac{5}{\sqrt{7}}\arctan\frac{x}{\sqrt{7}} - 8\int \frac{1}{(x^2+6x+9)+10}\,dx$$

$$= \frac{5}{\sqrt{7}}\arctan\frac{x}{\sqrt{7}} - 8\int \frac{1}{(x+3)^2+\left(\sqrt{10}\right)^2}\,dx$$

Now, evaluate the second integral:

$$= \frac{5}{\sqrt{7}}\arctan\frac{x}{\sqrt{7}} - \frac{8}{\sqrt{10}}\arctan\frac{x+3}{\sqrt{10}} + C$$

Don't worry if the second part of this problem seems confusing. If you just focus on applying the formula when it fits, you probably won't need the complicated stuff unless your teacher gives you a particularly difficult problem requiring integration by partial fractions.

Using algebra to solve more complex problems

When you know how to apply linear input functions to the set of functions that you already know how to integrate, you can evaluate a variety of more complex integrals. For example:

EXAMPLE

$$\int \frac{3x + \sqrt{3x}}{\sqrt[4]{3x}}\, dx$$

The repetition of *3x* in this example should alert you to the opportunity to use this method, but first you need to use algebra to simplify the function. To begin, split this function into the sum of two functions and express every rational element as a power:

$$= \int \frac{3x}{\sqrt[4]{3x}} dx + \int \frac{\sqrt{3x}}{\sqrt[4]{3x}}\, dx$$

$$= \int \frac{3x}{(3x)^{\frac{1}{4}}} dx + \int \frac{(3x)^{\frac{1}{2}}}{(3x)^{\frac{1}{4}}}\, dx$$

Next, use the rule for dividing exponents to simplify each resulting function:

$$= \int (3x)^{\frac{3}{4}} dx + \int (3x)^{\frac{1}{4}} dx$$

Now, use the variation on the Power rule to evaluate both integrals and then simplify the results:

$$= \frac{4}{7}\left(\frac{1}{3}\right)(3x)^{\frac{7}{4}} + \frac{4}{5}\left(\frac{1}{3}\right)(3x)^{\frac{5}{4}} + C$$

$$= \frac{4}{21}(3x)^{\frac{7}{4}} + \frac{4}{15}(3x)^{\frac{5}{4}} + C$$

Using trig identities to integrate more complex functions

In some cases, you can use identities to rewrite a trig function in a way that allows you to apply the formulas I provide earlier in this chapter. As I mention earlier, be sure to have the list of six basic trig antiderivatives handy (or, better yet, have them memorized!) so you know the result you're trying to achieve.

EXAMPLE

Here's a relatively quick example:

$$\int \frac{\pi}{\sin^2(\pi x)}\,dx$$

Don't let the presence of the constant π throw you off your game: In the numerator, π is just a constant, so you can safely transfer it outside the integral:

$$= \pi \int \frac{1}{\sin^2(\pi x)}\,dx$$

Now, you might be tempted to use the use the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$ in the denominator. However, the reciprocal identity $\frac{1}{\sin x} = \csc x$ leads to a better result:

$$= \pi \int \csc^2(\pi x)\,dx$$

You can integrate the function $\csc^2 x$ using anti-differentiation, so you can use the formula in Table 7-3 I gave you earlier in this chapter, this time treating π simply as a numerical coefficient to x:

$$= \pi\left(\frac{1}{\pi}\right)\cot(\pi x) + C = \cot(\pi x) + C$$

As you can see, some of the π values cancel as factors, but the input coefficient π remains in the final answer.

Here's a more difficult trig example, the toughest you've seen so far:

$$\int \frac{\cos\,(2x-\pi)}{1-\cos^2(2x-\pi)}\,dx$$

The repetition of the expression $2x-\pi$ is a clue to think of this function as a linear input (remember that π is just a constant!). The trick is to get the whole function to look like one of the six trig functions you can anti-differentiate. Begin by using the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$ to simplify the denominator:

$$= \int \frac{\cos\,(2x-\pi)}{\sin^2(2x-\pi)}\,dx$$

Now, by splitting the denominator into two factors, you can massage the result into a familiar expression:

$$= \int \frac{\cos\,(2x-\pi)}{\sin(2x-\pi)\sin(2x-\pi)}\,dx$$
$$= \int \cot(2x-\pi)\csc(2x-\pi)\,dx$$

You should recognize this result as a familiar function, so integrate:

$$= \frac{1}{2}\csc(2x - \pi) + C$$

Although this isn't an easy problem, if you review the steps, you'll probably see that everything here is within your grasp. The repetition of the linear input $2x - \pi$ should tip you off about the overall strategy: This repetition becomes useful only if you can find a clever way to use trig identities to rewrite the original function as a trig function that you already know how to integrate.

Understanding why integrating compositions of functions with linear inputs actually works

When you understand how to integrate compositions of functions with linear inputs, you're ready to understand why this trick works. And this understanding will help you grasp variable substitution (also called *u-substitution*), which is the topic of Chapter 8.

To give you this understanding, let's return to evaluating a relatively simple integral:

$$\int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1} + C$$

To see why this evaluation works, I reverse the process by finding the derivative of $\frac{1}{2}e^{2x+1}$:

$$\frac{d}{dx}\left(\frac{1}{2}e^{2x+1}\right)$$

As you can see, to keep the process simple, I've dropped the + C, which differentiates to 0. To begin, move the coefficient of $\frac{1}{2}$ outside the derivative:

$$= \frac{1}{2}\left(\frac{d}{dx}e^{2x+1}\right)$$

Now, use the Chain rule to differentiate. I do this in several steps to clarify the process:

$$= \frac{1}{2}e^{2x+1} \cdot \frac{d}{dx}(2x+1) = \frac{1}{2}e^{2x+1} \cdot 2 = e^{2x+1}$$

As you can see, when you apply the Chain rule to a function with a linear input of $2x+1$, the result includes multiplication by 2. So when you *undo* the Chain rule by integrating, you also need to *undo* this multiplication by including a coefficient of $\frac{1}{2}$.

This explanation also accounts for the presence of the exponent of $2x+1$ in both the original function e^{2x+1} and the result $\frac{1}{2}e^{2x+1}+C$. The Chain rule preserves this function during differentiation, so it's also unaffected when you reverse the process to integrate.

To sum up, this trick for integrating familiar functions that have linear inputs leverages the fact that every derivative of the form $\frac{d}{dx}(ax+b)$ evaluates to a, and can therefore be reversed by multiplying by $\frac{1}{a}$.

In Chapter 8, you extend this idea of reversing the Chain rule when integrating to a whole new level, which enables you to integrate much more complicated sets of functions.

IN THIS CHAPTER

» Understanding how variable substitution works

» Recognizing when variable substitution can help you

» Knowing a shortcut for using substitution with definite integrals

Chapter 8

Here's Looking at *U*-Substitution

In Chapters 6 and 7, you discover how to evaluate a variety of indefinite integrals. This promises to make solving area problems framed as definite integrals much easier. But it still begs the question of how to evaluate indefinite integrals that don't fit so nicely into the forms you know how to anti-differentiate.

In this chapter, you discover *variable substitution* (also called *u-substitution*), a very handy way to integrate functions that don't appear at all friendly. I first show you how to use this method step by step, then take you behind the scenes to understand when variable substitution is likely to work well so you can feel confident when solving problems.

To wrap up, I provide you with a shortcut for using *u*-substitution when evaluating definite integrals.

Knowing How to Use U-Substitution

REMEMBER

Variable substitution is especially useful when you're faced with the product of two functions. Without further ado, here are the five steps you need to know to evaluate integrals using variable substitution:

1. **Declare a variable *u* and use it to build part of the integral in terms of *u*.**
2. **Differentiate *u* and isolate the rest of the integral on the right side of the equals sign.**
3. **Rewrite the integral in terms of *u*.**
4. **Integrate in terms of *u*.**
5. **Substitute the original value of *u* into the result.**

These steps may seem odd at first, but with a few repetitions, I'm confident that they'll start to make sense. In this section, I walk you through the *how* of variable substitution so you can get a feel for it.

For example, imagine that you're faced with this integral:

EXAMPLE

$$\int \sin^3 x \cos x \, dx$$

The problem in this case is that the function that you're trying to integrate is the product of two functions — $\sin^3 x$ and $\cos x$. This would be simple to differentiate with the Product rule, but integration doesn't have a Product rule. Here's where variable substitution comes to the rescue:

1. **Declare a variable *u* and use it to build part of the integral in terms of *u*.**

   Here, I declare *u* as follows:

   $$\text{Let } u = \sin x$$

   Now use *u* to express $\sin^3 x$:

   $$u^3 = \sin^3 x$$

2. **Differentiate *u* and isolate the rest of the integral on the right side of the equals sign**:

   Notice that the expression cos *x dx* still remains and needs to be expressed in terms of *u*. To do this, differentiate *u*:

   $$\frac{du}{dx} = \cos x$$

   Now, to isolate cos *x dx* on the right side, multiply both sides by *dx*:

   $$du = \cos x \; dx$$

3. **Rewrite the integral in terms of *u*:**

   To perform this step, use the sidework you've done in the first two steps, substituting u^3 for $\sin^3 x$ and *du* for cos *x dx*:

   $$\int \sin^3 x \cos x \, dx = \int u^3 du$$

4. **Integrate in terms of *u*.**

   This turns out to be easy now:

   $$= \frac{1}{4}u^4 + C$$

5. **Substitute the original value of *u* into the result.**

   That is, substitute sin *x* for *u*:

   $$= \frac{1}{4}\sin^4 x + C$$

This is the final answer.

EXAMPLE

Here's another example of a product of functions that would be difficult to integrate without u-substitution:

$$\int x\sqrt{3x^2 + 7}\, dx$$

Here's how you integrate it step by step:

1. **Declare a variable *u* as follows and use it to build part of the integral in terms of *u*.**

   $$\text{Let } u = 3x^2 + 7$$

   Here, you may ask how I know what value to assign to *u*. Here's the short answer: *u* is the inner function, as you would identify if you were using the Chain rule. (See Chapter 3 for a review of the Chain rule.) I explain this more fully in the section, "Recognizing When to Use Substitution," later in this chapter.

2. **Differentiate $u = 3x^2 + 7$ and isolate x:**

$$\frac{du}{dx} = 6x$$
$$du = 6x\ dx$$

From Step 1, I know that I need to replace $x\,dx$ in the integral, so I divide both sides of this equation by 6:

$$\frac{1}{6}du = x\ dx$$

3. **Rewrite the integral in terms of u:**

Here I substitute $\sqrt{u}$ for $\sqrt{3x^2+7}$ and $\frac{1}{6}du$ for $x\ dx$:

$$\int x\sqrt{3x^2+7}\ dx = \int \sqrt{u}\left(\frac{1}{6}du\right)$$

Before integrating, you can move the fraction $\frac{1}{6}$ outside the integral:

$$= \frac{1}{6}\int \sqrt{u}\ du$$

4. **Integrate in terms of u:**

I've taken an extra step, putting the square root in exponential form, to make sure that you see how to do this:

$$= \frac{1}{6}\int u^{\frac{1}{2}}du$$
$$= \frac{1}{6}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C$$
$$= \frac{1}{9}u^{\frac{3}{2}} + C$$

5. **To finish up, substitute $3x^2+7$ for u:**

$$= \frac{1}{9}\left(3x^2+7\right)^{\frac{3}{2}} + C$$

EXAMPLE

Here's another example of a product of functions that responds well to integration with variable substitution:

$$\int xe^{x^2}dx$$

Before evaluating this integral, recall from Chapter 6 that the function e^{x^2} cannot be integrated as an elementary function. Yet the seemingly more complex function xe^{x^2} is relatively easy to integrate and does evaluate to an elementary function. The presence of that additional *x* makes all the difference. Keep an eye on this *x* as you work through the following steps:

1. **Declare a variable *u* and use it to build part of the integral in terms of *u*.**

   Here, I set *u* to be that troublesome x^2 exponent:

   Let $u = x^2$

2. **Differentiate *u* and isolate the rest of the integral on the right side of the equals sign.**

   In this case, I want to write *x dx* in terms of *u*:

$$\frac{du}{dx} = 2x$$
$$du = 2x\ dx$$
$$\frac{1}{2}du = x\ dx$$

3. **Rewrite the integral in terms of *u*:**

   To do this, substitute *u* for x^2 and $\frac{1}{2}du$ for $x\ dx$:

$$\int xe^{x^2}dx = \int e^u\left(\frac{1}{2}du\right)$$

   Before integrating, move the fraction $\frac{1}{2}$ outside the integral:

$$= \frac{1}{2}\int e^u du$$

4. **Integrate in terms of *u*.**

   Again, the integration itself is arguably the simplest step:

$$= \frac{1}{2}e^u + C$$

5. **Substitute the original value of *u* into the result:**

$$= \frac{1}{2}e^{x^2} + C$$

Did you catch how the presence of x in xe^{x^2} helped because the derivative of x^2 is $2x$? Here's one final example of a variable substitution that makes this connection glaringly obvious:

EXAMPLE

$$\int x^{99} e^{x^{100}} dx$$

This is where the magician moves their hands slowly so you catch the sleight of hand:

1. **Declare a variable *u* and use it to build part of the integral in terms of *u*.**

   This time, I strategically set u to x^{100}:

   $$\text{Let } u = x^{100}$$

   Now, it's clear that in the next step, I need to build $x^{99}dx$ in terms of u.

2. **Differentiate *u* and isolate the rest of the integral on the right side of the equals sign.**

   These steps are beginning to get repetitive, aren't they? That's actually a good sign — it means you're catching on to the process:

   $$\frac{du}{dx} = 100x^{99}$$
   $$du = 100x^{99}\ dx$$
   $$\frac{1}{100} du = x^{99}\ dx$$

3. **Rewrite the integral in terms of *u*:**

   This move should begin to look more obvious as well — substitute u for x^{100} and $\frac{1}{100} du$ for $x^{99}dx$, then pull the fraction out of the integral:

   $$\int x^{99} e^{x^{100}} dx = \int e^u \left(\frac{1}{100} du \right) = \frac{1}{100} \int e^u du$$

4. **Integrate in terms of *u*.**

   Once again, when integrating the *e* function, the *e* is for *easy:*

   $$= \frac{1}{100} e^u + C$$

5. **Substitute the original value of *u* into the result:**

   $$= \frac{1}{100} e^{x^{100}} + C$$

At this point, if you've followed these examples reasonably well, you're ready for the strategy behind u-substitution.

Recognizing When to Use U-Substitution

In the previous section, I show you the mechanics of variable substitution — that is, *how* to perform variable substitution when you're trying to integrate the product of functions.

In this section, I clarify *when* to use variable substitution. When you can see what makes a function conducive to this method, you'll find the strategy of choosing a *u*-value much easier.

Variable substitution works best in two cases. In this section, I first show you the simpler case, and then the more complex case.

The simpler case: $f(x) \cdot f'(x)$

The best case for variable substitution occurs when you want to integrate a function multiplied by its derivative. If you get good at spotting these opportunities, you can actually use the following formula:

$$= \int f(x) f'(x) dx = \frac{1}{2}[f(x)]^2 + C$$

Here are a few examples:

Integral	Evaluation
$\int \sin x \cos x \; dx$	$= \frac{1}{2}\sin^2 x + C$
$\int \tan x \sec^2 x \; dx$	$= \frac{1}{2}\tan^2 x + C$
$\int \frac{\ln x}{x} \; dx$	$= \frac{1}{2}(\ln x)^2 + C$
$\int (3x^2 - 7x + 4)(6x - 7) \; dx$	$= \frac{1}{2}(3x^2 - 7x + 4)^2 + C$

EXAMPLE

Here's an example:

$$\int \tan x \sec^2 x \; dx$$

The main thing to notice here is that the derivative of tan x is $\sec^2 x$. I do this in two ways, first using variable substitution and then using the formula, which is admittedly way easier.

1. **Declare *u* and use it to build part of the integral in terms of *u*.**

   Let $u = \tan x$

2. **Differentiate and isolate *x* as usual:**

$$\frac{du}{dx} = \sec^2 x$$
$$du = \sec^2 x\, dx$$

3. **Rewrite the integral in terms of *u*:**

$$\int \tan x \sec^2 x\, dx = \int u\, du$$

4. **This integration couldn't be much easier:**

$$= \frac{1}{2}u^2 + C$$

5. **Substitute back tan *x* for *u*:**

$$= \frac{1}{2}\tan^2 x + C$$

When you know how this variable substitution works, you can just use the formula that I gave you:

$$\int \tan x \sec^2 x\, dx = \frac{1}{2}\tan^2 x + C$$

In some cases, you may have an integral that you can tweak to make the formula work. An example should make this clear:

EXAMPLE

$$\int (10x^2 + 1)7x\, dx$$

In this case, if the contents outside the parentheses were $20x$ rather than $7x$, you'd be all set. Here, you can build the function you'd like to see. To begin, use the Constant Multiple rule to bring the 7 outside the integral:

$$= 7\int (10x^2 + 1)x\, dx$$

Now, you'd like to multiply *inside* the integral by a factor of 20, so multiply the *outside* by $\frac{1}{20}$:

$$= \frac{7}{20}\int (10x^2 + 1)20x\, dx$$

Now, the integral has the form $= \int f(x) f'(x) dx$, so you can use the formula and simplify:

$$= \frac{7}{20} \cdot \frac{1}{2} \left(10x^2 + 1\right)^2 + C$$
$$= \frac{7}{40} \left(10x^2 + 1\right)^2 + C$$

But if you find this method a little confusing and you're worried that you'll make a calculation error, you can always do a u-substitution, letting $u = 10x^2 + 1$ and $du = 20x\ dx$. Either method yields the same result.

The more complex case: g(*f*(*x*)) · *f* '(*x*) when you know how to integrate *g*(*x*)

When you've practiced integrating functions of the form $f(x) \cdot f'(x)$, you're ready for the more complex case $g(f(x)) \cdot f'(x)$, where $g(x)$ is a function that you know how to integrate.

The notation here is a little opaque, so here's an example:

$$\int x^3 \sqrt{x^4 - 1}\ dx$$

EXAMPLE

Notice that $x^4 - 1$, which is tucked inside the square root, has a derivative of $4x^3$, which looks a lot like another part of the integral. So here's the declaration, followed by the differentiation and some simple algebra:

$$\text{Let } u = x^4 - 1$$
$$\frac{du}{dx} = 4x^3$$
$$\frac{1}{4} du = x^3 dx$$

Now you can rewrite the whole integral in terms of u:

$$\int \sqrt{u} \left(\frac{1}{4} du\right) = \frac{1}{4} \int u^{\frac{1}{2}} du$$

At this point, you can solve the integral easily:

$$= \frac{1}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$$

To complete the problem, plug in $x^4 - 1$ for u and simplify:

$$= \frac{1}{6}\left(x^4 - 1\right)^{\frac{3}{2}} + C$$

As you can see, the trick to this technique is identifying one part of the function you're trying to integrate that differentiates to another part.

EXAMPLE

Here's a hairy-looking integral that also responds well to substitution:

$$\int \frac{2x+1}{\left(x^2 + x - 5\right)^{\frac{4}{3}}}\, dx$$

The key insight here is that the numerator of this fraction is the derivative of the inner function in the denominator. So set u to the denominator and differentiate:

$$\text{Let } u = x^2 + x - 5$$
$$\frac{du}{dx} = 2x + 1$$
$$du = (2x + 1)\, dx$$

Almost like magic, the results are ready to be substituted into the integral:

$$= \int \frac{1}{u^{\frac{4}{3}}}\, du$$
$$= \int u^{-\frac{4}{3}}\, du$$

Again, the integration step is relatively easy:

$$= -3u^{-\frac{1}{3}} + C = -3\left(x^2 + x - 5\right)^{-\frac{1}{3}} + C$$

By now, if you've worked through the examples in this chapter, you're probably beginning to see opportunities to make variable substitutions. Here's another example that looks entirely heinous until you see the opportunity:

EXAMPLE

$$\int e^{\cot x} \csc^2 x\, dx$$

Notice that the derivative of $\cot x$ is $-\csc^2 x$, so this looks like another good candidate for u-substitution:

$$\text{Let } u = \cot x$$
$$\frac{du}{dx} = -\csc^2 x$$
$$-du = \csc^2 x\, dx$$

This results in the following substitution:

$$= \int e^u (-du)$$
$$= -\int e^u du$$

Again, this is another integral that you can solve easily:

$$= -e^u + C = -e^{\cot x} + C$$

Using Substitution to Evaluate Definite Integrals

In the first two sections of this chapter, I cover how and when to evaluate indefinite integrals with variable substitution. All this information also applies to evaluating definite integrals, but I also have a time-saving trick that you should know.

When using variable substitution to evaluate a definite integral, you can save yourself some trouble at the end of the problem. Specifically, you can leave the solution in terms of *u* by changing the limits of integration.

EXAMPLE

For example, suppose that you're solving an area problem and need to evaluate the following definite integral:

$$\int_{x=0}^{x=1} x\sqrt{x^2+1}\, dx$$

Notice that I explicitly give the limits of integration as $x = 0$ and $x = 1$. This is just a notational change to remind you that the limits of integration are values of x. This fact becomes important later in the problem.

This is a great opportunity to use variable substitution as follows:

$$\text{Let } u = x^2 + 1$$
$$\frac{du}{dx} = 2x$$
$$\frac{1}{2} du = x\, dx$$

You're now ready to make the substitution in terms of *u* and integrate:

$$= \frac{1}{2}\int_{x=0}^{x=1} u^{\frac{1}{2}}\, du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{x=0}^{x=1}$$

Recall that when solving a definite integral, you use the notation shown here to indicate that you still need to crunch the numbers and solve for a value. Before continuing, I'll simplify:

$$= \frac{1}{3}u^{\frac{3}{2}}\Big|_{x=0}^{x=1}$$

If this were an indefinite integral, your next step would be to rewrite *u* in terms of *x*. But when using variable substitution with definite integrals, you can also choose to rewrite *x* in terms of *u*. To do this, substitute the limits of integration (0 and 1) for *x* into the substitution equation $u = x^2 + 1$:

$$u = 1^2 + 1 = 2$$

$$u = 0^2 + 1 = 1$$

Now use these values of *u* as your new limits of integration:

$$= \frac{1}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=2}$$

Evaluate this expression as I show you in Chapter 5:

$$= \frac{1}{3}\left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}(\sqrt{8} - 1)$$

$$= \frac{\sqrt{8} - 1}{3}$$

Thus, this value is the area you're looking for.

4 Advanced Integration Techniques

IN THIS PART . . .

Apply integration by parts as a consequence of the Product rule

Integrate a wide variety of trigonometric functions

Use trig substitution to evaluate integrals

Evaluate rational functions by integrating with partial fractions

IN THIS CHAPTER

» Making the connection between the Product rule and integration by parts

» Knowing how and when integration by parts works

» Integrating by parts by using the DI-agonal method

» Practicing the DI-agonal method on the four most common products of functions

Chapter 9

Parting Ways: Integration by Parts

In Calculus I, you find that the Product rule allows you to calculate the derivative of any two functions that are multiplied together. (I review this rule in Chapter 3, in case you need a refresher.) But integrating the product of two functions isn't quite as simple.

Unfortunately, no formula allows you to integrate the product of two indiscriminate functions. As a result, a variety of techniques have been developed to handle products of functions on a case-by-case basis.

In this chapter, I show you the most widely applicable technique for integrating products, called *integration by parts.* First, I demonstrate how the formula for integration by parts follows the Product rule. Then I show you how the formula works in practice. After that, I give you a list of the products of functions that are likely to yield to this method. I also give you a mnemonic to help you decide the best way to split up the function you're integrating.

After you understand the principle behind integration by parts, I give you a method — called the *DI-agonal method* — for performing this calculation efficiently and without errors. This method works especially well when you need to perform integration by parts two or more times to evaluate a single integral.

To finish up, I show you examples of how to use this method to integrate the four most common products of functions.

Introducing Integration by Parts

Integration by parts is a happy consequence of the Product rule (discussed in Chapter 3). In this section, I show you how to tweak the Product rule to derive the formula for integration by parts. I show you two versions of this formula — a complicated version and a simpler one — and then recommend that you memorize the second one. I show you how to use this formula, and then I give you a heads-up as to when integration by parts is likely to work best.

Reversing the Product rule

The Product rule (see Chapter 3) enables you to differentiate the product of two functions:

$$\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

Through a series of mathematical somersaults, you can turn this equation into a formula that's useful for integrating. This derivation doesn't have any truly difficult steps, but the notation along the way is mind-deadening, so don't worry if you have trouble following it. Knowing how to derive the formula for integration by parts is less important than knowing when and how to use it, which I focus on in the rest of this chapter.

The first step is simple: Just rearrange the right side of the equation into an equivalent form.

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Next, rearrange the terms of the equation:

$$f(x)\cdot g'(x) = \frac{d}{dx}[f(x)\cdot g(x)] - g(x)\cdot f'(x)$$

Now integrate both sides of this equation:

$$\int f(x)g'(x)\,dx = \int\left\{\frac{d}{dx}[f(x)g(x)] - g(x)f'(x)\right\}dx$$

Use the Sum rule to split the integral on the right in two:

$$\int f(x)g'(x)\,dx = \int\frac{d}{dx}[f(x)g(x)]\,dx - \int g(x)f'(x)\,dx$$

The first of the two integrals on the right undoes the derivative:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

This is the formula for integration by parts. But because it's so hairy looking, the following substitution is used to simplify it:

Let $u = f(x)$ Let $v = g(x)$

$du = f'(x)\,dx$ $dv = g'(x)\,dx$

REMEMBER

Here's the friendlier version of the same formula, which you should memorize:

$$\int u\,dv = uv - \int v\,du$$

That's not so bad, right? Memorize it! (Say after me: "The integral of *u dv* equals *uv* minus the integral of *v du*.")

TECHNICAL STUFF

Although the integrals in this formula may look like you're integrating one variable by a different variable, remember that both *u* and *v* are functions of *x*. So, you're integrating one function of *x* times another function of *x* by *dx*.

Knowing how to integrate by parts

The formula for integration by parts gives you the option to break the product of two functions down to its factors and integrate it in an altered form.

To integrate by parts:

1. **Decompose the entire integral (including *dx*) into two factors.**
2. **Let the factor without *dx* equal *u* and the factor with *dx* equal *dv*.**
3. **Differentiate *u* to find *du*, and integrate *dv* to find *v*.**
4. **Use the formula $\int u\,dv = uv - \int v\,du$.**
5. **Evaluate the right side of this equation to solve the integral.**

EXAMPLE

For example, suppose that you want to evaluate this integral:

$$\int x \ln x\,dx$$

In its current form, you can't perform this computation, so integrate by parts:

1. **Decompose the integral into ln *x* and *x dx*.**
2. **Let $u = \ln x$ and $dv = x\,dx$.**

3. **Differentiate ln *x* to find *du* and integrate *x dx* to find *v*:**

$$\text{Let } u = \ln x \qquad \text{Let } dv = x\,dx$$

$$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int x\,dx$$

$$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$$

4. **Using these values for *u*, *du*, *v*, and *dv*, you can use the formula $\int u\,dv = uv - \int v\,du$ to rewrite the integral as follows:**

$$\int x \ln x\,dx = (\ln x)\left(\frac{1}{2}x^2\right) - \int\left(\frac{1}{2}x^2\right)\frac{1}{x}dx$$

Notice that the integral you're trying to evaluate shows up on the left side of this equation. At this point, algebra is useful to simplify the right side:

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$$

5. **Evaluate the integral on the right:**

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}\right)x^2 + C$$

You can simplify this answer just a bit:

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

This version of the result is your final answer.

Knowing when to integrate by parts

After you know the basic mechanics of integrating by parts, as I show you in the previous section, it's important to recognize when integrating by parts is useful.

To start off, here are two important cases when integration by parts is definitely the way to go:

- The logarithmic function ln *x*
- The first four inverse trig functions (arcsin *x*, arccos *x*, arctan *x*, and arccot *x*)

Beyond these cases, integration by parts is useful for integrating the product of more than one function. For example:

- $x \ln x$
- $x \operatorname{arcsec} x$
- $x^2 \sin x$
- $e^x \cos x$

Notice that in each case, you can recognize the product of functions because the variable x appears more than once in the function.

TIP

Whenever you're faced with integrating the product of functions, consider variable substitution (which I discuss in Chapter 8) before you think about integration by parts. For example, $x \cos(x^2)$ is a job for variable substitution, not integration by parts.

You can use integration by parts to integrate any of the functions listed in Table 9-1.

TABLE 9-1 Knowing When to Integrate by Parts

Function	Example	Differentiate *u* to Find *du*	Integrate *dv* to Find *v*
Log function	$\int \ln x \, dx$	$\ln x$	dx
Log times algebraic	$\int x^4 \ln x \, dx$	$\ln x$	$x^4 \, dx$
Log composed with algebraic	$\int \ln x^3 dx$	$\ln x^3$	dx
Inverse trig forms	$\int \arcsin x \, dx$	$\arcsin x$	dx
Algebraic times sine	$\int x^2 \sin x \, dx$	x^2	$\sin x \, dx$
Algebraic times cosine	$\int 3x^5 \cos x \, dx$	$3x^5$	$\cos x \, dx$
Algebraic times exponential	$\int \frac{1}{2}x^2 e^{3x} dx$	$\frac{1}{2}x^2$	$e^{3x} \, dx$
Sine times exponential	$\int e^{\frac{x}{2}} \sin x \, dx$	$\sin x$	$e^{\frac{x}{2}} \, dx$
Cosine times exponential	$\int e^x \cos x \, dx$	$\cos x$	$e^x \, dx$

TIP

When you're integrating by parts, here's the most basic rule when deciding which term to integrate and which to differentiate: If you only know how to integrate one of the two, that's the one you integrate!

EXAMPLE

Here's how to (finally!) integrate the all-important natural log function:

$$\int \ln x \, dx$$

1. **Decompose the integral into ln *x* and *dx*.**
2. **Let *u* = ln *x* and *dv* = *dx*.**

   As you can see, I assign the variables *u* and *dv* as shown in Table 9-1.
3. **Differentiate ln *x* to find *du* and integrate *dx* to find *v*:**

$$\begin{aligned} &\text{Let } u = \ln x & &\text{Let } dv = dx \\ &\frac{du}{dx} = \frac{1}{x} & &\int dv = \int dx \\ &du = \frac{1}{x}dx & &v = x \end{aligned}$$

4. **Using these values for *u*, *du*, *v*, and *dv*, you can use the formula $\int u\,dv = uv - \int v\,du$ to rewrite the integral as follows:**

$$\int \ln x \, dx = (\ln x)(x) - \int (x)\frac{1}{x}dx$$

   As usual, a difficult integral now shows up on the right as something that's easier to evaluate. Begin by simplifying:

$$= \frac{1}{2}x \ln x - \int 1 \, dx$$

5. **Evaluate the integral on the right:**

$$= x \ln x - x + C$$

This trick of "differentiating to integrate" that I use for ln *x* carries over to the first four inverse trig functions. For example:

EXAMPLE

$$\int \arcsin x \, dx$$

1. **Decompose the integral into arcsin *x* and *dx*.**
2. **Let *u* = arcsin *x* and *dv* = *dx*.**

   Again, I assign the variables *u* and *dv* as shown in Table 9-1.

3. **Differentiate arcsin *x* to find *du* and integrate *dx* to find *v*:**

$$\text{Let } u = \arcsin x \qquad \text{Let } dv = dx$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \int dv = \int dx$$

$$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$$

4. **Using these values for *u*, *du*, *v*, and *dv*, you can use the formula $\int u\ dv = uv - \int v\ du$ to rewrite the integral as follows:**

$$\int \arcsin x\ dx = (\arcsin x)(x) - \int (x)\frac{1}{\sqrt{1-x^2}}dx$$

   As usual, take a moment to simplify the right side:

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$$

5. **Evaluate the integral on the right.**

   This looks like a casebook study in *u*-substitution (see Chapter 8 for more details):

$$u = 1 - x^2$$

$$du = -2x\ dx$$

$$-\frac{1}{2}dx = x\ dx$$

   Now, rewrite this integral in terms of *u*, evaluate it, and rewrite it in terms of *x* (again, these steps make more sense if you're familiar with Chapter 8):

$$= x\arcsin x - \left(-\frac{1}{2}\int \frac{1}{\sqrt{u}}du\right)$$

$$= x\arcsin x + \frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$= x\arcsin x + \frac{1}{2}(2u^{\frac{1}{2}}) + C$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

Like magic, integration by parts allows you to evaluate this useful integral.

Integrating by Parts with the DI-agonal Method

The *DI-agonal method* is basically integration by parts with a chart that helps you organize information. This method is especially useful when you need to integrate by parts more than once to solve a problem. In this section, I show you how to use the DI-agonal method to evaluate a variety of integrals.

Looking at the DI-agonal chart

The DI-agonal method avoids using *u* and *dv*, which are easily confused (especially if you write the letters *u* and *v* as sloppily as I do!). Instead, a column for *differentiation* is used in place of *u*, and a column for *integration* replaces *dv*.

Use the following chart for the DI-agonal method:

	I
D	
+	
–	

As you can see, the chart contains two columns: the *D* column for *differentiation*, which has a plus sign and a minus sign, and the *I* column for *integration*. You may also notice that the *D* and the *I* are placed diagonally in the chart — yes, the name *DI-agonal method* works on two levels (so to speak).

Using the DI-agonal method

Earlier in this chapter, I provide a list of functions that you can integrate by parts. The DI-agonal method works for all these functions. I also give you the mnemonic **L**ovely **I**ntegrals **A**re **T**errific (which stands for **L**ogarithmic, **I**nverse trig, **A**lgebraic, **T**rig) to help you remember how to assign values of *u* and *dv* — that is, what to differentiate and what to integrate.

To use the DI-agonal method:

1. **Write the value to differentiate in the box below the *D* and the value to integrate (omitting the *dx*) in the box below the *I*.**
2. **Differentiate down the *D* column and integrate down the *I* column.**
3. **Add the products of all *full* rows as terms.**

   I explain this step in further detail in the examples that follow.
4. **Add the integral of the product of the two lowest diagonally adjacent boxes.**

   I also explain this step in greater detail in the examples.

Don't spend too much time trying to figure this process out. The upcoming examples show you how it's done and give you plenty of practice. I show you how to use the DI-agonal method to integrate products that include logarithmic, inverse trig, algebraic, and trig functions.

L is for logarithm

You can use the DI-agonal method to evaluate the product of a log function and an algebraic function.

EXAMPLE

For example, suppose that you want to evaluate the following integral:

$$\int x^2 \ln x \, dx$$

TIP

Whenever you integrate a product that includes a log function, the log function always goes in the *D* column.

1. **Write the log function in the box below the *D* and the rest of the function value (omitting the *dx*) in the box below the *I*.**

	I
D	x^2
$+ \ln x$	
$-$	

2. Differentiate ln *x* and place the answer in the *D* column . . .

Notice that in this step, the minus sign already in the box attaches to $\frac{1}{x}$.

	I
D	x^2
$+\ln x$	
$-\frac{1}{x}$	

. . . and integrate x^2 and place the answer in the *I* column.

	I
D	x^2
$+\ln x$	$\frac{1}{3}x^3$
$-\frac{1}{x}$	

3. Add the product of the full row that's circled.

	I
D	x^2
$+\ \ln x$	$\frac{1}{3}x^3$
$-\ \frac{1}{x}$	

Here's what you write:

$$+\ln x\left(\frac{1}{3}x^3\right)$$

4. Add the integral of the two lowest diagonally adjacent boxes that are circled.

	I
D	x^2
+ $\ln x$	$\frac{1}{3}x^3$
− $\frac{1}{x}$	

Here's what you write:

$$(+\ln x)\left(\frac{1}{3}x^3\right)+\int\left(-\frac{1}{x}\right)\left(\frac{1}{3}x^3\right)dx$$

At this point, you can simplify both terms and then integrate the second term:

$$=\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2 dx$$
$$=\frac{1}{3}x^3\ln x-\left(\frac{1}{3}\right)\left(\frac{1}{3}x^3\right)+C$$
$$=\frac{1}{3}x^8\ln x-\frac{1}{9}x^3+C$$

Therefore, this is the correct answer:

$$\int x^2\ln x\ dx=\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$$

I is for inverse trig

As I mention earlier in this chapter, you can integrate four of the six inverse trig functions (arcsin *x*, arccos *x*, arctan *x*, and arccot *x*) using integration by parts. Before, I integrated arcsin *x*, so now I'll use the DI-agonal method to integrate arccos *x*. (By the way, if you haven't gotten around to memorizing the derivatives of the six inverse trig functions, which I give you in Chapter 3, tick tock . . .)

REMEMBER

Whenever you integrate a product that includes an inverse trig function, this function always goes in the *D* column.

EXAMPLE

For example, suppose that you want to integrate the following:

$$\int \arccos x \, dx$$

1. **Write the inverse trig function in the box below the *D*, and the rest of the function value (omitting the *dx*) in the box below the *I*.**

	I
D	1
+ arccos *x*	
–	

Note that the number 1 goes into the *I* column.

2. **Differentiate arccos *x* and place the answer in the *D* column, and then integrate 1 and place the answer in the *I* column.**

	I
D	1
+ arccos *x*	*x*
$-\left(-\frac{1}{\sqrt{1-x^2}}\right)$	

3. **Add the product of the full row that's circled.**

	I
D	1
+ arccos *x*	*x*
$-\left(-\frac{1}{\sqrt{1-x^2}}\right)$	

Here's what you write:

(+arccos *x*)(*x*)

4. **Add the integral of the lowest diagonal that's circled.**

	I
D	1
$+\arccos x$	x
$-\left(-\frac{1}{\sqrt{1-x^2}}\right)$	

Here's what you write:

$$(+\arccos x)(x)+\int -\left(-\frac{1}{\sqrt{1-x^2}}\right)(x)dx$$

Take a step to simplify this result:

$$x\arccos x+\int \frac{x}{\sqrt{1-x^2}}dx$$

Now, you can see that you need to do a variable substitution to integrate the second term (see Chapter 8 to find out why):

$$\text{Let } u = 1-x^2$$
$$du = -2x\ dx$$
$$-\frac{1}{2}du = x\ dx$$

WARNING

This variable substitution introduces a new variable *u*. Don't confuse this *u* with the *u* used for integration by parts.

$$= x\arccos x+\int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right)$$
$$= x\arccos x-\frac{1}{2}\int u^{-\frac{1}{2}}(du)$$
$$= x\arccos x-\frac{1}{2}(2u^{\frac{1}{2}})+C$$
$$= x\arccos x-\sqrt{u}+C$$

Substituting $1-x^2$ for *u* and simplifying gives you this answer:

$$= x\arccos x-\sqrt{1-x^2}+C$$

Therefore, $\int \arccos x\ dx = x\arccos x-\sqrt{1-x^2}+C$.

A is for algebraic

If you're a bit skeptical that the DI-agonal method is really worth the trouble, I guarantee you that you'll find it useful when handling algebraic factors.

EXAMPLE

For example, suppose that you want to integrate the following:

$$\int x^3 \sin x\, dx$$

This example is a product of functions, so integration by parts is an option. Going down the LIAT checklist, you notice that the product doesn't contain a log factor or an inverse trig factor. But it does include the algebraic factor x^3, so place this factor in the *D* column and the rest in the *I* column. By now, you're probably getting good at using the chart, so I've filled it in for you here:

	I
D	$\sin x$
$+x^3$	$-\cos x$
$-3x^2$	

Your next step is normally to write the following:

$$+\left(x^3\right)(-\cos x)+\int\left(-3x^2\right)(-\cos x)dx$$

But here comes trouble: The only way to calculate the new integral is by doing *another* integration by parts, and then *another*. And, peeking ahead a bit, here's what you have to look forward to:

$$=\left(x^3\right)(-\cos x)-\left[\left(3x^2\right)(-\sin x)-\int(6x)(-\sin x)dx\right]$$
$$=\left(x^3\right)(-\cos x)-\left\{\left(3x^2\right)(-\sin x)-\left[(6x)(\cos x)-\int 6\cos x\, dx\right]\right\}$$

At last, after integrating by parts *three times*, you finally have an integral that you can solve directly. If evaluating this expression looks like fun (and if you think you can do it quickly on an exam without dropping a minus sign along the way!), by all means go for it. If not, I show you a better way. Read on.

TIP

To integrate an algebraic function multiplied by a sine, a cosine, or an exponential function, place the algebraic factor in the *D* column and the other factor in the *I* column. Differentiate the algebraic factor down to zero, and then integrate the other factor the same number of times. You can then copy the answer directly from the chart.

Simply extend the DI chart as I show you here:

	I
D	$\sin x$
+ x^3	$-\cos x$
− $3x^2$	$-\sin x$
+ $6x$	$\cos x$
− 6	$\sin x$
+ 0	

Notice that you just continue the patterns in both columns. In the *D* column, continue alternating plus and minus signs and differentiate until you reach 0. And in the *I* column, continue integrating.

The very pleasant surprise is that you can now copy the answer from the chart. This answer contains four terms (+ *C*, of course), which I copy directly from the four circled rows in the chart:

$$x^3(-\cos x) - 3x^2(-\sin x) + 6x(\cos x) - 6(\sin x) + C$$

But wait! Didn't I forget the final integral on the diagonal? Actually, no; but this integral is $\int 0 \sin x \, dx$, which integrates to a constant — that is, + *C*.

EXAMPLE

Here's another example, just to show you again how easy the DI-agonal method is for products with algebraic factors:

$$\int 3x^5 e^{2x} dx$$

Without the DI chart, this problem is one gigantic miscalculation waiting to happen. But the chart keeps track of everything. Check it out:

	I
D	e^{2x}
$+$ $3x^5$	$\frac{1}{2}$ e^{2x}
$-$ $15x^4$	$\frac{1}{4}$ e^{2x}
$+$ $60x^3$	$\frac{1}{8}$ e^{2x}
$-$ $180x^2$	$\frac{1}{16}$ e^{2x}
$+$ $360x$	$\frac{1}{32}$ e^{2x}
$-$ 360	$\frac{1}{64}$ e^{2x}
$+$ 0	

Now just copy from the chart, add C, and simplify:

$$= +\left(3x^5\right)\left(\frac{1}{2}e^{2x}\right) - \left(15x^4\right)\left(\frac{1}{4}e^{2x}\right) + \left(60x^3\right)\left(\frac{1}{8}e^{2x}\right) - \left(180x^2\right)\left(\frac{1}{16}e^{2x}\right)$$

$$+ (360x)\left(\frac{1}{32}e^{2x}\right) - (360)\left(\frac{1}{64}e^{2x}\right) + C$$

$$= \frac{3}{2}x^5e^{2x} - \frac{15}{4}x^4e^{2x} + \frac{15}{2}x^3e^{2x} - \frac{45}{4}x^2e^{2x} + \frac{45}{4}xe^{2x} - \frac{45}{8}e^{2x} + C$$

This answer is perfectly acceptable, but if you want to get fancy, factor out $\frac{3}{8}e^{2x}$ and leave a reduced polynomial:

$$= \frac{3}{8}e^{2x}\left(4x^5 - 10x^4 + 20x^3 - 30x^2 + 30x - 15\right) + C$$

T is for trig

You can use the DI-agonal method to integrate the product of either a sine or a cosine and an exponential.

EXAMPLE

For example, suppose that you want to evaluate the following integral:

$$\int e^{\frac{x}{3}} \sin x \, dx$$

TIP

When integrating either a sine or cosine function multiplied by an exponential function, make your DI-agonal chart with five rows rather than four. Then place the trig function in the *D* column and the exponential in the *I* column.

	I
D	$e^{\frac{x}{3}}$
+ sin *x*	$3e^{\frac{x}{3}}$
– cos *x*	$9e^{\frac{x}{3}}$
+ –sin *x*	

This time, you have two rows to add as well as the integral of the product of the lowest diagonal:

$$(\sin x)\left(3e^{\frac{x}{3}}\right)+(-\cos x)\left(9e^{\frac{x}{3}}\right)+\int(-\sin x)\left(9e^{\frac{x}{3}}\right)dx$$

This may seem like a dead end because the resulting integral looks so similar to the one that you're trying to evaluate. Oddly enough, however, this similarity makes solving the integral possible. To see why this works, on the next step I simplify every term and also explicitly include the original integral back into the problem:

$$\int e^{\frac{x}{3}}\sin x\ dx = 3e^{\frac{x}{3}}\sin x - 9e^{\frac{x}{3}}\cos x - 9\int e^{\frac{x}{3}}\sin x\ dx$$

Can you now see that the integral $\int e^{\frac{x}{3}}\sin x\ dx$ appears on both sides of the equation? To clarify the next few steps, substitute the variable *I* for the integral that you're trying to solve. This action isn't strictly necessary, but it makes your course of action much more straightforward:

$$I = 3e^{\frac{x}{3}}\sin x - 9e^{\frac{x}{3}}\cos x - 9I$$

At this point, you now solve for *I* using a little basic algebra:

$$10I = 3e^{\frac{x}{3}}\sin x - 9e^{\frac{x}{3}}\cos x$$

$$I = \frac{3e^{\frac{x}{3}}\sin x - 9e^{\frac{x}{3}}\cos x}{10}$$

Finally, substitute the original integral back into the equation, and add *C*:

$$\int e^{\frac{x}{3}} \sin x \, dx = \frac{1}{10}\left(3e^{\frac{x}{3}} \sin x - 9e^{\frac{x}{3}} \cos x\right) + C$$

Optionally, you can clean up this answer a bit by factoring:

$$= \frac{3}{10} e^{\frac{x}{3}} (\sin x - 3\cos x) + C$$

IN THIS CHAPTER

» **Memorizing the basic trig integrals**

» **Integrating powers of sines and cosines, tangents and secants, and cotangents and cosecants**

» **Understanding the three cases for using trig substitution**

» **Avoiding trig substitution when possible**

Chapter 10

Trig Substitution: Knowing All the (Tri) Angles

Trig substitution is another technique to throw into your ever-expanding bag of integration tricks. It allows you to integrate functions that contain radicals of polynomials such as $\sqrt{4-x^2}$ and other similar difficult functions.

Trig substitution may remind you of variable substitution, which I discuss in Chapter 8. With both types of substitution, you break the function that you want to integrate into pieces and express each piece in terms of a new variable. With trig substitution, however, you express these pieces as trig functions.

So before you can do trig substitution, you need to be able to integrate a wider variety of products and powers of trig functions. The first few parts of this chapter give you the skills that you need. After that, I show you how to use trig substitution to express complicated-looking radical functions in terms of trig functions.

Integrating the Six Trig Functions

You already know how to integrate sin x and cos x from Chapter 6, but for completeness, here are the integrals of all six trig functions:

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \cot x\, dx = \ln|\sin x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\, dx = \ln|\csc x - \cot x| + C$$

Please commit these to memory — you need them! For practice, you can also try differentiating each result to show why each of these integrals is correct.

Integrating Powers of Sines and Cosines

Later in this chapter, when I show you trig substitution, you need to know how to integrate powers of sines and cosines in a variety of combinations. In this section, I show you what you need to know.

Odd powers of sines and cosines

You can integrate *any* function of the form $\sin^m x \cos^n x$ when m is odd, for any real value of n. For this procedure, keep in mind the handy trig identity: $\sin^2 x + \cos^2 x = 1$.

For example, here's how you integrate $\sin^7 x \cos^{\frac{1}{3}} x$:

1. **Peel off a sin *x* and place it next to the *dx*:**

   $$\int \sin^7 x \cos^{\frac{1}{3}} x\, dx = \int \sin^6 x \cos^{\frac{1}{3}} x \sin x\, dx$$

2. **Apply the trig identity $\sin^2 x = 1 - \cos^2 x$ to express the rest of the sines in the function as cosines:**

   $$= \int \left(1 - \cos^2 x\right)^3 \cos^{\frac{1}{3}} x \sin x\, dx$$

3. **Use the variable substitution $u = \cos x$ and $du = -\sin x\, dx$:**

$$= -\int (1-u^2)^3 u^{\frac{1}{3}} du$$

4. **Now that you have the function in terms of powers of u, the worst is over. You can expand the function out, turning it into a polynomial. This is just algebra:**

$$= -\int (1-u^2)(1-u^2)(1-u^2) u^{\frac{1}{3}} du$$
$$= -\int (1-3u^2+3u^4-u^6) u^{\frac{1}{3}} du$$
$$= -\int u^{\frac{1}{3}} - 3u^{\frac{7}{3}} + 3u^{\frac{13}{3}} - u^{\frac{19}{3}} du$$

5. **To continue, use the Sum rule and Constant Multiple rule to separate this into four integrals, as I show you in Chapter 6. Be careful — don't forget to distribute that minus sign to all four integrals!**

$$= -\int u^{\frac{1}{3}} du + 3\int u^{\frac{7}{3}} du - 3\int u^{\frac{13}{3}} du + \int u^{\frac{19}{3}} du$$

6. **At this point, you can evaluate each integral separately using the Power rule:**

$$= -\frac{3}{4}u^{\frac{4}{3}} + \frac{9}{10}u^{\frac{10}{3}} - \frac{9}{16}u^{\frac{16}{3}} + \frac{3}{22}u^{\frac{22}{3}} + C$$

7. **Finally, use $u = \cos x$ to reverse the variable substitution:**

$$= -\frac{3}{4}\cos^{\frac{4}{3}} x + \frac{9}{10}\cos^{\frac{10}{3}} x - \frac{9}{16}\cos^{\frac{16}{3}} x + \frac{3}{22}\cos^{\frac{22}{3}} x + C$$

Notice that when you substitute back in terms of x, the power goes next to the cos rather than next to the x, because you're raising the entire function $\cos x$ to a power. (See Chapter 2 if you're unclear about this point.)

Similarly, you integrate *any* function of the form $\sin^m x \cos^n x$ when n is odd, for any real value of m. These steps are practically the same as those in the previous example. For instance, here's how you integrate $\sin^{-4} x \cos^5 x$:

1. **Peel off a $\cos x$ and place it next to the dx:**

$$\int \sin^{-4} x \cos^5 x\, dx = \int \sin^{-4} x \cos^4 x \cos x\, dx$$

2. **Apply the trig identity $\cos^2 x = 1 - \sin^2 x$ to express the rest of the cosines in the function as sines:**

$$= \int \sin^{-4} x \left(1 - \sin^2 x\right)^2 \cos x \, dx$$

3. **Use the variable substitution $u = \sin x$ and $du = \cos x \, dx$:**

$$= \int u^{-4} \left(1 - u^2\right)^2 du$$

4. **Now, distribute, evaluate the integral, and reverse the variable substitution:**

$$= \int u^{-4} - 2u^{-2} + 1 \, du$$

$$= -\frac{1}{3} u^{-3} + 2u^{-1} + u + C$$

$$= -\frac{1}{3} (\sin x)^{-3} + 2(\sin x)^{-1} + \sin x + C$$

$$= -\frac{1}{3} \csc^3 x + 2 \csc x + \sin x + C$$

TIP

Note that the two terms that include $\sin x$ with a negative exponent get simplified to $\csc x$. This problem is a good example of why I always use the notation *arcsin x*. (and not the ambiguous $\sin^{-1} x$) to express the inverse sin function, to avoid unnecessary confusion between these two functions.

Even powers of sines and cosines

To integrate $\sin^2 x$ and $\cos^2 x$, use the two half-angle trig identities that I show you in Chapter 2:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

For example, here's how you integrate $\cos^2 x$:

1. **Use the half-angle identity for cosine to rewrite the integral in terms of $\cos 2x$:**

$$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx$$

2. **Use the Constant Multiple rule to move the denominator outside the integral:**

$$= \frac{1}{2} \int (1 + \cos 2x) \, dx$$

3. **Distribute the function and use the Sum rule to split it into several integrals:**

$$= \frac{1}{2}\left(\int 1\, dx + \int \cos 2x\, dx\right)$$

4. **Evaluate the two integrals separately:**

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

As a second example, here's how you integrate $\cos^4 x$:

1. **Use the two half-angle identities to rewrite the integral in terms of cos 2*x*:**

$$\begin{aligned}\int \cos^4 x\ dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\ &= \int \frac{1 + 2\cos 2x + \cos^2 2x}{4} dx\end{aligned}$$

2. **Use the Constant Multiple rule to move the denominators outside the integral:**

$$= \frac{1}{4}\int 1 + 2\cos 2x + \cos^2 2x\ dx$$

3. **Distribute the function and use the Sum rule to split it into several integrals:**

$$= \frac{1}{4}\left(\int 1\, dx + \int 2\cos 2x\, dx + \int \cos^2 2x\, dx\right)$$

4. **Evaluate the resulting odd-powered integrals by using the procedure from the earlier section, "Odd powers of sines and cosines," and evaluate the even-powered integrals by returning to Step 1 of the previous example.**

   In this case, evaluating the first two integrals is easy, but the third requires you to use the half-angle identity again:

$$= \frac{1}{4}\left(x + \sin 2x + \int \frac{1+\cos 4x}{2}\, dx\right)$$

Fortunately, this integration isn't too bad:

$$= \frac{1}{4}\left(x + \frac{1}{2}\sin 2x + \int \frac{1}{2}dx + \frac{1}{2}\int \cos 4x\, dx\right)$$

$$= \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Integrating Powers of Tangents and Secants

When you're integrating powers of tangents and secants, here's the rule to remember: *Eeeven* powers of *seeecants* are *eeeasy.* The threee Es in the keeey words should help you remember this rule. By the way, odd powers of tangents are also easy. You're on your own remembering this fact!

In this section, I show you how to integrate $\tan^m x \sec^n x$ for all positive integer values of m and n. You use this skill later in this chapter, when I show you how to do trig substitution.

Even powers of secants

To integrate $\tan^m x \sec^n x$ when n is even — for example, $\tan^8 x \sec^6 x$ — follow these steps:

1. **Peel off a $\sec^2 x$ and place it next to the dx:**

$$\int \tan^8 x \sec^6 x\, dx = \int \tan^8 x \sec^4 x \sec^2 x\, dx$$

2. **Use the trig identity $1 + \tan^2 x = \sec^2 x$ to express the remaining secant factors in terms of tangents:**

$$= \int \tan^8 x \left(1 + \tan^2 x\right)^2 \sec^2 x\, dx$$

3. **Use the variable substitution $u = \tan x$ and $du = \sec^2 x\ dx$, distribute, and then integrate as usual:**

$$= \int u^8 \left(1 + u^2\right)^2 du$$
$$= \int u^8 + 2u^{10} + u^{12} du$$
$$= \frac{1}{9}u^9 + \frac{2}{11}u^{11} + \frac{1}{13}u^{13} + C$$
$$= \frac{1}{9}\tan^9 x + \frac{2}{11}\tan^{11} x + \frac{1}{13}\tan^{13} x + C$$

To integrate $\tan^m x$ when m is even — for example, $\tan^4 x$ — follow these steps:

1. **Peel off a $\tan^2 x$ and use the trig identity $\tan^2 x = \sec^2 x - 1$ to express it in terms of tan x:**

$$\int \tan^4 x\ dx = \int \tan^2 x\left(\sec^2 x - 1\right) dx$$

2. **Distribute to split the integral into two separate integrals:**

$$= \int \tan^2 x \sec^2 x\ dx - \int \tan^2 x\ dx$$

3. **Evaluate the first integral using the procedure I show you earlier in this section: $u = \tan x$ and $du = \sec^2 x\ dx$, distribute, and then integrate as usual:**

$$= \int u^2\ du - \int \tan^2 x\ dx$$
$$= \frac{1}{3}u^3 - \int \tan^2 x\ dx$$
$$= \frac{1}{3}\tan^3 x - \int \tan^2 x\ dx$$

4. **Evaluate the second integral using the trig identity $\tan^2 x = \sec^2 x - 1$ to express it in terms of sec x.**

$$= \frac{1}{3}\tan^3 x - \int \sec^2 x - 1\ dx$$
$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

To integrate $\sec^n x$ when n is even — for example, $\sec^4 x$ — follow these steps:

1. **Peel off $\sec^2 x$ and use the trig identity $1 + \tan^2 x = \sec^2 x$ to express the function in terms of tangents:**

$$\int \sec^4 x\ dx = \int \left(1 + \tan^2 x\right) \sec^2 x\ dx$$

2. Distribute and split the integral into two integrals:

$$= \int \sec^2 x\, dx + \int \tan^2 x \sec^2 x\, dx$$

3. Integrate the first term easily as tan *x*, and use *u* = tan *x* and $du = \sec^2 x\, dx$ to integrate the second term:

$$= \tan x + \int u^2\, du$$
$$= \tan x + \frac{1}{3}u^3 + C$$
$$= \tan x + \frac{1}{3}\tan^3 + C$$

Odd powers of tangents

To integrate $\tan^m x \sec^n x$ when m is odd — for example, $\tan^7 x \sec^9 x$ — follow these steps:

1. Peel off a tan *x* and a sec *x* and place them next to the *dx*:

$$\int \tan^7 x \sec^9 x = \int \tan^6 x \sec^8 x \sec x \tan x\, dx$$

2. Use the trig identity $\tan^2 x = \sec^2 x - 1$ to express the remaining tangent factors in terms of secants:

$$= \int \left(\sec^2 x - 1\right)^3 \sec^8 x \sec x \tan x\, dx$$

3. Use the variable substitution *u* = sec *x* and *du* = sec *x* tan *x dx*:

$$= \int \left(u^2 - 1\right)^3 u^8 du$$
$$= \int \left(u^2 - 1\right)\left(u^2 - 1\right)\left(u^2 - 1\right) u^8 du$$
$$= \int u^{14} - 3u^{12} + 3u^{10} - u^8\, du$$

4. At this point, the integral is a polynomial, and you can evaluate it as I show you in Chapter 7.

$$= \frac{1}{15}u^{15} - \frac{3}{13}u^{13} + \frac{3}{11}u^{11} - \frac{1}{9}u^9 + C$$
$$= \frac{1}{15}\sec^{15} x - \frac{3}{13}\sec^{13} x + \frac{3}{11}\sec^{11} x - \frac{1}{9}\sec^9 x + C$$

To integrate $\tan^m x$ by itself when m is odd, use a trig identity to convert the function to sines and cosines as follows:

$$\int \tan^m x\, dx = \int \frac{\sin^m x}{\cos^m x} dx = \int \sin^m x \cos^{-m} x\, dx$$

After that, you can integrate using the procedure from the earlier section, "Odd powers of sines and cosines."

Other tangent and secant cases

For all other combinations of tangent and secant functions (apart from the two cases I mention earlier in this section), use integration by parts, as I discuss in Chapter 9.

This is the hardest case, so fasten your seat belt. To integrate $\sec^n x$ when n is odd — for example, $\sec^3 x$ — follow these steps:

1. **Peel off a sec *x*:**

   $$\int \sec^3 x\, dx = \int \sec^2 x \sec x\, dx$$

2. **Integrate by parts (see Chapter 9) as follows:**

   $$\begin{aligned} u &= \sec x & dv &= \sec^2 x\, dx \\ du &= \sec x \tan x\, dx & v &= \tan x \end{aligned}$$

   $$\int u\, dv = uv - \int v\, du$$

   $$\int \sec^3 x\, dx = \sec x \tan x - \int \tan^2 x \sec x\, dx$$

3. **Use the trig identity $\tan^2 x = \sec^2 x - 1$ to rewrite this integral:**

   $$\int \sec^3 x\, dx = \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x\, dx$$

   $$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

   $$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \ln|\sec x + \tan x|$$

4. **Follow the algebraic procedure that I outline in Chapter 9.**

   First, substitute the variable I for the integral on both sides of the equation:

   $$I = \ln|\sec x + \tan x| - I + \tan x \sec x$$

Now solve this equation for *I*:

$$2I = \ln|\sec x + \tan x| + \tan x \sec x$$
$$I = \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\tan x \sec x$$

Now you can substitute the integral back for *I*. Don't forget, however, that you need to add a constant to the right side of this equation, to cover all possible solutions to the integral:

$$\int \sec^3 x\, dx = \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\tan x \sec x + C$$

That's your final answer.

To integrate $\tan^m x \sec^n x$ when m is even and n is odd, transform the function into an odd power of secants, and then use the method that I outline in the previous example.

For example, here's how you integrate $\tan^2 x \sec x$:

1. **Use the trusty trig identity $\tan^2 x = \sec^2 x - 1$ to convert all the tangents to secants:**

$$\int \tan^2 x \sec x\, dx = \int (\sec^2 x - 1)\sec x\, dx$$

2. **Distribute the function and split the integral using the Sum rule:**

$$= \int \sec^3 x\, dx - \int \sec x\, dx$$
$$= \int \sec^3 x\, dx - \ln|\sec x + \tan x|$$

3. **Evaluate the first integral as shown in the previous example.**

I truly hope that you never have to integrate $\sec^5 x$, let alone higher odd powers of a secant. But if you do, the basic procedure I outline here will provide you with a value for $\int \sec^5 x\, dx$ in terms of $\int \sec^3 x\, dx$.

Good luck!

Integrating Powers of Cotangents and Cosecants

The methods for integrating powers of cotangents and cosecants are very close to those for tangents and secants, which I show you in the preceding section.

Here's how to integrate $\cot^8 x \csc^6 x$:

1. **Peel off a $\csc^2 x$ and place it next to the *dx*:**

   $$\int \cot^8 x \csc^6 x \, dx = \int \cot^8 x \csc^4 x \csc^2 x \, dx$$

2. **Use the trig identity $1 + \cot^2 x = \csc^2 x$ to express the remaining cosecant factors in terms of cotangents:**

   $$= \int \cot^8 x \left(1 + \cot^2 x\right)^2 \csc^2 x \, dx$$

3. **Use the variable substitution $u = \cot x$ and $du = -\csc^2 x \, dx$:**

   $$= -\int u^8 \left(1 + u^2\right)^2 du$$

At this point, the integral is a polynomial, and you can evaluate it as I show you in Chapter 7.

Notice that the steps here are virtually identical to those for tangents and secants. The biggest change here is the introduction of a minus sign in Step 3. So to find out everything you need to know about integrating cotangents and cosecants, try all the examples in the previous section, but switch every tangent to a cotangent and every secant to a cosecant.

TIP

Sometimes, knowing how to integrate cotangents and cosecants can be useful for integrating negative powers of other trig functions — that is, powers of trig functions in the denominator of a fraction.

For example, suppose that you want to integrate $\frac{\cos^2 x}{\sin^6 x}$. The methods that I outline earlier don't work very well in this case, but you can use trig identities to express it as cotangents and cosecants:

$$\frac{\cos^2 x}{\sin^6 x} = \frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\sin^4 x} = \cot^2 x \csc^4 x$$

I show you more about this method in the next section, "Integrating Weird Combinations of Trig Functions."

Integrating Weird Combinations of Trig Functions

You don't really have to know how to integrate *every* possible trig function to pass Calculus II. If you can do all the techniques that I introduce earlier in this chapter — and I admit that's a lot to ask! — you'll be able to handle most of what your professor throws at you with ease. You'll also have a good shot at hitting any curveballs that come at you on an exam.

But in case you're nervous about the exam and would rather study than worry, in this section I show you how to integrate a wider variety of trig functions. I don't promise to cover *all* possible trig functions exhaustively. But I do give you a few additional ways to think about and categorize trig functions that could help you when you're in unfamiliar territory.

You can express every product of powers of trig functions, no matter how weird, as the product of any pair of trig functions. The three most useful pairings (as you may guess from earlier in this chapter) are sine and cosine, tangent and secant, and cotangent and cosecant. Table 10-1 shows you how to express all six trig functions as each of these pairings.

TABLE 10-1 Expressing the Six Trig Functions as a Pair of Trig Functions

Trig Function	As Sines & Cosines	As Tangents & Secants	As Cotangents & Cosecants
tan *x*	$\frac{\sin x}{\cos x}$	tan *x*	$\frac{1}{\cot x}$
cot *x*	$\frac{\cos x}{\sin x}$	$\frac{1}{\tan x}$	cot *x*
sec *x*	$\frac{1}{\cos x}$	sec *x*	$\frac{\csc x}{\cot x}$
csc *x*	$\frac{1}{\sin x}$	$\frac{\sec x}{\tan x}$	csc *x*

For example, look at the following function:

$$\frac{\cos x \cot^3 x \csc^2 x}{\sin^2 x \tan x \sec x}$$

As it stands, you can't do much to integrate this monster. But try expressing it in terms of each of the three pairings of trig functions:

$$= \frac{\cos^6 x}{\sin^8 x}$$
$$= \frac{\sec^2 x}{\tan^8 x}$$
$$= \cot^6 x \csc^2 x$$

As it turns out, the most useful pairing for integration in this case is $\cot^6 x \csc^2 x$. No fraction is present — that is, both terms are raised to positive powers — and the cosecant term is raised to an even power, so you can use the same basic procedure that I show you in the earlier section, "Even powers of secants."

Using Trig Substitution

Trig substitution is similar to variable substitution (which I discuss in Chapter 8), using a change in variable to turn a function that you can't integrate into one that you can. With variable substitution, you typically use the variable u. With trig substitution, however, you typically use the variable θ.

Trig substitution allows you to integrate a whole slew of functions that you can't integrate otherwise. These functions have a special, uniquely scary look to them and are variations on these three themes:

$(a^2 - bx^2)^n$ $(a^2 + bx^2)^n$ $(bx^2 - a^2)^n$

TIP

Trig substitution is most useful when n is $\frac{1}{2}$ or a negative number — that is, for hairy square roots and polynomials in the denominator of a fraction. When n is a relatively low positive integer like 2, 3, or maybe 4, your best bet is to express the function as a polynomial and integrate it using the Power rule, as I show you in Chapter 6.

In this section, I show you how to use trig substitution to integrate functions like these. But, before you begin, take this simple test.

Trig substitution is:

A. Easy and *fun* — even a child can do it!

B. Not so bad when you know how.

C. About as attractive as drinking bleach.

I wish I could tell you that the answer is A, but then I'd be a big liarmouth and you'd never trust me again. So I admit that trig substitution is less fun than a toga party with a hot date. At the same time, your worst trig substitution nightmares don't have to come true, so please put the bottle of bleach back in the laundry room.

I have the system right here, and if you follow along closely, I give you the tool that you need to make trig substitution mostly a matter of filling in the blanks. Trust me — have I ever lied to you?

Distinguishing three cases for trig substitution

Trig substitution is useful for integrating functions that contain three very recognizable types of polynomials in either the numerator or denominator. Table 10-2 lists the three cases that you need to know about.

TABLE 10-2 The Three Trig Substitution Cases

Case	Radical of Polynomial	Example
Sine case	$(a^2 - bx^2)^n$	$\int \sqrt{4 - x^2}\, dx$
Tangent case	$(a^2 + bx^2)^n$	$\int \frac{1}{\left(4 + 9x^2\right)^2} dx$
Secant case	$(bx^2 - a^2)^n$	$\int \frac{1}{\sqrt{16x^2 - 1}}\, dx$

The first step to trig substitution is being able to recognize and distinguish these three cases when you see them.

TIP

Knowing the formulas for differentiating the inverse trig functions can help you remember these cases:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1 + x^2} \qquad \frac{d}{dx}\text{arcsec}\, x = \frac{1}{x\sqrt{x^2 - 1}}$$

Note that the differentiation formula for arcsin x contains a polynomial that looks like the sine case: a constant minus x^2. The formula for arctan x contains a polynomial that looks like the tangent case: a constant plus x^2. And the formula for arcsec x contains a polynomial that looks like the secant case: x^2 minus a constant. So if you already know these formulas, you don't have to memorize any additional information.

Integrating the three cases

Trig substitution is a five-step process:

1. **Draw the trig substitution triangle for the correct case.**
2. **Identify the separate pieces of the integral (including *dx*) that you need to express in terms of θ.**
3. **Express these pieces in terms of trig functions of θ.**
4. **Rewrite the integral in terms of θ and evaluate it.**
5. **Substitute *x* for θ in the result.**

Don't worry if these steps don't make much sense yet. In this section, I show you how to do trig substitution for each of the three cases.

The sine case

When the function you're integrating includes a term of the form $(a^2 - bx^2)^n$, draw your trig substitution triangle for the *sine case.* For example, suppose that you want to evaluate the following integral:

$$\int \sqrt{4 - x^2}\, dx$$

This is a sine case, because a constant minus a multiple of x^2 is being raised to a power $\left(\frac{1}{2}\right)$. Here's how you use trig substitution to handle the job:

1. **Draw the trig substitution triangle for the correct case.**

   Figure 10-1 shows you how to fill in the triangle for the sine case. Notice that the radical goes on the *adjacent* side of the triangle. Then, to fill in the other two sides of the triangle, I use the square roots of the two terms inside the radical — that is, 2 and *x*. I place 2 on the hypotenuse and *x* on the opposite side.

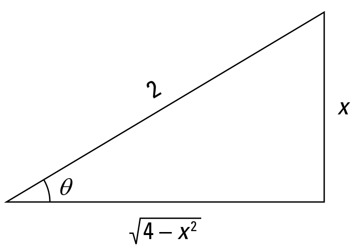

FIGURE 10-1: A trig substitution triangle for the sine case.

You can check to make sure that this placement is correct by using the Pythagorean Theorem:

$$x^2 + \left(\sqrt{4 - x^2}\right)^2 = 2^2$$

2. **Identify the separate pieces of the integral (including *dx*) that you need to express in terms of *θ*.**

   In this case, the function contains two separate pieces that contain *x:*

   $\sqrt{4 - x^2}$ and dx.

3. **Express these pieces in terms of trig functions of *θ*.**

   This is the real work of trig substitution, but when your triangle is set up properly, this work becomes a lot easier. In the sine case, *all* trig functions should be sines and cosines.

   To represent the radical portion as a trig function of *θ,* first build a fraction using the radical $\sqrt{4 - x^2}$ as the numerator and the constant 2 as the denominator. Then set this fraction equal to the appropriate trig function:

   $$\frac{\sqrt{4 - x^2}}{2} = \cos\theta$$

   Because the numerator is the adjacent side of the triangle and the denominator is the hypotenuse $\left(\frac{A}{H}\right)$, this fraction is equal to cos *θ.*

   Now a little algebra gets the radical alone on one side of the equation:

   $$\sqrt{4 - x^2} = 2\cos\theta$$

   Next, you want to express *dx* as a trig function of θ. To do so, build another fraction with the variable *x* in the numerator and the constant 2 in the denominator. Then set this fraction equal to the correct trig function.

   This time, the numerator is the opposite side of the triangle and the denominator is the hypotenuse $\left(\frac{O}{H}\right)$, so this fraction is equal to sin θ*:*

   $$\frac{x}{2} = \sin\theta$$

   Now solve for *x* and then differentiate:

   $$\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\ d\theta \end{aligned}$$

4. **Rewrite the integral in terms of θ and evaluate it:**

$$\int \sqrt{4-x^2}\,dx$$
$$= \int (2\cos\theta)(2\cos\theta)\ d\theta$$
$$= 4\int \cos^2\theta\ d\theta$$

Knowing how to evaluate trig integrals really pays off here. I cut to the chase in this example, but earlier in this chapter (in the section, "Integrating Powers of Sines and Cosines"), I show you how to integrate all sorts of trig functions like this one:

$= 2\theta + \sin 2\theta + C$

5. **To change those two θ terms into x terms, reuse the following equation:**

$$\frac{x}{2} = \sin\theta$$
$$\theta = \arcsin\frac{x}{2}$$

So here's a substitution that gives you an answer:

$$= 2\arcsin\frac{x}{2} + \sin\left(2\arcsin\frac{x}{2}\right) + C$$

This answer is perfectly valid, so, technically speaking, you can stop here. However, professors typically frown upon the nesting of trig and inverse trig functions, so they'll prefer a simplified version of $\sin\left(2\arcsin\frac{x}{2}\right)$. To find this simplified version, do the following:

1. **Start by applying the double-angle sine formula (see Chapter 2) to sin 2θ:**

   $\sin 2\theta = 2\sin\theta\cos\theta + C$

2. **Now use your trig substitution triangle to substitute values for sin θ and cos θ in terms of x:**

$$= 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) = \frac{1}{2}x\sqrt{4-x^2}$$

3. **To finish, substitute this expression for that problematic second term to get your final answer in a simplified form:**

$$2\theta + \sin 2\theta + C = 2\arcsin\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2} + C$$

The tangent case

When the function you're integrating includes a term of the form $(a^2 + x^2)^n$, draw your trig substitution triangle for the *tangent case.* For example, suppose that you want to evaluate the following integral:

$$\int \frac{1}{\left(4+9x^2\right)^2}\,dx$$

This is a tangent case, because a constant plus a multiple of x^2 is being raised to a power (-2). Here's how you use trig substitution to integrate:

1. **Draw the trig substitution triangle for the tangent case.**

   Figure 10-2 shows you how to fill in the triangle for the tangent case. Notice that the radical of what's inside the parentheses goes on the *hypotenuse* of the triangle. Then, to fill in the other two sides of the triangle, use the square roots of the two terms inside the radical — that is, 2 and $3x$. Place the constant term 2 on the adjacent side and the variable term $3x$ on the opposite side.

   WARNING

   With the tangent case, make sure not to mix up your placement of the variable and the constant. You really need to place that x term on the opposite side of the triangle!

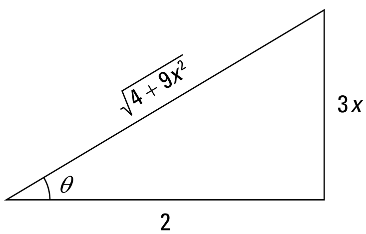

FIGURE 10-2: A trig substitution triangle for the tangent case.

2. **Identify the separate pieces of the integral (including *dx*) that you need to express in terms of *θ*.**

   In this case, the function contains two separate pieces that contain x:

   $\dfrac{1}{\left(4+9x^2\right)^2}$ and dx

3. **Express these pieces in terms of trig functions of *θ*.**

In the tangent case, *all* trig functions should be initially expressed as tangents and secants. To represent the rational portion as a trig function of *θ,* build a fraction using the radical $\sqrt{4+9x^2}$ as the numerator and the constant 2 as the denominator. Then set this fraction equal to the appropriate trig function:

$$\frac{\sqrt{4+9x^2}}{2}=\sec\theta$$

Because this fraction is the hypotenuse of the triangle over the adjacent side, it's equal to sec *θ*. Now use algebra and trig identities to tweak this equation into shape:

$$\sqrt{4+9x^2}=2\sec\theta$$

$$\frac{\left(4+9x^2\right)^2}{\left(4+9x^2\right)^2}=16\sec^4\theta$$

$$\frac{1}{\left(4+9x^2\right)^2}=\frac{1}{16\sec^4\theta}$$

Next, express *dx* as a trig function of *θ.* To do so, build another fraction with the variable 3*x* in the numerator and the constant 2 in the denominator:

$$\frac{3x}{2}=\tan\theta$$

This time, the fraction is the opposite side of the triangle over the adjacent side, so it equals tan *θ.* Now solve for *x* and then differentiate:

$$x=\frac{2}{3}\tan\theta \qquad dx=\frac{2}{3}\sec^2\theta\ d\theta$$

4. **Express the integral in terms of *θ* and evaluate it:**

$$\int\frac{1}{\left(4+9x^2\right)^2}\,dx=\int\frac{1}{16\sec^4\theta}\cdot\frac{2}{3}\sec^2\theta\ d\theta$$

Now some cancellation and reorganization turns this nasty-looking integral into something manageable:

$$\frac{1}{24}\int\cos^2\theta\ d\theta$$

At this point, use your skills from the earlier section, "Even powers of sines and cosines," to evaluate this integral:

$$\frac{1}{48}\theta+\frac{1}{96}\sin2\theta+C$$

5. **Change the two θ terms back into x terms:**

You need to find a way to express θ in terms of x. Here's the simplest way:

$$\tan\theta = \frac{3x}{2}$$
$$\theta = \arctan\frac{3x}{2}$$

So here's a substitution that gives you an answer:

$$\frac{1}{48}\theta + \frac{1}{96}\sin 2\theta + C = \frac{1}{48}\arctan\frac{3x}{2} + \frac{1}{96}\sin\left(2\arctan\frac{3x}{2}\right) + C$$

This answer is valid, but most professors won't be crazy about that ugly second term, with the sine of an arctangent. To simplify it, do the following:

1. **Apply the double-angle sine formula (see Chapter 2) to $\frac{1}{96}\sin 2\theta$:**

$$\frac{1}{96}\sin 2\theta = \frac{1}{48}\sin\theta\cos\theta$$

2. **Now use your trig substitution triangle to substitute values for sin θ and cos θ in terms of x:**

$$\frac{1}{48}\left(\frac{3x}{\sqrt{4+9x^2}}\right)\left(\frac{2}{\sqrt{4+9x^2}}\right) = \frac{6x}{48\left(4+9x^2\right)} = \frac{x}{\left(32+72x^2\right)}$$

3. **Finally, use this result to express the answer in terms of x:**

$$\frac{1}{48}\theta + \frac{1}{96}\sin 2\theta + C = \frac{1}{48}\arctan\frac{3x}{2} + \frac{x}{\left(32+72x^2\right)} + C$$

The secant case

When the function that you're integrating includes a term of the form $(bx^2 - a^2)^n$, draw your trig substitution triangle for the *secant case.* For example, suppose that you want to evaluate this integral:

$$\int \frac{1}{\sqrt{16x^2 - 1}}dx$$

This is a secant case, because a multiple of x^2 minus a constant is being raised to a power $\left(-\frac{1}{2}\right)$. Integrate using trig substitution as follows:

1. **Draw the trig substitution triangle for the secant case.**

   Figure 10-3 shows you how to fill in the triangle for the secant case. Notice that the radical goes on the opposite side of the triangle. Then, to fill in the other two sides of the triangle, use the square roots of the two terms inside the radical — that is, 1 and 4*x*. Place the constant 1 on the adjacent side and the variable 4*x* on the hypotenuse.

   You can check to make sure that this placement is correct by using the Pythagorean Theorem: $1^2 + \left(\sqrt{16x^2 - 1}\right)^2 = (4x)^2$.

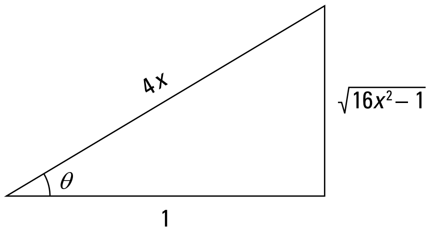

FIGURE 10-3: A trig substitution triangle for the secant case.

2. **Identify the separate pieces of the integral (including *dx*) that you need to express in terms of *θ*.**

   In this case, the function contains two separate pieces that contain *x:*

   $\frac{1}{\sqrt{16x^2 - 1}}$ and *dx*

3. **Express these pieces in terms of trig functions of *θ*.**

   In the secant case (as in the tangent case), *all* trig functions should be initially represented as tangents and secants.

   To represent the radical portion as a trig function of *θ,* build a fraction by using the radical $\sqrt{16x^2 - 1}$ as the numerator and the constant 1 as the denominator. Then set this fraction equal to the appropriate trig function:

   $$\frac{\sqrt{16x^2 - 1}}{1} = \tan\theta$$

   Notice that this fraction is the opposite side of the triangle over the adjacent side $\left(\frac{O}{A}\right)$, so it equals tan *θ.* Setting the reciprocals equal gives you this equation:

   $$\frac{1}{\sqrt{16x^2 - 1}} = \frac{1}{\tan\theta}$$

Next, express *dx* as a trig function of *θ.* To do so, build another fraction with the variable *x* in the numerator and the constant 1 in the denominator:

$$\frac{4x}{1} = \sec\theta$$

This time, the fraction is the hypotenuse over the adjacent side of the triangle $\left(\frac{H}{A}\right)$, which equals sec *θ.* Now solve for *x* and differentiate to find *dx:*

$$x = \frac{1}{4}\sec\theta \qquad dx = \frac{1}{4}\sec\theta\tan\theta\ d\theta$$

4. Express the integral in terms of *θ* and evaluate it:

$$\int \frac{1}{\sqrt{16x^2 - 1}}dx = \int \frac{1}{\tan\theta} \cdot \frac{1}{4}\sec\theta\tan\theta d\theta = \frac{1}{4}\int \sec\theta d\theta$$

Now use the formula for the integral of the secant function from the section, "Integrating the Six Trig Functions," earlier in this chapter:

$$= \frac{1}{4}\ln|\sec\theta + \tan\theta| + C$$

5. Change the two *θ* terms back into *x* terms:

In this case, you don't have to find the value of *θ* because you already know the values of sec *θ* and tan *θ* in terms of *x* from Step 3. So substitute these two values to get your final answer:

$$= \frac{1}{4}\ln\left|4x + \sqrt{16x^2 - 1}\right| + C$$

Knowing when to avoid trig substitution

Now that you know how to use trig substitution, I give you a skill that can be even more valuable: *avoiding* trig substitution when you don't need it. For example, look at the following integral:

$$\int (1 - 4x^2)^2 dx$$

This may look like a good place to use trig substitution, but it's an even better place to use a little algebra to expand the problem into a polynomial:

$$= \int (1 - 8x^2 + 16x^4)dx$$

Now, you can integrate each of these three terms separately using the Power rule, as I show you all the way back in Chapter 6.

Similarly, look at this integral:

$$\int \frac{x}{\sqrt{x^2 - 49}} dx$$

You can use trig substitution to evaluate this integral if you want to. (You can also walk to the top of the Empire State Building instead of taking the elevator if that tickles your fancy.) However, the presence of that little *x* in the numerator should tip you off that variable substitution will work just as well (flip to Chapter 8 for more on variable substitution):

$$\text{Let } u = x^2 - 49$$

$$du = 2x\,dx$$

$$\frac{1}{2} du = x\,dx$$

Using this substitution results in the following integral:

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + C = \sqrt{x^2 - 49} + C$$

Done! I probably don't need to tell you how much time and aggravation you can save by working smarter rather than harder. So I won't!

IN THIS CHAPTER

» Rewriting complicated fractions as the sum of two or more partial fractions

» Knowing how to use partial fractions in four distinct cases

» Integrating with partial fractions

» Using partial fractions with improper rational expressions

Chapter 11

Rational Solutions: Integration with Partial Fractions

Let's face it: At this point in your math career, you have bigger things to worry about than adding a couple of fractions. And if you've survived integration by parts (Chapter 9) and trig integration (Chapter 10), multiplying a few polynomials isn't going to kill you either.

So here's the good news about partial fractions: They're based on simple arithmetic and algebra. In this chapter, I introduce you to the basics of partial fractions and show you how to use them to evaluate integrals. I illustrate four separate cases in which partial fractions can help you integrate functions that would otherwise be a big ol' mess.

Now here's the bad news: Although the concept of partial fractions isn't difficult, using them to integrate is just about the most tedious thing you encounter in this book. And as if that weren't enough, partial fractions only work with *proper* rational functions, so I show you how to distinguish these from their ornery cousins, *improper* rational functions. I also give you a big blast from the past, a refresher on *polynomial division,* which I promise is easier than you remember it to be.

Strange but True: Understanding Partial Fractions

Partial fractions are useful for integrating *rational functions* — that is, functions in which a polynomial is divided by a polynomial. The basic tactic behind partial fractions is to split up a rational function that you *can't* integrate into two or more simpler functions that you *can* integrate.

In this section, I show you a simple analogy for partial fractions that involves only arithmetic. After you understand this analogy, partial fractions make a lot more sense. At the end of the section, I show you how to solve an integral using partial fractions.

Looking at partial fractions

EXAMPLE

To get a look at how to decompose a fraction to the sum partial fractions, suppose that you want to split the fraction $\frac{14}{15}$ into a sum of two smaller fractions.

Start by decomposing the denominator down to its factors — 3 and 5 — and setting the denominators of these two smaller fractions to these numbers:

$$\frac{14}{15} = \frac{A}{3} + \frac{B}{5}$$

Now, if you add these two fractions by using the obvious common denominator of 15, you get the following result:

$$\frac{14}{15} = \frac{A}{3} + \frac{B}{5} = \frac{5A + 3B}{15}$$

So, looking at the numerators, to find an A and a B, you want to find an integer solution to this equation:

$$5A + 3B = 14$$

Now, just by eyeballing this equation and noodling around, you can probably find the nice integer solution $A = 1$ and $B = 3$, so:

$$\frac{14}{15} = \frac{1}{3} + \frac{3}{5}$$

This procedure doesn't just work for $\frac{14}{15}$. Instead, it's *guaranteed* to work for all numerators that are integers. To solve for some numerators, however, you may need to use negative fractions.

EXAMPLE

For example, the fraction $\frac{1}{15}$ seems too small to be a sum of thirds and fifths, until you discover:

$$\frac{2}{3} - \frac{3}{5} = \frac{1}{15}$$

So, going forward, a "sum" of partial fractions may, in fact, be either a sum or a difference.

How does that sound so far? Make sense?

Using partial fractions with rational expressions

REMEMBER

The technique of breaking up fractions into sums of two or more fractions also works for rational expressions. Importantly, it can provide a strategy for integrating functions that you can't compute directly.

EXAMPLE

For example, suppose that you're trying to evaluate this integral:

$$\int \frac{6}{x^2 - 9} dx$$

You can't integrate this function directly, but if you break it into the sum of two simpler rational expressions, you can use the Sum rule to solve them separately. And, fortunately, the polynomial in the denominator factors easily:

$$\frac{6}{x^2 - 9} = \frac{6}{(x+3)(x-3)}$$

So set up this polynomial fraction just as I do with the regular fractions in the previous section:

$$\frac{6}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

Just as with fractions, this sum is guaranteed to work if you can find the values of A and B. To do this, add this pair of fractions using the common denominator you started with:

$$\frac{6}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3} = \frac{A(x-3) + B(x+3)}{(x+3)(x-3)}$$

This gives you the following equation in the numerators:

$$A(x-3) + B(x+3) = 6$$

This equation may look like it has too many variables. But the trick is that it works for *all* values of *x*. You can exploit this fact by picking helpful values of *x* to find the values of *A* and *B*. Watch what happens when you substitute the roots of the original polynomial (3 and −3) for *x*:

$$A(3-3)+B(3+3)=6 \qquad A(-3-3)+B(-3+3)=6$$
$$6B=6 \qquad -6A=6$$
$$B=1 \qquad A=-1$$

Now substitute these values of *A* and *B* back into the rational expressions:

$$\frac{6}{(x+3)(x-3)}=\frac{-1}{x+3}+\frac{1}{x-3}$$

Wait, what? Does this result really equal the original rational expression? Before moving on, I'll take a moment to verify it:

$$\frac{-1}{x+3}+\frac{1}{x-3}=\frac{-1(x-3)+1(x+3)}{(x+3)(x-3)}=\frac{-x+3+x+3}{(x+3)(x-3)}=\frac{6}{(x+3)(x-3)}$$

Yes! And this sum of two rational expressions is a whole lot friendlier to integrate than what you started with. To begin, use the Sum rule to split up the integral:

$$\int -\frac{1}{x+3}+\frac{1}{x-3}\,dx$$
$$=-\int\frac{1}{x+3}\,dx+\int\frac{1}{x-3}\,dx$$

Now, each of these integrals is simply a power function with a linear input of the form $ax + b$ (see Chapter 7):

$$=-\ln|x+3|+\ln|x-3|+C$$

If you've followed this example all the way to the end, you've now got the beginnings of a solid understanding of how integration with partial fractions works.

In the next section, I outline four distinct cases where you can use this technique to integrate complicated rational functions.

Solving Integrals by Using Partial Fractions

In the previous section, I show you how to use partial fractions to split a complicated rational function into several smaller and more-manageable functions. And although this technique will certainly amaze your friends at parties, you may be wondering why it's worth learning.

REMEMBER

When using partial fractions, the real payoff comes when you start integrating. Lots of times, you can integrate a big rational function by breaking it into the sum of several bite-sized chunks. Here's a bird's-eye view of how to use partial fractions to integrate a rational expression:

1. **Set up the rational expression as a sum of partial fractions with unknowns (*A, B, C,* and so forth) in the numerators.**

   I call these *unknowns* rather than variables to distinguish them from *x,* which remains a variable for the whole problem.

2. **Find the values of all the unknowns and plug them into the partial fractions.**
3. **Integrate the partial fractions separately by whatever method works.**

Setting up a sum of partial fractions isn't difficult, but you have four distinct cases to watch out for. Each case results in a different setup — some easier than others.

TIP

Try to become familiar with these four cases, because I use them throughout this chapter. Your first step in any problem that involves partial fractions is to recognize which case you're dealing with so that you can solve the problem.

Each of these cases is listed in Table 11-1.

TABLE 11-1 The Four Cases for Setting Up Partial Fractions

Case	Example	As Partial Fractions
Case 1: Distinct linear factors	$\frac{A}{\text{linear factor}}+\frac{B}{\text{linear factor}}+\ldots$	$\frac{1}{x(x+2)(x-5)}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-5}$
Case 2: Repeated linear factors	$\frac{A}{\text{linear factor}}+\frac{B}{(\text{linear factor})^2}+\ldots$	$\frac{x-3}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$
Case 3: Distinct irreducible quadratic factors	$\frac{A+Bx}{\text{quadratic factor}}+\frac{C+Dx}{\text{quadratic factor}}+\ldots$	$\frac{5x+4}{(x^2+3)(x^2-1)}=\frac{Ax+B}{x^2+3}+\frac{Cx+D}{x^2-1}$
Case 4: Repeated quadratic factors	$\frac{Ax+B}{\text{quadratic factor}}+\frac{Cx+D}{(\text{quadratic factor})^2}+\ldots$	$\frac{5x^3+x+7}{(x^2+x+1)^2}=\frac{A+Bx}{x^2+x+1}+\frac{C+Dx}{(x^2+x+1)^2}$

In the rest of this section, I focus on these four cases one by one.

Here's a warning: Some of this math is really tough! As you proceed, if you begin to feel like you're sailing into the heart of darkness, I encourage you to take a break and focus on something else. Many teachers give a nod to this material with one or two relatively simple problems. Others may ask you to set up an integral to be solved but *not* actually solve it. Still others may ask you for a partial solution, allowing you to leave the most difficult parts of the problem alone.

So, as you read, try to get a handle on whatever part of the process you can. As I mention in Chapter 19, your best is all you can do.

Case 1: Distinct linear factors

The simplest case in which partial fractions are helpful is when the denominator is the product of *distinct linear factors* — that is, linear factors that are non-repeating. This is the case I show you earlier in this chapter. It's also the most likely scenario on any test you may encounter.

REMEMBER

For each distinct linear factor in the denominator, add a partial fraction of the following form:

$$\frac{A}{\text{linear factor}}$$

Setting up partial fractions

EXAMPLE

For example, suppose that you want to evaluate the following integral:

$$\int \frac{1}{x(x+2)(x-5)}\,dx$$

The denominator is the product of three distinct linear factors — x, $(x + 2)$, and $(x - 5)$ — so it's equal to the sum of three fractions with these factors as denominators:

$$\frac{1}{x(x+2)(x-5)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-5}$$

TIP

The number of distinct linear factors in the denominator of the original expression determines the number of partial fractions. In this example, the presence of three factors in the denominator of the original expression yields three partial fractions.

Solving for unknowns *A*, *B*, and *C*

To find the values of the unknowns A, B, and C, first get a common denominator on the right side of this equation (the same denominator that's on the left side). To do this, multiply each unknown by the denominators of the other terms:

$$= \frac{A(x+2)(x-5) + Bx(x-5) + Cx(x+2)}{x(x+2)(x-5)}$$

The result is an equation with the same denominator on both sides, so you can multiply both sides by this denominator:

$$\frac{1}{x(x+2)(x-5)} = \frac{A(x+2)(x-5) + Bx(x-5) + Cx(x+2)}{x(x+2)(x-5)}$$
$$1 = A(x+2)(x-5) + Bx(x-5) + Cx(x+2)$$

The result seems to have too many variables to solve. But, as I show you in the previous section, to find the values of *A*, *B*, and *C*, substitute the roots of the three factors (0, −2, and 5) for *x* in three separate equations. When you do this, the equations magically solve themselves:

$$1 = A(2)(5) \qquad 1 = B(-2)(-2-5) \qquad 1 = C(5)(5+2)$$
$$1 = 10A \qquad 1 = 14B \qquad 1 = 35C$$
$$A = -\frac{1}{10} \qquad B = \frac{1}{14} \qquad C = \frac{1}{35}$$

Plugging these values back into the original equation gives you:

$$\frac{1}{x(x+2)(x-5)} = -\frac{1}{10x} + \frac{1}{14(x+2)} + \frac{1}{35(x-5)}$$

Evaluating the integral

This resulting expression is equivalent to what you started with, but it's much easier to integrate.

$$\int \frac{1}{x(x+2)(x-5)}\,dx = \int -\frac{1}{10x} + \frac{1}{14(x+2)} + \frac{1}{35(x-5)}\,dx$$

To do so, use the Sum rule to break it into three integrals, and the Constant Multiple rule to move fractional coefficients outside each integral:

$$= -\frac{1}{10}\int \frac{1}{x}\,dx + \frac{1}{14}\int \frac{1}{x+2}\,dx + \frac{1}{35}\int \frac{1}{x-5}$$

Now, each of these integrals is simply a linear input to the function $\frac{1}{x}$, which I show you how to integrate in Chapter 7:

$$= -\frac{1}{10}\ln x + \frac{1}{14}\ln(x+2) + \frac{1}{35}\ln(x-5) + C$$

TECHNICAL STUFF

The *C* here is distinct from the *C* used earlier in the problem. If your teacher is a stickler, you might need to call this integration constant *K* or something else to distinguish it from *C*.

Case 2: Repeated linear factors

Repeated linear factors are a bit more difficult to work with because each factor requires more than one partial fraction.

REMEMBER

For each squared linear factor in the denominator, add *two* partial fractions in the following form:

$$\frac{A}{\text{linear factor}} + \frac{B}{(\text{linear factor})^2}$$

For each quadratic factor in the denominator that's raised to the third power, add *three* partial fractions in the following form:

$$\frac{A}{\text{linear factor}} + \frac{B}{(\text{linear factor})^2} + \frac{C}{(\text{linear factor})^3}$$

Generally speaking, when a linear factor is raised to the *n*th power, add *n* partial fractions.

Setting up partial fractions

EXAMPLE

For example, suppose that you want to evaluate the following integral:

$$\int \frac{x-3}{(x-1)^2}\,dx$$

This expression contains all linear factors, but one of these factors $(x + 5)$ is non-repeating and the other $(x - 1)$ is raised to the third power. Set up your partial fractions this way:

$$\frac{x-3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Solving for unknowns *A* and *B*

Now, use the common denominator $(x-1)^2$ to sum up these two partial fractions into a single rational expression:

$$= \frac{A(x-1)+B}{(x-1)^2}$$

The result again is an equation with the same denominator on both sides, so you can simply drop the denominators:

$$\frac{x-3}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$$

$$x-3 = A(x-1)+B$$

In this case, the denominator has only one root (1), so plug this value in for x and solve:

$$1-3=A(1-1)+B$$
$$B=-2$$

Now, plug this value back into the previous equation and simplify:

$$x-3=A(x-1)-2$$
$$x-3=Ax-A-2$$
$$x-1=Ax-A$$

Although this equation has two variables, you can factor out $x-1$ on the right and then divide by this value to find A:

$$x-1=A(x-1)$$
$$A=1$$

Now, plug in 1 for A and -2 for B into the original equation:

$$\frac{x-3}{(x-1)^2}=\frac{1}{x-1}+\frac{-2}{(x-1)^2}$$

Evaluating the integral

This may not look like progress, but the resulting integral is much easier to evaluate:

$$\int \frac{x-3}{(x-1)^2}\,dx=\int \frac{1}{x-1}+\frac{-2}{(x-1)^2}\,dx$$

Now, you can change this integral to a pair of integrals and solve each one separately using the method I outline in Chapter 7. I do this in several steps so you can follow along:

$$=\int \frac{1}{x-1}\,dx-2\int \frac{1}{(x-1)^2}\,dx$$
$$=\int \frac{1}{x-1}\,dx-2\int (x-1)^{-2}\,dx$$
$$=\ln|x-1|-2(-1)(x-1)^{-1}+C$$
$$=\ln|x-1|+\frac{2}{x-1}+C$$

In my humble opinion, this is about as difficult a partial fractions problem as I would ever put on an exam.

Case 3: Distinct quadratic factors

A difficult case where you can use partial fractions is when the denominator is the product of *distinct quadratic factors* — that is, quadratic factors that are nonrepeating. Here, setting up the partial fractions isn't so bad, but the rest of the problem tends to be a lot of work. This is where you hope that, on an exam, your teacher assigns a problem like this but explicitly asks you to *set up but not evaluate the integral*. Oh, happy day!

REMEMBER

For each distinct quadratic factor in the denominator, add a partial fraction of the following form:

$$\frac{A + Bx}{\text{quadratic factor}}$$

Setting up partial fractions

EXAMPLE

For example, suppose that you want to evaluate this integral:

$$\int \frac{5x+4}{(x-2)\left(x^2+3\right)}\,dx$$

The first factor in the denominator is linear, but the second is quadratic and can't be decomposed to linear factors. So set up your partial fractions as follows:

$$\frac{5x+4}{(x-2)\left(x^2+3\right)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3}$$

TIP

As with distinct linear factors (Case 1), the number of distinct quadratic factors in the denominator tells you how many partial fractions you get. So in this example, two factors in the denominator yield two partial fractions.

Not so bad, right? This is more than I can promise for the rest of the problem.

Solving for unknowns *A*, *B*, and *C*

Now, as in the previous examples, your next step is to add the two partial fractions using the original denominator:

$$= \frac{A(x^2+3) + (Bx+C)(x-2)}{(x-2)(x^2+3)}$$

The result is an equation that, as usual, permits you to drop the denominator:

$$\frac{5x+4}{(x-2)\left(x^2+3\right)} = \frac{A(x^2+3) + (Bx+C)(x-2)}{(x-2)(x^2+3)}$$

$$5x + 4 = A(x^2+3) + (Bx+C)(x-2)$$

Now, letting $x = 2$ results in an equation you can solve for A:

$$\begin{aligned}5(2)+4 &= A(2^2+3)+[B(2)+C](2-2)\\ 14 &= 7A\\ A &= 2\end{aligned}$$

Plug in 2 for A:

$$5x+4 = 2(x^2+3)+(Bx+C)(x-2)$$

Unfortunately, you can't solve the remaining equation for B or C by plugging in other values of x. Instead, distribute and simplify this equation as much as you can:

$$\begin{aligned}5x+4 &= 2x^2+6+Bx^2-2Bx+Cx-2C\\ 5x-2 &= 2x^2+Bx^2-2Bx+Cx-2C\end{aligned}$$

The result may look hopeless. But remember, this equation is true for all values of x. Thus, you can separate it into three separate equations, each containing common powers of x:

$$\begin{aligned}0 &= 2x^2+Bx^2\\ 5x &= -2Bx+Cx\\ -2 &= -2C\end{aligned}$$

Now, solving the first equation gives you the value of B, and solving the third equation gives you the value of C (by the way, the second equation also confirms that these values are correct):

$$\begin{aligned}0 &= 2x^2+Bx^2\\ B &= -2\end{aligned} \qquad \begin{aligned}-2 &= -2C\\ C &= 1\end{aligned}$$

At last, you can now substitute these three values back into the original equation:

$$\frac{5x+4}{(x-2)(x^2+3)} = \frac{2}{x-2}+\frac{-2x+1}{x^2+3}$$

TIP

It's possible that, on a quiz or test, your teacher will ask you to go this far and no further with a problem of this difficulty. Be sure to keep an eye on directives like this before you burst into tears or quit university for a promising job in the merchant marine.

Evaluating the integral

Here, the hard work finally begins to pay off, because the new version of this function is much easier to integrate than the original:

$$\int \frac{5x+4}{(x-2)(x^2+3)}\,dx = \int \frac{2}{x-2}+\frac{-2x+1}{x^2+3}\,dx$$

As usual, break this integral in two:

$$= 2\int \frac{1}{x-2}\,dx + \int \frac{-2x+1}{x^2+3}\,dx$$

You can also further split up the second integral as follows:

$$= 2\int \frac{1}{x-2}\,dx + \int \frac{-2x}{x^2+3} + \frac{1}{x^2+3}\,dx$$

$$= 2\int \frac{1}{x-2}\,dx - 2\int \frac{x}{x^2+3}\,dx + \int \frac{1}{x^2+3}\,dx$$

At this point, you can evaluate the first two integrals by methods you already know. Evaluate the first as I've done throughout this chapter, as shown in Chapter 7:

$$= 2\ln|x-2| - 2\int \frac{x}{x^2+3}\,dx + \int \frac{1}{x^2+3}\,dx$$

Evaluate the second integral with variable substitution, as I show you in Chapter 8, using $u = x^2 + 3$ and $du = 2x\,dx$:

$$= 2\ln|x-2| - \ln|x^2+3| + \int \frac{1}{x^2+3}\,dx$$

Finally, you can solve the third integral using the following handy little formula that I provide you with in Chapter 7:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

In this specific case, $a = \sqrt{3}$:

$$= 2\ln|x-2| - \ln|x^2+3| + \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C$$

Speaking personally, I'm fine with radicals in the denominator, as are most professors as you ascend higher up the great math mountain.

Case 4: Repeated quadratic factors

And if you thought that last example was bad, your worst nightmare when it comes to partial fractions is when the denominator includes repeated quadratic factors. I really hope your professor doesn't force you to integrate one of these monsters all the way to the bitter end, but if so, here's what you do:

REMEMBER

For each squared quadratic factor in the denominator, add *two* partial fractions in the following form:

$$\frac{Ax+B}{\text{quadratic factor}}+\frac{Cx+D}{(\text{quadratic factor})^2}$$

For each quadratic factor in the denominator that's raised to the third power, add *three* partial fractions in the following form:

$$\frac{Ax+B}{\text{quadratic factor}}+\frac{Cx+D}{(\text{quadratic factor})^2}+\frac{Ex+F}{(\text{quadratic factor})^3}$$

Generally speaking, when a quadratic factor is raised to the *n*th power, add *n* partial fractions.

Setting up partial fractions

EXAMPLE

I'll keep the example as simple as I can, but you still won't like it:

$$\int \frac{5x^3+x+7}{(x^2+x+1)^2}\,dx$$

This denominator has one quadratic expression that's squared, but it's a doozy. (Stay tuned.) Here's how you set up the partial fractions:

$$\frac{5x^3+x+7}{(x^2+x+1)^2}=\frac{A+Bx}{x^2+x+1}+\frac{C+Dx}{(x^2+x+1)^2}$$

Solving for unknowns *A*, *B*, *C*, and *D*

Continue by rewriting the left side of this equation using a common denominator:

$$=\frac{(A+Bx)(x^2+x+1)+C+Dx}{(x^2+x+1)^2}$$

Now, set the two rational expressions equal, and allow the denominators to cancel out:

$$\frac{5x^3+x+7}{(x^2+x+1)^2}=\frac{(A+Bx)(x^2+x+1)+C+Dx}{(x^2+x+1)^2}$$

$$5x^3+x+7=(A+Bx)(x^2+x+1)+C+Dx$$

Distribute on the left side of this equation:

$$5x^3+x+7=Ax^2+Ax+A+Bx^3+Bx^2+Bx+C+Dx$$

The result looks to be unsolvable. But it holds for all values of x, so you can separate this equation into the following four equations based on the x values:

$$\begin{aligned} 5x^3 &= Bx^3 \\ 0 &= Ax^2 + Bx^2 \\ x &= Ax + Bx + Dx \\ 7 &= A + C \end{aligned}$$

This system of equations in four variables is actually pretty easy to solve: A = −5, B = 5, C = 12, and D = 1. Substitute all of these variables back into the original equation:

$$\frac{5x^3 + x + 7}{(x^2 + x + 1)^2} = \frac{-5 + 5x}{x^2 + x + 1} + \frac{12 + x}{(x^2 + x + 1)^2}$$

Evaluating the integral

As tedious as the previous step is, the most difficult part of this process is performing the actual evaluation. Here's where it stands so far:

$$\int \frac{5x^3 + x + 7}{(x^2 + x + 1)^2}\, dx = \int \frac{-5 + 5x}{x^2 + x + 1} + \frac{12 + x}{(x^2 + x + 1)^2}\, dx$$

Now, apply the Sum rule and Constant Multiple rule to this integral. (But *don't* split up the rational expressions! You see why in a moment):

$$= 5\int \frac{x - 1}{x^2 + x + 1}\, dx + \int \frac{x + 12}{(x^2 + x + 1)^2}\, dx$$

As you can see, I made a few small algebraic adjustments to these integrals. Now, you're at the end of all the tricks you already know. There's only one more I can show you, and then you will be enlightened. (Okay, maybe not — but you'll definitely have a good shot at acing your Calculus II midterm!)

At this point, your strategy is to do a variable substitution (see Chapter 8) on both integrals using $u = x^2 + x + 1$ and $du = 2x + 1$. To do this, multiply both integrals by ½ and then multiply both numerators by 2:

$$= \frac{5}{2}\int \frac{2x - 2}{x^2 + x + 1}\, dx + \frac{1}{2}\int \frac{2x + 24}{(x^2 + x + 1)^2}\, dx$$

Notice that this adjustment doesn't change the value of either integral. And neither do either of these next two steps:

$$= \frac{5}{2}\int \frac{2x + 1 - 3}{x^2 + x + 1}\, dx + \frac{1}{2}\int \frac{2x + 1 + 23}{(x^2 + x + 1)^2}\, dx$$

$$= \frac{5}{2}\int \frac{2x + 1}{x^2 + x + 1}\, dx + \frac{5}{2}\int \frac{-3}{x^2 + x + 1}\, dx + \frac{1}{2}\int \frac{2x + 1}{(x^2 + x + 1)^2}\, dx + \frac{1}{2}\int \frac{23}{(x^2 + x + 1)^2}\, dx$$

Now, the first and third of these four integrals are solved very easily using the u-substitution you set up (I skip the fine points here, but see Chapter 8 for more details):

$$= \frac{5}{2}\ln|x^2+x+1| + \frac{5}{2}\int \frac{-3}{x^2+x+1}\,dx + \frac{1}{2}\ln|x^2+x+1| + \frac{1}{2}\int \frac{23}{(x^2+x+1)^2}\,dx$$

What happens with the remaining integrals? Assuming you're still with me on this journey, first use the Constant Multiple rule to move the numerators outside the integrals:

$$= \frac{5}{2}\ln|x^2+x+1| - \frac{15}{2}\int \frac{1}{x^2+x+1}\,dx + \frac{1}{2}\ln|x^2+x+1| + \frac{23}{2}\int \frac{1}{(x^2+x+1)^2}\,dx$$

Now, recall this arctan formula from Chapter 7, which you used in the previous section:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

To use it in the second integral, you want to rewrite the denominator as the sum of squares by completing the square:

$$-\frac{15}{2}\int \frac{1}{x^2+x+1}\,dx = -\frac{15}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx = -\frac{15}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

Now, applying the arctan formula gives you the following result:

$$= -\frac{15}{2}\left(\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C = -\frac{15}{2}\left(\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right) + C$$

$$= -5\sqrt{3}\arctan\frac{2x+1}{\sqrt{3}} + C$$

To evaluate the integral in the fourth and final term, you need to apply the tangent case of trig substitution, as I show you in Chapter 10. To do this, use the work you just did on the second term to rewrite the fourth as follows:

$$\frac{23}{2}\int \left(\frac{1}{x^2+x+1}\right)^2 dx = \frac{23}{2}\int \frac{1}{\left(\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right)^2}\,dx$$

Now, your trig substitution triangle will have an opposite side of $x+\frac{1}{2}$, an adjacent side of $\frac{\sqrt{3}}{2}$, and a hypotenuse of $\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$. This integration is long and messy, but when the smoke clears, the answer is:

$$=\frac{46\sqrt{3}}{9}\left(\arctan\frac{2x+1}{\sqrt{3}}+\frac{\left(x+\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)+C$$

So, for the final answer to the problem, I patch in the completed integrals for the second and fourth terms:

$$=\frac{5}{2}\ln|x^2+x+1|-5\sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}+\frac{1}{2}\ln|x^2+x+1|$$
$$+\frac{46\sqrt{3}}{9}\left(\arctan\frac{2x+1}{\sqrt{3}}+\frac{\left(x+\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)+C$$

I sincerely hope that your professor doesn't spring something like this on you in an exam. In my opinion, a problem this difficult should be reserved for an extra-credit assignment, or at worst a group project.

Beyond the Four Cases: Knowing How to Set Up Any Partial Fraction

I have some great news: You'll probably never have to set up a partial fraction any more complex than those that I show you in the previous section. So relax.

I add this section because I'm aware that, while some students like to get this stuff on a case-by-case basis, others prefer to be shown an overall pattern so they can get the Zen math experience. If this is your path, read on. If not, feel free to skip ahead.

Additionally, it's possible that your teacher might ask you to *set up but not evaluate* a big, complicated integral using partial fractions. A question like this may seem difficult, but it could be a lot easier than a simpler-looking question where you actually have to evaluate an integral.

So here's the important information: You can break *any* rational function into a sum of partial fractions. You just need to understand the pattern for repeated higher-degree polynomial factors in the denominator. This pattern is simplest to understand with an example. Suppose that you're working with the following rational function:

$$\frac{5x+1}{\left(7x^4+1\right)^5(x+2)^2\left(x^2+1\right)^2}$$

In this function, the denominator includes a problematic factor that's a *fourth-degree polynomial* raised to the *fifth* power. You can't decompose this factor further, so the function falls outside the four cases I outline earlier in this chapter. Here's how you break this rational function into partial fractions:

$$=\frac{Ax^3+Bx^2+Cx+D}{7x^4+1}+\frac{Ex^3+Fx^2+Gx+H}{\left(7x^4+1\right)^2}+\frac{Ix^3+Jx^2+Kx+L}{\left(7x^4+1\right)^3}$$
$$+\frac{Mx^3+Nx^2+Ox+P}{\left(7x^4+1\right)^4}+\frac{Qx^3+Rx^2+Sx+T}{\left(7x^4+1\right)^5}$$
$$+\frac{U}{x+2}+\frac{V}{(x+2)^2}+\frac{Wx+X}{x^2+1}+\frac{Yx+Z}{\left(x^2+1\right)^2}$$

As you can see, I completely run out of capital letters. As you can also see, the problematic factor of $(7x^4 + 1)^5$ spawns *five partial fractions* — that is, the same number as the power it's raised to. Furthermore:

- The numerator of each of these fractions is a polynomial of *one degree less* than the exponent of 4.
- The denominator of each of these fractions is a carbon copy of the polynomial inside the first set of parentheses, but in each case raised to a different power up to and including the original exponent of 5.

The remaining two factors in the denominator — a repeated linear (Case 2 in the previous section) and a repeated quadratic (Case 4) — give you the remaining four fractions, which look tiny and simple by comparison.

Clear as mud? Spend a little time with this example and the pattern should become clearer. Notice, too, that the four cases that I outline earlier in this chapter all follow this same general pattern.

You'll probably never have to work with anything as complicated as this — let alone try to integrate it! — but when you understand the pattern, you can break any rational function into partial fractions without worrying which case it is.

Integrating Improper Rationals

Integration by partial fractions works with *proper rational expressions* but not with *improper rational expressions.* In this section, I show you how to tell these two beasts apart. Then I show you how to use polynomial division to turn improper rationals into more acceptable forms. Finally, I walk you through an example in which you integrate an improper rational expression by using everything in this chapter.

Distinguishing proper and improper rational expressions

Telling a proper fraction from an improper one is easy: A fraction is proper if the numerator (disregarding sign) is *less* than the denominator, and it's improper otherwise. With rational expressions, the idea is similar, but instead of comparing the value of the numerator and denominator, you compare their *degrees.* The degree of a polynomial is its highest power of *x.* (Flip to Chapter 2 for a refresher on polynomials.)

REMEMBER

A rational expression is proper if the degree of the numerator is less than the degree of the denominator, and it's improper otherwise.

For example, look at these three rational expressions:

$$\frac{x^2+2}{x^3} \qquad \frac{x^5}{3x^2-1} \qquad \frac{-5x^4}{3x^4-2}$$

In the first example, the numerator is a second-degree polynomial and the denominator is a third-degree polynomial, so the rational expression is *proper.* In the second example, the numerator is a fifth-degree polynomial and the denominator is a second-degree polynomial, so the expression is *improper.* In the third example, the numerator and denominator are both fourth-degree polynomials, so the rational expression is *improper.*

Trying out an example

In this section, I show you an example that walks you through just about everything in this chapter. Suppose that you want to integrate the following rational function:

$$\int \frac{x^4-x^3-5x+4}{(x-2)\left(x^2+3\right)}dx$$

This looks like a good candidate for partial fractions, as I show you earlier in the section, "Case 3: Distinct quadratic factors." But before you can express it as partial fractions, you need to determine whether it's proper or improper. The degree of the numerator is 4 and (because the denominator is the product of a linear and a quadratic) the degree of the entire denominator is 3. Thus, this is an improper polynomial fraction. (See the section, "Distinguishing proper and improper rational expressions," earlier in this chapter.) As a result, you can't integrate by parts.

However, you can use polynomial division to turn this improper polynomial fraction into an expression that includes a proper polynomial fraction. To begin, FOIL the denominator of the fraction:

$$(x-2)\left(x^2+3\right)=x^3-2x^2+3x-6$$

Now, divide the numerator by the FOILed version of the denominator – if you need more explanation of this process, check out *Algebra 2 For Dummies* by Mary Jane Sterling (Wiley, 2019):

$$\begin{array}{r} x+1 \\ x^3-2x^2+3x-6\overline{)x^4-x^3+0x^2-5x+4} \\ \underline{-\left(x^4-2x^3+3x^2-6x\right)} \\ x^3-3x^2+x+4 \\ \underline{-\left(x^3-2x^2+3x-6\right)} \\ x^2\quad -2x+10 \end{array}$$

Thus, you can rewrite the function you're trying to integrate as follows:

$$\frac{x^4-x^3-5x+4}{(x-2)\left(x^2+3\right)}=x+1+\frac{-x^2-2x+10}{(x-2)\left(x^2+3\right)}$$

As you can see, the first two terms of this expression are simple to integrate. (Don't forget about them!) To set up the remaining term for integration, use partial fractions:

$$\frac{-x^2-2x+10}{(x-2)\left(x^2+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$$

Get a common denominator on the right side of the equation:

$$\frac{-x^2-2x+10}{(x-2)\left(x^2+3\right)}=\frac{A\left(x^2+3\right)+(Bx+C)(x-2)}{(x-2)\left(x^2+3\right)}$$

Now multiply both sides of the equation by this denominator:

$$-x^2 - 2x + 10 = A(x^2 + 3) + (Bx + C)(x - 2)$$

Notice that $(x - 2)$ is a linear factor, so you can use the root of this factor to find the value of A. To find this value, let $x = 2$ and solve for A:

$$-(2^2) - 2(2) + 10 = A(2^2 + 3) + (B2 + C)(2 - 2)$$

$$2 = 7A$$

$$A = \frac{2}{7}$$

Substitute this value into the equation:

$$-x^2 - 2x + 10 = \frac{2}{7}\left(x^2 + 3\right) + (Bx + C)(x - 2)$$

At this point, to find the values of B and C, you need to split the equation into a system of two equations, as I show you earlier in this chapter:

$$-x^2 - 2x + 10 = \frac{2}{7}x^2 + \frac{6}{7} + Bx^2 + Cx - 2Bx - 2C$$

$$\left(\frac{2}{7} + B + 1\right)x^2 + (-2B + C + 2)x + \left(\frac{6}{7} - 2C - 10\right) = 0$$

This splits into three equations:

$$\frac{2}{7} + B + 1 = 0$$

$$-2B + C + 2 = 0$$

$$\frac{6}{7} - 2C - 10 = 0$$

The first and the third equations show you that $B = -\frac{9}{7}$ and $C = -\frac{32}{7}$. Now you can plug the values of A, B, and C back into the sum of partial fractions:

$$\frac{2}{7(x-2)} + \frac{-9x - 32}{7\left(x^2 + 3\right)}$$

Make sure that you remember to add in the two terms $(x + 1)$ that you left behind just after you finished your polynomial division:

$$\int \frac{x^4 - x^3 - 5x + 4}{(x-2)\left(x^2 + 3\right)}dx = \int \left[x + 1 + \frac{2}{7(x-2)} + \frac{-9x - 32}{7\left(x^2 + 3\right)} \right] dx$$

Thus, you can rewrite the original integral as the sum of five separate integrals:

$$\int x\,dx + \int dx + \frac{2}{7}\int \frac{1}{x-2}dx - \frac{9}{7}\int \frac{x}{x^2+3}dx - \frac{32}{7}\int \frac{1}{x^2+3}dx$$

You can solve the first two of these integrals by looking at them, and the next two by variable substitution. The last is done by using the following rule:

$$\int \frac{1}{x^2+n^2}dx = \frac{1}{n}\arctan\frac{x}{n} + C$$

Here's the solution so that you can work the last steps yourself:

$$\frac{1}{2}x^2 + x + \frac{2}{7}\ln|x-2| - \frac{9}{14}\ln\left|x^2+3\right| - \frac{32}{7\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C$$

5 Applications of Integrals

IN THIS PART . . .

Calculate the area between curves on the *xy*-graph

Find the arclength of functions

Solve tricky volume of revolution problems

Integrate to solve separable differential equations

IN THIS CHAPTER

- » Evaluating improper integrals
- » Solving area problems with more than one function
- » Measuring the area between functions
- » Finding unsigned areas
- » Understanding the Mean Value Theorem and calculating average value
- » Figuring out arc length

Chapter 12
Forging into New Areas: Solving Area Problems

With your toolbox now packed with the *hows* of calculating integrals, this chapter (along with Chapters 13 and 14) introduces you to some of the *whys* of calculating them.

I start with a simple rule for expressing an area as two separate definite integrals. Then I focus on improper integrals, which are integrals that are either horizontally or vertically infinite. Next, I give you a variety of practical strategies for measuring areas that are bounded by more than one function. I look at measuring areas between functions, and I also get you clear on the distinction between signed area and unsigned area.

After that, I introduce you to the Mean Value Theorem for Integrals, which provides the theoretical basis for calculating average value. Finally, I show you a formula for calculating arc length, which is the exact length between two points along a function.

Breaking Us in Two

Here's a simple but handy rule that looks complicated but is really very easy:

$$\int_a^b f(x)dx = \int_a^n f(x)dx + \int_n^b f(x)dx$$

This rule just says that you can split an area into two pieces and then add up the pieces to get the area that you started with.

For example, the entire shaded area in Figure 12-1 is represented by the following integral, which you can evaluate easily:

$$\begin{aligned}&\int_0^\pi \sin x\,dx\\&= -\cos x\Big|_{x=0}^{x=\pi}\\&= -\cos\pi - (-\cos 0)\\&= 1 + 1 = 2\end{aligned}$$

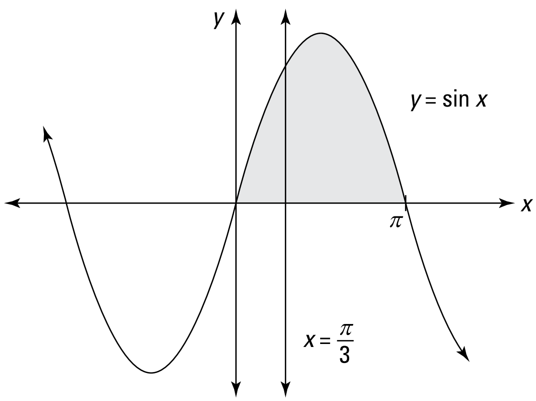

FIGURE 12-1: Splitting the area $\int_0^\pi \sin x\,dx$ into two smaller pieces.

Drawing a vertical line at $x = \frac{\pi}{3}$ and splitting this area into two separate regions results in two separate integrals:

$$\int_0^{\frac{\pi}{3}} \sin x\,dx + \int_{\frac{\pi}{3}}^{\pi} \sin x\,dx$$

It should come as no great shock that the sum of these two smaller regions equals the entire area:

$$= -\cos x\Big|_{x=0}^{x=\frac{\pi}{3}} - \cos x\Big|_{x=\frac{\pi}{3}}^{x=\pi}$$

$$= -\cos\frac{\pi}{3} + \cos 0 - \cos\pi + \cos\frac{\pi}{3}$$

$$= \cos 0 - \cos\pi$$

$$= 1 + 1 = 2$$

Although this idea is ridiculously simple, splitting an integral into two or more integrals becomes a powerful tool for solving a variety of the area problems in this chapter.

Improper Integrals

Improper integrals come in two varieties — horizontally infinite and vertically infinite:

- A *horizontally infinite* (or *Type 1*) improper integral contains either ∞ or $-\infty$ (or both) as a limit of integration. See the next section, "Getting horizontal," for examples of this type of integral.
- A *vertically infinite* (or *Type 2*) improper integral contains at least one vertical asymptote. I discuss this further in the later section, "Going vertical."

Improper integrals become useful for solving a variety of problems in Chapter 13. They're also useful for getting a handle on infinite series in Part 6. Evaluating an improper integral is a three-step process:

1. **Express the improper integral as the limit of an integral.**
2. **Evaluate the integral by whatever method works.**
3. **Evaluate the limit.**

In this section, I show you, step by step, how to evaluate both types of improper integrals.

Getting horizontal

The first type of improper integral occurs when a definite integral has a limit of integration that's either ∞ or $-\infty$ This type of improper integral is easy to spot because infinity is right there in the integral itself. You can't miss it.

For example, suppose that you want to evaluate the following improper integral:

$$\int_1^\infty \frac{1}{x^3}dx$$

Here's how you do it, step by step:

1. **Express the improper integral as the limit of an integral.**

   When the upper limit of integration is ∞ use this equation:

   $$\int_a^\infty f(x)dx = \lim_{c\to\infty}\int_a^c f(x)dx$$

   So here's what you do:

   $$\int_1^\infty \frac{1}{x^3}dx = \lim_{c\to\infty}\int_1^c \frac{1}{x^3}dx$$

2. **Evaluate the integral:**

   $$\lim_{c\to\infty}\left(-\frac{1}{2x^2}\Bigg|_{x=1}^{x=c}\right)$$
   $$= \lim_{c\to\infty}\left(-\frac{1}{2c^2}+\frac{1}{2}\right)$$

3. **Evaluate the limit.**

   Here, the first term tends to 0 and drops out of the expression:

   $$=\frac{1}{2}$$

Before moving on, reflect for one moment that the area under an *infinitely* long curve is actually *finite.* Ah, the magic and power of calculus!

Similarly, suppose that you want to evaluate the following:

$$\int_{-\infty}^0 e^{5x}dx$$

Here's how you do it:

1. **Express the integral as the limit of an integral.**

   When the lower limit of integration is –∞, use this equation:

   $$\int_{-\infty}^0 f(x)dx = \lim_{c\to-\infty}\int_c^b f(x)dx$$

So here's what you write:

$$\int_{-\infty}^{0} e^{5x}dx = \lim_{c\to-\infty}\int_{c}^{0} e^{5x}dx$$

2. **Evaluate the integral:**

$$= \lim_{c\to-\infty}\left(\frac{1}{5}e^{5x}\Big|_{x=c}^{x=0}\right)$$
$$= \lim_{c\to-\infty}\left(\frac{1}{5}e^{0} - \frac{1}{5}e^{5c}\right)$$
$$= \lim_{c\to-\infty}\left(\frac{1}{5} - \frac{1}{5}e^{5c}\right)$$

3. **Evaluate the limit.**

In this case, as c approaches $-\infty$, the first term, $\frac{1}{5}$, is unaffected. As for the second term, recall that the function e^x has an asymptote at 0 as x becomes more negative. Thus, as c approaches $-\infty$, the limit of $-\frac{1}{5}e^{5c}$ also approaches 0, so this term drops out:

$$= \frac{1}{5}$$

Again, calculus tells you that, in this case, the area under an infinitely long curve is convergent — that is, it's a finite area that can be expressed as a real number value.

Of course, sometimes the area under an infinitely long curve is divergent — it's infinitely large and cannot be expressed as a real number. In these cases, the improper integral can't be evaluated because the limit does not exist (DNE). Here's a quick example that illustrates this situation:

$$\int_{1}^{\infty} \frac{1}{x}dx$$

It may not be obvious that this improper integral represents an infinitely large area. After all, the value of the function approaches 0 as x increases. But watch how this evaluation plays out:

1. **Express the improper integral as the limit of an integral:**

$$\int_{1}^{\infty} \frac{1}{x}dx = \lim_{c\to\infty}\int_{1}^{c} \frac{1}{x}dx$$

2. **Evaluate the integral:**

$$= \lim_{c\to\infty} \ln x\Big|_{x=1}^{x=c}$$
$$= \lim_{c\to\infty}(\ln c - \ln 1)$$

At this point, you can see that the limit explodes to infinity, so it doesn't exist. Therefore, the improper integral is divergent, because the area that it represents is infinite.

Going vertical

Vertically infinite improper integrals are harder to recognize than those that are horizontally infinite. An integral of this type contains at least one vertical asymptote in the area that you're measuring. (A *vertical asymptote* is a value of x where $f(x)$ equals either ∞ or $-\infty$. See Chapter 2 for more on asymptotes.) The asymptote may be a limit of integration or it may fall someplace between the two limits of integration.

WARNING

Don't try to slide by and evaluate improper integrals as proper integrals. In most cases, you'll get the wrong answer!

In this section, I show you how to handle both cases of vertically infinite improper integrals.

Handling asymptotic limits of integration

Suppose that you want to evaluate the following integral:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

At first glance, you may be tempted to evaluate this as a proper integral. But this function has an asymptote at $x = 0$. The presence of an asymptote at one of the limits of integration forces you to evaluate this one as an improper integral.

1. **Express the integral as the limit of an integral:**

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c \to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx$$

Notice that in this limit, c approaches 0 from the right — that is, from the positive side — because this is the direction of approach from inside the limits of integration. (That's what the little plus sign ($^+$) in the limit means.)

2. **Evaluate the integral.**

This integral is easily evaluated as $x^{-\frac{1}{2}}$, using the Power rule as I show you in Chapter 7, so I spare you the details here:

$$\lim_{c \to 0^+} 2\sqrt{x}\,\Big|_{x=c}^{x=1}$$

3. Evaluate the limit:

$$\lim_{c \to 0^+} (2\sqrt{1} - 2\sqrt{c})$$

At this point, direct substitution provides you with your final answer:

= 2

Piecing together discontinuous integrands

In Chapter 6, I discuss the link between integrability and continuity: If a function is continuous on an interval, it's also integrable on that interval. (Flip to Chapter 6 for a refresher on this concept.)

Some integrals that are vertically infinite have asymptotes not at the edges but someplace in the middle. The result is a *discontinuous integrand* — that is, a function with a discontinuity on the interval that you're trying to integrate.

WARNING

Discontinuous integrands are the trickiest improper integrals to spot — you really need to know how the graph of the function that you're integrating behaves. (See Chapter 2 to see graphs of the elementary functions.)

To evaluate an improper integral of this type, separate it at each asymptote into two or more integrals, as I demonstrate earlier in the section, "Breaking Us in Two." Then evaluate each of the resulting integrals as an improper integral, as I show you in the previous section.

For example, suppose that you want to evaluate the following integral:

$$\int_0^{\pi} \sec^2 x \, dx$$

Because the graph of sec *x* contains an asymptote at $x = \frac{\pi}{2}$ (see Chapter 2 for a view of this graph), the graph of $\sec^2 x$ has an asymptote in the same place, as you see in Figure 12-2.

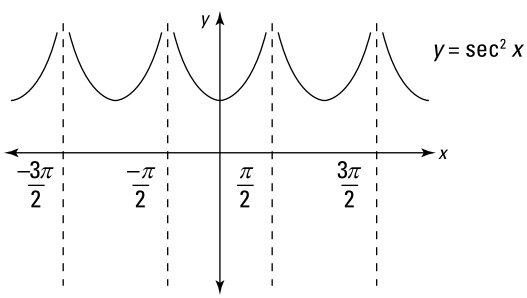

FIGURE 12-2: A graph of the improper integral $\int_0^{\pi} \sec^2 x \, dx$.

To evaluate this integral, break it into two integrals at the value of x where the asymptote is located:

$$\int_0^{\pi} \sec^2 x \, dx = \int_0^{\frac{\pi}{2}} \sec^2 x \, dx + \int_{\frac{\pi}{2}}^{\pi} \sec^2 x \, dx$$

Now evaluate the sum of the two resulting integrals.

TIP

You can save yourself a lot of work by noticing when two regions are symmetrical. In this case, the asymptote at $x = \frac{\pi}{2}$ splits the shaded area into two symmetrical regions. So you can find one integral and then double it to get your answer:

$$= 2\int_0^{\frac{\pi}{2}} \sec^2 x \, dx$$

Now use the steps from the previous section to evaluate this integral.

1. **Express the integral as the limit of an integral:**

   $$= 2 \lim_{c \to \frac{\pi}{2}} \int_0^{c} \sec^2 x \, dx$$

   In this case, the vertical asymptote is at the upper limit of integration, so c approaches $\frac{\pi}{2}$ from the left — that is, from inside the interval where you're measuring the area.

2. **Evaluate the integral:**

   $$= 2 \lim_{c \to \frac{\pi}{2}} \tan x \Big|_{x=0}^{x=c} = 2 \lim_{c \to \frac{\pi}{2}} \tan c - \tan 0 = 2 \lim_{c \to \frac{\pi}{2}} \tan c$$

3. **Evaluate the limit.**

   Now, when you try to plug in $\frac{\pi}{2}$ for c, you see that $\tan x = \frac{\pi}{2}$ is undefined, because the function tan x has an asymptote at $x = \frac{\pi}{2}$, so the limit does not exist (DNE). Therefore, the integral that you're trying to evaluate also does not exist because the area that it represents is infinite.

Finding the Unsigned Area of Shaded Regions on the *xy*-Graph

Recall that the definite integral allows you to find the signed area under any interval of a single function. But when you want to find the area of a shaded region on the *xy*-graph defined by more than one function, you sometimes need to be creative and piece together a solution. You also need to be aware of how to turn signed area into unsigned area — that is, flip negative area into positive area — because, in most cases, that's what the problem is asking for.

Professors love these problems as exam questions, because they test your reasoning skills as well as your calculus knowledge.

To solve problems like these, the trick is to break down the problem into two or more regions that you can measure by using the definite integral, and then use addition or subtraction to find the area of the shaded region that you're looking for.

In this section, I get you up to speed on problems that involve more than one definite integral. Throughout these examples, the problem is asking for the *unsigned area* of the shaded region.

Finding unsigned area when a region is separated horizontally

To solve some shaded-region problems, you need to separate a region into two smaller areas that line up horizontally on the *xy*-graph. In some cases, this type of problem gives you more than one function to integrate. In others, a single function crosses the *x*-axis, forcing you to integrate that function more than once. In this section, I show you how to solve both types of problems.

Crossing the line to find unsigned area

When a shaded region you're looking for is divided into two pieces because a function crosses the *x*-axis, the area below the *x*-axis is negative area as measured by the definite integral. But in most cases, a question of this type is asking you to find the *unsigned area* of the shaded region. So, to measure the area below the *x*-axis as positive, use the following formula:

REMEMBER

$$\text{Unsigned Area} = \text{Integral above } x\text{-axis} - \text{Integral below } x\text{-axis}$$

EXAMPLE

For example, in Figure 12-3, the goal is to find the area of the shaded region, which implies unsigned area. In this case, if you try to use the single integral $\int_0^2 x^3 - 1\,dx$, your result will be incorrect.

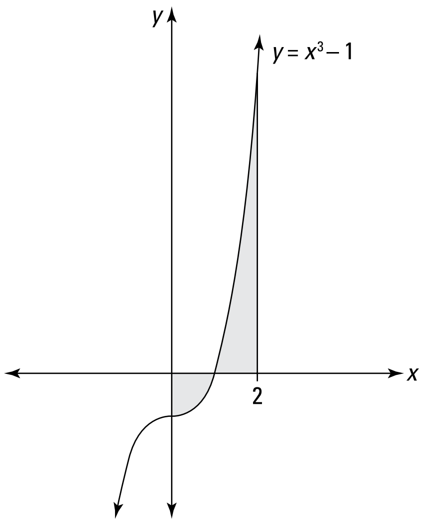

FIGURE 12-3: Finding the unsigned shaded area of $y = x^3 - 1$ between $x = 0$ and $x = 2$.

To handle this problem, use the following two integrals to measure each individual area:

$$\begin{aligned}\text{Unsigned Area} &= \text{Integral above } x\text{-axis} - \text{Integral below } x\text{-axis} \\ &= \int_1^2 x^3 - 1\,dx - \int_0^1 x^3 - 1\,dx\end{aligned}$$

The first integral in the formula measures the region above the x-axis (from 1 to 2), and the second measures the region below it (from 0 to 1). Evaluate both integrals as follows:

$$\begin{aligned}&= \frac{1}{4}x^4 - x\Big|_{x=1}^{x=2} - \frac{1}{4}x^4 - x\Big|_{x=0}^{x=1} \\ &= \left[\frac{1}{4}(2)^4 - 2\right] - \left[\frac{1}{4}(1)^4 - 1\right] - \left\{\left[\frac{1}{4}(1)^4 - 1\right] - [0]\right\} \\ &= (4-2) - \left(-\frac{3}{4}\right) - \left(-\frac{3}{4}\right) \\ &= \frac{7}{2}\end{aligned}$$

Therefore, the total unsigned area of the shaded region in Figure 12-3 is $\frac{7}{2}$.

In the previous problem, the point where the function crosses the x-axis is relatively easy to find. Sometimes, however, you need to find this point by setting the function to zero and solving it.

EXAMPLE

For example, the shaded region in Figure 12-4 is a combined unsigned area based on the function $y = 3x^2 - 12x + 9$. To solve this problem, you need to split the integral into two pieces, but you're not sure what limits of integration to use.

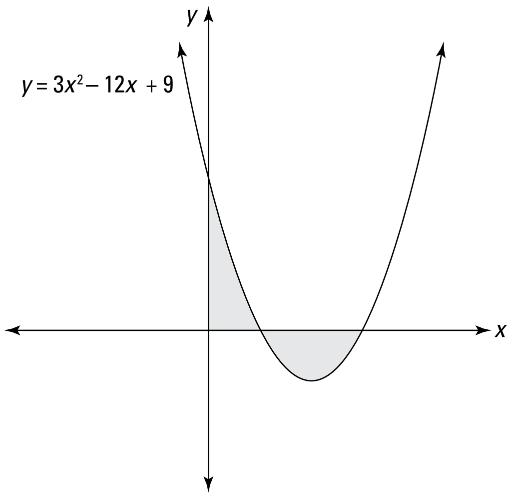

FIGURE 12-4: Finding the shaded area for $y = 3x^2 - 12x + 9$.

To begin, find the two roots of the function by setting it to zero and factoring:

$$0 = 3x^2 - 12x + 9$$
$$0 = x^2 - 4x + 3$$
$$0 = (x-1)(x-3)$$
$$x = 1 \text{ and } 3$$

Now, use these values to set up your integral, using a positive value for the first integral and a negative value for the second:

$$\text{Unsigned Area} = \text{Integral above } x\text{-axis} - \text{Integral below } x\text{-axis}$$
$$= \int_0^1 3x^2 - 12x + 9\ dx - \int_1^3 3x^2 - 12x + 9\ dx$$

Integrate:

$$\left[x^3 - 6x^2 + 9x\Big|_{x=0}^{x=1}\right] - \left[x^3 - 6x^2 + 9x\Big|_{x=1}^{x=3}\right]$$

Fortunately, the evaluation isn't so bad:

$$= [(1-6+9)-0]-[(27-54+27)-(1-6+9)]$$
$$= 4-(-4) = 8$$

Surprisingly, the two shaded regions are both the same size, each with an unsigned area of 4, so the combined area is 8.

Calculating the area under more than one function

Sometimes, a single geometric area is described by more than one function. To find the area of a shaded region defined in this way, separate the region into two (or more) sections and calculate the area of each.

EXAMPLE

For example, suppose that you want to find the shaded region shown in Figure 12-5.

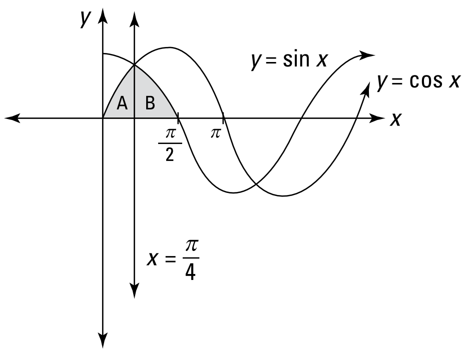

FIGURE 12-5: Finding the area under $f(x) = \sin x$ and $g(x) = \cos x$ from 0 to $\frac{\pi}{2}$.

The first thing to notice is that the shaded area isn't under a single function, so you can't expect to use a single integral to find it. Instead, the region labeled A is under $y = \sin x$ and the region labeled B is under $y = \cos x$. First, set up separate integrals to find the area of both of these regions:

$$\text{A} = \int_0^{\frac{\pi}{4}} \sin x \, dx \qquad \text{B} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

Now set up an equation to find their combined area:

$$\text{A} + \text{B} = \int_0^{\frac{\pi}{4}} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

At this point, you can evaluate each of these integrals separately. But there's an easier way.

TIP

Because region A and region B are symmetrical, they have the same area. So you can find their combined area by doubling the area of a single region:

$$= 2A = 2\int_0^{\frac{\pi}{4}} \sin x \, dx$$

I choose to double region A because the integral limits of integration are easier to calculate with, but doubling region B also works. Now integrate to find your answer:

$$= 2(-\cos x)\Big|_{x=0}^{x=\frac{\pi}{4}}$$

$$= 2\left(-\cos\frac{\pi}{4} - (-\cos 0)\right)$$

$$= 2\left(-\frac{\sqrt{2}}{2} + 1\right)$$

$$= 2 - \sqrt{2} \approx 0.586$$

In some problems, you may need to use algebra to find where a pair of functions intersect so you can split the integral into two pieces.

EXAMPLE

For example, to find the shaded region in Figure 12-6, you need to combine definite integrals of both $f(x)$ and $g(x)$. But where, exactly, does the first integral end and the second begin?

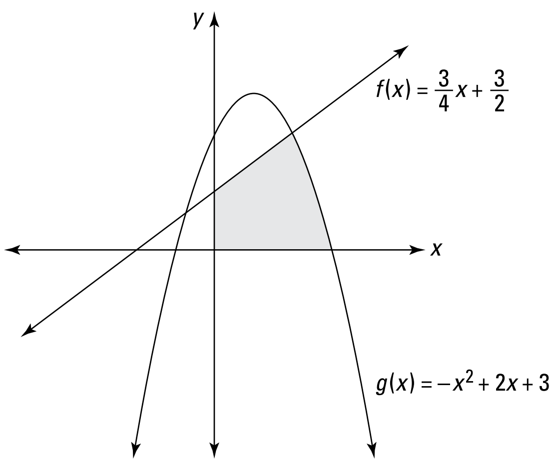

FIGURE 12-6: Finding the area of the shaded region under $f(x) = \frac{3}{4}x + \frac{3}{2}$ and $g(x) = -x^2 + 2x + 3$ between $x = 0$ and $x = 3$.

To find the point between $x = 0$ and $x = 3$ where the two functions intersect, set them equal and solve for x. Here, I use the quadratic formula:

$$\begin{aligned}
\frac{3}{4}x + \frac{3}{2} &= -x^2 + 2x + 3 \\
x^2 - \frac{5}{4}x - \frac{3}{2} &= 0 \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{\frac{5}{4} \pm \sqrt{\left(-\frac{5}{4}\right)^2 - 4(1)\left(-\frac{3}{2}\right)}}{2(1)} = \frac{\frac{5}{4} \pm \sqrt{\frac{121}{16}}}{2} = \frac{\frac{5}{4} \pm \frac{11}{4}}{2} = \frac{5}{8} \pm \frac{11}{8} = 2 \text{ and } -\frac{3}{4}
\end{aligned}$$

The value of x you're looking for is 2, so use this as the upper bound integral of $f(x)$ and the lower bound of $g(x)$:

$$\int_0^2 \frac{3}{4}x + \frac{3}{2}\,dx + \int_2^3 -x^2 + 2x + 3\,dx$$

Integrate:

$$= \left(\frac{3}{8}x^2 + \frac{3}{2}x \Big|_{x=0}^{x=2}\right) + \left(-\frac{1}{3}x^3 + x^2 + 3x \Big|_{x=2}^{x=3}\right)$$

At this point, if you've set up and evaluated the integral correctly, many professors will spare you the agony of the final evaluation:

$$\begin{aligned}
&= \left(\frac{3}{8}x^2 + \frac{3}{2}x \Big|_{x=0}^{x=2}\right) + \left(-\frac{1}{3}x^3 + x^2 + 3x \Big|_{x=2}^{x=3}\right) \\
&= \left[\frac{3}{8}(2)^2 + \frac{3}{2}(2)\right] - 0 + \left[-\frac{1}{3}(3)^3 + 3^2 + 3(3)\right] - \left[-\frac{1}{3}(2)^3 + 2^2 + 3(2)\right] \\
&= \left(\frac{3}{2} + 3\right) + (-9 + 9 + 9) - \left(-\frac{8}{3} + 4 + 6\right) \\
&= \frac{9}{2} + 9 - \frac{22}{3} \\
&= \frac{37}{6}
\end{aligned}$$

TIP On a test, you may get to the end of a problem like this and find that your answer is obviously incorrect — for example, if it's negative or an unrealistically great positive value. If this happens and you can't find the error, write your professor a little note acknowledging the problem. (For example: "Unsigned area should be positive — my arithmetic must be wrong!") Most teachers will appreciate not only the honesty, but that you could explain the discrepancy in your answer.

Measuring a single shaded region between two functions

As with the problems earlier in this section, when finding an area between two functions, you're looking for unsigned rather than signed area. To find the area of a single shaded region between two functions, use the following formula:

REMEMBER

$$\text{Unsigned Area} = \text{Integral of top function} - \text{Integral of bottom function}$$

EXAMPLE

For example, suppose that you want to calculate the shaded area in Figure 12-7.

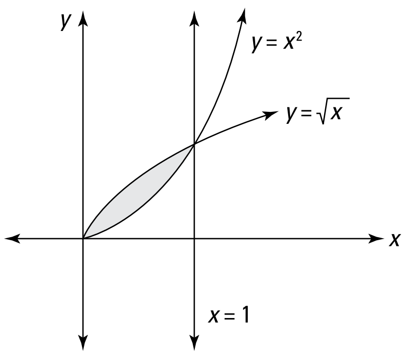

FIGURE 12-7: Finding the area between $f(x) = \sqrt{x}$ and $g(x) = x^2$.

First, notice that the two functions $f(x) = \sqrt{x}$ and $g(x) = x^2$ intersect where $x = 0$ and $x = 1$. Use these limits of integration in the Unsigned Area formula as follows:

$$\begin{aligned}\text{Unsigned Area} &= \text{Integral of top function} - \text{Integral of bottom function}\\ &= \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx\end{aligned}$$

With the problem set up properly, now all you have to do is evaluate the two integrals:

$$\begin{aligned}&= \left(\frac{2}{3}x^{\frac{3}{2}}\Bigg|_{x=0}^{x=1}\right) - \left(\frac{1}{3}x^3\Bigg|_{x=0}^{x=1}\right)\\ &= \left(\frac{2}{3} - 0\right) - \left(\frac{1}{3} - 0\right) = \frac{1}{3}\end{aligned}$$

So the area between the two curves is $\frac{1}{3}$.

EXAMPLE

In the previous example, the entire shaded region was above the x-axis. In Figure 12-8, however, the shaded region includes area that's both above and below the x-axis. Fortunately, the same formula works in this case as well.

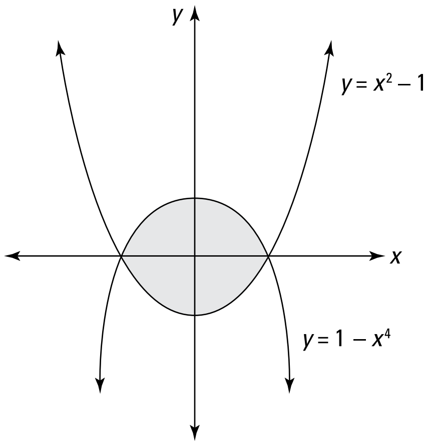

FIGURE 12-8: Finding the area between the top function $f(x) = 1 - x^4$ and the bottom function $g(x) = x^2 - 1$.

This time, both functions intersect each other and the x-axis at the same two values of x: $x = -1$ and $x = 1$. Use the Unsigned Area formula with -1 and 1 as the limits of integration, as follows:

$$\begin{aligned}\text{Unsigned Area} &= \text{Integral of top function} - \text{Integral of bottom function}\\ &= \int_{-1}^{1}(1 - x^4)\,dx - \int_{-1}^{1}(x^2 - 1)\,dx\end{aligned}$$

Solving this equation gives you the answer that you're looking for (be careful with all those minus signs!):

$$\begin{aligned}&= \left(x - \frac{1}{5}x^5 \Big|_{x=-1}^{x=1}\right) - \left(\frac{1}{3}x^3 - x\Big|_{x=-1}^{x=1}\right)\\ &= \left\{\left(1 - \frac{1}{5}\right) - \left[-1 - \left(-\frac{1}{5}\right)\right]\right\} - \left\{\left(\frac{1}{3} - 1\right) - \left[-\frac{1}{3} - (-1)\right]\right\}\\ &= \left(\frac{4}{5} + \frac{4}{5}\right) - \left(-\frac{2}{3} - \frac{2}{3}\right)\\ &= \frac{8}{5} + \frac{4}{3} = \frac{44}{15}\end{aligned}$$

Thus, the area of the shaded region in Figure 12-8 is $\frac{44}{15}$.

Finding the area of two or more shaded regions between two functions

In the previous section, you used the following formula for finding the area of a shaded region between a pair of functions:

$$\text{Unsigned Area} = \text{Integral of top function} - \text{Integral of bottom function}$$

In some cases, you may need to apply this formula more than once to get a final answer. This situation occurs when a pair of functions intersect, so that the top and bottom functions exchange positions.

EXAMPLE

For example, suppose that you want to find the shaded area in Figure 12-9. This time, the shaded area is two separate regions. Region A is bounded above by $g(x) = x^{\frac{1}{3}}$ and below by $f(x) = x$. However, for region B, the situation is reversed, and the region is bounded above by $f(x) = x$ and below by $g(x) = x^{\frac{1}{3}}$.

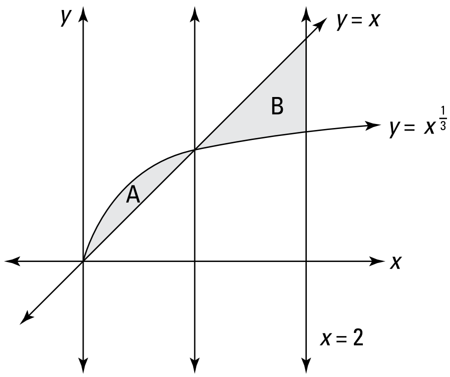

FIGURE 12-9: Finding the area of the combined shaded regions between $f(x) = x$ and $g(x) = x^{\frac{1}{3}}$.

The first step is finding where the two functions intersect — that is, where the following equation is true:

$$x = x^{\frac{1}{3}}$$

Fortunately, it's not too difficult to see that $x = 1$ satisfies this equation.

Now you need to apply the formula twice as follows:

$$\begin{aligned}\text{Unsigned Area A} &= \text{Integral of top function} - \text{Integral of bottom function}\\ &= \int_0^1 x^{\frac{1}{3}} dx - \int_0^1 x\ dx\end{aligned}$$

$$\begin{aligned}\text{Unsigned Area B} &= \text{Integral of top function} - \text{Integral of bottom function}\\ &= \int_1^2 x\ dx - \int_1^2 x^{\frac{1}{3}} dx\end{aligned}$$

TIP

At this point in the problem, take a step back to make sure you've set up the problem correctly. Yes, $g(x)$ is the top function from 0 to 1, and the bottom function from 1 to 2. Good to go!

Now you're forced to do some calculus:

$$
\begin{aligned}
&= \left(\frac{3}{4}x^{\frac{4}{3}}\Big|_{x=0}^{x=1}\right) - \left(\frac{1}{2}x^2\Big|_{x=0}^{x=1}\right) \\
&= \left(\frac{3}{4}(1)^{\frac{4}{3}} - 0\right) - \left(\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2\right) \\
&= 0.75 - 0.5 = 0.25
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2}x^2\Big|_{x=1}^{x=2}\right) - \left(\frac{3}{4}x^{\frac{4}{3}}\Big|_{x=1}^{x=2}\right) \\
&= \left(\frac{1}{2}(2)^2 - \frac{1}{2}(1)^2\right) - \left(\frac{3}{4}(2)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}}\right) \\
&= \left(2 - \frac{1}{2}\right) - \left(\frac{3}{4}16^{\frac{1}{3}} - \frac{3}{4}\right) \approx 1.5 - 1.89 + 0.75 = 0.36
\end{aligned}
$$

To complete the problem, add the areas of the two shaded regions:

$$0.25 + 0.36 = 0.61$$

Thus, the combined area of the two shaded regions A and B in Figure 12-9 is approximately 0.61.

The Mean Value Theorem for Integrals

The *Mean Value Theorem for Integrals* guarantees that for every definite integral, a rectangle with the same area and width exists. Moreover, if you superimpose this rectangle on the definite integral, the top of the rectangle intersects the function. This rectangle, by the way, is called the *mean-value rectangle* for that definite integral. Its existence allows you to calculate the *average value* of the definite integral.

The existence of the rectangle with the same area as the integral is a nice consequence of the MVT for Integrals.

WARNING

Calculus boasts *two* Mean Value Theorems — one for derivatives and one for integrals. This section discusses the Mean Value Theorem for Integrals. You can find out about the Mean Value Theorem for Derivatives in *Calculus All-In-One For Dummies* by Mark Ryan (Wiley).

The best way to see how this theorem works is with a visual example. The first graph in Figure 12-10 shows the region described by the definite integral $A = \int_1^2 x^2 dx$. This region obviously has a width of 1, and you can evaluate it easily:

$$\int_1^2 x^2 dx = \frac{1}{3}x^3 \Big|_{x=1}^{x=2} = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3 = \frac{7}{3}$$

Thus, the area is $\frac{7}{3}$.

The second graph in Figure 12-10 shows a rectangle with a width of 1 and an area of $\frac{7}{3}$. It should come as no surprise that this rectangle's height is also $\frac{7}{3}$, so the top of this rectangle intersects the original function.

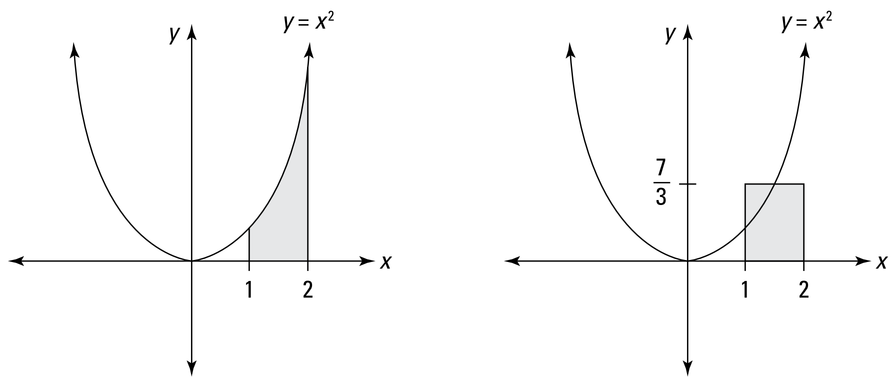

FIGURE 12-10: A definite integral and its mean-value rectangle have the same width and area.

The fact that the top of the mean-value rectangle intersects the function is mostly a matter of common sense. After all, the height of this rectangle represents the average value that the function attains over a given interval. This value must fall someplace between the function's maximum and minimum values on that interval.

Here's the formal statement of the Mean Value Theorem for Integrals: If $f(x)$ is a continuous function on the closed interval $[a, b]$, then there exists a number c in that interval such that:

$$\int_a^b f(x)dx = f(c) \cdot (b - a)$$

This equation may look complicated, but it's basically a restatement of this familiar equation for the area of a rectangle:

$$\text{Area} = \text{Height} \times \text{Width}$$

In other words, start with a definite integral that expresses an area, and then draw a rectangle of equal area with the same width ($b - a$). The height of that rectangle — $f(c)$ — is such that its top edge intersects the function where $x = c$.

The value $f(c)$ is the *average value* of $f(x)$ over the interval $[a, b]$. You can calculate it by rearranging the equation stated in the theorem:

$$f(c) = \frac{1}{b-a} \cdot \int_a^b f(x)dx$$

For example, here's how you calculate the average value of the shaded area in Figure 12-11:

$$f(c) = \frac{1}{4-2} \cdot \int_2^4 x^3 dx$$
$$= \frac{1}{2}\left(\frac{1}{4}x^4 \Big|_{x=2}^{x=4}\right)$$
$$= \frac{1}{2}\left(\frac{1}{4}(4)^4 - \frac{1}{4}(2)^4\right)$$
$$= \frac{1}{2}(64-4) = 30$$

Not surprisingly, the average value of this integral is 30, a value between the function's minimum of 8 and its maximum of 64.

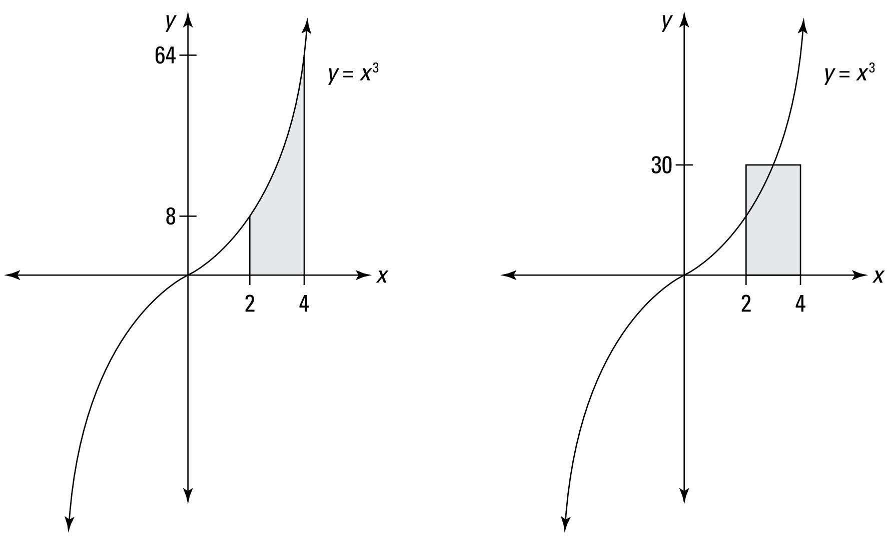

FIGURE 12-11: The definite integral $\int_2^4 x^3 dx$ and its mean-value rectangle.

Calculating Arc Length

The arc length of a function on a given interval is the length from the starting point to the ending point as measured along the graph of that function.

In a sense, arc length is similar to the practical measurement of driving distance. For example, you may live only 5 miles from work "as the crow flies," but when you check your odometer, you may find that the actual drive is closer to 7 miles. Similarly, the straight-line distance between two points is always less than the arc length along a curved function that connects them.

The formula for the arc length along a function $y = f(x)$ from a to b is as follows:

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Using the formula, however, often involves trig substitution. (See Chapter 10 for a refresher on this method of integration.)

EXAMPLE

For example, suppose that you want to calculate the arc length along the function $y = x^2$ from the point where $x = 0$ to the point where $x = 2$. (See Figure 12-12.)

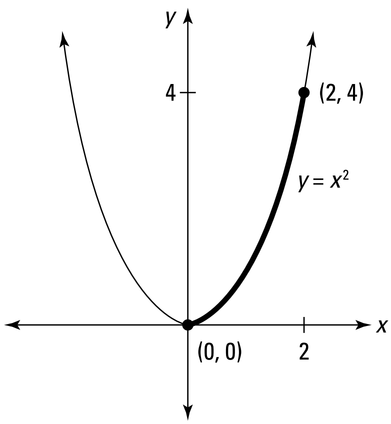

FIGURE 12-12: Measuring the arc length along $y = x^2$ from (0, 0) to (2, 4).

Before you begin, notice that if you draw a straight line between these two points, (0, 0) and (2, 4), its length is $\sqrt{20} \approx 4.4721$. So the arc length should be slightly greater.

To calculate the arc length, first find the derivative of the function x^2:

$$\frac{dy}{dx} = 2x$$

Now plug this derivative and the limits of integration into the formula as follows:

$$\int_0^2 \sqrt{1+(2x)^2}\,dx$$
$$=\int_0^2 \sqrt{1+4x^2}\,dx$$

Calculating arc length usually gives you an opportunity to practice trig substitution — in particular, the tangent case (see Chapter 10 for a more detailed explanation). When you draw your trig substitution triangle, place $\sqrt{1+4x^2}$ on the hypotenuse, $2x$ on the opposite side, and 1 on the adjacent side. This gives you the following substitutions:

$$\sqrt{1+4x^2} = \sec\theta$$
$$2x = \tan\theta$$
$$x = \frac{1}{2}\tan\theta$$
$$dx = \frac{1}{2}\sec^2\theta\ d\theta$$

The result is this integral:

$$\frac{1}{2}\int \sec^3\theta d\theta$$

Notice that I remove the limits of integration because I plan to change the variable back to x before computing the definite integral. I spare you the details of evaluating this integral, but you can see them in Chapter 10. Here's the result:

$$=\frac{1}{4}(\ln|\sec\theta+\tan\theta|+\tan\theta\sec\theta)+C$$

Now write each sec θ and tan θ in terms of x:

$$\frac{1}{4}\left(\ln\left|\sqrt{1+4x^2}+2x\right|+2x\sqrt{1+4x^2}\right)+C$$

At this point, I'm ready to evaluate the definite integral that I left off earlier:

$$\int_0^2 \sqrt{1+4x^2}\,dx$$
$$=\frac{1}{4}\left(\ln\left|\sqrt{1+4x^2}+2x\right|+2x\sqrt{1+4x^2}\right)\Bigg|_{x=0}^{x=2}$$
$$=\frac{1}{4}\left(\ln\left|\sqrt{1+4(2)^2}+2(2)\right|+2(2)\sqrt{1+4(2)^2}\right)-0$$

You can either take my word that the second part of this substitution works out to 0 or you can calculate it yourself. To finish up:

$$=\frac{1}{4}\ln|\sqrt{17}+4|+\sqrt{17}$$
$$\approx 0.5236+4.1231=4.6467$$

IN THIS CHAPTER

» Understanding the meat-slicer method for finding volume

» Using inverses to make a problem easier to solve

» Solving problems with solids of revolution and surfaces of revolution

» Finding the space between two surfaces

» Considering the shell method for finding volume

Chapter 13

Pump Up the Volume: Using Calculus to Solve 3-D Problems

In Chapter 12, I show you a bunch of different ways to use integrals to find area. In this chapter, you add a dimension by discovering how to use integrals to find volumes and surface areas of solids.

First, I show you how to find the volume of a solid by using the meat-slicer method, which is really a 3-D extension of the basic integration tactic you already know from Chapter 1: slicing an area into an infinite number of pieces and adding them up.

As with a real meat slicer, this method works best when the blade is slicing vertically — that is, perpendicular to the x-axis. So I also show you how to use inverses to rotate some solids into the proper position.

After that, I show you how to solve two common types of problems that calculus teachers just love: finding the volume of a solid of revolution and finding the area of a surface of revolution.

With these techniques in your back pocket, you move on to more complex problems, where a solid is described as the space between two surfaces. These problems are the 3-D equivalent of finding an area between two curves, which I discuss in Chapter 12.

To finish up, I give you an additional way to find the volume of a solid: the shell method. Then, I provide some practical perspective on all the methods in the chapter so that you know when to use them.

Slicing Your Way to Success

Did you ever marvel at the way in which a meat slicer turns an entire salami into dozens of tasty little paper-thin circles? Even if you're a vegetarian, calculus provides you with an animal-friendly alternative: the meat-slicer method for measuring the volume of solids.

The *meat-slicer method* works best with solids that have similar cross sections. (I discuss this further in the following section.) Here's the plan:

1. Find an expression that represents the area of a random cross section of the solid in terms of *x*.
2. Use this expression to build a definite integral (in terms of *dx*) that represents the volume of the solid.
3. Evaluate this integral.

Don't worry if these steps don't make a whole lot of sense yet. In this section, I show you when and how to use the meat-slicer method to find volumes that would be difficult or impossible without calculus.

Finding the volume of a solid with congruent cross sections

Before I get into calculus, I want to provide a little bit of background on finding the volume of solids. Spending a few minutes thinking about how volume is measured *without* calculus pays off big-time when you step into the calculus arena.

This is strictly no-brainer stuff — some basic, solid geometry that you probably know already. So just lie back and coast through this section.

One of the simplest solids to find the volume of is a prism. A *prism* is a solid that has all congruent cross sections in the shape of a polygon. That is, no matter how you slice a prism parallel to its base, its cross section is the same shape and area as the base itself.

The formula for the volume of a prism is simply the area of the base times the height:

$$V = A_b \times h$$

So if you have a triangular prism with a height of 3 inches and a base area of 2 square inches, its volume is 6 cubic inches.

This formula also works for cylinders — which are sort of prisms with a circular base — and generally any solid that has congruent cross sections. For example, the odd-looking solid in Figure 13-1 fits the bill nicely. In this case, you're given the information that the area of the base is 7 cm^2 and the height is 4 cm, so the volume of this solid is 28 cm^3.

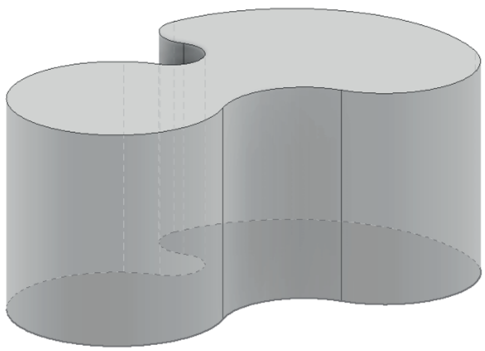

FIGURE 13-1: Finding the volume of an odd-looking solid with a constant height.

Finding the volume of a solid with congruent cross sections is always simple — well, simple as long as you know two things:

- The area of the base — that is, the area of any cross section
- The height of the solid

Finding the volume of a solid with similar cross sections

In the previous section, you didn't have to use any calculus brain cells. But now, suppose that you want to find the volume of the scary-looking hyperbolic cooling tower on the left side of Figure 13-2.

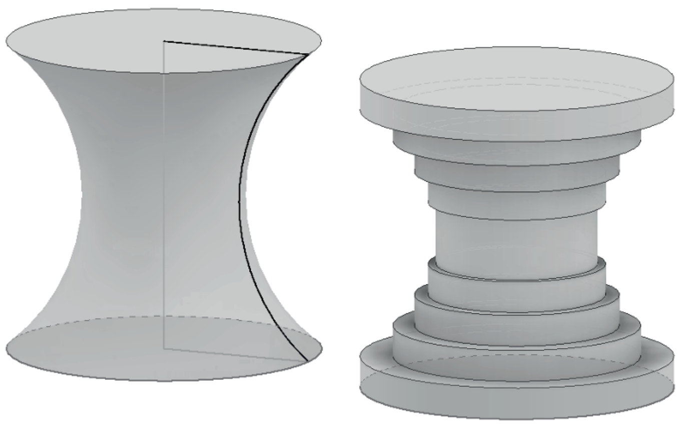

FIGURE 13-2: Estimating the volume of a hyperbolic cooling tower by slicing it into cylindrical sections.

What makes this problem out of the reach of the formula for prisms and cylinders? In this case, slicing parallel to the base always results in the same shape — a circle — but the area may differ. That is, the solid has *similar* cross sections rather than congruent ones.

You can estimate this volume by slicing the solid into numerous cylinders, finding the volume of each cylinder by using the formula for constant-height solids, and adding these separate volumes. Of course, making more slices improves your estimate. And, as you may already suspect, adding the limit of an infinite number of slices gives you the exact volume of the solid.

Hmmm . . . this is beginning to sound like a job for calculus. In fact, what I hint at in this section is the meat-slicer method, which works well for measuring solids that have similar cross sections.

TIP

When a problem asks you to find the volume of a solid, look at the picture of the solid and figure out how to slice it up so that all the cross sections are similar. This is a good first step in understanding the problem so that you can solve it.

TECHNICAL STUFF

To measure weird-shaped solids that *don't* have similar cross sections, you usually need multivariable calculus, which is the subject of Calculus III.

Measuring the volume of a pyramid

Suppose that you want to find the volume of a pyramid with a 6-x-6-unit square base and a height of 3 units. Geometry tells you that you can use the following formula:

$$V = \frac{1}{3}bh = \frac{1}{3}(36)(3) = 36$$

This formula works just fine, but it doesn't give you insight into how to solve similar problems; it works only for pyramids. The meat-slicer method, however, provides an approach to the problem that you can generalize to use for many other types of solids.

To start out, I skewer this pyramid on the x-axis of a graph, as shown in Figure 13-3. Notice that the vertex of the pyramid is at the origin, and the center of the base is at the point (3, 0).

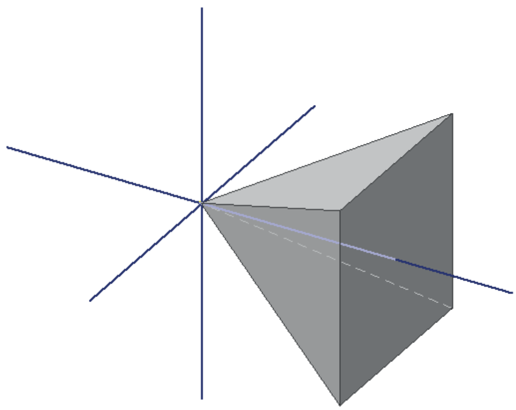

FIGURE 13-3: A pyramid skewered on the x-axis of a graph from $x = 0$ to $x = 3$.

To find the exact volume of the pyramid, here's what you do:

1. **Find an expression that represents the area of a random cross section of the pyramid in terms of *x*.**

   At $x = 1$, the cross section is $2^2 = 4$. At $x = 2$, it's $4^2 = 16$. And at $x = 3$, it's $6^2 = 36$. So generally speaking, the area of the cross section is:

   $A = (2x)^2 = 4x^2$

2. **Use this expression to build a definite integral that represents the volume of the pyramid.**

In this case, the limits of integration are 0 and 3, so:

$$V = \int_0^3 4x^2 dx$$

3. **Evaluate this integral:**

$$\frac{4}{3}x^3 \Big|_{x=0}^{x=3}$$
$$= \frac{4}{3}(3)^3 - 0 = 36$$

This is the same answer provided by the formula for the pyramid. But this method can be applied to a far wider variety of solids.

Measuring the volume of a weird solid

After you know the basic meat-slicer technique, you can apply it to any solid with a cross section that's a function of x. In some cases, these solids are harder to describe than they are to measure. For example, have a look at Figure 13-4.

The solid in Figure 13-4 consists of two exponential curves — one described by the equation $y = e^x$ and the other described by placing the same curve directly in front of the x-axis — joined by straight lines. The other sides of the solid are bounded planes slicing perpendicularly in a variety of directions.

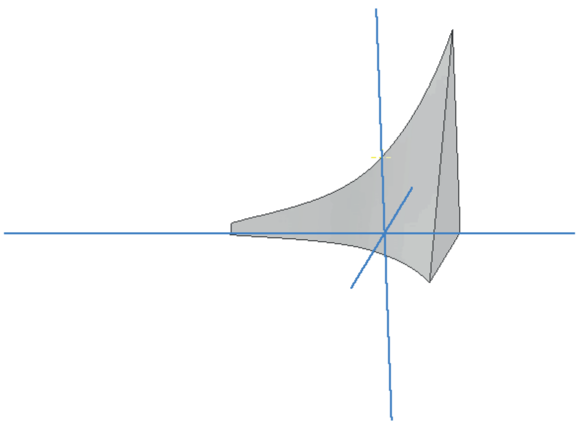

FIGURE 13-4: Measuring the volume of a solid based on two exponential curves in space from $x = 0$ to $x = 1$.

Notice that when you slice this solid perpendicular with the x-axis, its cross section is always an isosceles right triangle. This is an easy shape to measure, so the slicing method works nicely to measure the volume of this solid. Here are the steps:

1. **Find an expression that represents the area of a random cross section of the solid.**

   The triangle on the y-axis has a height and base of 1 — that is, e^0. And the triangle on the line $x = 1$ has a height and base of e^1, which is e. In general, the height and base of any cross-section triangle is e^x.

   So here's how to use the formula for the area of a triangle to find the area of a cross section in terms of x:

   $$A = \frac{1}{2}bh = \frac{1}{2}\left(e^x\right)\left(e^x\right) = \frac{1}{2}e^{2x}$$

2. **Use this expression to build a definite integral that represents the volume of the solid.**

   Now that you know how to measure the area of a cross section, integrate to add all the cross sections from $x = 0$ to $x = 1$:

   $$V = \int_0^1 \frac{1}{2}e^{2x}dx$$

3. **Evaluate this integral to find the volume.**

   $$= \frac{1}{2}\int_0^1 e^{2x}dx$$
   $$= \frac{1}{4}e^{2x}\Big|_{x=0}^{x=1}$$
   $$= \frac{1}{4}e^2 - \frac{1}{4}e^0$$
   $$\approx 1.597$$

Turning a Problem on Its Side

When using a real meat slicer, you need to find a way to turn whatever you're slicing on its side so that it fits in the machine. The same is true for calculus problems.

For example, suppose that you want to measure the volume of the solid shown in Figure 13-5. The base of this solid (light gray) is bounded on its sides by the function $y = x^4$ between the x-axis at the bottom and $y = 2$ across the top. The figure is 3 units high, such that the cross section when you slice parallel with the x-axis is a series of isosceles triangles, each with a height of 3 and a base that's the width across the function $y = x^4$.

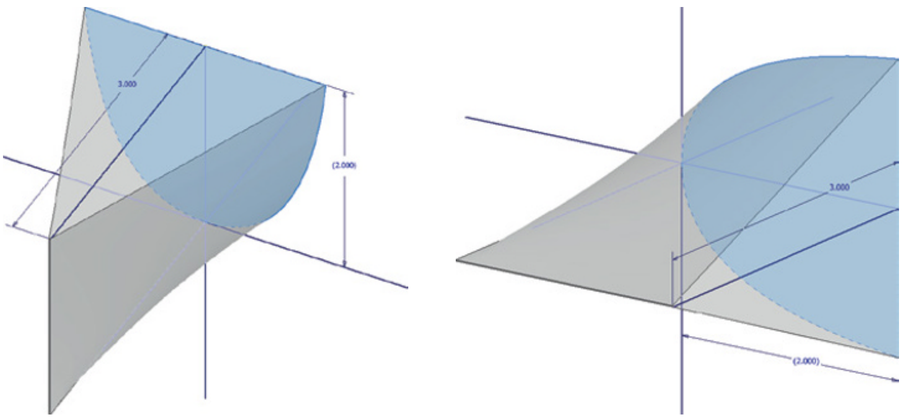

FIGURE 13-5: Using inverses to get a problem ready for the meat-slicer method.

The good news is that this solid has cross sections that are all similar triangles, so the meat-slicer method will work. Unfortunately, as the problem currently stands (upright), you'd have to make your slices perpendicular to the y-axis. But to use the meat-slicer method, you need to make your slices perpendicular to the x-axis.

To solve the problem, you first need to flip the solid over to the x-axis, as shown on the right side of Figure 13-5. The easiest way to do this is to use the inverse of the function $y = x^4$. To find the inverse, switch x and y in the equation and solve for y:

$$x = y^4$$

$$\pm\left(x^{\frac{1}{4}}\right) = y$$

WARNING

Note that the resulting equation, $\pm\left(x^{\frac{1}{4}}\right) = y$, in this case isn't a function of x because a single x-value can produce more than one y-value. However, you can use this equation in conjunction with the meat-slicer method to find the volume that you're looking for.

1. **Find an expression that represents the area of a random cross section of the solid.**

   The cross section is an isosceles triangle with a height of 3 and a base of $2x^{\frac{1}{4}}$, so use the formula for the area of a triangle:

   $$A = \frac{1}{2}bh = \frac{1}{2}\left(2x^{\frac{1}{4}}\right)(3) = 3x^{\frac{1}{4}}$$

2. **Use this expression to build a definite integral that represents the volume of the solid.**

   $$V = \int_0^2 3x^{\frac{1}{4}}dx$$

3. **Evaluate the integral.**

   $$3\left(\frac{4}{5}\right)x^{\frac{5}{4}}\Big|_{x=0}^{x=2}$$
   $$= \frac{12}{5}x^{\frac{5}{4}}\Big|_{x=0}^{x=2}$$

4. **Now evaluate this expression:**

   $$= \frac{12}{5}(2)^{\frac{5}{4}} - 0$$
   $$= \frac{12}{5}(32)^{\frac{1}{4}}$$
   $$\approx 5.7082$$

Two Revolutionary Problems

Calculus professors are always on the lookout for new ways to torture their students. Okay, that's a slight exaggeration. Still, sometimes it's hard to fathom exactly why a problem without much practical use makes the Calculus Hall of Fame.

In this section, I show you how to tackle two problems of dubious practical value (unless you consider the practicality of passing Calculus II!). First, I show you how to find the volume of a *solid of revolution,* which is a solid created by spinning a function around an axis. The meat-slicer method, which I discuss in the previous section, also applies to problems of this kind.

Next, I show you how to find the area of a *surface of revolution,* a surface created by spinning a function around an axis. Fortunately, a formula exists for solving this type of problem.

Solidifying your understanding of solids of revolution

A solid of revolution is created by taking a function, or part of a function, and spinning it around an axis — in most cases, either the x-axis or the y-axis.

For example, Figure 13-6 shows the function $y = 2\sin x$ between $x = 0$ and $x = \frac{\pi}{2}$.

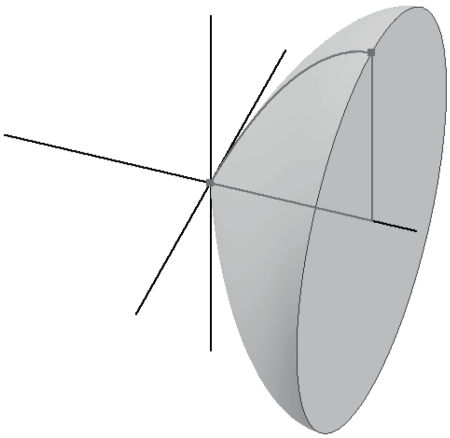

FIGURE 13-6: A solid of revolution of $y = 2\sin x$ around the x-axis.

TIP

Every solid of revolution has circular cross sections perpendicular to the axis of revolution. When the axis of revolution is the x-axis (or any other line that's parallel with the x-axis), you can use the meat-slicer method directly, as I show you earlier in this chapter.

However, when the axis of revolution is the y-axis (or any other line that's parallel with the y-axis), you need to modify the problem as I show you in the earlier section, "Turning a Problem on Its Side."

To find the volume of this solid of revolution, use the meat-slicer method:

1. **Find an expression that represents the area of a random cross section of the solid (in terms of x).**

   This cross section is a circle with a radius of 2 sin *x:*

   $$A = \pi r^2 = \pi(2\sin x)^2 = 4\pi\sin^2 x$$

2. **Use this expression to build a definite integral (in terms of *dx*) that represents the volume of the solid.**

This time, the limits of integration are from 0 to $\frac{\pi}{2}$:

$$V = \int_0^{\frac{\pi}{2}} 4\pi \sin^2 x \; dx$$
$$= 4\pi \int_0^{\frac{\pi}{2}} \sin^2 x \; dx$$

3. **Evaluate this integral by using the half-angle formula for sines, as I show you in Chapter 10**:

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{(1-\cos 2x)}{2} \; dx$$
$$= 2\pi \left(\int_0^{\frac{\pi}{2}} 1 \; dx - \int_0^{\frac{\pi}{2}} \cos 2x \; dx \right)$$
$$= 2\pi \left(x \Big|_{x=0}^{x=\frac{\pi}{2}} - \frac{1}{2} \sin 2x \Big|_{x=0}^{x=\frac{\pi}{2}} \right)$$

4. **Now evaluate (very carefully!):**

$$= 2\pi \left[\left(\frac{\pi}{2} - 0 \right) - \left(\frac{1}{2} \sin \pi - 0 \right) \right]$$
$$= 2\pi \left(\frac{\pi}{2} \right)$$
$$= \pi^2$$
$$\approx 9.8696$$

So the volume of this solid of revolution is approximately 9.8696 cubic units.

Later in this chapter, I give you more practice measuring the volume of solids of revolution.

Skimming the surface of revolution

The nice thing about finding the area of a surface of revolution is that there's a formula you can use. Memorize it and you're halfway done.

To find the area of a surface of revolution between *a* and *b*, use the following formula:

$$A = \int_a^b 2\pi r \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx$$

This formula looks long and complicated, but it makes more sense when you spend a minute thinking about it. The integral is made from two pieces:

- **The arc-length formula, which measures the length along the surface (see Chapter 12)**

  $$\text{Arclength} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

- **The formula for the circumference of a circle, which measures the length around the surface**

  $$C = 2\pi r$$

So multiplying these two pieces together is similar to multiplying length and width to find the area of a rectangle. In effect, the formula allows you to measure surface area as an infinite number of little rectangles.

When you're measuring the surface of revolution of a function $f(x)$ around the x-axis, substitute $r = f(x)$ into the formula I gave you:

$$A = \int_a^b 2\pi\, f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

For example, suppose that you want to find the surface of revolution that's shown in Figure 13-7.

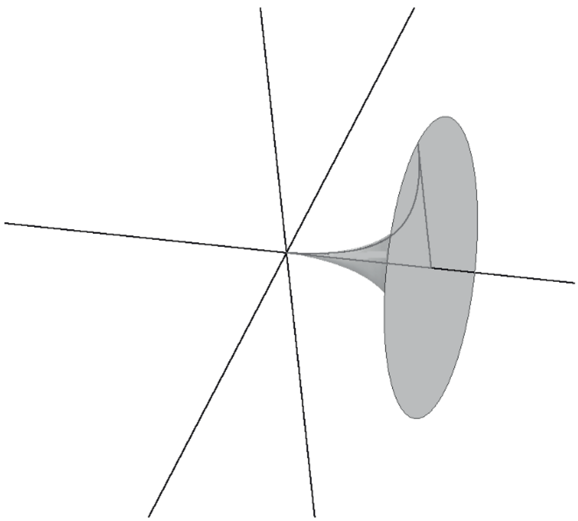

FIGURE 13-7: Measuring the surface of revolution of $y = x^3$ between $x = 0$ and $x = 1$.

To solve this problem, first note that for $f(x)=x^3$, the derivative $f'(x)=3x^2$. So set up the problem as follows:

$$A=\int_0^1 2\pi x^3\sqrt{1+\left(3x^2\right)^2}\,dx$$

To start off, simplify the problem a bit:

$$=2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx$$

You can solve this problem by using the following variable substitution:

$$\begin{aligned}\text{Let } u&=1+9x^4\\ du&=36x^3dx\end{aligned}$$

Now substitute u for $1+9x^4$ and $\frac{1}{36}du$ for $x^3\,dx$ into the equation:

$$=\frac{1}{36}\cdot 2\pi\int_1^{10}\sqrt{u}\,du$$

Notice that I've changed the limits of integration as follows: When $x=0$, $u=1$; and when $x=1$, $u=10$. Next, simplify a bit:

$$=\frac{1}{18}\pi\int_1^{10}\sqrt{u}\,du$$

Now you can perform the integration:

$$\begin{aligned}&=\frac{1}{18}\pi\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u-10}\\ &=\frac{1}{27}\pi u^{\frac{3}{2}}\Big|_{u=1}^{u-10}\end{aligned}$$

Finally, crunch the numbers:

$$\begin{aligned}&=\frac{1}{27}\pi(10)^{\frac{3}{2}}-\frac{1}{27}\pi(1)^{\frac{3}{2}}\\ &=\frac{1}{27}\pi 10\sqrt{10}-\frac{1}{27}\pi\\ &\approx 3.5631\end{aligned}$$

So the surface area for this solid of revolution is roughly 3.5631 square units.

Finding the Space Between

In Chapter 12, I show you how to find the area between two curves by subtracting one integral from another. This same principle applies in three dimensions to find the volume of a solid that falls between two different surfaces of revolution.

The meat-slicer method, which I describe earlier in this chapter, is useful for many problems of this kind. The trick is to find a way to describe the donut-shaped area of a cross section as the difference between two integrals: one integral that describes the whole shape minus another that describes the hole.

For example, suppose that you want to find the volume of the solid shown in Figure 13-8.

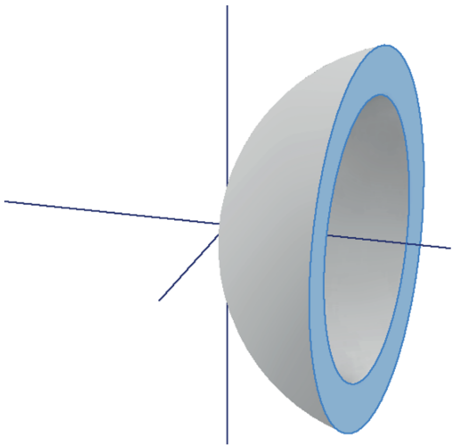

FIGURE 13-8: A vase-shaped solid between two surfaces of revolution.

This solid looks something like a bowl turned on its side. The outer edge is the solid of revolution around the x-axis for the function $x^{\frac{1}{3}}$. The inner edge is the solid of revolution around the x-axis for the function $\sqrt{x}$. And the resulting volume resides on the graph between $x = 0$ and $x = \frac{1}{2}$. Here's how to solve this problem:

1. **Find an expression that represents the area of a random cross section of the solid.**

   That is, find the area of the outer circle with a radius of $x^{\frac{1}{3}}$ and subtract the area of the inner circle with a radius of $\sqrt{x}$:

$$\text{Total Area} = \text{Area of outer circle} - \text{Area of inner circle}$$
$$= \pi\left(x^{\frac{1}{3}}\right)^2 - \pi(\sqrt{x})^2$$
$$= \pi\left(x^{\frac{2}{3}} - x\right)$$

2. **Use this expression to build a definite integral that represents the volume of the solid.**

   The limits of integration this time are 0 and $\frac{1}{2}$:

   $$V = \int_0^{\frac{1}{2}} \pi\left(x^{\frac{2}{3}} - x\right)dx$$

3. **Solve the integral:**

   $$= \pi\int_0^{\frac{1}{2}}\left(x^{\frac{2}{3}} - x\right)dx$$
   $$= \pi\left(\int_0^{\frac{1}{2}} x^{\frac{2}{3}}dx - \int_0^{\frac{1}{2}} x\ dx\right)$$
   $$= \pi\left(\frac{3}{5}x^{\frac{5}{3}}\Bigg|_{x=0}^{x=\frac{1}{2}} - \frac{1}{2}x^2\Bigg|_{x=0}^{x=\frac{1}{2}}\right)$$

4. **Now evaluate this expression:**

   $$= \pi\left[\left(\frac{3}{5}(\frac{1}{2})^{\frac{5}{3}} - 0\right) - \left(\frac{1}{2}(\frac{1}{2})^2 - 0\right)\right]$$
   $$= \pi\left(\frac{3}{5}(\frac{1}{32})^{\frac{1}{3}} - \frac{1}{8}\right)$$
   $$\approx \pi(0.189 - 0.125) = 0.064$$

So the volume in this case is approximately 0.064 cubic units.

Here's a problem that brings together everything you've worked with from the meat-slicer method: Find the volume of the solid shown in Figure 13-9. This solid falls between the surface of revolution $y = \ln x$ and the surface of revolution $y = x^{\frac{3}{4}}$, bounded below by $y = 0$ and above by $y = 1$. The cross section of this solid is shown on the right-hand side of Figure 13-9: a circle with a hole in the middle.

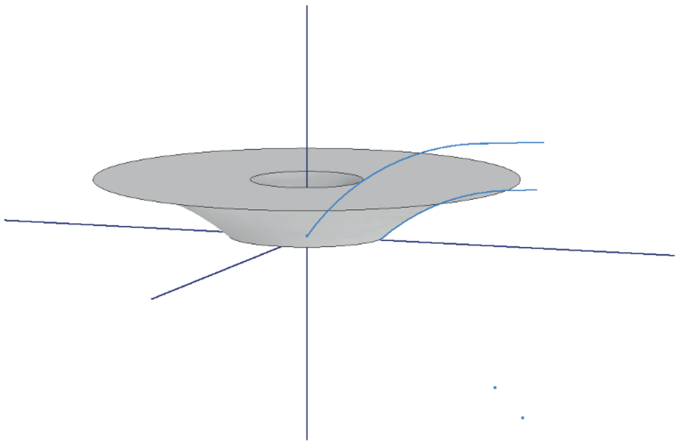

FIGURE 13-9: Another solid formed between two surfaces of revolution.

Notice, however, that this cross section is perpendicular to the y-axis. To use the meat-slicer method, the cross section must be perpendicular to the x-axis. Modify the problem using inverses, solving each equation for y as I show you in the section, "Turning a Problem on Its Side," earlier in this chapter:

$$x = \ln y \qquad x = y^{\frac{3}{4}}$$
$$e^x = y \qquad x^{\frac{4}{3}} = y$$

The resulting problem is shown in Figure 13-10.

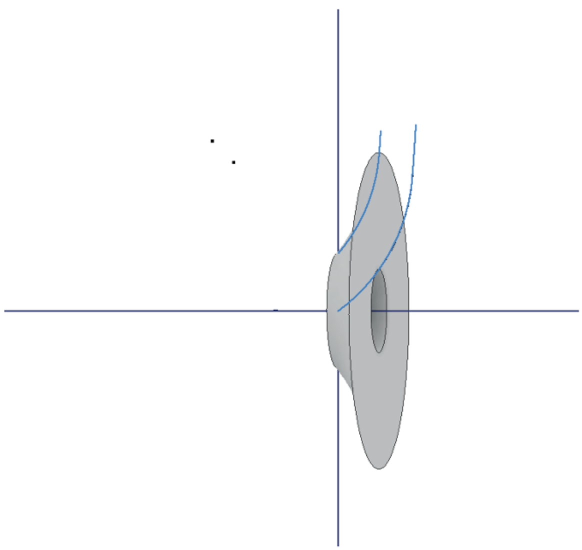

FIGURE 13-10: Use inverses to rotate the problem from Figure 13-9 so you can use the meat-slicer method.

Now you can use the meat-slicer method to solve the problem:

1. **Find an expression that represents the area of a random cross section of the solid.**

   That is, find the area of a circle with a radius of e^x and subtract the area of a circle with a radius of $x^{\frac{4}{3}}$. This is just geometry, but I take it slowly so you can see all the steps. Remember that the area of a circle is πr^2:

   $$\begin{aligned}&\text{Total area} = \text{Area of outer circle} - \text{Area of inner circle}\\&= \pi\left(e^x\right)^2 - \pi\left(x^{\frac{4}{3}}\right)^2\\&= \pi e^{2x} - \pi x^{\frac{8}{3}}\end{aligned}$$

2. **Use this expression to build a definite integral that represents the volume of the solid.**

   The limits of integration are 0 and 1:

   $$V = \int_0^1 \left(\pi e^{2x} - \pi x^{\frac{8}{3}}\right) dx$$

3. **Evaluate the integral:**

   $$\begin{aligned}&= \int_0^1 \pi e^{2x} dx - \int_0^1 \pi x^{\frac{8}{3}} dx\\&= \frac{\pi}{2} e^{2x}\Big|_{x=0}^{x=1} - \frac{3\pi}{11} x^{\frac{11}{3}}\Big|_{x=0}^{x=1}\\&= \left(\frac{\pi}{2}e^2 - \frac{\pi}{2}e^0\right) - \left(\frac{3\pi}{11}(1)^{\frac{11}{3}} - \frac{3\pi}{11}(0)^{\frac{11}{3}}\right)\\&= \frac{\pi}{2}e^2 - \frac{\pi}{2} - \frac{3\pi}{11}\\&\approx 2.9218\end{aligned}$$

   So the volume of this solid is approximately 2.9218 cubic units.

Playing the Shell Game

The *shell method* is an alternative to the meat-slicer method I discuss earlier in this chapter. It allows you to measure the volume of a solid by measuring the volume of many concentric surfaces of the volume, called "shells."

Although the shell method works only for solids with circular cross sections, it's ideal for solids of revolution around the y-axis, because you don't have to use inverses of functions — a method I show you how to use in the section, "Turning a Problem on Its Side," earlier in this chapter. Here's how the shell method works:

1. Find an expression that represents the area of a random shell of the solid in terms of x.
2. Use this expression to build a definite integral (in terms of dx) that represents the volume of the solid.
3. Evaluate this integral.

As you can see, this method resembles the meat-slicer method. The main difference is that you're measuring the area of shells instead of cross sections.

Peeling and measuring a can of soup

You can use a can of soup — or any other can that has a paper label on it — as a handy visual aid to give you insight into how the shell method works. To start out, go to the pantry and get a can of soup.

Suppose that your can of soup is industrial size, with a radius of 3 inches and a height of 8 inches. You can use the formula for a cylinder to figure out its volume as follows:

$$V = A_b \cdot h = 3^2\pi \cdot 8 = 72\pi$$

Another option for finding the volume is the meat-slicer method, as I show you earlier in this chapter. A third option, which I focus on here, is the shell method.

To understand the shell method, slice the can's paper label vertically, and carefully remove it from the can, as shown in Figure 13-11. (While you're at it, take a moment to read the label so that you're not left with "mystery soup.")

Notice that the label is simply a rectangle. Its shorter side is equal in length to the height of the can (8 inches) and its longer side is equal to the circumference ($C = 2\pi r = 2\pi \cdot 3 \text{ inches} = 6 \text{ inches}$). So the area of this rectangle is 48π square inches.

Now here's the crucial step: Imagine that the entire can is made up of infinitely many labels wrapped concentrically around each other, all the way to its core. The area of each of these rectangles is:

$$A = 2\pi x \cdot 8 = 16\pi x$$

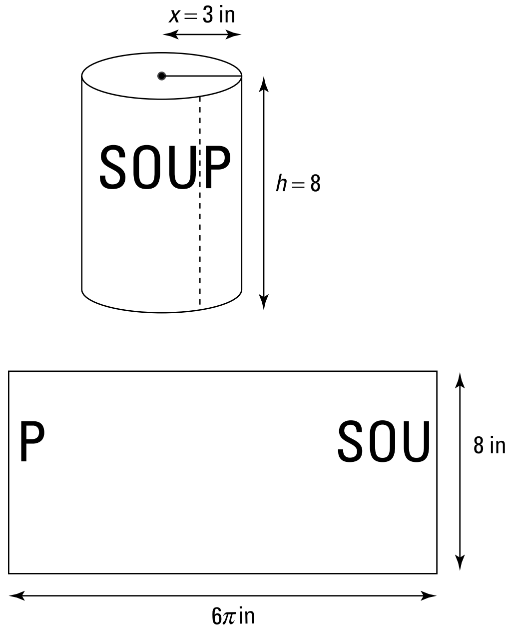

FIGURE 13-11: Removing the label from a can of soup can help you understand the shell method.

The variable *x* in this case is any possible radius, from 0 (the radius of the circle at the very center of the can) to 3 (the radius of the circle at the outer edge). Here's how you use the shell method, step by step, to find the volume of the can:

1. **Find an expression that represents the area of a random shell of the can (in terms of *x*).**

   You just did that:

   $A = 2\pi x \cdot 8 = 16\pi x$

2. **Use this expression to build a definite integral (in terms of *dx*) that represents the volume of the can.**

   Remember that, with the shell method, you're adding up all the shells from the center (where the radius is 0) to the outer edge (where the radius is 3). So use these numbers as the limits of integration:

   $$V = \int_0^3 16\pi x dx$$

3. Evaluate this integral:

$$= 16\pi \cdot \frac{1}{2}x^2 \Big|_{x=0}^{x=3}$$

$$= 8\pi x^2 \Big|_{x=0}^{x-3}$$

4. Now evaluate this expression:

$= 8\pi (3)^2 - 0 = 72\pi$

The shell method verifies that the volume of the can is 72π cubic inches.

Using the shell method without inverses

One advantage of the shell method over the meat-slicer method comes into play when you're measuring a volume of revolution around the *y*-axis.

Earlier in this chapter, I tell you that the meat-slicer method works best when a solid is on its side — that is, when you can slice it perpendicular to the *x*-axis. But when the similar cross sections of a solid are perpendicular to the *y*-axis, you need to use inverses to realign the problem before you can start slicing. (See the section, "Turning a Problem on Its Side," earlier in this chapter, for more details.)

This realignment step isn't necessary for the shell method. This makes the shell method ideal for measuring solids of revolution around the *y*-axis. For example, suppose that you want to measure the volume of the solid shown in Figure 13-12. This is a solid of revolution formed by sweeping the function cos *x* from $x = 0$ to $x = \frac{\pi}{2}$ around the *y*-axis.

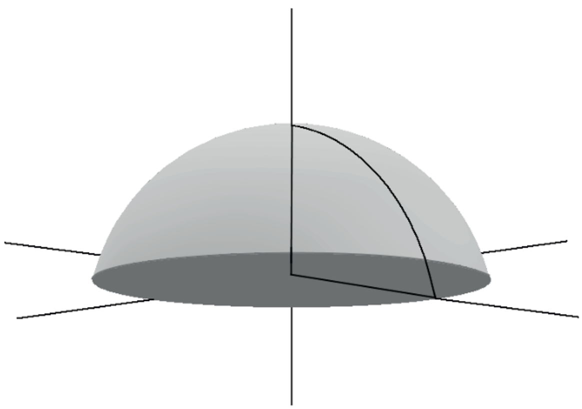

FIGURE 13-12: Using the shell method to find the volume of a solid of revolution.

Here's how the shell method can give you a solution without using inverses:

1. **Find an expression that represents the area of a random shell of the solid (in terms of *x*).**

   Remember that each shell is a rectangle with two different sides: One side is the height of the function at x — that is, $\cos x$. The other is the circumference of the solid at x — that is, $2\pi x$. So to find the area of a shell, multiply these two numbers together:

   $A = 2\pi x \cos x$

2. **Use this expression to build a definite integral (in terms of *dx*) that represents the volume of the solid.**

   In this case, remember that you're adding up all the shells from the center (at $x = 0$) to the outer edge (at $x = \frac{\pi}{2}$).

   $$= \int_0^{\frac{\pi}{2}} 2\pi x \cos x \, dx$$

   $$= 2\pi \int_0^{\frac{\pi}{2}} x \cos x \, dx$$

3. **Evaluate the integral.**

   This integral is pretty easy to solve using integration by parts (see Chapter 9):

   $$2\pi(x \sin x + \cos x)\Big|_{x=0}^{x-\frac{x}{2}}$$

4. **Now evaluate this expression:**

   $$= 2\pi\left[\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0\sin 0 + \cos 0)\right]$$

   $$= 2\pi\left[\left(\frac{\pi}{2} + 0\right) - (0 + 1)\right]$$

   $$= 2\pi\left[\frac{\pi}{2} - 1\right]$$

   $$\approx 3.5864$$

So the volume of the solid is approximately 3.5864 cubic units.

Knowing When and How to Solve 3-D Problems

Because students are so often confused when it comes to solving 3-D calculus problems, here's a final perspective on all the methods in this chapter and how to choose among them.

First, remember that every problem in this chapter falls into one of these two categories:

- Finding the area of a surface of revolution
- Finding a volume of a solid

In the first case, use the formula I provide in the section, "Skimming the surface of revolution," earlier in this chapter.

In the second case, remember that the key to measuring the volume of any solid is to slice it up in the direction where it has *similar cross sections* whose area can be measured easily — for example, a circle, a square, or a triangle. So your first question is whether these similar cross sections are arranged horizontally or vertically.

- **Horizontally** means that the solid is already in position for the meat-slicer method. (If it's helpful, imagine slicing salami in a meat-slicer. The salami must be aligned lying on its side — that is, horizontally — before you can begin slicing.)
- **Vertically** means that the solid is standing upright so that the slices are stacked on top of each other.

When the cross sections are arranged horizontally, the meat-slicer method is the easiest way to handle the problem (see the section, "Slicing Your Way to Success," earlier in this chapter).

When the cross sections are arranged vertically, however, your next question is whether these cross sections are circles:

- If the cross sections are *not* circles, you must use inverses to flip the solid in the horizontal direction (as I discuss in the section, "Turning a Problem on Its Side").
- If they *are* circles, you can either use inverses to flip the solid in the horizontal direction (as I discuss in the section, "Turning a Problem on Its Side") *or* use the shell method (as I discuss in the section, "Playing the Shell Game").

IN THIS CHAPTER

» Classifying different types of differential equations (DEs)

» Understanding the connection between DEs and integrals

» Checking a proposed solution to a DE

» Using a variety of methods to solve DEs

Chapter 14

What's So Different about Differential Equations?

The very mention of differential equations (DEs for short) strikes a spicy combination of awe, horror, and utter confusion into nonmathematical minds. Even intrepid calculus students have been known to consider a career in art history when these untamed beasts come into focus on the radar screen. Just what are differential equations? Where do they come from? Why are they necessary? And how in the world do you solve them?

In this chapter, I answer these questions and give you some familiarity with DEs. I show you how to identify the basic types of DEs so that if you're ever at a math department cocktail party (lucky you!), you won't feel completely adrift. I relate DEs to the integrals that you discover earlier in the book. I show you how to build your own DEs so you'll always have a hobby to pass the time, and I also demonstrate how to check DE solutions. In addition, you discover how DEs arise in physics. Finally, I give you a few simple methods for solving some basic differential equations.

Basics of Differential Equations

In a nutshell, a *differential equation,* or DE, is any equation that includes at least one derivative. For example:

$$\frac{dy}{dx} = \sin x e^y \qquad \frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 9y = 0 \qquad \frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos x$$

Solving a differential equation means finding the value of the dependent variable in terms of the independent variable. Throughout this chapter, I use y as the dependent variable, so the goal in each problem is to solve for y in terms of x.

In this section, I show you how to classify DEs. I also show you how to build DEs and check the solution to a DE.

Classifying DEs

As with other equations that you've encountered in this book, differential equations come in many varieties. And different varieties of DEs can be solved using different methods. In this section, I show you some important ways to classify DEs.

Ordinary and partial differential equations

An *ordinary differential equation* (ODE) has only derivatives of one variable — that is, it has no partial derivatives (derivatives with two or more variables). Here are a few examples of ODEs:

$$\frac{dy}{dx} = x\sin\left(x^2\right)\cos y \qquad \frac{dy}{dx} = y\csc x + e^x \qquad \frac{d^2y}{dx^2} + 4xy\frac{dy}{dx} + 5y = 0$$

In contrast, a *partial differential equation* (PDE) has at least one partial derivative. Here are a few examples of PDEs:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{x-y} \qquad 3\frac{\partial u}{\partial x^2} + 7\frac{\partial u}{\partial x\partial y} + 6\frac{\partial u}{\partial y^2} = 0 \qquad \frac{\partial v}{\partial t} - k\frac{\partial v}{\partial x^2} = v$$

Ordinary differential equations are usually the topic of a typical Differential Equations class in college. They're a step or two beyond what you're used to working with, but many students actually find Differential Equations an easier course than Calculus II (generally considered the most difficult class in the calculus series). However, ODEs are limited in how well they can actually express physical reality.

The real quarry is partial differential equations. A lot of physics gets done with these little gems. Unfortunately, solving PDEs is one giant leap forward in math from what the average calculus student is used to. Delving into the kind of math that makes PDEs come alive is typically reserved for graduate school.

Order of DEs

Differential equations are further classified according to their *order.* This classification is similar to the classification of polynomial equations by degree. (See Chapter 2 for more on polynomials.)

First-order ODEs contain only first derivatives. For example:

$$\frac{dy}{dx} = ye^x \qquad 3\frac{dy}{dx} = \sin y + 2e^{2x} \qquad \ln xy\frac{dy}{dx} = 2x^2 + y - \tan x$$

Higher-order ODEs are classified, as polynomials are, by the *greatest* order of their derivatives. Here are examples of second-, third-, and fourth-order ODEs.

Second-order ODE: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 10y = e^x$

Third-order ODE: $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + y = 0$

Fourth-order ODE: $x^2\frac{d^4y}{dx^4} + \cos y = x$

As with polynomials, generally speaking, a higher-order DE is more difficult to solve than one of lower order.

Linear DEs

What constitutes a linear differential equation depends slightly on who you ask. For practical purposes, a linear first-order DE fits into the following form:

$$\frac{dy}{dx} + a(x)y = b(x)$$

where $a(x)$ and $b(x)$ are functions of x. Here are a few examples of linear first-order DEs:

$$\frac{dy}{dx} + y = x \qquad \frac{dy}{dx} + 4xy = -\ln x \qquad \frac{dy}{dx} - y\sin x = e^x$$

A linear second-degree DE fits into the following form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where *a*, *b*, and *c* are all constants. Here are some examples:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0 \qquad 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \qquad \frac{d^2y}{dx^2} = 0$$

Note that the constant *a* can always be reduced to 1, resulting in adjustments to the other two coefficients. Linear second-degree DEs are usually an important topic in a college-level course in differential equations. Solving them requires knowledge of matrices and complex numbers that is beyond the scope of this book.

Looking more closely at DEs

You don't have to play professional baseball to enjoy baseball. Instead, you can enjoy the game from the bleachers or, if you prefer, from a nice cushy chair in front of the TV. Similarly, you don't have to get too deep into differential equations to gain a general understanding of how they work. In this section, I give you front-row seats to the game of differential equations.

How every integral is a DE

The integral is a particular example of a more general type of equation — the differential equation. To see how this is so, suppose that you're working with this nice little integral:

$$y = \int \cos x \, dx$$

Differentiating both sides turns it into a DE:

$$\frac{dy}{dx} = \cos x$$

Of course, you know how to solve this DE by thinking of it as an integral:

$$y = \sin x + C$$

So in general, when a DE is of the form

$$\frac{dy}{dx} = f(x)$$

with $f(x)$ an arbitrary function of x, you can express that DE as an integral and solve it by integrating.

Why building DEs is easier than solving them

The reason that the DE in the last section is so simple to solve is that the derivative is isolated on one side of the equation. DEs attain a new level of difficulty when the derivative isn't isolated.

A good analogy can be made in lower math, when you make the jump from arithmetic to algebra. For example, here's an arithmetic problem:

$$x = 20 - \left(4^2 + 3\right)$$

Even though this is technically an algebra problem, you can solve it without algebra because *x* is isolated at the start of the problem. However, the ballgame changes when *x* becomes more enmeshed in the equation. For example:

$$2x^3 - x^2 + 5x = 0$$

Arithmetic isn't strong enough for this problem, so algebra takes over. Similarly, when derivatives get entangled into the fabric of an equation — as in most of the DEs I show you earlier in the section, "Classifying DEs" — integrating is no longer effective and the search for new methods begins.

Although solving DEs is often tricky, building them is easy. For example, suppose you start with this simple quadratic equation:

$$y = 3x^2 + 4x - 5$$

Now find the first and second derivatives:

$$\frac{dy}{dx} = 6x + 4$$
$$\frac{d^2y}{dx^2} = 6$$

Adding up the left and right sides of all three equations gives you the following differential equation:

$$y + \frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x^2 + 10x + 5$$

Because you built the equation yourself, you know what *y* equals. But if you handed this equation off to some other students, they probably wouldn't be able to guess how you built it, so they would have to do some work to solve it for *y*. For example, here's another DE:

$$y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} = 0$$

This equation probably looks difficult because you don't have much information. And yet, after I tell you the solution, it appears simple:

$$y = \sin x \qquad \frac{dy}{dx} = \cos x \qquad \frac{d^2y}{dx^2} = -\sin x \qquad \frac{d^3y}{dx^3} = -\cos x$$

But even after you have the solution, how do you know whether it's the *only* solution? For starters, $y = -\sin x$, $y = \cos x$, and $y = -\cos x$ are also solutions. Do other solutions exist? How do you find them? And how do you know when you have them all?

Another difficulty arises when y itself becomes tangled up in the equation. For example, how do you solve this equation for y?

$$\frac{dy}{dx} = \frac{\sin x}{e^y}$$

As you can see, differential equations contain treacherous subtleties that you don't find in basic calculus.

Checking DE solutions

Even if you don't know how to find a solution to a differential equation, you can always check whether a proposed solution works. This is simply a matter of plugging the proposed value of the dependent variable — I use y throughout this chapter — into both sides of the equation to see whether equality is maintained.

For example, here's a DE:

$$\frac{dy}{dx} = 3y + 4e^{3x}\cos x$$

You may not have a clue how to begin solving this DE, but imagine that an angel lands on your pen and offers you this solution:

$$y = 4e^{3x}\sin x$$

You can check to see whether this angel really knows math by plugging in this value of y as follows:

$$\begin{aligned}
\frac{dy}{dx} &= 3y + 4e^{3x}\cos x \\
\frac{d}{dx}4e^{3x}\sin x &= 3\left(4e^{3x}\sin x\right) + 4e^{3x}\cos x \\
4\left(3e^{3x}\sin x + e^{3x}\cos x\right) &= 12e^{3x}\sin x + 4e^{3x}\cos x \\
12e^{3x}\sin x + 4e^{3x}\cos x &= 12e^{3x}\sin x + 4e^{3x}\cos x
\end{aligned}$$

Because the left and right sides of the equation are equal, the angel's solution checks out.

Solving Differential Equations

In this section, I show you how to solve a few types of DEs. First, you solve everybody's favorite DE, the *separable equation.* Next, you put this understanding to work to solve an *initial-value problem* (IVP). Finally, I show you how to solve a linear first-order DE using an integrating factor.

Solving separable equations

Differential equations become harder to solve the more entangled they become. In certain cases, however, an equation that looks all tangled up is actually easy to tease apart. Equations of this kind are called *separable equations* (or *autonomous equations*), and they fit into the following form:

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Separable equations are relatively easy to solve. For example, suppose that you want to solve the following problem:

$$\frac{dy}{dx} = \frac{\sin x}{e^y}$$

You can think of the symbol $\frac{dy}{dx}$ as a fraction and isolate the x and y terms of this equation on opposite sides of the equal sign:

$$e^y dy = \sin x \ dx$$

Now integrate both sides:

$$\int e^y dy = \int \sin x \ dx$$

$$e^y + C_1 = -\cos x + C_2$$

TECHNICAL STUFF

In an important sense, the previous step is questionable because the variable of integration is different on each side of the equal sign. You may think, "No problem, it's all integration!" But imagine if you tried to divide one side of an equation by 2 and the other by 3, and then laughed it off with "It's all division!" Clearly, you'd have a problem. The good news, however, is that, for technical reasons beyond the scope of this book, integrating both sides by different variables actually produces the correct answer.

C_1 and C_2 are both constants, so you can use the equation $C = C_2 - C_1$ to simplify the equation:

$$e^y = -\cos x + C$$

Next, use a natural log to undo the exponent, and then simplify:

$$\ln e^y = \ln(-\cos x + C)$$
$$y = \ln(-\cos x + C)$$

To check this solution, substitute this value for y into both sides of the original equation:

$$\frac{dy}{dx} = \frac{\sin x}{e^y}$$
$$\frac{d}{dx}\ln(-\cos x + C) = \frac{\sin x}{e^{\ln(-\cos x + C)}}$$
$$\frac{d}{dx}\ln(-\cos x + C) = \frac{\sin x}{-\cos x + C}$$
$$\frac{1}{-\cos x + C} \cdot \sin x = \frac{\sin x}{-\cos x + C}$$
$$\frac{\sin x}{-\cos x + C} = \frac{\sin x}{-\cos x + C}$$

Solving initial-value problems

In Chapter 5, I show you that the definite integral is a particular example of a whole family of indefinite integrals. In a similar way, an *initial-value problem* (IVP) is a particular example of a solution to a differential equation. Every IVP gives you extra information — called an *initial value* — that allows you to use the *general solution* to a DE to obtain a *particular solution.*

For example, here's an initial-value problem:

$$\frac{dy}{dx} = y\sec^2 x \quad y(0) = 5$$

This problem includes not only a DE but also an additional equation. To understand what this equation tells you, remember that y is a dependent variable, a function of x. So the notation $y(0) = 5$ means "when $x = 0$, $y = 5$." You see how this information comes into play as I continue with this example.

To solve an IVP, you first have to solve the DE. Do this by finding its general solution without worrying about the initial value. Fortunately, this DE is a separable equation, which you know how to solve from the last section:

$$\frac{1}{y}dy = \sec^2 x\,dx$$

Integrate both sides:

$$\int \frac{1}{y} dy = \int \sec^2 x\, dx$$
$$\ln y = \tan x + C$$

In this last step, I use C to consolidate the constants of integration from both sides of the equation into a single constant C. (If this doesn't make sense, I explain why this works in the section, "Solving separable equations," earlier in this chapter.) Next, I undo the natural log by using e:

$$e^{\ln y} = e^{\tan x + C}$$
$$y = e^{\tan x + C}$$
$$y = e^{\tan x} \cdot e^{C}$$

Because e^C is a constant, this equation can be further simplified using the substitution $D = e^C$:

$$y = De^{\tan x}$$

Before moving on, check to make sure that this solution is correct by substituting this value of y into both sides of the original equation:

$$\frac{dy}{dx} = y \sec^2 x$$
$$\frac{d}{dx} De^{\tan x} = De^{\tan x} \sec^2 x$$
$$De^{\tan x} \sec^2 x = De^{\tan x} \sec^2 x$$

This checks out, so $y = De^{\tan x}$ is, indeed, the *general solution* to the DE. To solve the initial-value problem, however, I need to find the specific value of the variable D by using the additional information I have: When $x = 0$, $y = 5$. Plugging both of these values into the equation makes it possible to solve for D:

$$5 = De^{\tan 0}$$
$$5 = De^{0}$$
$$5 = D$$

Now plug this value of D back into the general solution of the problem to get the IVP solution:

$$y = 5e^{\tan x}$$

This solution satisfies not only the differential equation $\frac{dy}{dx} = y \sec^2 x$ but also the initial value $y(0) = 5$.

6 Infinite Series

IN THIS PART . . .

Understand the relationships among an infinite series, its generating sequence, and its sequence generating of partial sums

Calculate the value of a convergent geometric series

Distinguish convergent and divergent series

Express functions as Taylor and Maclaurin series

IN THIS CHAPTER

- » Knowing a variety of notations for sequences
- » Telling whether a sequence is convergent or divergent
- » Expressing series in both sigma notation and expanded notation
- » Testing a series for convergence or divergence

Chapter 15
Following a Sequence, Winning the Series

Just when you think the semester is winding down, your Calculus II professor introduces a new topic: infinite series.

When you get right down to it, series aren't really all that difficult. After all, a series is just a bunch of numbers added together. Sure, it happens that this bunch is infinite, but addition is just about the easiest math on the planet.

But then again, the last month of the semester is crunch time. You're already anticipating final exams and looking forward to a break from studying. By the time you discover that the prof isn't fooling and really does expect you to know this material, infinite series can lead you down an infinite spiral of despair: Why *this*? Why *now*? Why *me*?

Don't worry. In this chapter, I show you the basics of series. First, you wade into these new waters slowly by examining infinite sequences. When you understand sequences, series make a whole lot more sense. Then I introduce you to infinite series. I discuss how to express a series in both expanded notation and sigma notation, and then I make sure you're comfortable with sigma notation. I also show you how every series is related to two sequences.

Next, I introduce you to the all-important topic of convergence and divergence. This concept looms large, so I give you the basics in this chapter and save the more complex information for Chapter 16. Finally, I introduce you to a few important types of series.

Introducing Infinite Sequences

A *sequence* of numbers is simply a bunch of numbers in a particular order. For example:

1, 4, 9, 16, 25, . . .	$\pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \ldots$	$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
2, 3, 5, 7, 11, 13, . . .	2, –2, 2, –2, . . .	0, 1, –1, 2 –2, 3, . . .

When a sequence goes on forever, it's an *infinite sequence.* Calculus — which focuses on all things infinite — concerns itself predominantly with infinite sequences.

Each number in a sequence is called a *term* of that sequence. So in the sequence 1, 4, 9, 16, . . ., the *first term* is 1, the *second term* is 4, and so forth.

Understanding sequences is an important first step toward understanding series, so read on to get started.

Understanding notations for sequences

The notation for defining a sequence is a variable with the subscript *n* surrounded by braces. For example:

$$\{a_n\} = \{1,\ 4,\ 9,\ 16,\ \ldots\} \quad \{b_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\right\} \quad \{c_n\} = \{4\pi,\ 6\pi,\ 8\pi,\ 10\pi,\ \ldots\}$$

You can reference a specific term in the sequence by using the subscript:

$$a_1 = 1 \qquad b_3 = \frac{1}{3} \qquad c_6 = 14\pi$$

WARNING

Make sure you understand the difference between notation with and without braces:

- The notation $\{a_n\}$ with braces refers to the entire sequence.
- The notation a_n without braces refers to the *n*th term of the sequence.

When defining a sequence, instead of listing the first few terms, you can state a rule based on *n*. (This is similar to how a function is typically defined.) For example:

$$\{a_n\},\ \text{where } a_n = n^2 \qquad \{b_n\},\ \text{where } b_n = \frac{1}{n} \qquad \{c_n\},\ \text{where } c_n = 2(n+1)$$

Sometimes, for increased clarity, the notation includes the first few terms plus a rule for finding the *n*th term of the sequence. For example:

$$\{a_n\} = \{1,\ 4,\ 9,\ \ldots,\ n^2,\ \ldots\} \qquad \{b_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \ldots,\ \frac{1}{n},\ \ldots\right\}$$

$$\{c_n\} = \{4\pi,\ 6\pi,\ 8\pi,\ \ldots,\ 2(n+1)\pi,\ \ldots\}$$

This notation can be made more concise by appending starting and ending values for *n:*

$$\{a_n\} = \{n^2\}_{n=1}^{\infty} \qquad \{b_n\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \qquad \{c_n\} = \{2n\pi\}_{n=2}^{\infty}$$

This last example points out the fact, although in most cases a sequence begins with $n = 1$ by convention, this starting value can be modified if needed.

TIP

Don't let the fancy notation for number sequences get to you. When you're faced with a new sequence that's defined by a rule, jot down the first four or five numbers in that sequence. After you see the pattern, you'll likely find that a problem is much easier to solve.

Looking at converging and diverging sequences

Every infinite sequence is either convergent or divergent. Here's what each means:

- A *convergent sequence* has a limit — that is, the limit of the *nth* term approaches a real number.
- A *divergent sequence* doesn't have a limit.

For example, here's a convergent sequence:

$$\{a_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots\right\}$$

This sequence approaches 0, so:

$$\lim \{a_n\} = 0$$

Thus, this sequence *converges to 0.*

Here's another convergent sequence:

$$\{b_n\} = \left\{7,\ 9,\ 7\frac{1}{2},\ 8\frac{1}{2},\ 7\frac{3}{4},\ 8\frac{1}{4},\ \ldots\right\}$$

This time, the sequence approaches 8 from above and below, so:

$$\lim \{b_n\} = 8$$

In many cases, however, a sequence *diverges* — that is, it fails to approach any real number. Divergence can happen in two ways. The most obvious type of divergence occurs when a sequence explodes to infinity or negative infinity — that is, it gets farther and farther away from 0 with every term. Here are a few examples:

–1, –2, –3, –4, –5, –6, –7, . . .   ln 1, ln 2, ln 3, ln 4, ln 5, . . .   2, 3, 5, 7, 11, 13, 17, . . .

In each of these cases, the sequence approaches either ∞ or $-\infty$, so the limit of the sequence *does not exist* (DNE). Therefore, the sequence is divergent.

A second type of divergence occurs when a sequence oscillates between two or among more than two values. For example:

0, 7, 0, 7, 0, 7, 0, 7, . . .   0, 1, 0, –1, 0, 1, 0, –1, . . .   1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, . . .

In these cases, the sequence bounces around indefinitely, never settling in on a value. Again, the limit of the sequence *does not exist*, so the sequence is divergent.

Introducing Infinite Series

In contrast to an infinite sequence (which is an endless list of numbers), an *infinite series* is an endless sum of numbers. You can change any infinite sequence to an infinite series simply by changing the commas to plus signs. Table 15-1 shows three examples of infinite sequences and infinite series:

TABLE 15-1 Infinite Sequences versus Infinite Series

Sequences	Series
$1, 2, 3, 4, 5, 6, \ldots$	$1+2+3+4+5+6+\ldots$
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\ldots$

The two principal notations for series are sigma notation and expanded notation. *Sigma notation* provides an explicit rule for generating the series (see Chapter 2 for the basics of sigma notation). *Expanded notation* gives enough of the first few terms of a series so that the pattern generating the series becomes clear.

For example, here are three series defined using both forms of notation:

$$\sum_{n=1}^{\infty} 2n = 2(1)+2(2)+2(3)+2(4)\ldots$$
$$= 2+4+6+8+\ldots$$

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{4^0}+\frac{1}{4^1}+\frac{1}{4^2}+\frac{1}{4^3}\ldots$$
$$= 1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$$

$$\sum_{n=3}^{\infty} \frac{n}{e^n} = \frac{3}{e^3}+\frac{4}{e^4}+\frac{5}{e^5}+\frac{6}{e^6}\ \ldots$$

As you can see, the *index* for a series — that is, the starting value of the variable n (or sometimes i) — can start at any integer. To generate the first term of the series, plug in the first index value for n. Add additional terms by increasing n by 1 repeatedly until you get a sense of how the series behaves.

As with sequences (see the section, "Introducing Infinite Sequences," earlier in this chapter), every series is either convergent or divergent:

- A *convergent series* evaluates to a real number.
- A *divergent series* doesn't evaluate to a real number.

To show how evaluation of a series connects with convergence and divergence, I give you a few examples. To start out, consider this convergent series:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

Notice that as you add this series from left to right, term by term, the running total is a sequence that approaches 2:

$$1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \ldots$$

This sequence is called the *sequence of partial sums* for this series. I discuss sequences of partial sums in greater detail later in the section, "Connecting a Series with Its Two Related Sequences."

REMEMBER

For now, please remember that the value of a series equals the limit of its sequence of partial sums. In this case, because the limit of the sequence is 2, you can evaluate the series as follows:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$$

Thus, this series *converges to 2.*

Often, however, a series *diverges* — that is, it doesn't equal any real number. As with sequences, divergence can happen in two ways. The most obvious type of divergence occurs when a series explodes to infinity or negative infinity. For example:

$$\sum_{n=1}^{\infty} -n = -1+(-2)+(-3)+(-4)+\ldots$$

This time, watch what happens as you add the series term by term:

–1, –3, –6, –10, . . .

Clearly, this sequence of partial sums diverges to negative infinity, so the series is divergent as well.

A second type of divergence occurs when a series alternates between positive and negative values in such a way that the series never approaches a value. For example:

$$\sum_{n=0}^{\infty}(-1)^n = 1 + -1 + 1 + -1 + \ldots$$

So here's the related sequence of partial sums:

1, 0, 1, 0, . . .

In this case, the sequence of partial sums alternates forever between 1 and 0, so it's divergent; therefore, the series is also divergent. This type of series is called, not surprisingly, an *alternating series.* I discuss alternating series in greater depth in Chapter 16.

TIP

Convergence and divergence are arguably the most important topics in your final weeks of Calculus II. Many of your exam questions will ask you to determine whether a given series is convergent or divergent.

Later in this chapter, I show you how to decide whether certain important types of series are convergent or divergent. Chapter 16 also gives you a ton of handy tools for answering this question more generally. For now, just keep this important idea of convergence and divergence in mind.

Getting Comfy with Sigma Notation

Sigma notation is a compact and handy way to represent series.

Okay — that's the official version of the story. What's also true is that sigma notation can be unclear and intimidating — especially when the professor starts scrawling it all over the blackboard at warp speed while explaining some complex proof. Lots of students get left in the chalk dust (or dry-erase marker fumes).

At the same time, sigma notation is useful and important because it provides a concise way to express series and mathematically manipulate them.

In this section, I give you a bunch of handy tips for working with sigma notation. Some of the uses for these tips become clearer as you continue to study series later in this chapter and in Chapters 16 and 17. For now, just add these tools to your toolbox and use them as needed.

Writing sigma notation in expanded form

TIP

When you're working with an unfamiliar series, begin by writing it out using both sigma and expanded notation. This practice is virtually guaranteed to increase your understanding of the series. For example:

$$\sum_{n=1}^{\infty}\frac{2^n}{3n}$$

As it stands, you may not have much insight into what this series looks like, so expand it out:

$$\sum_{n=1}^{\infty}\frac{2^n}{3n}=\frac{2}{3}+\frac{4}{6}+\frac{8}{9}+\frac{16}{12}+\frac{32}{16}+\ldots$$

As you spend a bit of time generating this series, it begins to grow less frightening. For one thing, you may notice that in a race between the numerator and denominator, the numerator starts out less than the denominator but eventually catches up and pulls ahead. Because the terms eventually grow greater than 1, the series explodes to infinity, so it *diverges.*

Seeing more than one way to use sigma notation

Virtually any series expressed in sigma notation can be rewritten in a slightly altered form. For example:

$$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots$$

You can express this series in sigma notation as follows:

$$\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots$$

Alternatively, you can express the same series in any of the following ways:

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n+1} \qquad = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+2} \qquad = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+3}$$

TECHNICAL STUFF

Depending on the problem that you're trying to solve, you may find one of these expressions more advantageous than the others. For example, the comparison tests that I introduce in Chapter 16 often require you to use a specific value of *n* for the starting value of the index. For now, just be sure to keep in mind the flexibility at your disposal when expressing a series in sigma notation.

Discovering the Constant Multiple rule for series

In Chapter 6, you discover that the Constant Multiple rule for integration allows you to simplify an integral by factoring out a constant. This option is also available when you're working with series. Here's the rule:

$$\sum ca_n = c \sum a_n$$

For example:

$$\sum_{n=1}^{\infty} \frac{7}{n^2} = 7 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

To see why this rule works, first expand the series so you can see what you're working with:

$$\sum_{n=1}^{\infty} \frac{7}{n^2} = 7 + \frac{7}{4} + \frac{7}{9} + \frac{7}{16} + \ldots$$

Working with the expanded form, you can factor a 7 from each term:

$$= 7\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\right)$$

Now express the contents of the parentheses in sigma notation:

$$= 7 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

As if by magic, this procedure demonstrates that the two sigma expressions are equal. But this magic is really nothing more exotic than your old friend from grade school, the distributive property.

Examining the Sum rule for series

Here's another handy tool for your growing toolbox of sigma tricks. This rule mirrors the Sum rule for integration (see Chapter 6), which allows you to split a sum inside an integral into the sum of two separate integrals. Similarly, you can break a sum inside a series into the sum of two separate series:

$$\sum (a_n + b_n) = \sum a_n + \sum b_n$$

EXAMPLE

For example:

$$= \sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

A little algebra allows you to split this fraction into two terms:

$$= \sum_{n=1}^{\infty} \left(\frac{n}{2^n} + \frac{1}{2^n} \right)$$

Now the rule allows you to split this result into two series:

$$= \sum_{n=1}^{\infty} \frac{n}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

This sum of two series is equivalent to the series that you started with. As with the Sum rule for integration, expressing a series as a sum of two simpler series tends to make problem-solving easier. Generally speaking, as you proceed onward with series, any trick you can find to simplify a difficult series is a good thing.

Connecting a Series with Its Two Related Sequences

Every series has two related sequences. Recall that the distinction between a sequence and a series is as follows:

- A sequence is a *list* of numbers separated by *commas* (for example: 1, 2, 3, . . .).
- A series is a *sum* of numbers separated by *plus* signs (for example: 1 + 2 + 3 + . . .).

When you see how a series and its two related sequences are distinct but also related, you gain a clearer understanding of how series work.

A series and its defining sequence

The first sequence related to a series is simply the sequence that defines the series in the first place. For example, Table 15-2 shows three series written in both sigma notation and expanded notation, each paired with its defining sequence.

When a sequence $\{a_n\}$ is already defined, you can use the notation $\Sigma\, a_n$ to refer to the related series starting at $n = 1$. For example:

$$\{a_n\} = \left\{\frac{1}{n^2}\right\} \qquad \sum a_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

Understanding the distinction between a series and the sequence that defines it is important for two reasons. First, and most basic, you don't want to get the concepts of sequences and series confused. But second, the sequence that defines a series can provide important information about the series. See Chapter 16 to find out about the *n*th-term test, which provides a connection between a series and its defining sequence.

TABLE 15-2 Series and Their Two Related Sequences

Series	Defining Sequence	Sequence of Partial Sums
$\sum_{n=1}^{\infty} n = 1+2+3+4+5+6\ldots$	$1, 2, 3, 4, 5, 6, \ldots$	$1, 3, 6, 10, 15, 21, \ldots$
$\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$	$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}, \ldots$
$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\ldots$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$	$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots$

A series and its sequences of partial sums

EXAMPLE

You can learn a lot about a series by finding the *partial sums* of its first few terms. For example, here's a series that you've seen before:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

And here are the first four partial sums of this series:

$$S_1 = \sum_{n=1}^{1}\left(\frac{1}{2}\right)^n = \frac{1}{2}$$

$$S_2 = \sum_{n=1}^{2}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \sum_{n=1}^{3}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \sum_{n=1}^{4}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

You can now turn the partial sums for this series into a sequence as follows:

$$\{S_n\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots, \frac{2^n - 1}{2^n}, \ldots\right\}$$

As you can see, the expression $\frac{2^n - 1}{2^n}$ generates this sequence of partial sums. Thus, you can calculate the value where the original series converges by calculating the limit of this sequence:

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty}\left[1 + \left(\frac{1}{2}\right)^n\right] = 1 + \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 1$$

Therefore:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$$

In general, every series $\Sigma\, a_n$ has a related sequence of partial sums $\{S_n\}$. For example, here are a few such pairings:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \qquad \{S_n\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\}$$

$$\sum_{n=1}^{\infty}\frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots \qquad \{S_n\} = \left\{\frac{1}{2}, \frac{7}{6}, \frac{23}{12}, \frac{163}{60}, \ldots\right\}$$

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ldots \qquad \{S_n\} = \left\{1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \ldots\right\}$$

REMEMBER

Every series and its related sequence of partial sums are either *both convergent* or *both divergent.* Moreover, if they're both convergent, both converge to the same number.

This rule should come as no big surprise. After all, a sequence of partial sums simply gives you a running total of where a series is going. Still, this rule can be helpful.

EXAMPLE

For example, suppose that you want to know whether the following sequence is convergent or divergent:

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{147}{60}, \ldots$$

What the heck is this sequence, anyway? Upon deeper examination, however, you discover that it's the sequence of partial sums for a very simple series:

$$1+\frac{1}{2}=\frac{3}{2}$$
$$1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6}$$
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}$$
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{137}{60}$$

This series, called the *harmonic series*, is divergent, so you can conclude that its sequence of partial sums also diverges.

Recognizing Geometric Series and p-Series

At first glance, many series look strange and unfamiliar. But a few big categories of series belong in the Hall of Fame. When you know how to identify these types of series, you have a big head start on discovering whether they're convergent or divergent. In some cases, you can also find out the exact value of a convergent series without spending all eternity adding numbers.

In this section, I show you how to recognize and work with two common types of series: geometric series and p-series.

Getting geometric series

A *geometric series* is any series of the following form:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots$$

EXAMPLE

Here are a few examples of geometric series:

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \ldots$$

$$\sum_{n=0}^{\infty} \frac{1}{10^n} = 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1{,}000} + \ldots$$

$$\sum_{n=0}^{\infty} \frac{3}{100^n} = 3 + \frac{3}{100} + \frac{3}{10{,}000} + \frac{3}{1{,}000{,}000} + \ldots$$

In the first series, $a = 1$ and $r = 2$. In the second, $a = 1$ and $r = \frac{1}{10}$. And in the third, $a = 3$ and $r = \frac{1}{100}$.

If you're unsure whether a series is geometric, you can test it as follows:

1. Let a equal the first term of the series.
2. Let r equal the second term divided by the first term.
3. Check to see whether the series fits the form $a + ar^2 + ar^3 + ar^4 + \ldots$.

EXAMPLE

For example, suppose that you want to find out whether the following series is geometric:

$$\frac{8}{5} + \frac{6}{5} + \frac{9}{10} + \frac{27}{40} + \frac{81}{160} + \frac{243}{640} + \ldots$$

Use the procedure I outline as follows:

1. **Let *a* equal the first term of the series:**

$$a = \frac{8}{5}$$

2. **Let *r* equal the second term divided by the first term:**

$$r = \frac{6}{5} \div \frac{8}{5} = \frac{3}{4}$$

3. **Check to see whether the series fits the form $a + ar^2 + ar^3 + ar^4 + \ldots$:**

$$a = \frac{8}{5} \qquad ar = \frac{8}{5}\left(\frac{3}{4}\right) = \frac{6}{5} \qquad ar^2 = \frac{8}{5}\left(\frac{3}{4}\right)^2 = \frac{9}{10} \qquad ar^3 = \frac{8}{5}\left(\frac{3}{4}\right)^3 = \frac{27}{40}$$

As you can see, this series is geometric. To find the limit of a geometric series $a + ar + ar^2 + ar^3 + \ldots$, use the following formula:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

So the limit of the series in the previous example is:

$$\sum_{n=0}^{\infty}\frac{8}{5}\left(\frac{3}{4}\right)^n=\frac{\frac{8}{5}}{1-\frac{3}{4}}=\frac{8}{5}\cdot\frac{4}{1}=\frac{32}{5}$$

When the limit of a series exists, as in this example, the series is called *convergent.* So you say that this series *converges* to $\frac{32}{5}$. In some cases, however, the limit of a geometric series does not exist (DNE). In that case, the series is *divergent.*

REMEMBER

Here's the complete rule that tells you whether a series is convergent or divergent:

> For any geometric series $a + ar + ar^2 + ar^3 + \ldots$, if r falls in the open set (–1, 1), the series converges to $\frac{a}{1-r}$; otherwise, the series diverges.

EXAMPLE

An example makes clear why this is so. Look at the following geometric series:

$$1+\frac{5}{4}+\frac{25}{16}+\frac{125}{64}+\frac{625}{256}+\ldots$$

In this case, $a = 1$ and $r=\frac{5}{4}$. Because $r > 1$, each term in the series is greater than the term that precedes it, so the series grows at an ever-accelerating rate.

This series illustrates a simple but important rule of thumb for deciding whether a series is convergent or divergent: A series can be convergent only when its related sequence *converges to zero.* I discuss this important idea (called the *nth-term test*) further in Chapter 16.

EXAMPLE

Similarly, look at this example:

$$1+\left(-\frac{5}{4}\right)+\frac{25}{16}+\left(-\frac{125}{64}\right)+\frac{625}{256}+\ldots$$

This time, $a = 1$ and $r=-\frac{5}{4}$. Because $r < -1$, the odd terms grow increasingly positive and the even terms grow increasingly negative. So the related sequence of partial sums alternates wildly from the positive to the negative, with each term further from zero than the preceding term.

A series in which alternating terms are positive and negative is called an *alternating series.* I discuss alternating series in greater detail in Chapter 16.

TIP

Generally speaking, the geometric series is the only type of series that has a simple formula to calculate its value. So when a problem asks for the value of a series, try to put it in the form of a geometric series.

For example, suppose that you're asked to calculate the value of this series:

$$\frac{5}{7}+\frac{10}{21}+\frac{20}{63}+\frac{40}{189}+\ldots$$

The fact that you're being asked to calculate the value of the series should tip you off that it's geometric. Use the procedure I outline earlier to find a and r:

$$a=\frac{5}{7} \qquad r=\frac{10}{21}\div\frac{5}{7}=\frac{2}{3}$$

So here's how to express the series in sigma notation as a geometric series in terms of a and r:

$$\sum_{n=0}^{\infty}\frac{5}{7}\left(\frac{2}{3}\right)^n=\frac{5}{7}+\frac{10}{21}+\frac{20}{63}+\frac{40}{189}+\ldots$$

At this point, you can use the formula for calculating the value of this series:

$$\frac{a}{1-r}=\frac{\frac{5}{7}}{1-\frac{2}{3}}=\frac{5}{7}\cdot\frac{3}{1}=\frac{15}{7}$$

Pinpointing *p*-series

Another important type of series is called the p-series. A *p-series* is any series in the following form, where each successive term is a fraction whose denominator is a counting number raised to a constant power p:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$$

Here's a common example of a *p-series*, when $p = 2$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$$

Here are a few other examples of p-series:

$$\sum_{n=1}^{\infty}\frac{1}{n^5}=1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1{,}024}+\ldots$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{2}+\frac{1}{\sqrt{5}}+\ldots$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{-1}}=1+2+3+4+\ldots$$

WARNING

Don't confuse p-series with geometric series (which I introduce in the previous section). Here's the difference:

- A geometric series has the index n in the exponent — for example, $\sum \left(\frac{1}{2}\right)^n$.
- A p-series has the index n in the base — for example, $\sum \frac{1}{n^2}$.

As with geometric series, a simple rule exists for determining whether a p-series is convergent or divergent.

> A p-series converges when $p > 1$ and diverges when $p \leq 1$.

I give you a proof of this theorem in Chapter 16. In this section, I show you why a few important examples of p-series are either convergent or divergent.

Harmonizing with the harmonic series

When $p = 1$, the p-series takes the following form:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

This p-series is important enough to have its own name: the *harmonic series.* The harmonic series is *divergent.*

Testing p-series when $p = 2$, $p = 3$, and $p = 4$

Here are the p-series when p equals the first few counting numbers greater than 1:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots \qquad \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots$$

Because $p > 1$, these series are all *convergent.*

Testing p-series when $p = \frac{1}{2}$

When $p = \frac{1}{2}$, the p-series looks like this:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}} + \frac{1}{3} + \frac{1}{\sqrt{10}} + \ldots$$

Because $p \leq 1$, this series *diverges.* To see why it diverges, notice that when n is a square number, say $n = k^2$, the nth term equals $\frac{1}{k}$. So this p-series includes every term in the harmonic series — $1, \frac{1}{2}, \frac{1}{3}$ — plus many more terms. Because the harmonic series is divergent, this series is also divergent.

IN THIS CHAPTER

» Understanding convergence and divergence

» Using the *n*th-term test to prove that a series diverges

» Applying the versatile integral test, ratio test, and root test

» Distinguishing absolute convergence and conditional convergence

Chapter 16

Where Is This Going? Testing for Convergence and Divergence

Testing for convergence and divergence is The Main Event in your Calculus II study of series. In Chapter 15, I mention that when a series *converges,* it can be evaluated as a real number. However, when a series *diverges,* it can't be evaluated as a real number, because it either explodes to positive or negative infinity or fails to settle in on a single value.

In Chapter 15, I give you two tests for determining whether specific types of series (geometric series and *p*-series) are convergent or divergent. In this chapter, I give you seven more tests that apply to a much wider range of series.

The first of these is the *n*th-term test, which is sort of a no-brainer. With this test under your belt, I move on to two comparison tests: the direct comparison test and the limit comparison test. These are relatively simple to use, but each hinges on your finding a useful series for comparison (called a benchmark series), which isn't always easy. Next, I show you three more difficult tests: the integral test, the ratio test, and the root test.

Finally, you work with alternating series, in which terms are alternately positive and negative (as I discuss in Chapter 15). I contrast alternating series with positive series (which are the series you're already familiar with), and I show you how to turn a positive series into an alternating series and vice versa. Then I show you how to prove whether an alternating series is convergent or divergent by using the alternating series test. To finish up, I introduce you to the important concepts of absolute convergence and conditional convergence.

Whew. You better get started!

Starting at the Beginning

When testing for convergence or divergence, don't get too hung up on where the series starts. For example:

$$\sum_{n=1,001}^{\infty} \frac{1}{n}$$

This is just a harmonic series with the first 1,000 terms lopped off:

$$= \frac{1}{1,001} + \frac{1}{1,002} + \frac{1}{1,003} + \dots$$

These fractions may look tiny, but the harmonic series diverges (see Chapter 15) and removing a finite number of terms from the beginning of this series doesn't change this fact.

The lesson here is that, when you're testing for convergence or divergence, what's going on at the beginning of the series is irrelevant. Feel free to lop off the first few billion or so terms of a series if it helps you to prove that the series is convergent or divergent.

Similarly, in most cases you can add on a few terms to a series without changing whether it converges or diverges. For example, you can start this series anywhere from $n = 1$ to $n = 999$ without changing the fact that it diverges (because it's a harmonic series). Just be careful, because if you try to start the series from $n = 0$, you're adding the term $\frac{1}{0}$, which is a big no-no. However, in most cases you can extend an infinite series without causing problems or changing the convergence or divergence of the series.

WARNING

Although eliminating terms from the beginning of a series doesn't affect whether the series is convergent or divergent, it *does* affect the sum of a convergent series. For example:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = 2$$

Lopping off the first few terms of this series — say, 1, $\frac{1}{2}$, and $\frac{1}{4}$ — doesn't change the fact that it's convergent. But it does change the value that the series converges to. For example:

$$\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{8}+\frac{1}{16}+\ldots = \frac{1}{4}$$

Using the *n*th-Term Test for Divergence

The *n*th-term test for divergence is the first test that you need to know. It's easy, and it enables you to identify lots of series as divergent. It can be summarized as follows:

> If the limit of sequence $\{a_n\}$ doesn't equal 0, then the series $\Sigma\, a_n$ is *divergent.*

To show you why this test works, I define a sequence that meets the necessary condition — that is, a sequence that doesn't approach 0:

$$\{a_n\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}\ldots$$

Notice that the limit of the sequence is 1 rather than 0. So here's the related series:

$$\sum_{n=1}^{\infty}\frac{n}{n+1} = \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots$$

Because this series is the sum of an infinite number of terms that are very close to 1, it naturally produces an infinite sum, so it's divergent.

WARNING

The fact that the limit of a sequence $\{a_n\}$ equals 0 doesn't necessarily imply that the series $\Sigma\, a_n$ is convergent.

For example, the harmonic sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ approaches 0, but (as I tell you in Chapter 15) the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is divergent.

When testing for convergence or divergence, always perform the *n*th-term test first. It's a simple test, and plenty of teachers include it on exams because it's easy to grade but still catches the unwary student.

REMEMBER

If the defining sequence of a series doesn't approach 0, the series diverges; otherwise, you need to move on to other tests.

Let Me Count the Ways

Tests for convergence or divergence tend to fall into two categories: one-way tests and two-way tests. I explain both in the following sections.

One-way tests

A *one-way test* allows you to draw a conclusion only when a series passes the test, not when it fails. Typically, *passing the test* means that a given *necessary* condition has been met, but that this condition isn't *sufficient* to show that the opposite is true.

As a somewhat silly example that has nothing to do with math, if you're in Dallas, then you're *necessarily* in Texas. But if you're *not* in Dallas, this isn't *sufficient* information to conclude that you're *not* in Texas. (You could be in Houston, or Chicago, or Singapore, so you don't know whether or not you're in Texas.)

The *n*th-term test for divergence is a perfect example of a one-way test: If a series *passes* the test — that is, if the limit of its defining sequence equals something other than 0 — the series is *divergent.* But if the series *fails* the test, you can draw no conclusion — it may be convergent or divergent.

Later in this chapter, you discover two more one-way tests: the direct comparison test and the limit comparison test.

Two-way tests

A *two-way test* allows you to draw one conclusion when a series passes the test and the opposite conclusion when a series fails the test. As with a one-way test, *passing the test* means that a given condition has been met. *Failing the test* means that the negation of that condition has been met.

For example, the test for geometric series is a two-way test. (See Chapter 15 to find out more about testing geometric series for convergence and divergence.) If a series passes the test — that is, if r falls in the open set $(-1, 1)$ — the series is convergent. And if the series fails the test — that is, if $r \leq -1$ or $r \geq 1$ — the series is divergent.

Similarly, the test for p-series is also a two-way test. (See Chapter 15 for more on this test.)

WARNING

Keep in mind that no test — even a two-way test — is *guaranteed* to give you an answer. Think of each test as a tool. If you run into trouble trying to cut a piece of wood with a hammer, it's not the hammer's fault: You just chose the wrong tool for the job. Likewise, if you can't find a clever way to demonstrate either the condition or its negation required by a specific test, you're out of luck. In that case, you may need to use a different test that's better suited to the problem.

Later in this chapter, I show you three more two-way tests: the integral test, the ratio test, and the root test.

Choosing Comparison Tests

Comparison tests allow you to use stuff that you know to find out stuff that you want to know. The stuff that you know is more eloquently called a *benchmark series* — a series whose convergence or divergence you've already proven. The stuff that you want to know is, of course, whether an unfamiliar series converges or diverges.

As with the nth-term test, comparison tests are one-way tests: When a series passes the test, you prove what you've set out to prove (that is, either convergence or divergence). But when a series fails the test, the result of a comparison test in inconclusive.

In this section, I show you two basic comparison tests: the direct comparison test and the limit comparison test.

Getting direct answers with the direct comparison test

You can use the direct comparison test to prove either convergence or divergence, depending on how you set up the test.

To prove that a series converges:

1. Find a benchmark series that you know converges.
2. Show that each term of the series you're testing is less than or equal to the corresponding term of the benchmark series.

To prove that a series diverges:

1. Find a benchmark series that you know diverges.
2. Show that each term of the series you're testing is greater than or equal to the corresponding term of the benchmark series.

EXAMPLE

For example, suppose that you're asked to determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \ldots$$

It's hard to tell just by looking at it whether this particular series is convergent or divergent. However, it looks a bit like a p-series with $p = 2$:

$$\text{Benchmark series: } \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

You know that this p-series converges (see Chapter 15 if you're not sure why), so use it as your benchmark series. Now your task is to show that every term in the series you're testing is less than the corresponding term of the benchmark series:

$$\text{First term: } \frac{1}{2} < 1 \qquad \text{Second term: } \frac{1}{5} < \frac{1}{4} \qquad \text{Third term: } \frac{1}{10} < \frac{1}{9}$$

This looks good, but to complete the proof formally, here's what you want to show:

$$\frac{1}{n^2+1} \leq \frac{1}{n^2}$$

You can cross-multiply to simplify this inequality, because both denominators are greater than 1, and then subtract n^2 from both sides:

$$n^2 \leq n^2 + 1$$
$$0 \leq 1$$

This statement is clearly true, which verifies the original statement, $\frac{1}{n^2+1} \leq \frac{1}{n^2}$. Thus, every term in the series being tested is less than the corresponding term in the convergent benchmark series. Therefore, both series are convergent.

EXAMPLE

As another example, suppose that you want to test the following series for convergence or divergence:

$$\sum_{n=1}^{\infty} \frac{3}{n} = 3 + \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5} + \dots$$

This time, the series reminds you of the trusty harmonic series, which you know is divergent:

$$\text{Benchmark series: } \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

Using the harmonic series as your benchmark, compare the two series term by term:

First term: $3 > 1$ Second term: $\frac{3}{2} > \frac{1}{2}$ Third term: $1 > \frac{1}{3}$

Again, you have reason to be hopeful, but to complete the proof formally, you want to show the following:

$$\frac{3}{n} \geq \frac{1}{n}$$

This time, you can simply multiply both sides by n, which is justified because n is positive:

$$3 \geq 1$$

In this case, you've shown that every term in the test series is greater than the corresponding term in the divergent benchmark series, so both series are divergent.

EXAMPLE

As a third example, suppose that you're asked to show whether this series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$$

In this case, multiplying out the denominators is a helpful first step:

$$= \sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$$

Now the series looks a little like a p-series with $p = 2$, so make this your benchmark series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

The benchmark series converges, so you want to show that every term of the test series is less than the corresponding term of the benchmark. This looks likely because:

$$\text{First term}: \frac{1}{6} < 1 \qquad \text{Second term}: \frac{1}{12} < \frac{1}{4} \qquad \text{Third term}: \frac{1}{20} < \frac{1}{9}$$

However, to convince the professor, you want to show that *every* term of the test series is less than the corresponding term:

$$\frac{1}{n^2 + 3n + 2} \le \frac{1}{n^2}$$

As with the first example in this section, you can cross-multiply because both denominators are greater than 0, and then subtract:

$$\begin{aligned} n^2 &\le n^2 + 3n + 2 \\ 0 &\le 3n + 2 \end{aligned}$$

This verifies the original inequality, so the test series is, indeed, less than the benchmark series, which means that the test series is also convergent.

Testing your limits with the limit comparison test

As with the direct comparison test, the *limit comparison test* works by choosing a benchmark series whose behavior you know and using it to provide information about a test series whose behavior you don't know.

Here's the limit comparison test: Given a test series $\Sigma\, a_n$ and a benchmark series $\Sigma\, b_n$, find the following limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

If this limit evaluates as a positive number, then either both series converge or both diverge.

As with the direct comparison test, when the test succeeds, what you learn depends on what you already know about the benchmark series. If the benchmark series converges, so does the test series. However, if the benchmark series diverges, so does the test series.

Remember, however, that this is a one-way test: If the test fails, you can draw no conclusion about the test series.

The limit comparison test is especially good for testing infinite series based on *rational expressions.*

EXAMPLE

For example, suppose that you want to see whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n-5}{n^2+1}$$

TIP

When testing an infinite series based on a rational expression, choose a benchmark series that's proportionally similar — that is, whose numerator and denominator differ by the same number of degrees.

In this example, the numerator is a first-degree polynomial, and the denominator is a second-degree polynomial. (For more on polynomials, see Chapter 2.) So the denominator is one degree greater than the numerator. Therefore, I choose a benchmark series that's proportionally similar — the trusty harmonic series:

$$\text{Benchmark series: } \sum_{n=1}^{\infty} \frac{1}{n}$$

Before you begin, take a moment to get clear on what you're testing, and jot it down. In this case, you know that the benchmark series diverges. So if the test succeeds, you prove that the test series also diverges. (If it fails, however, you're back to square one because this is a one-way test.)

Now set up the limit (by the way, it doesn't matter which series you put in the numerator and which in the denominator):

$$\lim_{n \to \infty} \frac{\frac{n-5}{n^2+1}}{\frac{1}{n}}$$

At this point, you just crunch the numbers:

$$= \lim_{n \to \infty} \frac{(n-5)n}{n^2+1} = \lim_{n \to \infty} \frac{n^2-5n}{n^2+1}$$

Notice at this point that the numerator and denominator are both second-degree polynomials, so the limit of this rational function is simply the ratio of the two leading coefficients:

$$= \frac{1}{1} = 1$$

As if by magic, the limit evaluates to a positive number, so the test succeeds. Therefore, the test series diverges. Remember, however, that you made this magic happen by choosing a benchmark series in proportion to the test series.

Another example should make this crystal clear. Discover whether this series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{n^3 - 2}{4n^5 - n^3 - 2}$$

When you see that this series is based on a rational expression, you immediately think of the limit comparison test. Because the denominator is two degrees higher than the numerator, choose a benchmark series with the same property:

$$\text{Benchmark series}: \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Before you begin, jot down the following: The benchmark converges, so if the test succeeds, the test series also converges. Next, set up your limit:

$$\lim_{n \to \infty} \frac{\frac{n^3 - 2}{4n^5 - n^3 - 2}}{\frac{1}{n^2}}$$

Now just solve the limit:

$$= \lim_{n \to \infty} \frac{(n^3 - 2)n^2}{4n^5 - n^3 - 2} = \lim_{n \to \infty} \frac{n^5 - 2n^2}{4n^5 - n^3 - 2}$$

Again, the numerator and denominator have the same degree, so the limit of this rational function is the ratio of the two leading coefficients:

$$= \frac{1}{4}$$

The test succeeds, so the test series converges. As you can see, a well-chosen benchmark series can be a very powerful tool for proving convergence or divergence.

Two-Way Tests for Convergence and Divergence

In the previous section, I give you a variety of tests for convergence or divergence that work in one direction at a time. That is, passing the test gives you an answer, but failing it provides no information.

The tests in this section all have one important feature in common: Regardless of whether the series passes or fails, whenever the test gives you an answer, that answer tells you whether the series is convergent or divergent.

Integrating a solution with the integral test

Just when you thought that you wouldn't have to think about integration until two days before your final exam, here it is again. The good news is that the *integral test* gives you a two-way test for convergence or divergence.

TIP

Here's the integral test:

> For any series of the form $\sum_{x=a}^{\infty} f(x)$, consider its associated integral $\int_a^{\infty} f(x)dx$. If this integral converges, the series also converges; however, if this integral diverges, the series also diverges.

In most cases, you use this test to find out whether a series converges or diverges by testing its associated integral. Of course, changing the series to an integral makes all the integration tricks that you already know and love available to you.

For example, here's how to use the integral test to show that the harmonic series is divergent. First, here's your series:

$$\sum_{x=1}^{\infty} \frac{1}{x} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

The integral test tells you that this series converges or diverges depending on whether the following definite integral converges or diverges:

$$\int_1^{\infty} \frac{1}{x} dx$$

To evaluate this improper integral, express it as a limit, as I show you in Chapter 12:

$$= \lim_{c \to \infty} \int_1^{c} \frac{1}{x} dx$$

This is simple to integrate and evaluate:

$$= \lim_{c \to \infty} \left(\ln x \Big|_{x=1}^{x=c} \right)$$
$$= \lim_{c \to \infty} (\ln c - \ln 1)$$
$$= \lim_{c \to \infty} \ln c - 0 = \infty$$

Because the limit explodes to infinity, the integral doesn't exist. Therefore, the integral test verifies that the harmonic series is divergent.

As another example, suppose that you want to discover whether the following series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Notice that this series starts at $n = 2$, because $n = 1$ would produce the term $\frac{1}{0}$.

To use the integral test, transform the sum into this definite integral, using 2 as the lower limit of integration:

$$\int_2^{\infty} \frac{1}{x \ln x} dx$$

Again, rewrite this improper integral as the limit of an integral (see Chapter 12):

$$\lim_{c \to \infty} \int_2^{c} \frac{1}{x \ln x} dx$$

To solve the integral, use the following variable substitution:

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

So you can rewrite the integral as follows:

$$\lim_{c \to \infty} \int_{\ln 2}^{\ln c} \frac{1}{u} du$$

Note that as the variable changes from x to u, the limits of integration change from 2 and c to ln 2 and ln c. This change arises when I plug the value $x = 2$ into the equation $u = \ln x$, so $u = \ln 2$. (For more on using variable substitution to evaluate definite integrals, see Chapter 8.)

At this point, you can evaluate the integral:

$$\lim_{c \to \infty} \left(\ln u \Big|_{u = \ln 2}^{u = \ln c} \right) = \lim_{c \to \infty} [\ln(\ln c) - \ln(\ln 2)] = \infty$$

You can see without much effort that as c approaches infinity, so does ln c, and the rest of the expression doesn't affect this. Therefore, the series that you're testing is divergent.

Rationally solving problems with the ratio test

The *ratio test* is especially good for handling series that include factorials. Recall that the factorial of a counting number, represented by the symbol !, is that number multiplied by every counting number less than itself. For example:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

TIP

Flip to Chapter 2 for some handy tips on factorials that may help you in this section.

To use the ratio test, take the limit (as n approaches ∞) of the $(n + 1)$th term divided by the nth term of the series:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

At the risk of destroying all the trust that you and I have built between us over these pages, I must confess that there are not two, but *three* possible outcomes to the ratio test:

- If this limit is less than 1, the series converges.
- If this limit is greater than 1, the series diverges.
- If this limit equals 1, the test is inconclusive.

But I'm sticking to my guns and calling this a two-way test, because — depending on the outcome — it can potentially prove either convergence or divergence.

For example, suppose that you want to find out whether the following series is convergent or divergent:

$$\sum_{n=1}^{\infty}\frac{2^n}{n!}$$

Before you begin, expand the series so you can get an idea of what you're working with. I do this in two steps to make sure the arithmetic is correct:

$$= \frac{2}{1} + \frac{2\cdot 2}{2\cdot 1} + \frac{2\cdot 2\cdot 2}{3\cdot 2\cdot 1} + \frac{2\cdot 2\cdot 2\cdot 2}{4\cdot 3\cdot 2\cdot 1} + \ldots$$

$$= 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + \ldots$$

To find out whether this series converges or diverges, set up the following limit:

$$\lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$$

For starters, notice that I'm free to omit the absolute value bars because the values here are all positive. As you can see, I place the function that defines the series in the denominator. Then I rewrite this function, substituting $n + 1$ for n, and I place the result in the numerator. Now evaluate the limit:

$$= \lim_{n\to\infty}\frac{\left(2^{n+1}\right)(n!)}{(n+1)!\left(2^n\right)}$$

At this point, to see why the ratio test works so well for exponents and factorials, factor out a 2 from 2^{n+1} and an $n + 1$ from $(n + 1)!$:

$$= \lim_{n\to\infty}\frac{2\left(2^n\right)(n!)}{(n+1)(n!)\left(2^n\right)}$$

This trick allows you to simplify the limit greatly:

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$$

Because the limit is less than 1, the series converges.

Rooting out answers with the root test

The *root test* works best with series that have powers of n in both the numerator and denominator.

To use the root test, take the limit (as n approaches ∞) of the nth root of the nth term of the series:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

As with the ratio test, even though I call this a two-way test, there are really three possible outcomes:

- If the limit is less than 1, the series converges.
- If the limit is greater than 1, the series diverges.
- If the limit equals 1, the test is inconclusive.

For example, suppose that you want to decide whether the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$$

This would be a very hairy problem to try to solve using the ratio test. To use the root test, take the limit of the nth root of the nth term:

$$\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}}$$

Notice that, as often happens with the ratio test (see the previous section), here I omitted the absolute value bars because the terms of the series are all positive. Although this expression looks worse than what you started with, it begins to look better when you separate the numerator and denominator into two roots:

$$= \lim_{n\to\infty} \frac{\sqrt[n]{(\ln n)^n}}{\sqrt[n]{n^n}}$$

Now a lot of cancellation is possible:

$$= \lim_{n\to\infty} \frac{\ln n}{n}$$

Suddenly, the problem doesn't look so bad. The numerator and denominator both approach ∞, so you'd love to apply L'Hôpital's rule. One small problem, however, is that n originated as an index of summation, so it's a discrete variable that only accepts positive integers as inputs. Thus, technically speaking, you can't differentiate functions of n because they aren't continuous. A workaround here is to declare a similar limit using x, a variable whose domain is explicitly the real numbers:

$$\lim_{x\to\infty} \frac{\ln x}{x}$$

I know . . . this limit looks the same as the previous one, except every n is has been replaced by an x. This change, however, permits you to apply L'Hôpital's rule to complete the problem.

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)} = \lim_{x\to\infty} \frac{1}{x} = 0 < 1$$

Because the limit is less than 1, the same is true of the limit in n that you were originally working with; therefore, the series is convergent.

Looking at Alternating Series

Each of the series that I discuss earlier in this chapter (and most of those in Chapter 15) have one thing in common: Every term in the series is positive. So each of these series is a *positive series.* In contrast, a series that has infinitely many positive and infinitely many negative terms is called an *alternating series.*

Most alternating series flip back and forth between positive and negative terms so that every odd-numbered term is positive and every even-numbered term is negative, or vice versa. This feature adds another spin onto the whole question of convergence and divergence. In this section, I show you what you need to know about alternating series.

Eyeballing two forms of the basic alternating series

The most basic alternating series comes in two forms. In the first form, the odd-numbered terms are negated; in the second, the even-numbered terms are negated.

Without further ado, here's the first form of the basic alternating series:

$$\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-\ldots$$

As you can see, in this series the odd terms are all negated. And here's the second form, whose even terms are negated:

$$\sum_{n=1}^{\infty}(-1)^{n-1} = 1-1+1-1+\ldots$$

Obviously, in whichever form it takes, the basic alternating series is divergent because it never converges on a single sum but instead jumps back and forth between two sums for all eternity. Although the functions that produce these basic alternating series aren't of much interest by themselves, they get interesting when they're multiplied by an infinite series.

Making new series from old ones

You can turn any positive series into an alternating series by multiplying the series by $(-1)^n$ or $(-1)^{n-1}$. For example, here's an old friend, the harmonic series:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

To negate the odd terms, multiply by $(-1)^n$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$$

To negate the even terms, multiply by $(-1)^{n-1}$:

$$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Alternating series based on convergent positive series

REMEMBER

If you know that a positive series converges, any alternating series based on this series also converges. This simple rule allows you to list a ton of convergent alternating series. For example:

$$\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{2^n}{n!} = 2 - 2 + \frac{4}{3} - \frac{2}{3} + \frac{4}{15} - \ldots$$

The first series is an alternating version of a geometric series with $r = \frac{1}{2}$. The second is an alternating variation on the familiar p-series with $p = 2$. The third is an alternating series based on a series that I introduce in the section, "Rationally solving problems with the ratio test," earlier in this chapter. In each case, the non-alternating version of the series is convergent, so the alternating series is also convergent.

To see why this works, consider the first of these three series, and calculate the first few partial sums:

$$S_1 = \sum_{n=0}^{1}(-1)^n \left(\frac{1}{2}\right)^n = 1$$

$$S_2 = \sum_{n=0}^{2}(-1)^n \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2} = \frac{1}{2}$$

$$S_3 = \sum_{n=0}^{3}(-1)^n \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_4 = \sum_{n=0}^{4}(-1)^n \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$$

Notice that the partial sums for this series alternately increase and decrease. Additionally, because the terms of the original series approach 0, the partial sums tend to alternate less and less erratically — that is, they hone in on a specific value. You may not know how to calculate this value, but you can still state that such a value exists, so the series is convergent.

As you see in the next section, "Checking out the alternating series test," this is a specific case of a broader test for convergence. For now, just remember that if a positive series converges, the alternating version of this series also converges.

Checking out the alternating series test

As I discuss in the previous section, when you know that a positive series is convergent, you can assume that any alternating series based on that series is also convergent. In contrast, some divergent positive series become convergent when transformed into alternating series.

Fortunately, I can give you a simple test to decide whether an alternating series is convergent or divergent.

REMEMBER

An alternating series converges if these two conditions are met:

- **Its defining sequence converges to zero — that is, it passes the *n*th-term test.**
- **Its terms are non-increasing (ignoring minus signs) — that is, each term is less than or equal to the term before it.**

These conditions are fairly easy to test for, making the alternating series test one of the easiest tests in this chapter. For example, here are three alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{2}+\dots$$

$$\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{n\ln n}=\frac{1}{2\ln 2}-\frac{1}{3\ln 3}+\frac{1}{4\ln 4}-\frac{1}{5\ln 5}+\dots$$

Just by eyeballing them, you can see that each of these series meets both criteria of the alternating series test, so they're all convergent. Notice, too, that in each case, the positive version of the same series is divergent. This underscores an important point: When a positive series is convergent, an alternating series based on it is also necessarily convergent; but when a positive series is divergent, an alternating series based on it may be either convergent or divergent.

Technically speaking, the alternating series test is a one-way test: If the series passes the test — that is, if both conditions hold — the series is convergent. However, if the series fails the test — that is, if either condition isn't met — you can draw no conclusion.

In practice, however — and I'm going out on a thin mathematical limb here — I'd say that when a series fails the alternating series test, you have strong circumstantial evidence that the series is divergent.

Why do I say this? First of all, notice that the first condition is the good old-fashioned *n*th-term test. If any series fails this test, you can just chuck it on the divergent pile and get on with the rest of your day.

Second, it's rare when a series — *any series* — meets the first condition but fails to meet the second condition. Sure, it happens, but you really have to hunt around to find a series like that. And even when you find one, the series usually settles down into an ever-decreasing pattern fairly quickly.

For example, take a look at the following alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{2^n}=\frac{1}{2}-1+\frac{9}{8}-1+\frac{25}{32}-\frac{9}{16}+\frac{49}{128}-\ldots$$

Clearly, this series passes the first condition of the alternating series test — the *n*th-term test — because the denominator explodes to infinity at a much faster rate than the numerator.

What about the second condition? Well, the first three terms are increasing (disregarding sign), but beyond these terms the series settles into an ever-decreasing pattern. So you can chop off the first few terms and express the same series in a slightly different way:

$$=\frac{1}{2}-1+\frac{9}{8}-1+\sum_{n=5}^{\infty}(-1)^{n-1}\frac{n^2}{2^n}$$

This version of the series is non-increasing, so now it passes the two conditions of the alternating series test with flying colors; therefore it's convergent. Obviously, adding a few constants to this series doesn't make it divergent, so the original series is also convergent.

So when you're testing an alternating series, here's what you do:

- **Test for the first condition — that is, apply the *n*th-term test.**

  If the series fails, it's divergent by the *n*th-term test, so you're done.

» **If the series passes the *n*th-term test, test for the second condition — that is, see whether its terms *eventually* settle into a constantly decreasing pattern (ignoring their sign, of course).**

In most cases, you'll find that a series that meets the first condition also meets the second, which means that the series is convergent.

In the rare cases when an alternating series meets the first condition of the alternating series test but doesn't meet the second condition, you can draw no conclusion about whether that series converges or diverges.

WARNING

These cases really are rare, but I show you one so you know what to do in case your professor decides to get cute on an exam:

$$-\frac{1}{10}+\frac{1}{9}-\frac{1}{100}+\frac{1}{99}-\frac{1}{1{,}000}+\frac{1}{999}-\ldots \qquad -\frac{1}{10}+\frac{1}{2}-\frac{1}{100}+\frac{1}{3}-\frac{1}{1{,}000}+\frac{1}{4}-\ldots$$

Both of these series meet the first criteria of the alternating series test but fail to meet the second, so you can draw no conclusion based on this test. In fact, the first series is convergent and the second is divergent. Spend a little time studying them and I believe that you'll see why. (***Hint:*** Try to break each series apart into two separate series.)

Understanding absolute and conditional convergence

In the previous two sections, I demonstrate this important fact: When a positive series is convergent, an alternating series based on it is also necessarily convergent; but when a positive series is divergent, an alternating series based on it may be either convergent or divergent.

So for any alternating series, you have three possibilities:

- An alternating series is convergent, and the positive version of that series is also convergent.
- An alternating series is convergent, but the positive version of that series is divergent.
- An alternating series is divergent, so the positive version of that series must also be divergent.

The existence of three possibilities for alternating series makes a new concept necessary: the distinction between *absolute convergence* and *conditional convergence.*

Table 16-1 tells you when an alternating series is absolutely convergent, conditionally convergent, or divergent.

TABLE 16-1 Understanding Absolute and Conditional Convergence of Alternating Series

An Alternating Series Is:	When That Series Is:	And Its Related Positive Series Is:
Absolutely Convergent	Convergent	Convergent
Conditionally Convergent	Convergent	Divergent
Divergent	Divergent	Divergent

Here are a few examples of alternating series that are absolutely convergent:

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n!} = 2-2+\frac{4}{3}-\frac{2}{3}+\frac{4}{15}-\ldots$$

I pulled these three examples from the section, "Alternating series based on convergent positive series," earlier in this chapter. In each case, the positive version of the series is convergent, so the related alternating series must be convergent as well. Taken together, these two facts mean that each series converges absolutely.

And here are a few examples of alternating series that are conditionally convergent:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}} = 1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{2}+\ldots$$

$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln n} = \frac{1}{2\ln 2}-\frac{1}{3\ln 3}+\frac{1}{4\ln 4}-\frac{1}{5\ln 5}+\ldots$$

I pulled these examples from the section, "Checking out the alternating series test," earlier in this chapter. In each case, the positive version of the series diverges, but the alternating series converges (by the alternating series test). So each of these series converges conditionally.

Finally, here are some examples of alternating series that are divergent:

$$\sum_{n=1}^{\infty}(-1)^{n-1}n = 1-2+3-4+\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1} = \frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\ldots$$

$$-\frac{1}{10}+\frac{1}{2}-\frac{1}{100}+\frac{1}{3}-\frac{1}{1{,}000}+\frac{1}{4}-\ldots$$

As you can see, the first two series fail the *n*th-term test, which is also the first condition of the alternating series test, so these two series diverge. As for the third series, it's basically a divergent harmonic series *minus* a convergent geometric series — that is, a divergent series with a finite number subtracted from it — so the entire series diverges.

Testing alternating series

Suppose that somebody (like your professor) hands you an alternating series that you've never seen before and asks you to determine whether it's absolutely convergent, conditionally convergent, or divergent. Here's what you do:

1. **Apply the alternating series test.**

   In most cases, this test tells you whether the alternating series is convergent or divergent:

   - If it's divergent, you're done! (The alternating series is divergent.)
   - If it's convergent, the series is either absolutely convergent or conditionally convergent. Proceed to Step 2.
   - If the alternating series test is inconclusive, you can't rule any option out. Proceed to Step 2.

2. **Rewrite the alternating series as a positive series by:**

   - Removing $(-1)^n$ or $(-1)^{n-1}$ when you're working with sigma notation.
   - Changing the minus signs to plus signs when you're working with expanded notation.

3. **Test this positive series for convergence or divergence by using any of the tests in this chapter or Chapter 15:**

   - If the positive series is convergent, the alternating series is absolutely convergent.
   - If the positive series is divergent *and* the alternating series is convergent, the alternating series is conditionally convergent.
   - If the positive series is divergent *but* the alternating series test is inconclusive, the series is either conditionally convergent or divergent, but you still can't tell which.

In most cases, you're not going to get through all these steps and still have a doubt about the series. In the unlikely event that you do find yourself in this position, see whether you can break the alternating series into two separate series — one with positive terms and the other with negative terms — and study these two series for whatever clues you can.

IN THIS CHAPTER

» Understanding elementary functions

» Seeing power series as polynomials with infinitely many terms

» Expressing functions as a Maclaurin series

» Discovering the Taylor series as a generalization of the Maclaurin series

» Approximating expressions with the Taylor and Maclaurin series

Chapter 17

Dressing Up Functions with the Taylor Series

The infinite series known as the *Taylor series* is one of the most brilliant mathematical achievements you'll ever come across. It's also quite a lot to get your head around. Although many calculus books tend to throw you in the deep end with the Taylor series, I prefer to take you by the hand and help you wade in slowly.

The Taylor series is a specific form of the power series. In turn, it's helpful to think of a power series as a polynomial with an infinite number of terms. So, in this chapter, I begin with a discussion of polynomials. I contrast polynomials with other elementary functions, pointing out a few reasons mathematicians like polynomials so much (often to the exclusion of their families and friends).

Then I move on to power series, showing you how to discover when a power series converges or diverges. I also discuss the interval of convergence for a power series, which is the set of *x* values for which that series converges. After that, I introduce you to the Maclaurin series — a simplified, but powerful, version of the Taylor series.

Finally, the main event: the Taylor series. First, I show you how to use the Taylor series to evaluate other functions; you'll most likely need that for your final exam. I introduce you to the Taylor remainder term, which allows you to find the margin of error when making an approximation. To finish up the chapter, I show you why the Taylor series works, which helps to make sense of the series, but may not be strictly necessary for passing an exam.

Elementary Functions

In Chapter 6, I discuss elementary functions, which are those familiar functions that you work with all the time in calculus. You discover that every elementary function is infinitely differentiable — that is, its derivative is an elementary function that is also differentiable.

In this section, I discuss some of the difficulties of working with elementary functions. In contrast, I show you why a small subset of elementary functions — the polynomials — is much easier to work with. To finish, I consider the advantages of expressing elementary functions as polynomials when possible.

Identifying two drawbacks of elementary functions

Differentiating elementary functions tends to be relatively simple, and always produces a result that is also an elementary function. Unfortunately, integration is another matter. For example, here's an integral that can't be evaluated as an elementary function (the proof of this fact is pretty hairy and depends on something called Risch's algorithm, so I hope you'll take my word on it):

$$\int e^{x^2} dx$$

So even though the set of elementary functions is large and complex enough to confuse most math students, for you — the emerging calculus guru — it's a rather small pool.

Another problem with elementary functions is that many of them are difficult to evaluate for a given value of x. Even the simple function $\sin x$ isn't so simple to evaluate because (except for 0) every integer input value results in an irrational output for the function. For example, what's the value of $\sin 3$?

Appreciating why polynomials are so friendly

In contrast to other elementary functions, polynomials are just about the friendliest functions around. Here are just a few reasons:

- Polynomials are easy to integrate (see Chapter 6 to see how to compute the integral of every polynomial).
- Polynomials are easy to evaluate for any value of *x*.
- Polynomials are infinitely differentiable — that is, you can calculate the value of the first derivative, second derivative, third derivative, and so on, infinitely.

Representing elementary functions as series

The tactic of expressing complicated functions as polynomials (and other simple functions) motivates much of the study of infinite series.

Although series may seem difficult to work with — and, admittedly, they do pose their own specific set of challenges — they have two great advantages that make them useful for integration:

- **An infinite series breaks easily into terms.** So in most cases, you can use the Sum rule to break a series into separate terms and evaluate these terms individually.
- **Series tend to be built from a recognizable pattern.** So if you can figure out how to integrate one term, you can usually generalize this method to integrate every term in the series.

Specifically, power series include many of the features that make polynomials easy to work with. I discuss power series in the next section.

Power Series: Polynomials on Steroids

In Chapter 15, I introduce the geometric series:

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \ldots$$

I also show you a simple formula to figure out whether the geometric series converges or diverges.

The geometric series is a simplified form of a larger set of series called the *power series.*

REMEMBER

A *power series* is any series of the following form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

Notice how the power series differs from the geometric series:

- In a geometric series, every term has the same coefficient.
- In a power series, the coefficients may be different — usually according to a rule that's specified in the sigma notation.

Here are a few examples of power series:

$$\sum_{n=0}^{\infty} n x^n = x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} x^n = \frac{1}{4} + \frac{1}{8}x + \frac{1}{16}x^2 + \frac{1}{32}x^3 + \ldots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$$

You can think of a power series as a polynomial with an infinite number of terms. For this reason, many useful features of polynomials (which I describe earlier in this chapter) carry over to power series.

REMEMBER

The most general form of the power series is as follows:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

This form is for a power series that's centered at a. Notice that when $a = 0$, this form collapses to the simpler version I introduce earlier in this section. So a power series in this form is centered at 0.

Integrating power series

In Chapter 6, I show you a three-step process for integrating polynomials. Because power series resemble polynomials, they're simple to integrate using the same basic process:

1. Use the Sum rule to integrate the series term by term.
2. Use the Constant Multiple rule to move each coefficient outside its respective integral.
3. Use the Power rule to evaluate each integral.

For example, take a look at the following integral:

$$\int \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} x^n dx$$

At first glance, this integral of a series may look scary. But to give it a chance to show its softer side, I expand the series out as follows:

$$= \int \left(\frac{1}{4} + \frac{1}{8}x + \frac{1}{16}x^2 + \frac{1}{32}x^3 + \ldots \right) dx$$

Now you can apply the three steps for integrating polynomials to evaluate this integral.

1. **Use the Sum rule to integrate the series term by term:**

$$= \int \frac{1}{4} dx + \int \frac{1}{8} x\, dx + \int \frac{1}{16} x^2 dx + \int \frac{1}{32} x^3 dx + \ldots$$

2. **Use the Constant Multiple rule to move each coefficient outside its respective integral:**

$$= \frac{1}{4} \int dx + \frac{1}{8} \int x\, dx + \frac{1}{16} \int x^2 dx + \frac{1}{32} \int x^3 dx + \ldots$$

3. **Use the Power rule to evaluate each integral:**

$$= \frac{1}{4}x + \frac{1}{16}x^2 + \frac{1}{48}x^3 + \frac{1}{128}x^4 + \ldots$$

Notice that this result is another power series, which you can turn back into sigma notation:

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)2^{n+2}} x^{n+1}$$

Thus, you can evaluate the integral of one power series as a second power series:

$$\int \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} x^n dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)2^{n+2}} x^{n+1}$$

Stand back a minute and admire that you're actually able to make sense of this morass of mathematical symbols. While you're at it, also notice that you've integrated a power series — a polynomial with infinitely many terms — and produced another power series.

This is an example of how the set of power series is closed under the operation of integration - that is, every power series integrates to another power series. And this closure is very important because, if you can count on the integral of a power series behaving in a predetermined way, you can use it in a variety of abstract and even creative ways without worrying too much about the details.

Understanding the interval of convergence

As with geometric series and *p*-series (which I discuss in Chapter 15), an advantage to power series is that they converge or diverge according to a well-understood pattern.

Unlike these simpler series, however, a power series often converges or diverges based on its *x* value. This leads to a new concept when dealing with power series: the interval of convergence.

The *interval of convergence* for a power series is the set of *x* values for which that series converges.

The interval of convergence is never empty

REMEMBER

Every power series converges for some value of *x*. That is, the interval of convergence for a power series is never the empty set.

Although this fact has useful implications, it's actually pretty much a no-brainer. For example, take a look at the following power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

When $x = 0$, this series evaluates to $1 + 0 + 0 + 0 + \ldots$, so it obviously converges to 1. Similarly, take a peek at this power series:

$$\sum_{n=0}^{\infty} n(x+5)^n = (x+5) + 2(x+5)^2 + 3(x+5)^3 + 4(x+5)^4 + \ldots$$

This time, when $x = -5$, the series converges to 0, just as trivially as the last example — that is, it converges, but not in a way that's particularly surprising or helpful.

Note that in both of these examples, the series converges trivially at $x = a$ for a power series centered at a. (See the beginning of the section, "Power Series: Polynomials on Steroids.") As you can see, every series of this type has to converge for at least one value of x. Series that converge for only one value of x, however, aren't particularly useful or interesting.

Three varieties for the interval of convergence

REMEMBER

Three possibilities exist for the interval of convergence of any power series:

- The series converges only when $x = a$.
- The series converges on some interval (open or closed at either end) centered at a.
- The series converges for all real values of x.

EXAMPLE

For example, suppose that you want to find the interval of convergence for:

$$\sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

This power series is centered at 0, so it converges trivially when $x = 0$. Using the ratio test (see Chapter 16), you can find out whether it converges for any other values of x. To start out, set up the following limit:

$$\lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right|$$

To evaluate this limit, start out by canceling x^n in the numerator and denominator:

$$= \lim_{n\to\infty}\frac{(n+1)|x|}{n}$$

This step simplifies the expression considerably. Furthermore, remember that n is always positive, so the absolute value pertains only to x. Fortunately, this value can be pulled outside the limit, and then the limit that remains equals 1:

$$= |x|\lim_{n\to\infty}\frac{(n+1)}{n} = |x|\cdot 1 = |x|$$

From this result, the ratio test tells you that the series:

- Converges when $-1 < x < 1$
- Diverges when $x < -1$ and $x > 1$
- May converge or diverge when $x = 1$ and $x = -1$

Fortunately, it's easy to see what happens in these two remaining cases. Here's what the series looks like when $x = 1$:

$$\sum_{n=0}^{\infty} n(1)^n = 1 + 2 + 3 + 4 + \ldots$$

Clearly, the series diverges. Similarly, here's what it looks like when $x = -1$:

$$\sum_{n=0}^{\infty} n(-1)^n = -1 + 2 - 3 + 4 - \ldots$$

This alternating series swings wildly between negative and positive values, so it also diverges.

EXAMPLE

As a final example, suppose that you want to find the interval of convergence for the following series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

As in the last example, this series is centered at 0, so it converges trivially when $x = 0$. The real question is whether it converges for other values of x. Because this is an alternating series, I apply the ratio test to the positive version of it to see whether I can show that it's absolutely convergent:

$$\lim_{n \to \infty} \left| \frac{\frac{x^{2(n+1)}}{(2(n+1))!}}{\frac{x^{2n}}{(2n)!}} \right|$$

First off, I want to simplify this a bit:

$$= \lim_{n \to \infty} \left| \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

Next, I expand out the exponents and factorials, as I show you in Chapter 16:

$$= \lim_{n \to \infty} \left| \frac{x^{2n} x^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

At this point, a lot of canceling is possible, enabling me to evaluate the limit:

$$= \lim_{n \to \infty} \frac{|x^2|}{(2n+2)(2n+1)} = |x^2| \lim_{n \to \infty} \frac{1}{(2n+2)(2n+1)}$$

At last, the x-term, which requires the absolute value bars, is isolated in the numerator. This allows you to move it outside the limit, making the limit itself very simple to evaluate:

$$= \left| x^2 \right| \cdot 0 = 0$$

This time, the limit is 0, which falls between −1 and 1 for all values of x. This result tells you that the series converges absolutely for all values of x, so the alternating series also converges for all values of x.

Expressing Functions as Series

In this section, you begin to explore how to express functions as infinite series. I begin by showing some examples of formulas that express sin x and cos x as series. These examples lead to a more general formula for expressing a wider variety of elementary functions as series.

This formula is the Maclaurin series, a simplified but powerful version of the more general Taylor series, which I introduce later in this chapter.

Expressing sin *x* as a series

Here's an odd formula that expresses the sine function as an alternating series:

$$\sin x = \sum_{x=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To make sense of this formula, use expanded notation:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Notice that this is a power series (which I discuss earlier in this chapter). To get a quick sense of how it works, here's how you can find the value of sin 0 by substituting 0 for x:

$$\sin 0 = 0 - \frac{0^3}{3!} + \frac{0^5}{5!} - \frac{0^7}{7!} + \ldots = 0$$

As you can see, the formula verifies what you already know: sin 0 = 0.

You can use this formula to approximate sin *x* for any value of *x* to as many decimal places as you like. For example, look what happens when you substitute 1 for *x* in the first four terms of the formula:

$$\begin{aligned}\sin 1 &\approx 1-\frac{1}{6}+\frac{1}{120}-\frac{1}{5{,}040}\\ &\approx 0.841468\end{aligned}$$

Note that the actual value of sin 1 to six decimal places is 0.841471, so this estimate is correct to five decimal places — not bad! And, of course, this estimate could be improved by generating more terms of the series, plugging in 1 for *x*, and crunching the numbers.

Table 17-1 shows the value of sin 3 approximated to six terms. Note that the actual value of sin 3 is approximately 0.14112, so the six-term approximation is correct to three decimal places. Again, not bad. Though, this one wasn't quite as good as the estimate for sin 1.

TABLE 17-1 Approximating the Value of sin 3

# of Terms	Substitution	Approximation
1	3	3
2	$3-\frac{3^3}{3!}$	–1.5
3	$3-\frac{3^3}{3!}+\frac{3^5}{5!}$	0.525
4	$3-\frac{3^3}{3!}+\frac{3^5}{5!}-\frac{3^7}{7!}$	0.09107
5	$3-\frac{3^3}{3!}+\frac{3^5}{5!}-\frac{3^7}{7!}+\frac{3^9}{9!}$	0.14531
6	$3-\frac{3^3}{3!}+\frac{3^5}{5!}-\frac{3^7}{7!}+\frac{3^9}{9!}-\frac{3^{11}}{11!}$	0.14087

As a final example, Table 17-2 shows the value of sin 10 approximated out to eight terms. The true value of sin 10 is approximately −0.54402, so by any standard this is a poor estimate. Nevertheless, if you continue to generate terms, this estimate continues to get better and better, to any level of precision you like. If you doubt this, notice that after five terms, after swinging around wildly, the approximations are beginning to settle in closer to the actual value, even though they're still way off. Each successive approximation from now on will land closer and closer to −0.54402, to any level of precision you'd like.

TABLE 17-2 Approximating the Value of sin 10

# of Terms	Substitution	Approximation
1	10	10
2	$10-\frac{10^3}{3!}$	–156.66667
3	$10-\frac{10^3}{3!}+\frac{10^5}{5!}$	676.66667
4	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}$	–1307.460317
5	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}$	1448.272
6	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}-\frac{10^{11}}{11!}$	–1056.938
7	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}-\frac{10^{11}}{11!}+\frac{10^{13}}{13!}$	548.966
8	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}-\frac{10^{11}}{11!}+\frac{10^{13}}{13!}-\frac{10^{15}}{15!}$	–215.750

Expressing cos *x* as a series

In the previous section, I show you a formula that expresses the value of sin *x* for all values of *x* as an infinite series. Differentiating both sides of this formula leads to a similar formula for cos *x:*

$$\frac{d}{dx}\sin x=\frac{d}{dx}x-\frac{d}{dx}\frac{x^3}{3!}+\frac{d}{dx}\frac{x^5}{5!}-\frac{d}{dx}\frac{x^7}{7!}+\ldots$$

Now evaluate these derivatives:

$$\cos x=1-3\frac{x^2}{3!}+5\frac{x^4}{5!}-7\frac{x^6}{7!}+\ldots$$

Finally, simplify the result a bit:

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

As you can see, the result is another power series (which I discuss earlier in this chapter). Here's how you write it with sigma notation:

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

To gain some confidence that this series really works as advertised, note that the substitution $x = 0$ provides the correct equation $\cos 0 = 1$. Furthermore, substituting $x = 1$ into the first four terms gives you the following approximation:

$$\cos 1 \approx 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = 0.5402\overline{777}$$

This estimate is accurate to four decimal places.

Introducing the Maclaurin Series

In the previous two sections, I show you formulas for expressing both $\sin x$ and $\cos x$ as infinite series. You may begin to suspect that there's some sort of method behind these formulas. Without further ado, here it is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Behold the *Maclaurin series*, a simplified version of the much-heralded Taylor series, which I introduce in the next section.

The notation $f^{(n)}$ means "the nth derivative of f." This should become clearer in the expanded version of the Maclaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

The Maclaurin series is the template for the two formulas I introduce earlier in this chapter. It allows you to express many other functions as power series by following these steps:

1. Find the first few derivatives of the function until you recognize a pattern.
2. Substitute 0 for x into each of these derivatives.
3. Plug these values, term by term, into the formula for the Maclaurin series.
4. If possible, express the series in sigma notation.

For example, suppose that you want to find the Maclaurin series for e^x.

1. **Find the first few derivatives of e^x until you recognize a pattern:**

   $f'(x) = e^x$

   $f''(x) = e^x$

$f'''(x) = e^x$

$\ldots$

$f^{(n)}(x) = e^x$

2. Substitute 0 for *x* into each of these derivatives.

$f'(0) = e^0$

$f''(0) = e^0$

$f'''(x) = e^0$

$\ldots$

$f^{(n)}(x) = e^0$

3. Plug these values, term by term, into the formula for the Maclaurin series:

$$e^x = e^0 + e^0 x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

4. If possible, express the series in sigma notation:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To check this formula, use it to estimate e^0 and e^1 by substituting 0 and 1, respectively, into the first six terms:

$$e^0 = 1 + 0 + 0 + 0 + 0 + 0 + \ldots = 1$$
$$e^1 \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.71\overline{66}$$

This exercise nails e^0 exactly and approximates e^1 to two decimal places. And, as with the formulas for sin *x* and cos *x* that I show you earlier in this chapter, the Maclaurin series for e^x allows you to calculate this function for any value of *x* to any number of decimal places.

As with the other formulas, however, the Maclaurin series for e^x works best when *x* is close to 0. As *x* moves away from 0, you need to calculate more terms to get the same level of precision.

But now, you can begin to see why the Maclaurin series tends to provide better approximations for values close to 0: The number 0 is "hardwired" into the formula as $f(0)$, $f'(0)$, $f''(0)$, and so forth.

Figure 17-1 illustrates this point. The first graph shows sin *x* approximated by using the first two terms of the Maclaurin series — that is, as the third-degree polynomial $x-\frac{x^3}{3!}$. The subsequent graph shows an approximation of sin *x* with eight terms as a 15th-degree polynomial.

As you can see, the second approximation greatly improves on the previous one. Furthermore, each equation tends to provide its best approximation when *x* is close to 0.

A TALE OF THREE SERIES

It's easy to get confused about the three categories of series that I discuss in this chapter. Here's a helpful way to think about them:

- The *power series* is a subcategory of infinite series.
- The *Taylor series* (named for mathematician Brook Taylor) is a subcategory of power series.
- The *Maclaurin series* (named for mathematician Colin Maclaurin) is a subcategory of the Taylor series.

After you have that down, consider that the power series has two basic forms:

- The *specific form*, which is centered at zero, so *a* drops out of the expression.
- The *general form*, which isn't centered at zero, so *a* is part of the expression.

Furthermore, each of the other two series uses one of these two forms of the power series:

- The Maclaurin series uses the specific form, so it's less powerful and simpler to work with.
- The Taylor series uses the general form, so it's more powerful and harder to work with.

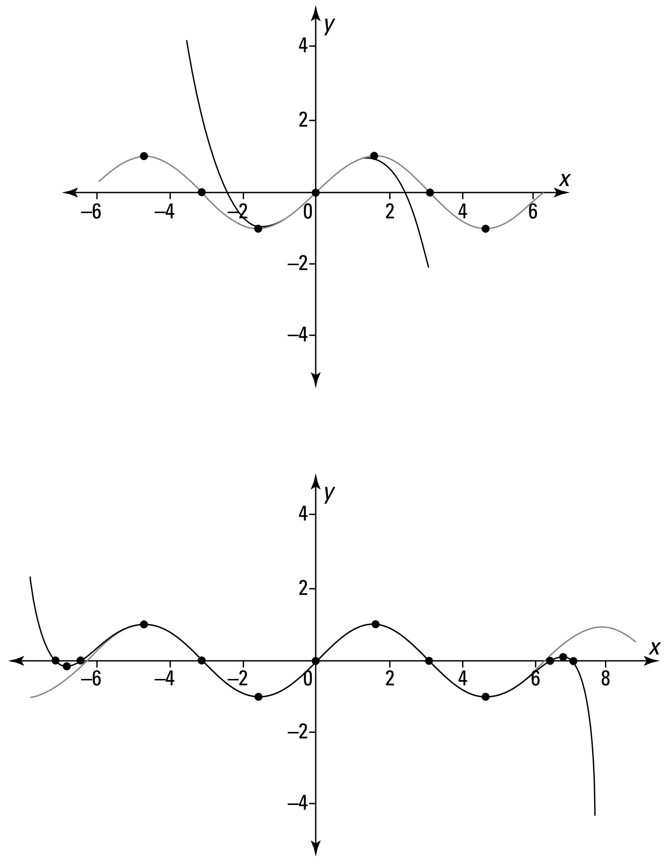

FIGURE 17-1: Approximating sin *x* by using the Maclaurin series.

Introducing the Taylor Series

Like the Maclaurin series (which I introduce in the previous section), the Taylor series provides a template for representing a wide variety of functions as power series.

In fact, the Taylor series is really a more general version of the Maclaurin series. The advantage of the Maclaurin series is that it's a bit simpler to work with. The advantage to the Taylor series is that you can tailor it to obtain a better approximation of many functions.

Here's the Taylor series in all its glory:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with the Maclaurin series, the Taylor series uses the notation $f^{(n)}$ to indicate the *n*th derivative. Here's the expanded version of the Taylor series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Notice that the Taylor series includes the variable *a*, which isn't found in the Maclaurin series. Or, more precisely, in the Maclaurin series, $a = 0$, so it drops out of the expression.

The explanation for this variable can be found earlier in this chapter, in the section, "Power Series: Polynomials on Steroids." In that section, I show you two forms of the power series:

- A simpler form centered at 0, which corresponds to the Maclaurin series
- A more general form centered at *a,* which corresponds to the Taylor series

In the next section, I show you the advantages of working with this extra variable.

Computing with the Taylor series

The presence of the variable *a* makes the Taylor series more complex to work with than the Maclaurin series. But this variable provides the Taylor series with greater flexibility, as the next example illustrates.

In the section, "Expressing Functions as Series," earlier in this chapter, I attempt to approximate the value of sin 10 with the Maclaurin series. Unfortunately, taking this calculation out to eight terms still results in a poor estimate. This problem occurs because the Maclaurin series always takes a default value of $a = 0$, and 0 isn't close enough to 10.

This time, I use only four terms of the Taylor series to make a much better approximation. The key to this approximation is a shrewd choice for the variable *a:*

Let $a = 3\pi$

This choice has two advantages: First, this value of *a* is close to 10 (the value of *x*), which makes for a better approximation. Second, it's an easy value for calculating sines and cosines, so the computation shouldn't be too difficult.

To start off, substitute 10 for *x* and 3π for *a* in the first four terms of the Taylor series:

$$\sin 10 \approx \sin 3\pi + \left(\sin' 3\pi\right)(10-3\pi) + \frac{\left(\sin'' 3\pi\right)(10-3\pi)^2}{2!} + \frac{\left(\sin''' 3\pi\right)(10-3\pi)^3}{3!}$$

Next, substitute in the first, second, and third derivatives of the sine function and simplify:

$$= \sin 3\pi + (\cos 3\pi)(0.5752) - \frac{(\sin 3\pi)(0.5752)^2}{2!} - \frac{(\cos 3\pi)(0.5752)^3}{3!}$$

The good news is that sin $3\pi = 0$, so the first and third terms fall out:

$$= (\cos 3\pi)(0.5752) - \frac{(\cos 3\pi)(0.5752)^3}{3!}$$

At this point, you probably want to grab your calculator:

$$= -1(0.5752) - \left[-\frac{1}{6}(0.5752)^3 \right]$$
$$= -0.5752 + 0.0317 = -0.5434$$

This approximation is correct to two decimal places — quite an improvement over the estimate from the Maclaurin series!

Examining convergent and divergent Taylor series

Earlier in this chapter, I show you how to find the interval of convergence for a power series — that is, the set of x values for which that series converges.

REMEMBER

Because the Taylor series is a form of power series, you shouldn't be surprised that every Taylor series also has an interval of convergence. When this interval is the entire set of real numbers, you can use the series to find the value of $f(x)$ for every real value of x.

WARNING

However, when the interval of convergence for a Taylor series is bounded — that is, when it diverges for some values of x — you can use it to find the value of $f(x)$ *only* on its interval of convergence.

For example, here are the three important Maclaurin series I've introduced so far in this chapter:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

All three of these series converge for all real values of *x* (you can check this by using the ratio test, as I show you earlier in this chapter), so each equals the value of its respective function.

Now consider the following function:

$$f(x)=\frac{1}{1-x}$$

I express this function as a Maclaurin series, using the steps that I outline earlier in this chapter in the section, "Expressing Functions as Series."

1. **Find the first few derivatives of $f(x)=\frac{1}{1-x}$ until you recognize a pattern:**

$$f'(x)=\frac{1}{(1-x)^2}$$
$$f''(x)=\frac{2}{(1-x)^3}$$
$$f'''(x)=\frac{6}{(1-x)^4}$$
$$\ldots$$
$$f^{(n)}(x)=\frac{n!}{(1-x)^{n+1}}$$

2. **Substitute 0 for *x* into each of these derivatives:**

$f'(0) = 1$

$f''(0) = 2$

$f'''(0) = 6$

. . .

$f^{(n)}(0) = n!$

3. **Plug these values, term by term, into the formula for the Maclaurin series:**

$$\frac{1}{1-x}=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$
$$=1+x+x^2+x^3+\ldots$$

4. **If possible, express the series in sigma notation:**

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3$$

To test this formula, I use it to find $f(x)$ when $x = \frac{1}{2}$.

$$f\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

You can test the accuracy of this expression by substituting $\frac{1}{2}$into $\frac{1}{1} - x$:

$$f\left(\frac{1}{2}\right) = \frac{1}{1 - \frac{1}{2}} = 2$$

As you can see, the formula produces the correct answer. Now I try to use it to find $f(x)$ when $x = 5$, noting that the correct answer should be $\frac{1}{1} - 5 = -\frac{1}{4}$:

$$f(5) = 1 + 5 + 25 + 125 + \ldots = \infty \qquad \text{WRONG!}$$

What happened? This series converges only on the interval $(-1, 1)$, so the formula produces only the value $f(x)$ when x is in this interval. When x is outside this interval, the series diverges, so the formula is invalid.

Expressing functions versus approximating functions

It's important to be crystal clear in your understanding about the difference between two key mathematical practices:

- *Expressing* a function as an infinite series
- *Approximating* a function by using a finite number of terms of series

Both the Taylor series and the Maclaurin series are variations of the power series. You can think of a power series as a polynomial with infinitely many terms. Also, recall that the Maclaurin series is a specific form of the more general Taylor series, arising when the value of a is set to 0.

Every Taylor series (and, therefore, every Maclaurin series) provides the exact value of a function for all values of x where that series converges. That is, for any value of x on its interval of convergence, a Taylor series converges to $f(x)$.

In practice, however, adding up an infinite number of terms simply isn't possible. Nevertheless, you can approximate the value of $f(x)$ by adding a finite number from the appropriate Taylor series. You do this earlier in the chapter to estimate the value of sin 10 and other expressions.

An expression built from a finite number of terms of a Taylor series is called a *Taylor polynomial*, $T_n(x)$. Like other polynomials, a Taylor polynomial is identified by its degree. For example, here's the fifth-degree Taylor polynomial, $T_5(x)$, that approximates e^x:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

Generally speaking, a higher-degree polynomial results in a better approximation. And because this polynomial comes from the Maclaurin series, where $a = 0$, it provides a much better estimate for values of e^x when x is near 0. For the value of e^x when x is near 100, however, you get a better estimate by using a Taylor polynomial for e^x with $a = 100$:

$$e^x \approx e^{100} + e^{100}(x-100) + \frac{e^{100}}{2!}(x-100)^2 + \frac{e^{100}}{3!}(x-100)^3 + \frac{e^{100}}{4!}(x-100)^4 + \frac{e^{100}}{5!}(x-100)^5$$

To sum up, remember the following:

- A convergent Taylor series expresses the exact value of a function.
- A Taylor polynomial, $T_n(x)$, from a convergent series approximates the value of a function.

Understanding Why the Taylor Series Works

The best way to see why the Taylor series works is to see how it's constructed in the first place. If you've read through this chapter until this point, you should be ready to go.

To make sure that you understand every step along the way, however, I construct the Maclaurin series, which is just a tad more straightforward. This construction begins with the key assumption that a function can be expressed as a power series in the first place:

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

The goal now is to express the coefficients on the right side of this equation in terms of the function itself. To do this, I make another relatively safe assumption that 0 is in the domain of $f(x)$. So when $x = 0$, all but the first term of the series equal 0, leaving the following equation:

$$f(0) = c_0$$

This process gives you the value of the coefficient c_0 in terms of the function. Now differentiate $f(x)$:

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 \ldots$$

At this point, when $x = 0$, all the x terms drop out:

$$f'(0) = c_1$$

So you have another coefficient, c_1, expressed in terms of the function. To continue, differentiate $f'(x)$:

$$f''(x) = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots$$

Again, when $x = 0$, the x terms disappear:

$$f''(0) = 2c_2$$
$$\frac{f''(0)}{2} = c_2$$

By now, you're probably noticing a pattern: You can always get the value of the next coefficient by differentiating the previous equation and substituting 0 for x into the result:

$$f'''(x) = 6c_3 + 24c_4x + 60c_5x^2 + 120c_6x^3 + \ldots$$
$$f'''(0) = 6c_3$$
$$\frac{f'''(0)}{6} = c_3$$

Furthermore, the coefficients also have a pattern:

$$c_0 = f(0)$$
$$c_1 = f'(0)$$
$$c_2 = \frac{f''(0)}{2!}$$
$$c_3 = \frac{f'''(0)}{3!}$$
$$\ldots$$
$$c_n = \frac{f^{(n)}(0)}{n!}$$

Substituting these coefficients into the original equation results in the familiar Maclaurin series from earlier in this chapter:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

TECHNICAL STUFF

To construct the Taylor series, use a similar line of reasoning, starting with the more general form of the power series:

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \ldots$$

In this case, setting $x = a$ gives you the first coefficient:

$$f(a) = c_0$$

Continue to find coefficients by differentiating $f(x)$ and then repeating the process.

7 The Part of Tens

IN THIS PART . . .

Review the big-picture Calculus II topics

Remember that integration is just fancy addition

Start a test by breathing, reading, and answering the easiest question first

Find your way out of a stuck place when taking a test

IN THIS CHAPTER

» Understanding the key concepts of integration

» Distinguishing the definite integral from the indefinite integral

» Knowing the basics of infinite series

Chapter 18
Ten "Aha!" Insights in Calculus II

Okay, here you are near the end of the book. You've read every single word that I wrote, memorized the key formulas, and worked through all the problems. You're all set to ace your final exam, and you've earned it. Good for you! (Or maybe you just picked up the book and skipped to the end. That's fine, too! This is a great place to get an overview of what this Calculus II stuff is all about.)

But still, you have this sneaking suspicion that you're stuck in the middle of the forest and can't see it because of all those darn trees. Forget the equations for a moment and spend five minutes looking over these top ten "Aha!" insights. When you understand them, you will have a solid conceptual framework for Calculus II.

Integrating Means Finding the Area

Finding the area of a polygon or circle is easy. Integration is all about finding the area of shapes with weird edges that are hard to work with. These edges may be the curves that result from polynomials, exponents, logarithms, trig functions, or inverse trig functions, or the products and compositions of these functions.

Integration gives you a concrete way to look at this question, known as the area problem. No matter how complicated integration gets, you can always understand what you're working on in terms of this simple question: "How does what I'm doing help me find an area?"

See Chapter 1 for more about the relationship between integration and area.

When You Integrate, Area Means Signed Area

In the real world, area is always positive. For example, there's no such thing as a piece of land that's −4 square miles in area. This (real world) concept of area is called unsigned area.

But on the *xy*-graph in the context of integration, area is measured as signed area, with area below the *x*-axis considered to be negative area.

In this context, a 2-x-2-unit square below the *x*-axis is considered to be −4 square units in signed area. Similarly, a 2-x-2-unit square that's divided in half by the *x*-axis is considered to have an area of 0.

The definite integral always produces the signed area between a curve and the *x*-axis, within the limits of integration. So if an application calls for the unsigned area, you need to measure the positive area and negative area separately, change the sign of the negative area, and add these two results together.

See Chapter 5 for more about signed area.

Integrating Is Just Fancy Addition

To measure the area of an irregularly shaped polygon, a good first step is to cut it into smaller shapes that you know how to measure — for example, triangles and rectangles — and then add up the areas of these shapes.

Integration works on the same principle. It allows you to slice a shape into smaller shapes that approximate the area that you're trying to measure, and then add up the pieces. In fact, the integral sign itself is simply an elongated *S*, which stands for *sum.*

When integrating, that sum is called a *Riemann sum*. But please remember that at the end of the day, a Riemann sum is just a fancy way to estimate area by adding up the areas of a bunch of rectangles.

See Chapters 1 and 4 for more about how Riemann sums connect integration with addition.

Integration Uses Infinitely Many Infinitely Thin Slices

Here's where integration differs from other methods of measuring area: Integration allows you to slice an area into infinitely many pieces, all of which are infinitely thin, and then add up these pieces to find the total area.

Or, to put a slightly more mathematical spin on it: The definite integral is the limit of the total area of all these slices as the number of slices approaches infinity and the thickness of each slice approaches 0.

That is, when you calculate a definite integral, you set up a Riemann sum to approximate area and then apply a limit to it so that this approximation turns into an exact measurement.

This concept is also useful when you're trying to find volume, as I show you in Chapter 13.

See Chapter 4 for more about how this concept of infinite slicing relates to integration.

Integration Contains a Slack Factor

To paraphrase science fiction writer Robert Heinlein, math is a harsh mistress. A small error at the beginning of a problem often leads to a big mistake by the end.

So finding out that you can thin-slice an area in a bunch of different ways and still get the correct answer is refreshing. Some of these methods for thin-slicing include left rectangles, right rectangles, and midpoint rectangles. I cover them all in Chapter 4.

This *slack factor,* as I call it, comes about because integration exploits an infinite sequence of successive approximations. Each approximation brings you closer to the answer that you're seeking. So no matter what route you take to get there, an infinite number of such approximations brings you to the answer.

A Definite Integral Evaluates to a Number

A definite integral represents the well-defined area of a shape on an xy-graph. You can represent any such area as a number of square units, so the definite integral is a number.

See Chapters 1 and 4 for more about the definite integral.

An Indefinite Integral Evaluates to a Function

An indefinite integral is a template that allows you to calculate infinitely many related definite integrals by plugging in some numbers. In math, such a template is called a *function.*

The input values to an indefinite integral are the two limits of integration. Specifying these two values turns the indefinite integral into a definite integral, which then outputs a number representing an area.

But if you don't specify the limits of integration, you can still evaluate an indefinite integral as a function. The process of finding an indefinite integral turns an input function (for example, $\cos x$) into an output function ($\sin x + C$).

See Chapter 5 for more about the indefinite integral and Part 3 for a variety of techniques for evaluating indefinite integrals.

Integration Is Inverse Differentiation

Integration and differentiation are inverse operations: Either of these operations undoes the other (up to a constant C). Another way to say this is that integration is *anti-differentiation.*

Here's an example of how differentiation undoes integration:

$$\int 5x^3 dx = \frac{5}{4}x^4 + C$$
$$\frac{d}{dx}\left(\frac{5}{4}x^4 + C\right) = 5x^3$$

As you can see, integrating a function and then differentiating the result produces the function that you started with.

Now here's an example of how integration undoes differentiation:

$$\frac{d}{dx}\sin x = \cos x$$
$$\int \cos x \, dx = \sin x + C$$

As you can see, differentiating a function and then integrating the result produces the function that you started with, plus a constant C.

See Parts 2 and 3 for more on how this inverse relationship between integration and differentiation provides a variety of clever methods for integrating complicated functions.

Every Infinite Series Has Two Related Sequences

Every infinite series has two related sequences that are important for understanding how that series works: its defining sequence and its sequence of partial sums.

The *defining sequence* of a series is simply the sequence that defines the series in the first place. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

has the defining sequence

$$\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$$

Notice that the same function — in this case, $\frac{1}{n}$ — appears in the shorter notation for both the series and its defining sequence.

The *sequence of partial sums* of a series is the sequence that results when you successively add a finite number of terms. For example, the previous series has the following sequence of partial sums:

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$$

Notice that a series may diverge while its defining sequence converges, as in this example. However, a series and its sequence of partial sums always converge or diverge together. In fact, the definition of convergence for a series is based on the behavior of its sequence of partial sums (see the next section for more on convergence and divergence).

See Part 6 for more about infinite series.

Every Infinite Series Either Converges or Diverges

Every infinite series either converges or diverges, with no exceptions.

A series *converges* when it evaluates to (equals) a real number. For example:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = 1$$

On the other hand, a series *diverges* when it doesn't evaluate to a real number. Divergence can happen in two different ways. The more common type of divergence is when the series explodes to ∞ or $-\infty$. For example:

$$\sum_{n=1}^{\infty} n = 1+2+3+4+\ldots$$

Clearly, this series doesn't add up to a real number — it just keeps getting bigger and bigger forever.

Another type of divergence occurs when a series bounces forever among two or more values. This happens only when a series is *alternating* (see Chapter 16 for more on alternating series). For example:

$$\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-\ldots$$

The sequence of partial sums (see the previous section) for this series alternates forever between -1 and 0, never settling into a single value, so the series diverges.

Convergent series are especially helpful when working with integrals we don't know how to evaluate with algebraic methods, such as u-substitution and integration by parts. Taylor/Maclaurin series are essentially polynomial approximations of functions that, like all polynomials, are simple to integrate.

See Part 6 for more about infinite series.

IN THIS CHAPTER

» Doing a memory dump and a quick read-through before you begin

» Getting unstuck

» Checking for mistakes

Chapter 19

Ten Tips to Take to the Test

I've never met anyone who loved taking a math test. The pressure is on, the time is short, and that formula that you can't quite remember is out of reach. Unfortunately, exams are a part of every student's life. Here are my top ten suggestions to make test-taking just a little bit easier.

Breathe

A lot of what you may feel when facing a test — for example, butterflies in your stomach, sweaty palms, or trembling — is simply a physical reaction to stress that's caused by adrenalin. Your body is preparing you for a fight-or-flight response, but with a test, you have nothing to fight and nowhere to fly.

A few slow, deep breaths can help calm you down. If you like, picture serenity and deep knowledge of all things mathematical entering your body on the in-breath, and all the bad stuff exiting on the out-breath.

Start by Doing a Memory Dump as You Read through the Exam

When you receive your exam, do a memory dump: Write down all the formulas you're worried you may forget before you begin the test.

Next, take a minute to read through it so you know what you're up against. This practice starts your brain working (consciously or not) on the problems. As you do this, write down any other formulas that may occur to you.

Solve the Easiest Problem First

After the initial read-through, turn to the page with the easiest problem and solve it. This warm-up gets your brain working and usually reduces your anxiety.

Don't Forget to Write *dx* and + *C*

Remember to include those pesky little *dx*s in every integration statement. They need to be there, and some professors take it very personally when you don't include them. You have absolutely no reason to lose points over something so trivial.

And don't forget that the solution to every *indefinite* integral ends with $+ C$ (or whatever constant you choose). No exceptions! As with the *dx*s, omitting this constant can cost you points on an exam, so get in the habit of including it.

Take the Easy Way Out Whenever Possible

In Chapters 6 through 11, I introduce a bunch of integration techniques. These run the gamut from the laughably simple plug-and-play anti-differentiation formulas to the infuriatingly complicated integration with partial fractions. Before you jump in to a calculation, take a moment to walk through all the methods you know, from easiest to hardest.

Always check first to see whether you know a simple formula: For example, $\int \frac{1}{x\sqrt{x^2-1}}\,dx$ may cause you to panic until you remember that the answer is simply arcsec $x + C$. (See Chapter 6 for a list of easy-to-use anti-differentiation formulas.)

If no direct formula exists, think through whether the function you're trying to integrate includes a linear input to something you know how to integrate. For example, $\int \sec^2 \pi x\,dx$ looks difficult, but it evaluates easily as $\frac{1}{\pi}\tan \pi x + C$. (See Chapter 7 to see how this method works.)

Next, ask whether a simple variable substitution is possible. For example, $\int \frac{x^2}{x^3+2}\,dx$ yields nicely to $u = x^3 + 2$, giving a result of $\frac{1}{3}\ln\left(x^3+2\right)+C$. (See Chapter 8 to discover how to do this.)

If all of these methods fail, try integration by parts (Chapter 9). Your last resorts are always trig substitution (Chapter 10) and integration with partial fractions (Chapter 11).

When you're working on solving area problems, stay open to the possibility that calculus may not be necessary. For example, you don't need calculus to find the area under a straight line or semicircle. So before you start integrating, step back for a moment to see whether you can spot an easier way to find the answer.

If You Get Stuck, Scribble

When you look at a problem and you just don't know which way to go, grab a piece of scratch paper and scribble everything you can think of, without trying to make sense of it.

Use algebra, trig identities, and variable substitutions of all kinds. Write series in both sigma notation and expanded notation. Draw pictures and graphs. Write it all down, even the ideas that seem worthless.

You may find that this process jogs your brain. Even copying the problem — equations, graphs, and all — onto some scratch paper can sometimes help you to notice something important that you missed in your first reading of the question.

If You Really Get Stuck, Move On

I see no sense in beating your head against a brick wall, unless you like getting brick dust in your hair. Likewise, I see no sense in spending the whole exam frozen in front of one problem.

So after you scribble and scribble some more (see the previous section) and you're still getting nowhere with a problem, move on. You may as well make the most of the time you're given by solving the problems that you can solve. What's more, many problems seem easier on the second try. And working on other areas of the test may remind you of some important information that you'd forgotten.

Check Your Answers

Toward the end of the test, especially if you're stuck, take a moment to check some of the problems that you've already completed. Does what you've written still make sense? If you see any missing *dx*s or + *C*s, fill them in. Make sure you didn't drop any minus signs. Most important, do a reality check of your answer compared with the original problem to see whether it makes sense.

For example, suppose that you're integrating to find an area someplace inside a 2 x 2 region on a graph, and your answer is 7 trillion. Obviously, something went wrong. If you have time to find out what happened, trace back over your steps.

Although fixing a problem on an exam can be tedious, it usually takes less time than starting (and maybe not finishing) a brand-new problem from scratch.

If an Answer Doesn't Make Sense, Acknowledge It

Suppose you're integrating to find an area someplace inside a 2 x 2 region on a graph, and your answer is 7 trillion. Obviously, something went wrong. If you have time, try to find out what went wrong and fix the problem (see the preceding section). However, if you *don't* have time to find out what happened, write a note to the professor acknowledging the problem.

Writing such a note lets your professor know that your conceptual understanding of the problem is okay — that is, you get the idea that integration means area. So if it turns out that your calculation got messed up because of a minor mistake like a misplaced decimal point, you'll probably lose only a couple of points for the error.

Repeat the Mantra, "I'm Doing My Best," and Then Do Your Best

All you can do is your best, and even the best math student occasionally forgets a formula or stares at an exam question and goes "Huh?"

When these moments arrive, gently remind yourself, "I'm doing my best." And then do your best with what you have. Perfection is not of this world, but if you can cut yourself a bit of slack when you're under pressure, you'll probably end up doing better than you would've otherwise.

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About the Author

Mark Zegarelli is the author of *Basic Math and Pre-Algebra For Dummies* (Wiley), *SAT Math For Dummies* (Wiley), and eight other books in the *For Dummies* series. He holds degrees in both English and math from Rutgers University. Most recently, he provides online lessons for kids as young as 4 years old, showing them easy ways to understand — and enjoy! — multiplication and division, square numbers and square roots, negative numbers, factors, prime numbers, fractions, and even basic algebra concepts.

Dedication

For my brilliant and beautiful sister, Tami. You are still an inspiration!

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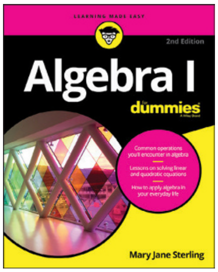

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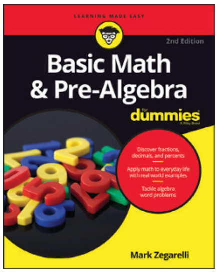

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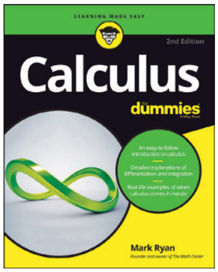

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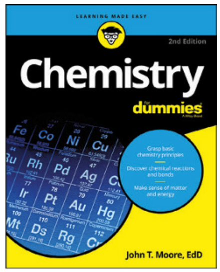

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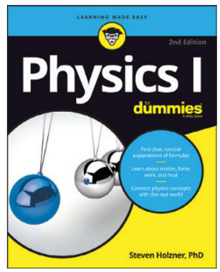

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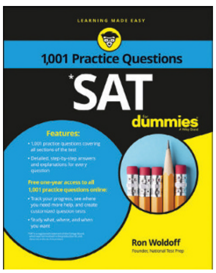

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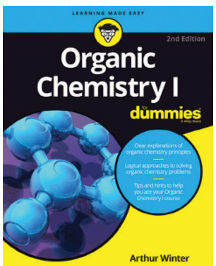

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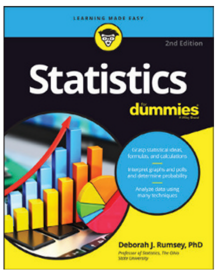

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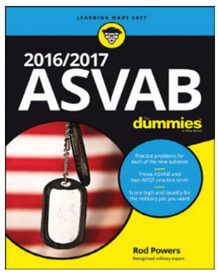

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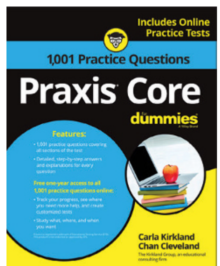

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